p-polarized optical properties of a metal with a diffusely scattering surface*

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A theory is developed for the p-polarized optical properties of a semi-infinite electron gas with a surface that scatters the electrons diffusely. The electron gas response is described by the Boltzmann equation. An important ingredient of the theory is the use of a legitimate distribution function for electrons leaving the surface, one which permits the normal component of the current density to vanish at the surface. It is found that the optical absorptance below the plasma frequency is an order of magnitude too large if this boundary condition on the normal component of the current is ignored by simply using the unperturbed distribution function for electrons leaving the surface. The calculated absorptance is compared for diffusely and specularly scattering surfaces. Below the plasma frequency the absorptance is higher for diffuse scattering, as is also true for s polarization, while above the plasma frequency the absorptance is essentially the same for diffuse and specular scattering. Interesting structure occurs in the vicinity of the plasma frequency.

I. INTRODUCTION

Two of us have been involved for some time now with the problem of the optical properties of a metal when the light is incident at an oblique angle. Our studies have dealt principally with the electron gas, for which the nonlocal effects attending oblique incidence when the light is p polarized can be understood in detail. These investigations began with an extension of the work of Reuter and Sondheimer[1] on the surface impedance for a semi-infinite electron gas with specular surface scattering to include oblique incidence for both s and p polarizations.[2] This work was then generalized to the case of a slab of arbitrary thickness.[3] To this point in the development, the theory was based upon the Boltzmann equation, with the most significant elements in the theory being the nonlocal transverse and longitudinal dielectric functions resulting from the Boltzmann equation. Recognizing that the major effects of nonlocality arise from wave vectors so large that the Boltzmann-type dielectric functions cease to be valid, we developed self-consistent-field dielectric functions, both transverse and longitudinal, which included a finite electron lifetime,[4,5] and used these dielectric functions in the optical theory.[6]

Turning our attention toward diffuse electron scattering, we generalized the diffuse scattering theory of Reuter and Sondheimer[1] and Dingle[7] for a semi-infinite medium to include oblique incidence for s-polarized incident light.[8] In addition, a theory of Dingle[9] for the optical properties of a metallic slab of arbitrary thickness when the surface scattering is diffuse and the light normally incident was extended to include oblique incidence for s polarization.[8]

Additional early studies relating to these problems include those of Collins[10] and Silin and Fetisov,[11] dealing with both specular and diffuse scattering for a semi-infinite electron gas. Sauter[12] and Forstmann[13] employed a multiple-wave formalism, equivalent to a specular scattering assumption, to include longitudinal effects in an optical theory for p-polarized light incident upon a half-space. This procedure was then used in a study of the optical properties of a film by Melnyk and Harrison.[14] These latter authors also investigated the question of boundary conditions appropriate within a nonlocal theory.[15] An extension of the nonlocal theory to include Fermi-liquid effects has been given by Silin and Fetisov.[16]

Conspicuous by its absence in the work on this problem is a generally valid theory for the optical properties when p-polarized light is incident on a surface for which the electron scattering is diffuse. Reuter and Sondheimer[1] suggested a scheme for generalizing their normal-incidence expression for the surface impedance to the case of oblique incidence. However, their scheme is invalid since they did not allow for the unique physical effects arising from the longitudinal field which results when p-polarized light is incident on a metallic surface. The same is true of the work of Collins.[10] Silin and Fetisov[11,16] recognized the importance of longitudinal effects but their diffuse scattering theory is incomplete.

In this paper we present an exact solution for the surface impedance for p-polarized light incident upon a semi-infinite electron gas for which the surface electron scattering is diffuse. The solution is based upon the Boltzmann equation and the results thus reflect the inherent limitations of a Boltzmann-equation-based theory.

The nonlocal nature of the present problem results in electron distribution functions within the metal which are integrals over the electric fields existing throughout the metal. The corresponding Maxwell equations are thus integral (or integro-differential) equations. For p polarization, there

exist electric field components both perpendicular and parallel to the surface. Physically this results in current components both perpendicular and parallel to the surface; mathematically it leads to coupled integro-differential equations. The basic formulation of the problem leading to these coupled equations together with a discussion of the nature of diffuse scattering for the present problem are presented in Sec. II. The coupled integro-differential equations are uncoupled in Sec. III, and the uncoupled equations are solved using the Wiener-Hopf technique in Sec. IV. Results are presented and discussed in Sec. V.

