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Logarithmic corrections to the mean-field theory of tricritical points

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The logarithmic corrections to the mean-field theory of symmetrical tricritical points in three dimensions are derived using a graphical method. The free energy, equation of state, and other thermodynamic quantities are obtained in the disordered and ordered phases. The difference in thermodynamic potential between the ordered and disordered states takes the form $G = \mu^{3/2}L^{1/2}(r) \times g(Q\mu^{-1/2}L^{1/2-p}(r), \zeta\mu^{-5/4}L^{-1/4}(r))$, where the fields μ and Q are measured normal and tangential to the critical line, respectively, ζ is the field which couples to the order parameter, r is the inverse susceptibility, and $L(r) \sim \ln r$. The exponent p depends on the number of components of the order parameter. This form for the free energy differs from that found by Wegner and Riedel.

I. INTRODUCTION

The phenomenological theory for thermodynamic systems exhibiting tricritical points was first given by Landau.[1] This theory was extended by Griffiths[2] who showed that a tricritical point occurs at the intersection of three lines of second-order transitions. In this paper we will be concerned with symmetrical tricritical points such as are found in ^{3}He-^{4}He mixtures or in certain magnetic materials when there are competing antiferromagnetic and ferromagnetic interactions. At such tricritical points positive and negative values of the order parameter are equivalent. A spin model appropriate to ^{3}He-^{4}He mixtures of such tricritical points has been given by Blume, Emery, and Griffiths.[3] Nelson and Fisher[4] have discussed the case of metamagnets and shown that the reduced Hamiltonian in this case has the same form as in the Landau phenomenological theory. Unsymmetrical tricritical points can occur in mixtures of three or more substances and a phenomenological theory of such tricritical points has recently been presented by Griffiths.[5] A recent review of the phenomenological theories of tricritical and higher-order critical points has been given by Widom.[6] A third type of tricritical point apparently occurs in the Potts model (Straley and Fisher[7]; Mittag and Stephen[8]).

A scaling theory of tricritical behavior has been developed by Riedel[9] and Riedel and Wegner.[10,11] One important observation of these authors is that the mean-field theory of tricritical points is modified by logarithmic corrections in three dimensions. Similar logarithmic corrections occur in four dimensions in the mean-field theory of the Ising model and in three dimensions for a uniaxial ferroelectric. In the latter two cases, the logarithmic corrections were obtained by Larkin and Khmelnitski[12] using graphical methods. The logarithmic corrections to the tricritical behavior in three dimensions and to the Ising model in four dimensions have been discussed by Wegner and Riedel[10] using the approximate recursion relations of Wilson.[13] In the four-dimensional Ising model the logarithmic factors arise from those graphs with two internal lines, the so-called parquet graphs. In the case of the tricritical point in three dimensions the logarithmic factors arise from those graphs with three internal lines [see Fig. 2(a)]. The approximate recursion relations do not treat diagrams with an odd number of internal lines correctly.[14] A discussion of this point has been given by Wilson and Kogut.[15] It is thus desirable to derive the logarithmic corrections to the mean-field theory of tricritical points by another method and in this paper we use a graphical method[16] similar to that of Larkin and Khmelnitski.[12] We obtain the equation of state, free energy, and other thermodynamic properties in the ordered and disordered states. Our results differ in several respects from those of Wegner and Riedel.[11] The methods of this paper can easily be extended to determine the logarithmic corrections to the mean-field theory of unsymmetrical tricritical points in three dimensions and to the mean-field theory of the Potts model in four dimensions. This will be given elsewhere.

The discussion of symmetrical tricritical points is based on a reduced Hamiltonian

$$\frac{H}{kT}=\int d^3x\left[\frac{1}{2}\sum_i\{r_0 s_i^2(x)+[\nabla s_i(x)]^2\} + \frac{u_4}{4!}\left(\sum_i s_i^2(x)\right)^2+\frac{u_6}{6!}\left(\sum_i s_i^2(x)\right)^3\right], \qquad (1)$$

