Parametric interaction processes in acoustical noise

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The amplification of acoustical noise in a piezosemiconductor is considered. A system of nonlinear equations for amplitudes of many interacting sound waves is derived, and a physical interpretation to the various terms is given. On the basis of this, the equation describing the evolution of the spectral composition of the amplified noise is derived. An analytical solution (which is valid for the early nonlinear stage of the amplification) of this equation is obtained and a comparison with the available experimental results is made.

INTRODUCTION

In recent years a considerable number of works have appeared which discuss parametric interaction of sound waves in piezoelectric semiconductors. A detailed investigation of parametric effects for three interacting waves was presented by Conwell and Ganguly[1] (there is also an extensive bibliography there). This paper contains the derivation of a system of equations governing the spatial variation of the amplitudes of three monochromatic interacting sound waves. On the basis of these equations various parametric effects have been investigated (up and down conversion, second-harmonic generation).

The problem as formulated in Ref. 1 corresponds, strictly speaking, to an experimental situation in which the sound waves are introduced into the crystal from outside (through the boundary). However, in many experiments (for example Refs. 2–8) it is not monochromatic sound waves that are amplified, but acoustical noise. In this case the amplified waves have a wide range of frequencies and propagate in directions within a cone around the direction of the drift of the electrons (the "amplification cone"). This means that the interaction of what is essentially a continuous spectrum of noncoherent waves (modes) must be considered.

A detailed description of energetic characteristics of the acoustical noise may be given in terms of the quantity $U_{\vec{q}}(\vec{r}, t)$ which is defined in such a way that $U_{\vec{q}} d^3q$ is the density of acoustical energy (at a given place and time) in the interval d^3q of the wave vectors. Such quantities as the density of the energy in a given range of frequencies, or the total density of energy, or the energy flux in a given direction can be easily expressed in terms of $U_{\vec{q}}$. Because of the interaction between different modes the energy may be transferred from one part of the spectrum to another. The task of a theory is to derive a nonlinear equation for $U_{\vec{q}}$ governing the variation of $U_{\vec{q}}$ in space and time.

The theoretical works devoted to amplification of acoustical noise differ from one another both in their approach to the problem and in their results. Yamada[9] and Gurevich *et al.*,[10] using a method developed in works[11,12] on turbulent plasma, gave a rather formal derivation of an equation for a quantity connected to $U_{\vec{q}}$. The equation derived in Ref. 9 contains a nonlinear term, which has the same structure as the collision term in the well-known Peierls equation for phonons.[13] In contrast to Ref. 9, the equation derived in Ref. 10 contains, in addition to the "Peierls" term, another nonlinear term, which is called in Ref. 10 the "non-Peierls" term. This term contains the small electromechanical coupling constant χ raised to a lower power than in the Peierls term, and therefore it has been suggested that the Peierls term may usually be neglected.

On the other hand, in the recent work of Ridley[14] the problem was formulated in the simple language of parametric interaction of many waves. However, an equation describing the evolution of acoustical noise in space and time was not obtained in this work. Furthermore, in the equations of Ref. 14 for the amplitudes of interacting waves only nonlinear terms containing products of two amplitudes were taken into account; however (as will be seen below), in the case of many incoherent waves also nonlinear terms containing products of three amplitudes must be taken into account.

In the present work we derive an equation for $U_{\vec{q}}$, using the clear and transparent picture accepted in Refs. 1 and 14. First, we derive a system of equations for amplitudes of many interacting sound waves and give a physical interpretation to the various terms. Then on the basis of this we derive the final equation for $U_{\vec{q}}$. This equation is similar to that of Ref. 10, and using the narrowness of the "amplification cone" one can bring both equations to the same form. Our analysis of this equation shows that at the early stage of the nonlinear regime the Peierls term is always important. It is this term which gives rise to the up and down conversion at this early stage. For this case we obtain an analytical expression for $U_{\vec{q}}$ and compare it with experimental results. At the later stage of amplification the non-Peierls term becomes important. In particular, it provides the stabilization of the sound amplification in the low-frequency region.

EQUATION FOR THE AMPLITUDES

There are two characteristic time-scales in our problem: a short scale associated with the period of sound waves T, and a big scale associated with a characteristic time t_0, during which the amplitudes of the sound waves (and $U_{\vec{q}}$) change essentially. To these times two lengths correspond—the wavelength Tv_s (v_s—velocity of sound) and the characteristic "amplification length" $t_0 v_s$. The strong inequality $T \ll t_0$ allows averaging of the quantities (such as the electric field, concentration of electrons, the lattice displacement) over the time interval t, where $T \ll t \ll t_0$ (or over a length l, where $Tv_s \ll l \ll t_0 v_s$). Quantities averaged in such a manner may still vary on the long-time (space) scale. The instantaneous local values of any quantity can be represented as a sum of its average value and a rapidly oscillating part (the latter is obviously zero on the average). For the rapidly oscillating parts of the electric field $-\nabla\varphi$, lattice displacement $\vec{u}$ and electron concentration n, the following system of equations holds (see, for example, Ref. 15):

$$\rho\frac{\partial^2 u_i}{\partial t^2} = c_{iklm}\frac{\partial^2 u_m}{\partial x_k \partial x_l} - \beta_{lki}\frac{\partial^2\varphi}{\partial x_k\partial x_l} + \eta_{iklm}\frac{\partial^3 u_l}{\partial x_k \partial x_m \partial t}, \tag{1a}$$

$$\epsilon\frac{\partial^2\varphi}{\partial x_i\partial x_i} + 4\pi\beta_{ikl}\frac{\partial^2 u_k}{\partial x_i \partial x_l} + 4\pi e n = 0, \tag{1b}$$

$$\frac{\partial n}{\partial t} + v_i\frac{\partial n}{\partial x_i} - \mu n_0\frac{\partial^2\varphi}{\partial x_i\partial x_i} - \mathfrak{D}\frac{\partial^2 n}{\partial x_i \partial x_i} - \mu\frac{\partial}{\partial x_i}\left(n\frac{\partial\varphi}{\partial x_i}\right) = 0. \tag{1c}$$

Latin indices denote components of vectors and tensors and the summation from 1 to 3 over each repeated index is implied. The drift velocity is $v_i = \mu\bar{E}_i$, where $\bar{E}_i$ is the field, averaged in the above sense; n_0 is the equilibrium concentration of electrons, ρ is density, ϵ, μ, $\mathfrak{D}$ are the dielectric constant, carrier mobility, and diffusion coefficient, respectively; they are all assumed to be scalars. c_{iklm}, β_{ikl}, and η_{iklm} are elastic, piezoelectric, and viscous tensors, respectively; e is the electron charge. The condition $ql_e \ll 1$ is supposed to hold (q—the acoustic wave number, l_e—the electron mean free path).