II. BASIC FORMULATION

The basic procedure in this section follows closely the development of Ref. 2. We take the metal-vacuum surface to be the $z=0$ plane, with the semi-infinite metal on the positive z side. The optical plane of incidence is chosen to be the x-z plane. All fields and currents are assumed to have a space-time dependence

$$E(x,\ y,\ z,\ t)=E(z)\,e^{i(kx-\omega t)},$$

where k is that component of the incident wave vector parallel to the metal surface. The metal is treated as a free-electron gas, whose properties are described by a linearized Boltzmann equation. We write the one-particle distribution function as

$$f=f_0+f_1(\vec{v},\ z)\,e^{i(kx-\omega t)}, \tag{2.1}$$

where f_0 is the equilibrium distribution (in principle a Fermi function, but actually a step function is the usual zero-temperature limit). The theory includes an elastic scattering or relaxation mechanism, described by a characteristic time τ, but the relaxation is to a local oscillating distribution different, in general, from the equilibrium distribution.

The traditional strategy in using the Boltzmann equation to study nonlocal electromagnetic properties is first to treat the fields as known functions of position. The Boltzmann equation is solved formally in terms of these fields, and expressions for current density are written from the resulting distribution function. This apparatus is then used to construct Maxwell's equations, which must be solved to obtain the actual fields.

Much of the present work concerns properties of the kernels appearing in the expressions for currents. At this stage in the program just outlined, the fundamental parameter describing the (free) electrons in the metal, besides the relaxation time τ, is the Fermi velocity $v_F=\beta c$. The ratio v_F/ω, roughly the distance an electron at the Fermi surface travels during an electromagnetic oscillation, forms a natural unit of length for this part of the work. So we shall use coordinates x and z, and corresponding variables of integration, as dimensionless quantities (distances multipled by ω/v_F). This implies that for an angle of incidence θ, the tangential component of the incident propagation vector is

$$k=\beta\sin\theta\,.$$

In the integrals describing currents, τ only occurs in the combination

$$a=(\omega\tau)^{-1}\,.$$

For p polarization there are x and z components of the electric field $\vec{E}$, and the magnetic field $\vec{H}$ has only a y component. From Ref. 2, with slight changes in notation, we have, for $f_1(\vec{v},\ z)$, the correction to the distribution function for an electron of velocity $\vec{v}$,

$$f_1(\vec{v},\ z)=\frac{ev}{\omega}\left(\frac{\partial f_0}{\partial\epsilon}\right)e^{-\xi z}\left(F(\vec{v})+\frac{1}{v_z}\int_0^z ds\,e^{\xi s}[v_x E_x(s)+v_z E_z(s)+vG(s)]\right). \tag{2.2}$$

Here e is the magnitude of the electron charge and ϵ the single-electron energy. The quantity ξ is defined by

$$\xi=-i(1+ia)(v/v_z)+ik(v_x/v_z)\,. \tag{2.3}$$

As already mentioned, allowance has been made for the possibility that the electron distribution relaxes to a local distribution different from the equilibrium distribution. The quantity G is a measure of the difference between these two distributions.[17] The function $F(\vec{v})$ is a (velocity-dependent) integration constant. It must be determined so that the function f_1 vanishes (the electron distribution approaches that corresponding to equilibrium) at a sufficiently great depth in the metal. This requirement defines $F(\vec{v})$ for v_z negative. For v_z positive (electron flow away from the surface) f_1 vanishes automatically at large z because of damping. But in this domain $F(\vec{v})$ will contain information about electron reflection at the surface $z=0$.

What should be implied by diffuse reflection of electrons at a surface? It is not sufficient to say that electrons are reflected with no "memory" of the direction of impact. A more precise traditional statement is that the velocity distribution of electrons leaving the surface is the equilibrium distribution at the appropriate temperature. But this statement ignores conservation of the number of particles, the requirement that as many particles leave the surface in unit time as strike it. We choose to define a diffusely reflecting surface as one in which the velocity distribution of the reflected electrons may differ from the equilibrium distribution by a quantity proportional to $\partial f_0/\partial\epsilon$ and independent of direction over a hemisphere. Thus, for $v_z>0$,

$$f_1(\vec{v},\ z=0)=\left(\frac{ev}{\omega}\right)A\left(\frac{\partial f_0}{\partial \epsilon}\right), \tag{2.4}$$

where $A=F(\vec{v})|_{v_z>0}$ is a constant. The usual definition of diffuse scattering would correspond to $A=0$. Below we shall choose A so that the z component of the current density is zero at the surface. This important physical requirement cannot be satisfied if $A=0$. It is clear that more complicated forms for the reflected distribution function could be used. However, the above is the simplest choice which is meaningful physically and manifests the diffuse scattering concept.[18]