where s_i is an n-component spin ($-\infty < s_i < \infty$). The

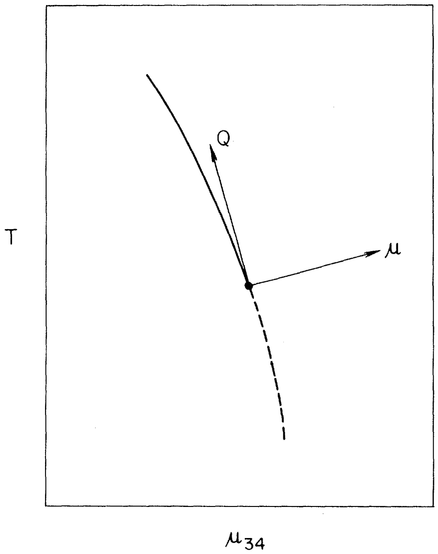

FIG. 1. Critical (solid) and coexistence (dashed) lines near the tricritical point. Q and μ are the mean-field scaling variables and are, respectively, tangential and perpendicular to the critical line at the tricritical point.

parameters r_0, u_4, and u_6 in (1) depend on the thermodynamic fields in the problem. We will use use a language appropriate to ^{3}He-^{4}He mixtures in which the thermodynamic fields are temperature T, the difference of the chemical potentials $\mu_{34} = \mu_3 - \mu_4$, of the ^{3}He and ^{4}He atoms and a field ζ which couples to the order parameter. It is more convenient to measure quantities from the tricritical point T_0, μ_{340}, and $\zeta = 0$ and to introduce variables $t = T/T_0 - 1$ and $\Delta = \mu_{34}/\mu_{340} - 1$. In the case of metamagnets μ_{34} and ζ are replaced by H, the internal field, and H_s, the staggered field, respectively. We will assume that the parameters r_0 and μ_4 in (1) vary linearly with t and Δ to a first approximation. The experimental thermodynamic fields t and Δ are not the scaling fields and Riedel[8] has shown that a convenient choice of scaling fields is obtained by introducing new variables Q and μ which are measured tangentially and normally to the critical line at the tricritical point (see Fig. 1). These variables will appear naturally in our theory. Actually, μ can be chosen in any direction oblique to Q. In perturbation theory μ and Q are given to a first approximation by

$$\begin{aligned} \mu &= r_0 + \frac{n+2}{6} u_4 F_1(0) + \frac{(n+2)(n+4)}{5!} u_6 F_1^2(0) , \\ Q &= u_4 + \tfrac{1}{10}(n+4) u_6 F_1(0) , \end{aligned} \tag{2}$$

where $F_1(0)$ is defined by Eq. (5) below and is proportional to a cutoff k_c, which is introduced into all momentum integrals. The terms in F_1 in (2) are the Hartree contributions to r_0 and u_4, respectively. In mean-field theory (including the Hartree terms), the tricritical point is the point $\mu = Q = 0$. The critical line is the line $\mu = 0$, $Q > 0$. Thus, to a first approximation, Q is tangential and μ is normal to the critical line at the tricritical point. This same variable Q was introduced by Stephen and McCauley[17] who derived the ϵ expansion in 3-ϵ dimensions for the tricritical exponents and by Amit and de Dominicis[18] who considered the $1/n$ expansion.

The region of validity of the present theory is best discussed after we have derived the results. For the analysis of the terms in the perturbation theory we will assume

$$u_6 < 1 , \tag{3}$$

$$Q^2/r < 1 , \tag{4}$$

where r is the inverse susceptibility. We will see that the actual expansion is not in terms of u_6 but in terms of the physical interaction Γ_6 which becomes small near the tricritical point. It can be expected that the nature of the solution does not change essentially in going over from small to large values of u_6. The magnitude of u_6 then determines the size of the critical region. Equation (4) defines the small-Q region according to the mean-field theory (the fluctuations lead to a logarithmic correction to this condition, as we shall see below). The inverse susceptibility r vanishes at the tricritical point and along the critical line. From (4) it follows that the present theory is not valid close to the critical line and the tricritical point cannot be approached along curves which violate (4).

In this paper various thermodynamic quantities of interest are conveniently expressed in terms of the quantities μ, Q, r, and u_6. In each case the corrections to the mean-field results of all orders in $u_6|\ln r|$ are determined. The inverse susceptibility r is also determined in terms of the fields μ and Q which then enables us to express all quantities in terms of μ and Q.