To become observable the acoustical noise has to be amplified several orders of magnitude from the thermal level. Usually only sound waves of one branch are amplified, so that $\vec{u}$ can be represented as a sum of the modes connected only to this branch:

$$\vec{u}(\vec{r}, t) = \mathrm{Re}\sum_{\vec{q}} \vec{e}_{\vec{q}} A_{\vec{q}}(x, t) e^{-i\omega_{\vec{q}} t + i\vec{q}\cdot\vec{r}}. \tag{2}$$

We use the discrete spectrum for convenience, but in the final formulas we shall go over to the continuous spectrum, according to $\sum_{\vec{q}} \to [\Omega/(2\pi)^3]\int d^3q$, where Ω is a normalizing volume (for instance the volume of the crystal). The summation in (2) may be taken over the half-space $q_x > 0$, where q_x is the $\vec{q}$ component in the direction of the drift (x axes). However, only modes propagating in the "amplification cone" give an essential contribution to this sum. The width of the cone may change during the amplification. $\vec{e}_{\vec{q}}$ is the polarization vector of the mode $\vec{q}$; $\omega_{\vec{q}}$ its frequency. By the frequency $\omega_{\vec{q}}$ we mean the real positive frequency of a "pure" sound wave, i.e., a wave which would propagate in a corresponding ideal dielectric without lattice attenuation and piezointeraction. Thus $\omega_{\vec{q}}$ and $\vec{e}_{\vec{q}}$ are the eigenvalue and the eigenvector of the equation

$$\rho\omega_{\vec{q}}^2 e_{\vec{q}i} = c_{iklm} q_k q_l e_{\vec{q}m}. \tag{3}$$

The complex amplitudes $A_{\vec{q}}$ depend on the coordinate x and on time because of the piezointeraction and lattice attenuation. We neglect the possible dependence of $A_{\vec{q}}$ on the transverse coordinates y and z which may occur because of the surface effects or anisotropy of the crystal. In particular cases $A_{\vec{q}}$ may depend only on x or only on t. For example, in a region inside a sufficiently long crystal the noise is spatially homogeneous, i.e., $U_{\vec{q}}$ and $A_{\vec{q}}$ do not depend on x for $t < t_s$, where t is the time elapsed from the switching on of the external field, t_s is the time required by sound to arrive from the boundary of the crystal to the region under consideration. A contrary extreme case, stationary noise, when $U_{\vec{q}}$ changes along the sample but does not depend on time. This case is realized if $t > \tau_s$ (τ_s—sound propagation time through the sample) and reflection of sound from the boundaries are negligible. We consider below the general case when $U_{\vec{q}}$ depends on both x and t.

It is convenient to rewrite expression (2) in the form

$$\vec{u} = \frac{1}{2}\sum_{\vec{q}}{}' \vec{e}_{\vec{q}} A_{\vec{q}} e^{-i\omega_{\vec{q}} t + i\vec{q}\cdot\vec{r}}, \tag{4}$$

where the summation is now extended also over the region $q_x < 0$. This is denoted by the prime. By definition $A_{-\vec{q}} = A_{\vec{q}}^*$, $\omega_{-\vec{q}} = -\omega_{\vec{q}}$, $\vec{e}_{-\vec{q}} = \vec{e}_{\vec{q}}$.

With any sound wave a wave of electron density and electric potential is associated. Hence

$$n(\vec{r}, t) = \frac{1}{2}\sum_{\vec{q}}{}' B_{\vec{q}}(x, t) e^{-i\omega_{\vec{q}} t + i\vec{q}\cdot\vec{r}},$$
$$\varphi(\vec{r}, t) = \frac{1}{2}\sum_{\vec{q}}{}' C_{\vec{q}}(x, t) e^{-i\omega_{\vec{q}} t + i\vec{q}\cdot\vec{r}}. \tag{5}$$

It is essential that the complex amplitudes $A_{\vec{q}}$, $B_{\vec{q}}$, and $C_{\vec{q}}$ be slow functions of x and t compared with the rapidly oscillating exponents, i.e.,

$$\left|\frac{\partial A_{\vec{q}}}{\partial t}\right| \ll |\omega_{\vec{q}} A_{\vec{q}}|, \quad \left|\frac{\partial A_{\vec{q}}}{\partial x}\right| \ll |q_x A_{\vec{q}}| \tag{6}$$

and analogously for $B_{\vec{q}}$ and $C_{\vec{q}}$. Inequalities (6) allow the derivation from (1) of a system of equations for the complex amplitudes by (4), (5) being substituted in (1) and the terms with identical exponents being collected. In (1b), (1c) it is sufficient to keep only the dominant terms obtained by differentiating the exponents. However, in (1a) the dominant terms are cancelled, as according to (3) they should be, and only the terms connected to piezointeraction and lattice attenuation remain. Therefore, in (1a) also the terms containing the derivatives of $A_{\vec{q}}$ must be kept. Eliminating $C_{\vec{q}}$ we get the following equations:

$$B_{\vec{q}} = \frac{\beta_{\vec{q}} q^2}{e a_{\vec{q}}} A_{\vec{q}} - \frac{1}{2en_0 a_{\vec{q}}} \sum_{\vec{q}_1+\vec{q}_2=\vec{q}}{}' \vec{q}\cdot\vec{q}_1 \times [(e/q_1^2) B_{\vec{q}_1} B_{\vec{q}_2} - \beta_{\vec{q}_1} A_{\vec{q}_1} B_{\vec{q}_2}] \exp(i\omega_{\vec{q}\vec{q}_1\vec{q}_2} t), \tag{7a}$$

$$\frac{\partial A_{\vec{q}}}{\partial t} + v_{s\vec{q}} \frac{\partial A_{\vec{q}}}{\partial x} = -\tfrac{1}{2} i\omega_{\vec{q}} \chi_{\vec{q}} A_{\vec{q}} + i\frac{2\pi e \beta_{\vec{q}}}{\epsilon\rho\omega_{\vec{q}}} B_{\vec{q}} - \tfrac{1}{2}\alpha_{\vec{q}}^{(l)} A_{\vec{q}}. \tag{7b}$$

Here

$$\beta_{\vec{q}} = \frac{1}{q^2}\beta_{ikl} q_i q_k e_{\vec{q}l}, \quad \chi_{\vec{q}} = 4\pi\beta_{\vec{q}}^2 q^2/\rho\epsilon\omega_{\vec{q}}^2,$$

$$a_{\vec{q}} = (1+q^2R^2) + i(\vec{q}\cdot\vec{v} - \omega_{\vec{q}})\tau_c,$$

where

$$\tau_c = \frac{\epsilon}{4\pi e \mu n_0},$$

dielectric relaxation time.

$$R = (\mathfrak{D}\tau_c)^{1/2} = \left(\frac{\epsilon T}{4\pi e^2 n_0}\right)^{1/2}$$

is the Debye length (T—temperature in energetic units).

$$v_{s\vec{q}} = \frac{1}{2\rho\omega_{\vec{q}}}(c_{ikxm} + c_{ixkm}) q_k e_{\vec{q}m} e_{\vec{q}i},$$

i.e., the group velocity of the sound wave in the x direction.

$$\alpha_{\vec{q}}^{(l)} = (1/\rho)\eta_{iklm} q_k q_m e_{\vec{q}i} e_{\vec{q}l}$$

describes the lattice attenuation. Further,

$$\omega_{\vec{q}\vec{q}_1\vec{q}_2} = \omega_{\vec{q}} - \omega_{\vec{q}_1} - \omega_{\vec{q}_2}.$$

Equations similar to (7) for the amplitudes of electromagnetic waves in plasma were derived by Tsitowich.[16]

It can be seen from Eqs. (7) that the inequalities (6) hold because the parameters $\chi_{\vec{q}}$, $\alpha_{\vec{q}}^{(l)}/\omega_{\vec{q}} \ll 1$. In the nonlinear regime [where the second term on the right-hand side of (7a) is important], the condition $|\omega_{\vec{q}} - \omega_{\vec{q}'} - \omega_{\vec{q}-\vec{q}'}| \ll \omega_{\vec{q}}$ is also required; this condition follows from the narrowness of the "amplification cone."