The function $F(\vec{v})$ for $v_z<0$ can now be determined, and the solution to the Boltzmann equation completed for $f_1(\vec{v},\ z)$, the excess of the one-particle distribution function over the Fermi function. Thus, for $v_z<0$,

$$f_1(\vec{v},\ z)=-\frac{ev}{\omega v_z}\left(\frac{\partial f_0}{\partial \epsilon}\right)\int_z^\infty ds\,[v_x E_x(s) + v_z E_z(s)+vG(s)]e^{\xi(s-z)}, \tag{2.5a}$$

and for $v_z>0$,

$$f_1(\vec{v},\ z)=\frac{ev}{\omega}\left(\frac{\partial f_0}{\partial \epsilon}\right)\left(Ae^{-\xi z}+\frac{1}{v_z}\int_0^z ds\,[v_x E_x(s) + v_z E_z(s)+vG(s)]e^{-\xi(z-s)}\right). \tag{2.5b}$$

Current densities are obtained by multiplying the distribution function by the electron charge and the appropriate velocity component, then integrating over velocities. The quantity G is determined (self-consistently) by a similar integration over the distribution function.[2] The basic expressions which result are

$$G(z)(2/a)=-AK_z(z)+\int_0^\infty ds\,[iE_x(s)K_x(z-s) - E_z(s)K_z(z-s)+G(s)K(z-s)], \tag{2.6}$$

$$J_x(z)[2/(3a\sigma_0)]=iAK_{xz}(z)+\int_0^\infty ds\,[-E_x(s)K_{xx}(z-s) + iE_z(s)K_{xz}(z-s)+iG(s)K_x(z-s)], \tag{2.7a}$$

and

$$J_z(z)[2/(3a\sigma_0)]=AK_{zz}(z)+\int_0^\infty ds\,[iE_x(s)K_{xz}(z-s) + E_z(s)K_{zz}(z-s)-G(s)K_z(z-s)]. \tag{2.7b}$$

σ_0 is formally the standard low-frequency conductivity of the metal, expressed in terms of the (free) electron density n or the plasma frequency ω_p as

$$\sigma_0=ne^2\tau/m=\omega_p^2\tau/4\pi.$$

This relation is perhaps more formal or dimensional than physical, since the low-frequency conductivity cannot play a physical role for optical frequencies.

The kernels appearing in Eqs. (2.6) and (2.7) are identical, except for notation, with those of Ref. 2. Their definitions will be repeated and their properties discussed in Appendix A.

We introduce the notation[19]

$$C(z)=\frac{dE_x(z)}{dz}-ikE_z(z), \tag{2.8}$$

$$D(z)=ikE_x(z)+\frac{dE_z(z)}{dz} \tag{2.9}$$

for the y component of the curl and the divergence, respectively, of the electric field in the metal. For p polarization, Faraday's law gives

$$H(z)\equiv H_y(z)=-i\beta^{-1}C(z). \tag{2.10}$$

The remaining Maxwell equations are

$$\frac{dC(z)}{dz}+\beta^2E_x(z)=-4\pi i\beta^2\omega^{-1}J_x(z) \tag{2.11a}$$

and

$$-ikC(z)+\beta^2E_z(z)=-4\pi i\beta^2\omega^{-1}J_z(z). \tag{2.11b}$$

Within the metal, the current densities are given by expressions (2.7). Equations (2.6), (2.7), and (2.11) thus form a set of coupled equations for E_x, E_z, and G. Along with the boundary condition that J_z must vanish at the boundary $z=0$, these equations describe completely the variation of the fields within the metal.

III. UNCOUPLING THE INTEGRAL EQUATIONS

The uncoupling of these equations followed from the realization that equations for $C(z)$ and $D(z)$ derived from Eqs. (2.11) are naturally uncoupled. In addition, the electric fields themselves can be obtained from C and D together with one integration constant, as will be seen below.