It is convenient to define certain integrals involving the Green's function $G(k,r) = (k^2 + r)^{-1}$,[19]

$$F_1(r) = \frac{1}{(2\pi)^3} \int d^3k \, G(k,r) , \tag{5}$$

$$F_n(r) = \frac{1}{(2\pi)^{3n-3}} \int d^3k_1 \cdots d^3k_n \, G(k_1, r) \cdots G(k_n, r) \, \delta(\vec{k}_1 + \vec{k}_2 \cdots \vec{k}_n) . \tag{6}$$

All momentum integrals are cut off in some way for $k > k_c$. Some integrals of particular interest are

$$\begin{aligned} &F_1(r) \simeq F_1(0) - r^{1/2}/4\pi, \quad F_2(r) \simeq 1/8\pi r^{1/2} , \\ &F_3(r) \simeq -(1/32\pi^2) \ln r . \end{aligned} \tag{7}$$

II. DISORDERED PHASE

A. Graphical analysis

We begin by discussing the six-point vertex Γ_6 when the momenta in all the channels are zero. In perturbation theory, a typical graph contributing a logarithmic factor of order $u_6^2 \ln r$ to Γ_6 is shown in Fig. 2(a). It is those graphs which break into two pieces when three internal lines are cut which contribute the leading logarithmic factors in each order of perturbation theory. These graphs make up a generalized set of parquet graphs. We calculate the one-particle irreducible graphs of Γ_6, Γ_4, etc., as only these are required in the equation of state. Therefore the graph of Fig. 2(b) may be omitted (actually it is easy to find once Γ_4 is known). Furthermore, it can be shown that those higher-order graphs which contain one particle reducible parts of Γ_6 [such as in Fig. 2(d) where the right-hand vertex of Fig. 2(a) is replaced by the one-particle reducible graph Fig. 2(b)] give lower order contributions as a result of (3) and (4). The graph of Fig. 2(c) is of order $u_6 Q/r^{1/2}$ and from (4) is smaller by a factor $|\ln r|^{-1/2}$ than that of Fig. 2(a). It should be noted that in perturbation theory each four-point vertex contributes a factor

$$u_4 + \tfrac{1}{10}(n+4)u_6 F_1(r) \ .$$

There are ten channels of Γ_6 for the transition $3 \leftrightarrow 3$ [Fig. 2(a)]. For each of these channels we can write a Bethe-Salpeter (BS) equation. An example is given in Fig. 2(e), where the kernel $I^{(1)}$ is irreducible in channel 1. A differential equation for $\Gamma_6^{(1)}$ is then obtained by differentiating, in turn, every internal triple line in the BS equation with respect to r. This procedure gives the leading contribution to Γ_6. If we sum the results of this differentiation we get a full Γ_6 on each side of the triple line which has been differentiated. In general, these factors of Γ_6 will depend on the momentum of the internal lines. It can be shown, however, that this momentum dependence can be ignored in determining the leading logarithmic corrections to Γ_6. This procedure has been discussed in more detail by Tsuneto and Abrahams[20] in the case of the Ising model. We sum over the spin components and obtain the equation for channel 1,

$$\Gamma_6^{(1)\prime} = -\tfrac{1}{150}(3n+22)\,\Gamma_6 F_3' \Gamma_6 \ ,$$

where a prime indicates differentiation with respect to r. The numerical factor $\frac{1}{150}(3n+22)$ reduces to $(3!)^{-1}$ for $n=1$, where 3 is the number of internal lines. For $n>1$, the factor $(3!)^{-1}$ is increased by $(3/5^2)(n-1)+1$, where 3 is the number of ways of choosing 2 internal lines in Fig. 2(a) differently from the external lines and $\frac{1}{5}$ is the reduction in the number of ways of assigning the lines at each vertex. These factors are thus determined by the topology of the diagrams and other such factors which appear below may be obtained in a similar manner.