From (7a) one can express via iteration the amplitude of the electron wave $B_{\vec{q}}$ through the amplitudes of the sound waves. In the linear approximation

$$B_{\vec{q}} = \beta_{\vec{q}}(q^2/e a_{\vec{q}}) A_{\vec{q}}. \tag{8}$$

Substituting (8) in the nonlinear term of (7a) we get $B_{\vec{q}}$ to the accuracy of quadratic terms with respect to the amplitudes of sound waves. This iteration procedure can be continued (the condition for its validity is given below). We restrict ourselves to the terms containing products of three amplitudes of the acoustical waves. Substituting the resultant expression for $B_{\vec{q}}$ into (7b) we get

$$\frac{\partial A_{\vec{q}}}{\partial t} + v_{s\vec{q}}\frac{\partial A_{\vec{q}}}{\partial x} = \tfrac{1}{2}(\nu_{\vec{q}} - \alpha_{\vec{q}}^{(l)}) A_{\vec{q}} + \frac{1}{\omega_{\vec{q}}}\beta_{\vec{q}} \sum_{\vec{q}_1+\vec{q}_2=\vec{q}}{}' \beta_{\vec{q}_1}\beta_{\vec{q}_2} P_{\vec{q}\vec{q}_1\vec{q}_2} A_{\vec{q}_1} A_{\vec{q}_2} e^{i\omega_{\vec{q}\vec{q}_1\vec{q}_2} t} + \frac{1}{\omega_{\vec{q}}}\beta_{\vec{q}} \sum_{\vec{q}'+\vec{q}''=\vec{q}}{}' Q_{\vec{q}\vec{q}'\vec{q}''} A_{\vec{q}'} e^{i\omega_{\vec{q}\vec{q}'\vec{q}''} t} \sum_{\vec{q}_1+\vec{q}_2=\vec{q}''}{}' \beta_{\vec{q}_1}\beta_{\vec{q}_2} P_{\vec{q}''\vec{q}_1\vec{q}_2} A_{\vec{q}_1} A_{\vec{q}_2} e^{i\omega_{\vec{q}''\vec{q}_1\vec{q}_2} t}, \tag{9}$$

where

$$\nu_{\vec{q}} = \tfrac{1}{2} i\omega_{\vec{q}} \chi_{\vec{q}} (1 - a_{\vec{q}})/a_{\vec{q}},$$

$$P_{\vec{q}\vec{q}_1\vec{q}_2} = \frac{i\pi}{\rho\epsilon e n_0} \frac{\vec{q}\cdot\vec{q}_1 q_2^2 (a_{\vec{q}_1} - 1)}{a_{\vec{q}} a_{\vec{q}_1} a_{\vec{q}_2}}, \tag{10}$$

$$Q_{\vec{q}\vec{q}'\vec{q}''} = \frac{1}{2en_0 a_{\vec{q}} a_{\vec{q}'}}\left(\vec{q}\cdot\vec{q}'(a_{\vec{q}'} - 1) - \vec{q}\cdot\vec{q}''\frac{q'^2}{q''^2}\right).$$

Equation (9) describes the variation of $A_{\vec{q}}$ in space and time. The first term on the right-hand side is the usual linear term and

$$\tfrac{1}{2}\mathrm{Re}\,\nu_{\vec{q}} \equiv \tfrac{1}{2}\alpha_{\vec{q}}^{(e)} = \tfrac{1}{2}\chi_{\vec{q}}\omega_{\vec{q}}\tau_c\left(\frac{\vec{q}\cdot\vec{v} - \omega_{\vec{q}}}{(1+q^2R^2)^2 + (\vec{q}\cdot\vec{v} - \omega_{\vec{q}})^2\tau_c^2}\right) \tag{11}$$

is the linear amplification (attenuation) coefficient (with respect to the amplitude) connected to the piezointeraction.

The imaginary part of $\nu_{\vec{q}}$ leads to a change of phase of the complex amplitude $A_{\vec{q}}$, i.e., to the dispersion of the sound velocity due to the piezointeraction. It is more natural to describe this effect by renormalizing the frequency. To do this we introduce

$$A'_{\vec{q}} = A_{\vec{q}} \exp(-\tfrac{1}{2} i \,\mathrm{Im}\,\nu_{\vec{q}} t)\,, \quad \omega'_{\vec{q}} = \omega_{\vec{q}} - \tfrac{1}{2}\mathrm{Im}\,\nu_{\vec{q}}\,.$$

Then

$$\vec{u} = \frac{1}{2} \sum_{\vec{q}}{}' \vec{e}_{\vec{q}} A'_{\vec{q}} e^{-i\omega'_{\vec{q}} t + i\vec{q}\vec{r}}\,,$$

and according to (9)

$$\frac{\partial A'_{\vec{q}}}{\partial t} + v_{s\vec{q}} \frac{\partial A'_{\vec{q}}}{\partial x} = \tfrac{1}{2}\alpha_{\vec{q}} A'_{\vec{q}} + \frac{1}{\omega_{\vec{q}}} \beta_{\vec{q}} \sum_{\vec{q}_1+\vec{q}_2=\vec{q}}{}' \beta_{\vec{q}_1} \beta_{\vec{q}_2} P_{\vec{q}\vec{q}_1\vec{q}_2} A'_{\vec{q}_1} A'_{\vec{q}_2} e^{i\omega'_{\vec{q}\vec{q}_1\vec{q}_2} t}$$
$$+ (1/\omega_{\vec{q}})\beta_{\vec{q}} \sum_{\vec{q}'+\vec{q}''=\vec{q}}{}' \beta_{\vec{q}'} Q_{\vec{q}\vec{q}'\vec{q}''} A'_{\vec{q}'} e^{i\omega'_{\vec{q}\vec{q}'\vec{q}''} t} \sum_{\vec{q}_1+\vec{q}_2=\vec{q}''}{}' \beta_{\vec{q}_1} \beta_{\vec{q}_2} P_{\vec{q}''\vec{q}_1\vec{q}_2} A'_{\vec{q}_1} A'_{\vec{q}_2} e^{i\omega'_{\vec{q}''\vec{q}_1\vec{q}_2} t}\,, \qquad (12)$$

where

$$\alpha_{\vec{q}} = \alpha^{(e)}_{\vec{q}} - \alpha^{(l)}_{\vec{q}}\,.$$

The difference between $\omega_{\vec{q}}$ and $\omega'_{\vec{q}}$ is small (of the order of $\chi_{\vec{q}}\omega_{\vec{q}}$). This difference is negligible everywhere except, perhaps, in the exponential factors, where the dispersion may be essential.