We proceed to calculate the divergence and curl of the current densities. If we use Eqs. (A15) to eliminate the derivatives of the kernels that arise, we find from (2.7)

$$\operatorname{div}\vec{J}=ikJ_x+\frac{dJ_z}{dz}$$
$$=\tfrac{3}{2}ia\sigma_0(1+ia)\left(-AK_z(z)+\int_0^\infty ds\,[iE_x(s)K_x(z-s) - E_z(s)K_z(z-s)+G(s)K(z-s)]\right)+3a\sigma_0G(z). \tag{3.1}$$

The integral in this expression is precisely that of Eq. (2.6). So

$$\operatorname{div}\vec{J}=3i\sigma_0G, \tag{3.2}$$

and, from Eqs. (2.11),

$$D=-4\pi i\omega^{-1}\operatorname{div}\vec{J}=3\omega^{-1}\omega_p^2\tau G. \tag{3.3}$$

G is just a multiple $(v_F/3\sigma_0)$ of the charge density, and its inclusion in the Boltzmann equation is equiv-

alent to allowing relaxation to a distribution with a nonvanishing local charge density.

Now use Eq. (A15b) to eliminate K_z from (3.1). An integration by parts then yields

$$\operatorname{div}\vec{J} - 3a\sigma_0 G$$

$$= 3i(1+ia)\sigma_0 G = \tfrac{3}{2}\sigma_0 ai(1+ia)\Big[-AK_z(z) + k^{-1}E_z(0)K_x(z) + \frac{1}{k}\int_0^\infty ds\left(ikE_x(s) + \frac{dE_z(s)}{ds}\right)K_x(z-s) + \int_0^\infty ds\, G(s)K(z-s)\Big]. \tag{3.4}$$

The divergence of the current density is expressed by an integral involving the electric field only through its divergence. So the divergence of the field satisfies an integral equation that does not involve the transverse part of the field. A slight manipulation of Eqs. (3.3) and (3.4) puts this integral equation in the form

$$2kD(z) = 3(\omega_b/\omega)^2\left(-kAK_z(z) + E_z(0)K_x(z) + \int_0^\infty ds\, D(s)K_x(z-s)\right) + ka\int_0^\infty ds\, D(s)K(z-s). \tag{3.5}$$

Now we calculate the y component of the curl of $\vec{J}$ from Eqs. (2.7). In the resulting integral expression, the coefficient of G vanishes by virtue of Eq. (A15b). The derivative of K_{xz}, which appears in the coefficient of E_z within the integral and in the coefficient of A, can be eliminated by use of Eq. (A15e) or (A15f). Then the coefficient of E_z within the integral takes the form

$$i[k(K_{xx} - K_{zz}) + i(1+ia)K_x] = iz^{-1}K_{xz},$$

the final form by Eqs. (A13) and (A14). For convenience, we define

$$K_c = (kz)^{-1}K_{xz}. \tag{3.6}$$

Equations (A15b) and (A15g) can be used to show that the coefficient of E_x in the integrand of $(\operatorname{curl}\vec{J})_y$ is proportional to the derivative of K_c. As a result

$$(\operatorname{curl}\vec{J})_y = \frac{dJ_x(z)}{dz} - ikJ_z(z)$$

$$= \tfrac{3}{2}a\sigma_0\Big[iAkK_c(z) + \int_0^\infty ds\left(E_x(s)\frac{dK_c(z-s)}{ds} + ikE_z(s)K_c(z-s)\right)\Big]$$

$$= \tfrac{3}{2}a\sigma_0\left([iAk - E_x(0)]K_c(z) - \int_0^\infty ds\, C(s)K_c(z-s)\right). \tag{3.7}$$

Thus $(\operatorname{curl}\vec{J})_y$ is expressed in terms of an integral that involves the electric field only through its curl. So Maxwell's equation for the curl of the electric field becomes an integro-differential equation that does not involve the longitudinal part of the field:

$$\left(\frac{d^2}{dz^2} + g^2\right)C(z) = -4\pi i\beta^2\omega^{-1}(\operatorname{curl}\vec{J})_y = ib\left([E_x(0) - iAk]K_c(z) + \int_0^\infty ds\, C(s)K_c(z-s)\right). \tag{3.8}$$