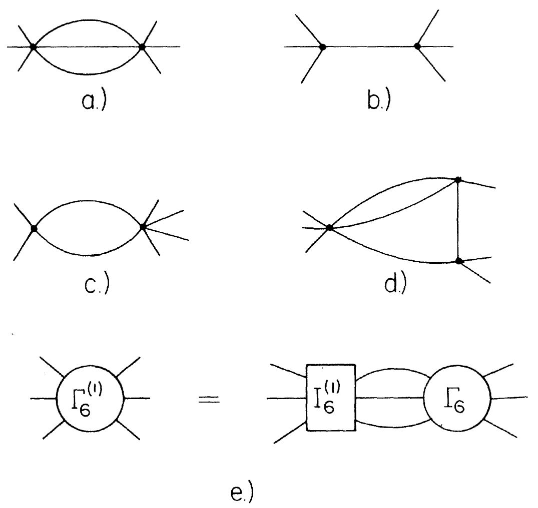

FIG. 2. Graphs for the six-point vertex Γ_6.

The equation for Γ_6 is obtained by adding the contributions from all of the ten channels:

$$\Gamma_6' = -\tfrac{1}{15}(3n+22)\Gamma_6^2 F_3' \ . \tag{8}$$

We integrate this equation with the boundary condition that $\Gamma_6 = u_6$, the bare interaction, when $r=1$,

$$\Gamma_6 = u_6/L(r) \ , \tag{9a}$$

$$L(r) = 1 - [(3n+22)/480\pi^2]u_6 \ln r \ . \tag{9b}$$

For sufficiently small r, i.e., close to the tricritical point, $\Gamma_6 \sim |\ln r|^{-1}$, and is small and independent of u_6.

For the four-point vertex Γ_4, the leading logarithmic factors again come from graphs with three internal lines. Γ_4 is of order Q and a graph of order $Qu_6 \ln r$ is shown in Fig. 3(a). The graphs of Figs. 3(b) and 3(c) are of order $Q^2/r^{1/2}$ and $u_6^2 r^{1/2} \ln r$, respectively, and can be omitted. The differential equation for Γ_4 is obtained as above by differentiating the internal triple lines and adding the contribution from the four channels of Fig. 3(a),

$$\Gamma_4' = -\tfrac{2}{15}(n+4)\Gamma_4 F_3' \Gamma_6 \ , \tag{10}$$

The boundary condition for Γ_4 is that $\Gamma_4 = Q$, its bare or mean-field value, when $r=1$ [see Eq. (2)]. Then using (9) we find

$$\Gamma_4 = Q/L^p(r) \ , \tag{11}$$

where

$$p = 2(n+4)/(3n+22) \ .$$

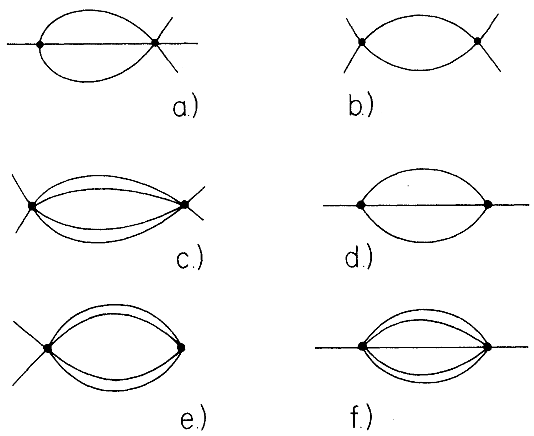

FIG. 3. (a)–(c) Graphs for the four-point vertex Γ_4. (d)–(f) Graphs for the self-energy Σ.

For the self-energy Σ, the graphs giving the leading logarithmic corrections are of the same type as those considered above. One such graph of order $Q^2 \ln r$ is shown in Fig. 3(d). Some other graphs of Σ are shown in Figs. 3(e) and 3(f). They are of order $Q u_6 r^{1/2} \ln r$ and $u_6^2 r \ln r$, respectively, and can be omitted. From Fig. 3(d) the differential equation for Σ is

$$\Sigma' = \tfrac{1}{18}(n+2)\Gamma_4^2 F_3' \, . \tag{12}$$

The inverse susceptibility is determined from $r = r_0 - \Sigma(r)$ and we impose the boundary condition on (12) that as $r \to 1$ the inverse susceptibility becomes equal to its mean-field value μ. Then using Eq. (11) in (12) and this boundary condition we find the inverse susceptibility is

$$r = \mu - \tfrac{5}{6}[(n+2)/(6-n)]\,(Q^2/u_6)[L^{1-2p}(r) - 1] \, . \tag{13}$$