The nonlinear terms in (12) [or (9)] describe the parametric interaction between different modes. A simple physical interpretation can be given to these terms. In the linear approximation an electron density wave with an amplitude proportional to $\beta_{\vec{q}} A_{\vec{q}}$ is associated with every sound wave $A_{\vec{q}}$ (first-order electron wave). Parametric interaction of two such waves with wave vectors $\vec{q}_1$ and $\vec{q}_2$ gives rise to a new wave with a wave vector $\vec{q} = \vec{q}_1 + \vec{q}_2$ and an amplitude proportional to $\beta_{\vec{q}_1}\beta_{\vec{q}_2} A_{\vec{q}_1} A_{\vec{q}_2}$ (second-order electron wave). This new electron wave causes, in turn, a sound wave with a wave vector $\vec{q}$ and an amplitude proportional to $\beta_{\vec{q}}\beta_{\vec{q}_1}\beta_{\vec{q}_2} A_{\vec{q}_1} A_{\vec{q}_2}$. This process is described by the second term on the right-hand side of (12). Analogously, the third term describes a process of the next order, i.e., the parametric interaction between first- and second-order electron waves. This gives a third-order electron wave with an amplitude proportional to $\beta^3 A^3$, and the amplitude of the corresponding sound wave is proportional to $\beta^4 A^3$. The exponential factors describe the phase shift that occurs because of the parametric interaction.

Equation for $U_{\vec{q}}$

The slowly changing quantities characterizing the intensity of sound are obtained by taking the time average of the squares of the rapidly oscillating quantities. For example, we have the density of the acoustical energy $\mathcal{E} = \rho (d\vec{u}/dt)^2_{\mathrm{av}}$ where av means the average over the time interval between t and $t + \Delta t$. This interval must satisfy an inequality

$$1/\omega \ll \Delta t \ll t_0\,, \qquad (13)$$

where ω is the characteristic frequency of sound (for instance the maximum amplification frequency) and t_0, the time during which the "slow" quantities themselves change significantly. This time t_0 is the lesser of the two times: the linear amplification time $1/\alpha$ and the characteristic time of the nonlinear interaction τ_n (the estimate of τ_n is given below).

If there is no interference between different modes, the terms containing products of two different modes vanish after averaging. Hence $\mathcal{E} = \frac{1}{2}\sum_{\vec{q}} \rho\omega^2_{\vec{q}} |A'_{\vec{q}}|^2$, i.e., the energy is simply a sum of the energies of all modes. In such a case the average over the time may be replaced by the average over the phases of the complex amplitudes $A'_{\vec{q}}$, assuming all the phases to be completely random (the phase average is understood in the usual sense of an average over a statistical ensemble). This random phase approximation (RPA) is usually used in problems in which a wide spectrum of interacting modes exists.[9–12,14] In Ref. 10 some criteria of RPA validity are given. In our case the RPA means

$$\langle A'_{\vec{q}} A'^{*}_{\vec{q}'} \rangle \equiv \langle A'_{\vec{q}} A'_{-\vec{q}'} \rangle = |A'_{\vec{q}}|^2 \Delta(\vec{q} - \vec{q}')\,, \qquad (14)$$

where the brackets denote phase average and

$$\Delta(\vec{q} - \vec{q}') = \begin{cases} 1 \text{ if } \vec{q} = \vec{q}' \\ 0 \text{ if } \vec{q} \neq \vec{q}'\,. \end{cases}$$

The density of the acoustical energy in RPA is

$$\mathcal{E} = \rho \left\langle \left(\frac{d\vec{u}}{dt} \right)^2 \right\rangle = \tfrac{1}{4}\rho \sum_{\vec{q},\vec{q}'}{}' \vec{e}_{\vec{q}} \cdot \vec{e}_{\vec{q}'}\, \omega_{\vec{q}}\omega_{\vec{q}'} \langle A'_{\vec{q}} A'^{*}_{\vec{q}'} \rangle e^{-i(\omega'_{\vec{q}} - \omega'_{\vec{q}'})t + i(\vec{q} - \vec{q}')\vec{r}}$$

$$=\tfrac{1}{4}\rho \sum_{\vec{q}}{}' \omega_{\vec{q}}^2 |A'_{\vec{q}}|^2 = \tfrac{1}{2}\rho \sum_{\vec{q}} \omega_{\vec{q}}^2 |A'_{\vec{q}}|^2 = \sum_{\vec{q}} \mathcal{E}_{\vec{q}},$$

i.e., equal to the sum of the energy densities of different modes, as it should be when there is no correlation between modes.

Now from (12) using (14) one can derive an equation for the quantity $|A'_{\vec{q}}|^2$ (written below as $S_{\vec{q}}$). For that one has to calculate $\partial S_{\vec{q}}/\partial t$, i.e., $1/\Delta t[S_{\vec{q}}(x, t+\Delta t) - S_{\vec{q}}(x,t)]$, where Δt obviously must satisfy the condition (13). It is convenient to integrate (12) over time from t to $t+\Delta t$, which leads to the integral equation:

$$\begin{aligned} A'_{\vec{q}}(t+\Delta t) &= A'_{\vec{q}}(t) - v_{s\vec{q}} \int_t^{t+\Delta t} \frac{\partial A'_{\vec{q}}(t')}{\partial x} dt' + \tfrac{1}{2}\alpha_{\vec{q}} \int_t^{t+\Delta t} A'_{\vec{q}}(t')\, dt' \\ &+ \frac{1}{\omega_{\vec{q}}} \beta_{\vec{q}} \sum_{\vec{q}_1+\vec{q}_2=\vec{q}}{}' \beta_{\vec{q}_1}\beta_{\vec{q}_2} F_{\vec{q}\vec{q}_1\vec{q}_2} \int_t^{t+\Delta t} dt' A'_{\vec{q}_1}(t') A'_{\vec{q}_2}(t')\, e^{i\omega'_{\vec{q}\vec{q}_1\vec{q}_2} t'} \\ &+ \frac{1}{\omega_{\vec{q}}} \beta_{\vec{q}} \sum_{\vec{q}'+\vec{q}''=\vec{q}}{}' \beta_{\vec{q}'} Q_{\vec{q}\vec{q}'\vec{q}''} \sum_{\vec{q}_1+\vec{q}_2=\vec{q}''}{}' \beta_{\vec{q}_1}\beta_{\vec{q}_2} F_{\vec{q}''\vec{q}_1\vec{q}_2} \int_t^{t+\Delta t} dt' A'_{\vec{q}'}(t') A'_{\vec{q}_1}(t') A'_{\vec{q}_2}(t')\, e^{i(\omega'_{\vec{q}\vec{q}'\vec{q}''} + \omega'_{\vec{q}''\vec{q}_1\vec{q}_2})t'} . \end{aligned} \tag{15}$$

For brevity the argument x of each A is omitted. We have introduced $F_{\vec{q}\vec{q}_1\vec{q}_2} = \tfrac{1}{2}(P_{\vec{q}\vec{q}_1\vec{q}_2} + P_{\vec{q}\vec{q}_2\vec{q}_1})$, which has the symmetry property $F_{\vec{q}\vec{q}_1\vec{q}_2} = F_{\vec{q}\vec{q}_2\vec{q}_1}$.