The constants in this equation are

$$g^2 = \beta^2 - k^2 = \beta^2\cos^2\theta, \tag{3.9a}$$

the square of the z component of the incident wave vector, and

$$b = \tfrac{3}{2}(\beta\omega_b/\omega)^2. \tag{3.9b}$$

The kernel K_c relates $(\operatorname{curl}\vec{J})_y$ to $(\operatorname{curl}\vec{E})_y = C$. Properties of K_c follow from the relation

$$K_c(z) = -\tfrac{1}{2}[(1+ia)^2F_2(z) + F_1(z)], \tag{3.10}$$

the F functions defined in Eqs. (A8). The asymptotic behavior for large argument is

$$K_c(z) \sim i(1+ia)^{-3}Qz^{-2}e^{-Q|z|}, \tag{3.11}$$

with Q defined by (A5). For small values of z,

$$K_c(z) = \tfrac{1}{2}(\ln\{\tfrac{1}{2}[Q - i(1+ia)]|z|\} + \gamma - ik^{-2}(1+ia)[Q + i(1+ia)]) + O(z), \tag{3.12}$$

where γ, Euler's constant, is equal to $0.577\cdots$.

Equations (3.5) and (3.8) are the two uncoupled integral equations that describe the behavior of the fields within the metal.

IV. SOLUTIONS TO THE INTEGRAL EQUATIONS

Equations (3.5) and (3.8) can be solved by the Wiener-Hopf technique. Since these equations are valid only for z positive, it is convenient here to define $C(z)$ and $D(z)$ to be zero for z negative. Consider first Eq. (3.8).

Define a new function $M_1(z)$ by

$$M_1(z) = \begin{cases} 0, & z > 0 \\ ib\left([E_x(0) - iAk]K_c(z) + \int_0^\infty ds\, C(s)K_c(z-s)\right), & z < 0. \end{cases} \tag{4.1}$$

Add Eqs. (3.8) and (4.1) and take the two-sided Laplace transform of the sum. In terms of the transform variable p, the result is[20]

$$\mathfrak{M}_1(p) - h = ib[E_x(0) - iAk]\mathfrak{K}_c(p) + [ib\mathfrak{K}_c(p) - p^2 - g^2]\mathfrak{C}(p), \tag{4.2}$$

where we have defined

$$h = \left(\frac{dC(z)}{dz} + pC(z)\right)\Big|_{z=0}. \tag{4.3}$$

$\mathfrak{M}_1(p)$ is a function regular in the half-space to the left of Q. $C(z)$ must fall off for large z as some power of z times e^{-Qz} and so $\mathfrak{C}(p)$ must be regular in the half-space to the right of $-Q$. $\mathfrak{K}_c(p)$ is known to be regular in the strip $-\mathrm{Re}Q < \mathrm{Re}p < \mathrm{Re}Q$. Thus Eq. (4.2) is valid in this strip.

The Wiener-Hopf method requires breaking the coefficient of $\mathfrak{C}(p)$ into two factors, each of which is regular in a right or left half-plane. For this purpose, write

$$ib\mathfrak{K}_c(p) - p^2 - g^2 = (N^2 - p^2)\mathfrak{U}(p), \tag{4.4}$$

where N is any number with a positive real part. For the present it is convenient to think of N as real and $N > \mathrm{Re}Q$. $\mathfrak{U}(p)$ approaches unity as p becomes arbitrarily large in any direction. In addition, we shall now show that $\mathfrak{U}(p)$ has no roots in a strip of finite width centered about the imaginary axis.

There is no question of roots of $\mathfrak{U}(p)$ for arbitrarily large imaginary values of p. For finite pure-imaginary values of p, the real part of $\mathfrak{K}_c(p)$, and hence the imaginary part of $\mathfrak{U}(p)$, is negative and bounded away from zero. Consider $\mathfrak{K}_c(p)$ as p starts from the value k, goes along the real axis to the origin, and then goes up or down the imaginary axis to $\pm i\infty$. Along this entire path P [defined by Eq. (A3a)] is pure imaginary, say $P = i\rho$, where ρ is real, starts out at zero along this path, and increases monotonically.[21] For small P and ρ

$$\mathfrak{K}_c(p) = -\tfrac{2}{3}i(1 + ia)^{-1} + O(P),$$

which has a negative real part as long as $a = (\omega\tau)^{-1}$ does not vanish. By differentiating $\rho^3\mathfrak{K}_c(p)$ one can show that the real part of this function for pure-imaginary P (real ρ) is a monotonic decreasing function of ρ. Thus, except possibly in the limit $P \to i\infty$ ($\rho \to \infty$), the imaginary part of $\mathfrak{U}(p)$ is bounded away from zero, which demonstrates the existence of a strip of finite width centered on the imaginary axis in which $\mathfrak{U}(p)$ has no roots as long as a is finite.[22]