The case $n=6$ requires a separate discussion. The exponent of the logarithmic $1-2p = (6-n)/(3n+22)$ and is positive in the interesting case $n<6$. A similar relation for the inverse susceptibility was found by Wegner and Riedel.[11] From (13) the lines of constant r are simple parabolas in terms of the mean field variables μ, Q. When r is small the second term in (13) can be large (when $n<6$) and the parabolas of constant r tend to become concave towards the disordered phase. However, we are not able to determine the critical line $r=0$ exactly as Eq. (13) breaks down for small r [see Eq. (4)]. Along the line $Q=0$, $r=\mu$ so that the susceptibility exponent is $\gamma = 1$ and there are no corrections to the mean-field behavior.[21]

B. Thermodynamics

We now consider some thermodynamic quantities of interest. We assume that the parameter r_0 depends linearly on the physical fields t and Δ. It is simplest to express the thermodynamic quantities in terms of r with the understanding that r is to be determined from (13).

a. Entropy. We define a thermodynamic potential by $e^{-G} = \mathrm{Tr}\, e^{-H/kT}$. The contribution of the fluctuations to S is

$$S = -k_B \frac{\partial G}{\partial t} = \text{const} + \frac{n k_B}{8\pi}\,\frac{dr_0}{dt}\, r^{1/2} \, , \tag{14}$$

where k_B in Boltzmann's constant.

b. Specific heat. The contribution of the fluctuations to the specific heat at constant μ_{34} is

$$C = -k_B \frac{\partial^2 G}{\partial t^2} = \frac{n k_B}{16\pi}\left(\frac{dr_0}{dt}\right)^2 r^{-1/2} \, . \tag{15}$$

There are logarithmic corrections to this result but they are of order $Q/rL^p(r)$ and thus small.

c. ^{3}He concentration. The contribution of the fluctuations to the ^{3}He concentration x is

$$x = -\frac{k_B T_0}{\mu_{340}}\,\frac{\partial G}{\partial \Delta} = \text{const} + \frac{n k_B T_0}{8\pi \mu_{340}}\left(\frac{dr_0}{d\Delta}\right) r^{1/2} \, . \tag{16}$$

d. Free energy and equation of state. In the presence of a field $\vec{\zeta}$ conjugate the order parameter a term $-\vec{\zeta}\cdot\vec{s}$ is included in the reduced Hamiltonian (1). The expectation value of the order parameter is denoted by $\vec{M} = \langle \vec{s} \rangle$. We then introduce a free energy $F(M) = G + M\zeta$ and it is not difficult to show[22] that

$$\frac{\partial F}{\partial M} = \zeta, \quad \frac{d^m F}{dM^m} = \Gamma_m, \quad m > 2, \tag{17}$$

when Γ_m is the m-point vertex (excluding one-particle reducible parts). For $M=0$, Γ_6 and Γ_4 are given by (9) and (11), and $\Gamma_2 = r$, which is determined by (13). The free energy expanded in a Taylor series around $M=0$ is then given by

$$F(M) = F(0) + \sum_{m=2}^{\infty} \frac{1}{m!}\,\Gamma_m(r) M^m \, . \tag{18}$$

As the order parameter is of order $M^4 \sim r/u_6$ it is only necessary to retain terms with $m \leq 6$ in (18) and then

$$F(M) = F(0) + \tfrac{1}{2}\Gamma_2(r)M^2 + (1/4!)\Gamma_4(r)M^4 + (1/6!)\Gamma_6(r)M^6 \, . \tag{19}$$

The neglected terms are of order $(u_6/L)^{1/2}$ smaller than those retained. We use Eqs. (9a), (11), and (13) to eliminate Γ_6, Γ_4, and $\Gamma_2(r) = r$ and find for the free-energy difference

$$\Delta F = \frac{1}{2}\left(\mu - \frac{5}{6}\,\frac{n+2}{6-n}\,\frac{Q^2}{u_6}\,[L^{1-2p}(r) - 1]\right)M^2 + \frac{Q}{4!}\,L^{-p}(r)M^4 + \frac{u_6}{6!}\,L^{-1}(r)M^6 \, . \tag{20}$$