Equation (15) is more convenient than (12) for solving by iterations. As a zero-order approximation it is natural to take $A'_{\vec{q}}(t)$. Iterating up to terms containing products of three amplitudes and restricting ourselves to the first approximation with respect to the small parameter $\alpha_{\vec{q}}\Delta t$ we get

$$\begin{aligned} A'_{\vec{q}}(t+\Delta t) &= A'_{\vec{q}}(t) - v_{s\vec{q}}\Delta t \frac{\partial A'_{\vec{q}}(t)}{\partial x} + \tfrac{1}{2}\alpha_{\vec{q}}\Delta t A'_{\vec{q}}(t) \\ &+ (\beta^3 A'(t)A'(t)) + (\beta^6 A'(t)A'(t)A'(t)) \\ &+ (\beta^4 A'(t)A'(t)A'(t)) . \end{aligned} \tag{16}$$

For brevity we have written the nonlinear terms in (16) schematically, denoting only the power of β and the number of multiplied amplitudes. The last term in (16) is obtained from the last term in (15) by replacing all $A'(t')$ by $A'(t)$. Two other nonlinear terms in (16) are generated by the term before last in (15) in the first and second approximation, respectively. Multiplying (16) by its complex conjugate and averaging over phases we get

$$\begin{aligned} S_{\vec{q}}(t+\Delta t) &= S_{\vec{q}}(t) + \alpha_{\vec{q}}\Delta t S_{\vec{q}}(t) - v_{s\vec{q}}\Delta t \frac{\partial S_{\vec{q}}(t)}{\partial x} + 4\Delta t \,\mathrm{Re} \frac{1}{\omega_{\vec{q}}} \sum_{\vec{q}'}{}' \beta_{\vec{q}}^2 \beta_{\vec{q}'}^2 Q_{\vec{q},\vec{q}',\vec{q}-\vec{q}'} \\ &\times F_{\vec{q}-\vec{q}',\vec{q},-\vec{q}'} S_{\vec{q}}(t) S_{\vec{q}'}(t) + \frac{2}{\omega_{\vec{q}}} \sum_{\vec{q}'}{}' \beta_{\vec{q}}^2 \beta_{\vec{q}'}^2 \beta_{\vec{q}-\vec{q}'}^2 \Bigg(\frac{1}{\omega_{\vec{q}}} |F_{\vec{q},\vec{q}',\vec{q}-\vec{q}'}|^2 S_{\vec{q}'}(t) S_{\vec{q}-\vec{q}'}(t) \\ &\times \left| \int_0^{\Delta t} e^{i\omega'_{\vec{q},\vec{q}',\vec{q}-\vec{q}'} t'} dt' \right|^2 + \frac{2}{\omega_{\vec{q}'}} \mathrm{Re} F_{\vec{q},\vec{q}',\vec{q}-\vec{q}'} F_{\vec{q}',\vec{q},\vec{q}'-\vec{q}} S_{\vec{q}}(t) S_{\vec{q}'}(t) \int_0^{\Delta t} dt'\, e^{i\omega'_{\vec{q},\vec{q}',\vec{q}-\vec{q}'} t'} \int_0^{t'} dt''\, e^{-i\omega_{\vec{q},\vec{q}',\vec{q}-\vec{q}'} t''} \Bigg). \end{aligned} \tag{17}$$

The first nonlinear term in (17) ($\sim\beta^4$) appears after averaging the product $(\beta^4 A'A'A')A'^*_{\vec{q}}$ (plus c.c.); the other nonlinear term ($\sim\beta^6$) appears after averaging $(\beta^3 A'A')(\beta^3 A'A')^*$ and also $(\beta^6 A'A'A')A'^*_{\vec{q}}$ (plus c.c.). Although the second term contains β to a higher power than the first term we still keep it because the structures of these terms are different, and one cannot say in advance which term is bigger. In the final equation we shall make an estimate of the value of the ratio of these terms.

During the derivation of the nonlinear terms in (17) one has to calculate the average of the product of four amplitudes $\langle A'_{\vec{q}_1} A'_{\vec{q}_2} A'_{\vec{q}_3} A'_{\vec{q}_4}\rangle$ which splits into $\langle A'_{\vec{q}_1} A'_{\vec{q}_2}\rangle \langle A'_{\vec{q}_3} A'_{\vec{q}_4}\rangle$, plus two other terms corresponding to the two other ways of pairing. During the calculation some ways of pairing lead to a condition $\vec{q}=0$ (or $\vec{q}''=0$). Such pairings do not have to be considered because all the above equations describe only the rapidly oscillating waves.

Now it becomes clear why in (9) it is necessary to keep the last term. Although it is of higher order (with respect to both β and A) in comparison with the previous term, after averaging it gives a contribution in (17) proportional to β^4, while the contribution of the first nonlinear term in (9) is proportional to β^6.

One obtains the final equation for $S_{\vec{q}}$ by calculating $(1/\Delta t)[S_{\vec{q}}(t+\Delta t) - S_{\vec{q}}(t)]$ and substituting the expressions (10) for P and Q. The sums $\sum'$ must be transformed to sums $\sum$, which means summation only over the $\vec{q}'$, which correspond to positive frequencies (i.e., over the "physical" region in $\vec{q}$ space). Finally, replacing $S_{\vec{q}}$ by

$$U_{\vec{q}}=\mathcal{E}_{\vec{q}}\frac{\Omega}{(2\pi)^3}=\frac{1}{2}\frac{\Omega}{(2\pi)^3}\rho\omega_{\vec{q}}^2 S_{\vec{q}}$$

we obtain the following:

$$\frac{\partial U_{\vec{q}}}{\partial t}+v_{s\vec{q}}\frac{\partial U_{\vec{q}}}{\partial x}=\alpha_{\vec{q}}U_{\vec{q}}+\frac{\omega_{\vec{q}}}{Tn_0}\int d^3q'[\Phi_{q,\vec{q}'}+\Phi_{\vec{q}-\vec{q}'}]\,U_{\vec{q}}\,U_{\vec{q}'}$$
$$+\frac{\omega_{\vec{q}}}{Tn_0}\int d^3q'\,[\Psi_{\vec{q},-\vec{q}'}(\omega_{\vec{q}}U_{\vec{q}'}U_{\vec{q}-\vec{q}'}-\omega_{\vec{q}-\vec{q}'}U_{\vec{q}}U_{\vec{q}'}-\omega_{\vec{q}'}U_{\vec{q}}U_{\vec{q}-\vec{q}'})\,\delta(\omega'_{\vec{q}}-\omega'_{\vec{q}'}-\omega'_{\vec{q}-\vec{q}'})$$
$$+2\Psi_{\vec{q},\vec{q}'}(\omega_{\vec{q}}U_{\vec{q}'}U_{\vec{q}+\vec{q}'}-\omega_{\vec{q}+\vec{q}'}U_{\vec{q}}U_{\vec{q}'}+\omega_{\vec{q}'}U_{\vec{q}}U_{\vec{q}+\vec{q}'})\,\delta(\omega'_{\vec{q}}+\omega'_{\vec{q}'}-\omega'_{\vec{q}+\vec{q}'})]\,, \tag{18}$$

where

$$\Phi_{\vec{q},\vec{q}'}=\tfrac{1}{2}\chi_{\vec{q}}\chi_{\vec{q}'}\,\mathrm{Re}\,\frac{q'^2R^2}{ia_{\vec{q}}^2\,|a_{\vec{q}'}|^2a_{\vec{q}+\vec{q}'}}\left(\frac{\vec{q}\cdot\vec{q}'}{q'^2}(a^*_{\vec{q}'}-1)+\frac{\vec{q}\cdot(\vec{q}+\vec{q}')}{(\vec{q}+\vec{q}')^2}\right)\left(\frac{(\vec{q}+\vec{q}')\cdot\vec{q}}{q^2}(a_{\vec{q}}-1)+\frac{(\vec{q}+\vec{q}')\cdot\vec{q}'}{q'^2}(a_{\vec{q}'}-1)\right)\,, \tag{19}$$

$$\Psi_{\vec{q},\vec{q}'}=\frac{\pi}{8}\chi_{\vec{q}}\chi_{\vec{q}'}\chi_{\vec{q}+\vec{q}'}\frac{(q^2R^2)(q'^2R^2)[(\vec{q}+\vec{q}')^2R^2]}{(1+q^2R^2)^2(1+q'^2R^2)^2[1+(\vec{q}+\vec{q}')^2R^2]^2}\,. \tag{20}$$

Equation (18) may be rewritten in terms of the phonon distribution function $N_{\vec{q}}$ using the connection $\hbar\omega_{\vec{q}}N_{\vec{q}}=U_{\vec{q}}$. In (18) we went to the limit of a continuous spectrum. During the derivation of $\Psi_{\vec{q},\vec{q}'}$ we used the condition $\omega\tau_c\,|v/v_s-1|\ll 1$. This is the condition for the weak activity regime[14] which is usually satisfied in sufficiently conducting materials. Furthermore the factor

$$\left(\frac{2}{\omega'_{\vec{q},\vec{q}',\vec{q}-\vec{q}'}}\right)^2\sin^2(\tfrac{1}{2}\omega'_{\vec{q},\vec{q}',\vec{q}-\vec{q}'}\Delta t)$$

appearing during the derivation of the second nonlinear term in (18) (the Peierls term), was replaced by $2\pi\Delta t\,\delta(\omega'_{\vec{q},\vec{q}',\vec{q}-\vec{q}'})$. For this replacement to be valid, the factor must have a resonance property, i.e., the argument of the sine must change rapidly in the integration region. This implies a certain condition on the width of the amplification cone, namely: $\omega_{\vec{q}}\Delta t\Delta O\gg 1$, where ΔO is the solid angle occupied by the cone. That means that the cone must not be too narrow. For example, if all the waves were to propagate only in one direction, the argument of the sine (in the absence of dispersion) would be identically equal to zero and, obviously, the above factor could not be considered as a resonance. The small dispersion does not essentially change the situation, because for the dispersion to manifest itself significantly (i.e., to change the argument of the sine by order of 1) the time $\Delta t\gtrsim 1/\chi_{\vec{q}}\omega_{\vec{q}}$ is required. The physical meaning of the condition $\omega_{\vec{q}}\Delta t\Delta O\gg 1$ is the following: for RPA to be valid it is necessary for the nonlinear interaction to randomize the phases in the time Δt. Thus, even if there were a correlation between different modes at the beginning, the nonlinear interaction would destroy this correlation during the time $\Delta t\simeq 1/\omega_{\vec{q}}\Delta O$.

The first integral on the right-hand side of (18) represents the non-Peierls term. As we have seen, this term is due to the second nonlinear term in (9). This non-Peierls term does not appear in Yamada's equation[9] (or in the more general equation of Nakamura,[17]) because these authors considered only the first nonlinear term in the lattice motion equation. For the same reason this non-Peierls term cannot be obtained from the nonlinear equations derived in Refs. 14 and 18 or (for the case $ql_e\gg 1$) in Ref. 19.

The quantities Φ and Ψ were calculated to the lowest nonvanishing order of χ. This means that a small correction term ($\sim\chi^3$) to the non-Peierls term in (18) exists. This correction is always small in comparison with the leading term ($\sim\chi^2$) and can be neglected. On the other hand the second nonlinear term in (18), which is also proportional to χ^3, cannot, generally speaking, be neglected, as has already been noted. A rough estimate shows that the ratio of the first nonlinear term in (18) to the second is of the order $(1/\chi)\,\omega\tau_c\,|(v/v_s)-1|\,\Delta O$. This parameter can be either larger or smaller than 1. But (what is more important) is that these two terms have a different structure and therefore (as we are going to show) play a different role at various stages of the amplification.

To obtain (18) we used iterations with respect to the nonlinear terms in (9) and (15). This is certainly correct at the early stage of the nonlinear regime, when the main part of the energy is still concentrated in the region of the linear amplification, and up and down conversions just begin to manifest themselves. A careful examination is required to investigate the limit of validity of (18) in the general case. A very rough estimate (based on comparison of the mean squares of the successive terms) gives a criterion $\chi\mathcal{E}/Tn_0 \ll 1$ for the validity of the expansion of $B_{\vec{q}}$ in terms of $A_{\vec{q}}$ in (9) ($\mathcal{E}$—the acoustical energy density in the amplification cone). The corresponding criterion for the validity of the iteration of the nonlinear terms in (15) is weaker, and contains χ to a higher power (χ^3).

Now we estimate τ_n. According to its definition τ_n is the ratio between $U_{\vec{q}}$ and the nonlinear term in (18). If the first nonlinear term is dominant the ratio gives

$$\tau_n \simeq \frac{1}{\chi}\,\frac{Tn_0}{\mathcal{E}}\,\frac{1}{\alpha}$$

(i.e., larger than $1/\alpha$). If the second nonlinear term in (18) dominates, one can show that τ_n is at least not smaller than $1/\chi\omega$.

Equation (18) describes the regime of a "developed" instability when the acoustical energy is concentrated in a narrow cone and the nonlinear effects may be important. To obtain an equation describing the amplification of acoustical noise from the very beginning (i.e., from the thermal noise level) one has to add to the right-hand side of (18) a term $G_{\vec{q}}$, which describes the spontaneous thermal generation of noise. Obviously, in equilibrium the thermal generation is exactly compensated for by attenuation. This defines the form of $G_{\vec{q}}$ (see, for example Ref. 2): $G_{\vec{q}} = -\alpha_{\vec{q}}^{(0)} U_{\vec{q}}^{(0)}$. Here $U_{\vec{q}}^{(0)} = T/(2\pi)^3$ and $\alpha_{\vec{q}}^{(0)}$ is the value of $\alpha_{\vec{q}}$ when $\vec{v}=0$ (equilibrium); in other words, $-\alpha_{\vec{q}}^{(0)}$ is the equilibrium coefficient of sound attenuation.

The term $G_{\vec{q}}$ is essential only at the very first state of amplification, when the level of noise is comparable to the thermal noise. At this stage the nonlinear terms in (18) are negligibly small. In the regime of a "developed" instability, when the noise is several orders higher than the thermal noise, the term $G_{\vec{q}}$ may be neglected.

If the amplification begins from the thermal noise, the initial condition for Eq. (18) is $U_{\vec{q}}(x,0) = T(2\pi)^3$. In the case of spatially inhomogeneous noise one needs also boundary conditions to solve (18). These boundary conditions depend on reflective properties of the contacts.

UP AND DOWN CONVERSION ON THE EARLY STAGE OF AMPLIFICATION

In this section we obtain an approximate expression for $U_{\vec{q}}$ which is valid at the early stage of the nonlinear regime, and compare this expression with the available experimental results. The evolution of the spectral composition of the amplified sound is studied experimentally by Brillouin scattering techniques.[2–5,7,8] The experimentally measured quantity is the intensity $I_{\vec{q}}\,d^3q$ of the light scattered by the acoustical waves, with wave vectors in the region d^3q near $\vec{q}$. Since the light probes a constant volume in $\vec{q}$ space, $I_{\vec{q}}$ is proportional to $U_{\vec{q}}$.