When $L(r) > 1$ this free-energy difference is of the form

$$\Delta F = \mu^{3/2} L^{1/2}(r) \times F\left(Q u^{-1/2} L^{1/2-p}(r), M \mu^{-1/4} L^{-1/4}(r)\right) . \tag{21}$$

Alternatively, we can write the difference in thermodynamic potential between the ordered and disordered states in the form[23]

$$\Delta G = \mu^{3/2} L^{1/2} \times g\left(Q u^{-1/2} L^{1/2-p}(r), \zeta\, \mu^{-5/4} L^{-1/4}(r)\right) . \tag{22}$$

The logarithmic factors destroy the scaling behavior of the free energy.

The equation of state is obtained by differentiating (20) with respect to M,

$$\zeta = \left(\mu - \frac{5}{6}\frac{n+2}{6-n}\frac{Q^2}{u_6}\left[L^{1-2p}(r) - 1\right]\right)M + \frac{Q}{3!} L^{-p}(r) M^3 + \frac{u_6}{5!} L^{-1}(r) M^5 . \tag{23}$$

III. ORDERED PHASE

In the ordered phase the order parameter has a nonzero value. It is simplest to first consider the case $n=1$. We discuss below what changes occur for $n>1$. The order parameter is given by $M=\langle s\rangle$ and it is convenient to make a change of variables in (1) by introducing $\phi = s - M$. After collecting terms we may write the reduced Hamiltonian including the field ζ as

$$\frac{H}{kT} = \text{const} + \int d^3x \left[\left(r_0 M + \frac{u_4}{3!} M^3 + \frac{u_6}{5!} M^5 - \zeta\right)\phi + \tfrac{1}{2}\left(r_0 + \frac{u_4}{2} M^2 + \frac{u_6}{24} M^4\right)\phi^2 + \tfrac{1}{2}(\nabla\phi)^2 + \frac{1}{3!}(u_4 M + \tfrac{1}{6} u_6 M^3)\phi^3 + \frac{1}{4!}(u_4 + \tfrac{1}{2} u_6 M^2)\phi^4 + \frac{1}{5!} u_6 M \phi^5 + \frac{u_6}{6!}\phi^6\right] . \tag{24}$$

It is now possible to write differential equations for the m-point vertices exactly as before. It is also necessary to consider vertices with m odd. We will only give the results for these vertices:

$$\overline{\Gamma}_6 = u_6 L^{-1}(r), \quad \overline{\Gamma}_5 = u_6 M L^{-1}(r), \quad \overline{\Gamma}_4 = Q L^{-2/5}(r) + \tfrac{1}{2} u_6 M^2 L^{-1}(r), \quad \overline{\Gamma}_3 = Q M L^{-2/5}(r) + \tfrac{1}{6} u_6 M^3 L^{-1}(r) , \tag{25}$$

and the susceptibility r is determined by

$$r = \mu - (Q^2/2u_6)\left[L^{1/5}(r) - 1\right] + \tfrac{1}{2} M^2 Q L^{-2/5}(r) + \tfrac{1}{24} M^4 u_6 L^{-1}(r) . \tag{26}$$

The equation of state is easily obtained by writing a differential equation for $\Gamma_1 = \langle\phi\rangle / r$. Retaining only those graphs with three internal lines as before this equation is

$$\Gamma_1' = \tfrac{1}{6}\overline{\Gamma}_3 F_3' \overline{\Gamma}_4 . \tag{27}$$

We impose the boundary condition that as $r \to 1$, Γ_1 reduce to its mean-field value, i.e.,

$$\Gamma_1(r=1) = \zeta - \mu M - (Q/3!)M^3 - (u_6/5!)M^5 .$$

The equation of state is $\Gamma_1 = 0$, and from (25) and (27) with the above boundary condition we find

$$\zeta = \left(\mu - \frac{Q^2}{2u_6}\left[L^{1/5}(r) - 1\right]\right)M + \frac{Q}{3!} L^{-2/5}(r) M^3 + \frac{u_6}{5!} L^{-1}(r) M^5 . \tag{28}$$