We shall be interested in the spectral composition of the "on-axis" flux only (this corresponds to the simplest scattering geometry[3,5]). Hence we take $\vec{q}=\{q,0,0\}$. We also assume that $U_{\vec{q}}$ does not depend on x. This corresponds either to the case of spatially homogeneous noise or to the case where the observer "moves" with the acoustic flux. The latter case corresponds to the usual experimental situation,[2–4,7] when the acoustical energy is concentrated in a traveling (with the velocity of sound) domain.

In spherical coordinates (with $\vec{q} \parallel Ox$ as a polar axis) $U_{\vec{q}'}$ depends on the modulus of $\vec{q}'$ and on the angle ϑ between $\vec{q}'$ and $\vec{q}$, i.e., $U_{\vec{q}'} = U_{q',\vartheta}$. Similarly Ψ and Φ are functions of q, q', and ϑ. The δ function in the Peierls term provides (neglecting dispersion) that only $\vec{q}' \parallel \vec{q}$ give a contribution to the integral, i.e., only the interaction between collinear phonons is possible. On the contrary the non-Peierls term allows also interaction between the noncollinear phonons. Under the above conditions we obtain from (18) the following:

$$\frac{dU_q(t)}{dt} = [\alpha_q + \alpha_q^{(n)}(t) + \alpha_q^{(p)}(t)]\,U_q(t) + \frac{\pi\omega_q}{Tn_0}\int_0^q dq'q'(q-q')\,\Psi_{q,-q'}U_{q'}(t)U_{q-q'}(t)$$
$$+2\,\frac{\pi\omega_q}{Tn_0}\int_0^\infty dq'q'(q+q')\,\Psi_{q,q'}U_{q'}(t)\,U_{q+q'}(t) + G_q \quad . \tag{21}$$

Here $U_{q'}$ corresponds to $\vartheta=0$. Also $\Psi_{q,q'}$ is the value of $\Psi_{\vec{q},\vec{q}'}$ for $\vec{q}' \parallel \vec{q}$. The nonlinear terms which are proportional to U_q we have written in the form $[\alpha_q^{(n)}(t)+\alpha_q^{(p)}(t)]\,U_q$, where $\alpha_q^{(n)}$ and $\alpha_q^{(p)}$ are due to the non-Peierls and Peierls terms, respectively:

$$\alpha_q^{(n)}(t) = \frac{\omega_q}{Tn_0}\int d^3q'\,[\Phi_{\vec{q},\vec{q}'} + \Phi_{\vec{q},-\vec{q}'}]\,U_{\vec{q}'}(t)\,, \tag{22}$$

$$\alpha_q^{(p)}(t) = -\frac{\pi}{Tn_0}\int_0^q dq'q'(q-q')\,\Psi_{q,-q'}[\omega_{q-q'}U_{q'}(t) + \omega_{q'}U_{q-q'}(t)]$$

$$+\frac{2\pi}{Tn_0}\int_0^\infty dq'q'(q+q')\Psi_{q,q'}[\omega_{q'}U_{q+q'}(t)-\omega_{q+q'}U_{q'}(t)]\,. \tag{23}$$

The second and third terms on the right-hand side of (21) describe the up and down conversion, respectively. A formal solution of (21) [with the initial condition $U_q(0)=T/(2\pi)^3$] leads to the following integral equation:

$$U_q(t)=e^{\alpha_q t+r_q(t)}\times\left(\frac{T}{(2\pi)^3}+\int_0^t dt' e^{-\alpha_q t'-r_q(t')}[L_q(t')+M_q(t')+G_q]\right). \tag{24}$$

Here for brevity we have made the following notations: $L_q(t')$ and $M_q(t')$ are the second and third terms (with t' instead of t) on the right-hand side of (21);

$$r_q(t)=\int_0^t dt'[\alpha_q^{(n)}(t')+\alpha_q^{(p)}(t')]\,.$$

On the early nonlinear stage $r_q \ll 1$ (one can show that r_q may not become small only in the strongly nonlinear regime, when the acoustoelectric current becomes comparable with the ohmic current). Hence we may neglect r_q in the exponents in (24).

Now we solve (24) by iterations. Neglecting the nonlinear terms L and M in (24) we obtain the usual linear solution,[2] which in the case of the developed instability ($\alpha_q t \gg 1$) is

$$U_q^{(l)}(t)=\frac{T}{(2\pi)^3}\left(1-\frac{\alpha_q^{(0)}}{\alpha_q}\right)e^{\alpha_q t}=\frac{T}{(2\pi)^3}\left(1+\frac{1}{\gamma}\right)e^{\alpha_q t} \tag{25}$$

(in the weak-activity regime $\alpha_q=\gamma\,|\,\alpha_q^{(0)}\,|$, where $\gamma=v/v_s-1$). Now substituting (25) in the L and M of (24) and integrating over t', one obtains

$$U_q(t)=U_q^{(l)}(t)+U_q^{(u)}(t)+U_q^{(d)}(t)\,, \tag{26}$$

where

$$U_q^{(u)}(t)=\frac{T}{(2\pi)^3}\,\frac{1}{8\pi^2 n_0}\left(1+\frac{1}{\gamma}\right)^2\omega_q\int_0^q dq'q'(q-q')\Psi_{q,-q'}\,\frac{\exp[(\alpha_{q'}+\alpha_{q-q'})t]}{\alpha_{q'}+\alpha_{q-q'}-\alpha_q}\;, \tag{27}$$

$$U_q^{(d)}(t)=\frac{T}{(2\pi)^3}\,\frac{1}{4\pi^2 n_0}\left(1+\frac{1}{\gamma}\right)^2\omega_q\int_0^\infty dq'q'(q+q')\Psi_{q,q'}\,\frac{\exp[(\alpha_{q'}+\alpha_{q+q'})t]}{\alpha_{q'}+\alpha_{q+q'}-\alpha_q}\;. \tag{28}$$

$U_q^{(u)}$ and $U_q^{(d)}$ describe the up and down conversion in the early stage of the amplification.

To perform the integration in (27), (28) we shall use an approximate expression[2] for the linear amplification coefficient.

$$\alpha_q \approx \alpha_m[1-(1-q/q_m)^2]\,, \tag{29}$$

where α_m is the maximum value of α_q, which corresponds to $q=q_m=1/R$ (or to the frequency

$$f_m=v_s q_m/2\pi=(f_c f_D)^{1/2},$$

where $f_c=1/2\pi\tau_c$, $f_D=v_s^2/2\pi\mathfrak{D}$). Using (29) and also the fact that the preexponential functions in the integrals vary slowly in comparison with the exponents, one may evaluate approximately the integrals in (27), (28). This leads to

$$U_q^{(u)}(t)=\frac{T}{(2\pi)^3}\frac{1}{8\pi^2}\left(1+\frac{1}{\gamma}\right)^2\frac{q_m^3}{8n_0}\left(\frac{q}{q_m}\right)^2\left(\frac{\pi}{2\alpha_m t}\right)^{1/2}\frac{\omega_q}{2\alpha_{q/2}-\alpha_q}\Psi_{q,-q/2}\exp\left[2\alpha_m t\,\frac{q}{q_m}\left(1-\frac{1}{4}\frac{q}{q_m}\right)\right], \tag{30}$$

$$U_q^{(d)}(t)=\frac{T}{(2\pi)^3}\,\frac{1}{4\pi^2}\left(1+\frac{1}{\gamma}\right)^2\frac{q_m^3}{2n_0}\left|1-\frac{q^2}{4q_m^2}\right|\left(\frac{\pi}{2\alpha_m t}\right)^{1/2}\frac{\omega_q}{\alpha_{q_m-q/2}+\alpha_{q_m+q/2}-\alpha_q}\Psi_{q,q_m-q/2}\exp\left[2\alpha_m t\left(1-\frac{1}{4}\frac{q^2}{q_m^2}\right)\right]. \tag{31}$$

One obtains the $U_q(t)$ by summing (25), (30), and (31) [with α_q given by (29)].