This is of exactly the same form as (23) (for $n=1$), but in this case the inverse susceptibility r, which is the argument of the logarithmic factor, is to be determined from (26). This difference is important: for example, at the tricritical point $\mu = Q = 0$ and from (26) we have $r \sim u_6 M^4$. Thus, from (28) the order parameter varies with field at the tricritical point as

$$\sim M^5 u_6 L^{-1}(u_6 M^4) . \tag{29}$$

As a second example, in the ordered phase along the line $Q=0$ the spontaneous order is determined by the equations

$$r = \mu + \tfrac{1}{24} u_6 L^{-1}(r) M_0^4 , \qquad 0 = \zeta + (u_6/5!) L^{-1}(r) M_0^4 . \tag{30}$$

From these equations we find

$$r = 4|\mu| , \quad M_0^4 = (5!\,|\mu|/u_6) L(|\mu|) . \tag{31}$$

The free-energy difference between the ordered and disordered phases is obtained by integrating (28). To logarithmic accuracy the logarithmic factors may be regarded as constant in this integration. Thus

$$F(M) - F(0) = \frac{1}{2}\left(\mu - \frac{Q^2}{2u_6}\left[L^{1/5}(r) - 1\right]\right)M^2 + \frac{Q}{4!} L^{-2/5}(r) M^4 + \frac{u_6}{6!} L^{-1}(r) M^6 . \tag{32}$$

Again this only differs from (20) in that the argument r of the logarithms is to be determined from (26).

Other thermodynamic quantities of interest are

$$S = \text{const} - \frac{k_B}{2}\frac{dr_0}{dt}\left(M^2 - \frac{r^{1/2}}{4\pi}\right) , \tag{33}$$

$$C = \text{const} + k_B\left(\frac{dr_0}{dt}\right)^2\left(\frac{M^2}{r} + \frac{1}{16\pi r^{1/2}}\right) , \tag{34}$$

$$x = \text{const} - \frac{k_B T_0}{2\mu_{340}} \frac{dr_0}{d\Delta}\left(M^2 - \frac{r^{1/2}}{4\pi}\right) . \tag{35}$$

As the spontaneous order $M_0^2 \sim [(|\mu|/u_6)L(|\mu|)]^{1/2}$, the first term dominates sufficiently close to the tricritical point in Eqs. (33)–(35). A more complete discussion of the various densities and susceptibilities will be given in another publication.[24]

We now consider how the above results will be changed when $n > 1$. This case is slightly more complicated in the ordered phase than the case $n = 1$ because of the two possible susceptibilities $r_\parallel$ and $r_\perp$ parallel and perpendicular to the direction of ordering, respectively. For $n > 1$, the free-energy difference and equation of state in both the ordered and disordered phase are given by (20) and (23) provided the argument of the logarithmic factors is determined correctly. It does not appear possible when $n > 1$ to express the logarithmic corrections generally in terms of $r_\parallel$ and $r_\perp$. In the case $n = 1$ above this was possible because there is only one susceptibility $r_\parallel = r$. For $n > 1$ it is necessary to determine $r_\parallel$ and $r_\perp$ in terms of the variables μ or M and then the results are simply stated. At the tricritical point $r \sim r_\parallel \sim r_\perp \sim u_6 M^4$ and $L(r)$ in (20) and (23) is replaced by $L(u_6 M^4)$. In the ordered or disordered states away from the tricritical point $r \sim r_\parallel \sim r_\perp \sim |\mu|$ and $L(r)$ is replaced by $L(|\mu|)$ in (20) and (23) We now briefly consider some of the consequences of these results.

(i) From (20) and (23) the first-order coexistence line is determined by

$$\mu = \frac{5Q^2}{6u_6(6-n)}\left(\frac{26+n}{4} L^{1-2p}(r) - (n+2)\right) , \tag{36}$$

and the spontaneous order on the coexistence line is

$$M_{0c}^2 = -(15Q/u_6)L^{1-p}(r) \quad . \tag{37}$$

The susceptibility in the ordered phase on the coexistence line is

$$r = 4\left(\mu - \frac{5}{6}\,\frac{n+2}{6-n}\,\frac{Q^2}{u_6}\,[L^{1-2p}(r) - 1]\right) \tag{38}$$

and is four times larger than in the disordered phase.