The iteration procedure which we have used to solve Eq. (24) is valid only if the amount of energy concentrated in the up and down conversion regions is small in comparison with the amount of the energy in the linear amplification region. To obtain a quantitative criterion we compare the energy spectral density U_f (which is proportional to q^2U_q) at $f=2f_m$ (which corresponds to the maximum of up

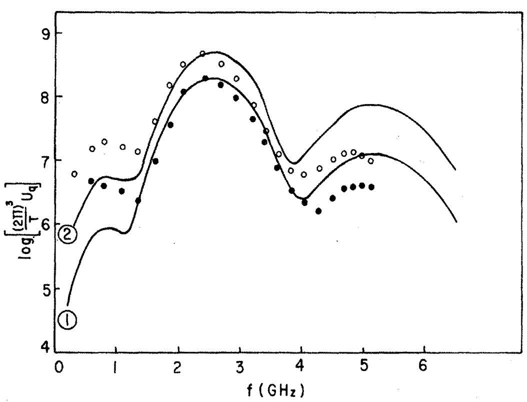

FIG. 1. Plot of $\log\{[(2\pi)^3/T]U_q\}$ vs $f=qv_s/2\pi$. Curves 1 and 2 correspond to $t_1=3.07$ μsec and $t_2=3.22$ μsec. The full and open circles are the experimental points from Ref. 7 corresponding to t_1 and t_2, respectively.

conversion) with its value at $f=f_m$. We require the ratio U_{2f_m}/U_{f_m} (denoted below by δ) to be small. We choose U_{2f_m} because it is much larger than U_f in the down conversion region. From (30), (25), and (20) we obtain the required criterion:

$$\delta \approx 8\times 10^{-4}\chi^3\left(1+\frac{1}{\gamma}\right)\frac{q_m^3}{n_0}\,\frac{f_m}{\alpha_m}\,\frac{1}{(\alpha_m t)^{1/2}}\,e^{\alpha_m t} \ll 1\,. \tag{32}$$

We shall compare our calculation with the experimental results of Parker and Bray.[7] In Ref. 7 are presented the first (and as far as we know up to now the only) quantitative observation of up and down conversion at the early stage of the acoustical noise amplification. The scattered light intensity as a function of acoustic wave frequency was measured for different times following the onset of amplification. Two of them ($t_1=3.07$ μsec and $t_2=3.22$ μsec) correspond to the early nonlinear stage. For the sample of GaAs used in Ref. 7 $\alpha_m\approx 6.3\times10^6$ sec^{-1}, $f_m\approx 2.6$ GHz; to such an f_m corresponds $q_m=5\times10^4$ cm^{-1} ($v_s=3.35\times10^5$ cm/sec). From q_m we estimated $n_0\approx 4\times10^{14}$ cm^{-3}. The electromechanical coupling constant $\chi\approx 3.8\times10^{-3}$. The experimental conditions in Ref. 7 are such that $v\gg v_s$, hence we have neglected $1/\gamma$ with respect to the unity.

In Fig. 1 the $\log\{[(2\pi)^3/T]U_q\}$ is plotted as a function of the frequency $f=qv_s/2\pi$ for two instances of time: $t_1=3.07$ μsec (curve 1) and $t_2=3.22$ μsec (curve 2). The experimental points of Ref. 7 are given in the same plot. In Ref. 7 an arbitrary logarithmic scale was used. That means that for one experimental point the ordinate may be chosen arbitrarily. We remove this arbitrariness by bringing together the theoretical and experimental points for $f=f_m$ on the curve corresponding to t_1.

One can see that quite a good qualitative agreement between experiment and theory exists, but the quantitative agreement is not so good. The main reason for the quantitative discrepancy is apparently that in the case under consideration the iteration procedure which we have used to obtain (30) and (31) begins to fail for $t\gtrsim 3.0$ μsec. From (32) it follows that when $t=3.3$ μsec the parameter $\delta\approx 1$ (at $t=3.0$ μsec $\delta\approx 0,2$). This agrees with the experimental result of Ref. 7, according to which the flux near the up conversion maximum grows proportionally to $\exp 2\alpha_m t$ (which is typical for the early nonlinear stage) only for $t<3.0$ μsec. Hence one can not expect (for $t>3.0$ μsec) the expressions (30), (31) to give a good quantitative description of the up and down conversion. For this purpose the integral equation (24) has to be solved numerically.

In the low-frequency region there is also an additional reason for the quantitative disagreement between experiment and theory, namely the approximate expression (29) for α_q. This approximation is good for the up conversion which (near its maximum) is due to the interaction between pairs of linearly amplified waves with $q\approx q_m$. But for the down conversion the situation is somewhat different. The maximum for down conversion is not at $q\approx 0$ (as one would expect), but is shifted to some value $q=q_d$. The reason for this is that $\Psi_{q,q'}\to 0$ for $q\to 0$, i.e., the "interaction parameter" becomes very small when the wave numbers (q' and $q'+q$) of the two interacting waves become close. Therefore two waves from the vicinity of q_m cannot interact effectively in the down conversion process, and a wider ($\simeq q_d$) spectrum of wave numbers is involved in this process. For such waves (29) may not be a good approximation. A rough estimate [based on neglect of the lattice attenuation and using (11) for $\alpha_q^{(e)}$ instead of (29)] gives an increase of the down conversion by a factor of about 3.

It must also be noticed that for the up conversion region the parameter ql_e for the material used in Ref. 7 is not small, so it is not clear to what extent the phenomenological theory is applicable.

In the later stage of amplification ($t\gtrsim 3.3$ μsec for the experimental conditions in Ref. 7) the specific up and down conversion peaks disappear, and the maximum of U_q is shifted from q_m to the lower values of q. At this stage the nonlinear corrections $\alpha_q^{(n)}$ and $\alpha_q^{(p)}$ to the linear amplification coefficient α_q play an important role. The equations (30), (31) [and the more general (27), (28)] do not hold any more (even qualitatively). Numerical solution of Eq. (24) would allow comparison with experiment also at this later stage of the amplification.

Finally we would like to make a remark in connection with the latest stage of the amplification, when one expects $U_q(t)$ to saturate. Yamada[9] pointed out an essential difficulty in this problem, namely, that U_q is not stabilized in the low-frequency region. This is due to the fact that α_q for small q is proportional to q^2, while $\alpha_q^{(p)}$ (which is the only nonlinear correction to α_q in Ref. 9) is proportional to q^3; hence one cannot satisfy (for $q \to 0$) the condition $\alpha_q + \alpha_q^{(p)} = 0$ for any finite U_q. We want to point out that taking $\alpha_q^{(n)}$ (which is due to the non-Peierls term) into account solves this difficulty, since as follows from (22) and (19), $\alpha_q^{(n)}$ for small q is proportional to q^2.

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