(ii) The contribution of the fluctuations to the ^{3}He concentration in the disordered phase on the coexistence line is

$$x - x_0 = cr^{1/2} = c(5/8u_6)^{1/2}\,|Q|\,L^{1/2-p}(r) \ , \tag{39}$$

where $c = (k_B T_0/8\pi\mu_{340})\partial r_0/\partial\Delta$, r is given by (13), and we have used (36). On the critical line where $r = 0$ there is no such contribution. In the ordered phase on the coexistence line we find, from (35),

$$x - x_0 = -c M_{0c}^2 = (15c/u_6)QL^{1-p}(r) \ , \tag{40}$$

where r is determined by (38) and (36).

IV. CONCLUSIONS

The logarithmic corrections to the mean-field theory of symmetrical tricritical points have been derived on the assumption $u_6 < 1$. However, the actual expansion is in terms of Γ_6 which becomes small near the tricritical point and thus we expect the results also to be valid for $u_6 > 1$. The magnitude of u_6 determines the size of the critical region. The logarithmic corrections are all expressed in terms of $L(r) = 1 - [(3n+22)/480\pi^2]u_6 \ln r$ and then the critical region is determined by

$$|\ln r| > 480\pi^2/u_6(3n+22) \ . \tag{41}$$

From the spin model of Blume *et al.*[3] we estimate $u_6 \sim 100$ which leads to an appreciable critical region in this model. For large n we expect $u_6 \sim 1/n^2$ and the critical region vanishes as $n \to \infty$. The $1/n$ expansion for this model has recently been considered by Emery[25] and has shown to exhibit no logarithmic corrections in the limit $n \to \infty$ in agreement with the present results.

The other assumption that was made in analyzing the perturbation series, $Q^2/r < 1$, also requires some discussion. An examination of Eq. (13) shows that the correct condition is $Q^2/u_6 < rL^{2p-1}(r)$. This is an important restriction on the results and in particular prevents us from determining the critical line and the behavior close to this line. There exists a crossover region in which the behavior changes from the tricritical form near the tricritical point to ordinary critical point behavior near the critical line. The tricritical point behavior can be discussed in three dimensions and the critical behavior can be discussed in systems close to four dimensions. A quantitative discussion of the crossover behavior in three dimensions has not yet been given.

V. SUMMARY OF IMPORTANT RESULTS

The results are most concisely expressed in terms of

$$\Gamma_2(r) = \mu - \frac{5}{6}\,\frac{n+2}{6-n}\,\frac{Q^2}{u_6}\,[L^{1-2p}(r) - 1] , \tag{42}$$

$$\Gamma_4(r) = Q/L^p(r) \ , \quad \Gamma_6(r) = u_6/L(r) \ , \tag{43}$$

where

$$L(r) = 1 - \frac{3n+22}{480\pi^2}\,u_6 \ln r \ , \qquad p = \frac{2(n+4)}{3n+22} \ . \tag{44}$$

(a) Inverse parallel susceptibility:

$$r = \Gamma_2(r) + \tfrac{1}{2}\Gamma_4(r)M^2 + (1/4!)\,\Gamma_6(r)M^4 \ . \tag{45}$$

(b) Equation of state:

$$\zeta = \Gamma_2(r)M + (1/3!)\Gamma_4(r)M^3 + (1/5!)\Gamma_6(r)M^5 \ . \tag{46}$$

(c) Free-energy difference between ordered and disordered states:

$$\Delta F = \tfrac{1}{2}\Gamma_2(r)M^2 + (1/4!)\Gamma_4(r)M^4 + (1/6!)\Gamma_6(r)M^6\ . \tag{47}$$

(d) Entropy:

$$S \sim M^2 - r^{1/2}/4\pi\ . \tag{48}$$

(e) Specific heat:

$$C \sim M^2/r + 1/16\pi r^{1/2}\ . \tag{49}$$

(f) ^{3}He concentration:

$$x \sim M^2 - r^{1/2}/4\pi\ . \tag{50}$$

In each case the inverse susceptibility is determined by (45) and the order parameter by (46). In the case $n > 1$, in the ordered state, the argument of the logarithmic factor has been written as r with the understanding that r is to expressed in terms of μ and M using (45). Thus at the tricritical point $\mu = Q = 0$, $r \sim M^4$, while along the line $Q = 0$, $r \sim |\mu|$.

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