In this strip, we write

$$\mathfrak{U}(p) = \mathfrak{U}_R(p)/\mathfrak{U}_L(p), \tag{4.5}$$

so that $\ln\mathfrak{U}(p)$, an analytic function, can be written[23]

$$\ln\mathfrak{U}(p) = -\ln\mathfrak{U}_L(p) + \ln\mathfrak{U}_R(p) = \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} dq\,\frac{\ln\mathfrak{U}(q)}{q-p} + \frac{1}{2\pi i}\int_{-d-i\infty}^{-d+i\infty} dq\,\frac{\ln\mathfrak{U}(q)}{p-q}, \tag{4.6}$$

where d is a positive quantity limited by the half-width of the strip. We have thus defined the function $\mathfrak{U}_R(p)$ to be regular in the half-plane $\mathrm{Re}p > -d$, and $\mathfrak{U}_L(p)$ to be regular in the half-plane $\mathrm{Re}p < d$.[24]

Now multiply Eq. (4.2) by $\mathfrak{U}_L(p)/(N-p)$. We find

$$(N-p)^{-1}\mathfrak{U}_L(p)\{\mathfrak{M}_1(p) - h - (p^2+g^2)[E_x(0) - iAk]\} = (N+p)\mathfrak{U}_R(p)[E_x(0) - iAk + \mathfrak{C}(p)]. \tag{4.7}$$

The equality in Eq. (4.7) is valid in the strip of width $2d$ about the imaginary axis. But by the usual Wiener-Hopf argument the left member of (4.7) is analytic in a left half-plane, likewise the right member is analytic in a right half-plane, and the two half-planes overlap. Thus, the two members of Eq. (4.7) define a single function analytic everywhere in the finite p plane.

The precise form of this function can be determined by inspecting conditions for large values of $\mathrm{Re}p$. In this limit, $\mathfrak{C}(p) \sim C(z=0)/p \equiv C(0)/p$, and $\ln\mathfrak{U}_R(p) \sim u/p$, where u is a constant. So

$$(N+p)\mathfrak{U}_R(p)[E_x(0) - iAk + \mathfrak{C}(p)] = (p+N+u)[E_x(0) - iAk] + C(0), \tag{4.8}$$

or, rearranging,

$$\mathfrak{C}(p) = \frac{C(0)}{(p+N)\mathfrak{U}_R(p)} + \left(\frac{p+N+u}{(p+N)\mathfrak{U}_R(p)} - 1\right) \times [E_x(0) - iAk]. \tag{4.9}$$

Previously we asserted that the value of N was immaterial. It is easy to show, by differentiating, that in fact $(N+p)\mathfrak{U}_R(p)$ (and, consequently, $N+u$) is independent of N.

Equation (3.5) for the divergence of the field is treated in a very similar fashion. Define a new function $M_2(z)$ by

$$M_2(z) = \begin{cases} 0, & z > 0 \\ 3(\omega_p/\omega)^2\left(E_z(0)K_x(z) - kAK_z(z) + \int_0^\infty ds\,D(s)K_x(z-s)\right) + ak\int_0^\infty ds\,D(s)K(z-s), & z < 0. \end{cases} \tag{4.10}$$

Add Eqs. (3.5) and (4.10) and take the two-sided Laplace transform of the sum. The result is

$$\mathfrak{M}_2(p) = 3(\omega_p/\omega)^2[E_z(0)\mathcal{K}_x(p) - kA\mathcal{K}_z(p)] + [3(\omega_p/\omega)^2\mathcal{K}_x(p) + ak\mathcal{K}(p) - 2k]\mathfrak{D}(p). \quad (4.11)$$

$\mathfrak{M}_2(p)$ is a function regular in the half of p space to the left of Q. $\mathfrak{D}(p)$ is regular in some right half-space, presumably in the half-space to the right of $-Q$. So Eq. (4.11) is valid in the strip $-\mathrm{Re}Q < \mathrm{Re}p < \mathrm{Re}Q$.

Define a function $\mathcal{T}(p)$ by

$$2k\mathcal{T}(p) = 2k - 3(\omega_p/\omega)^2\mathcal{K}_x(p) - ak\mathcal{K}(p). \quad (4.12)$$

$\mathcal{T}(p)$ approaches 1 when p is large in magnitude, and it has no roots on or arbitrarily close to the imaginary axis for finite a, as is demonstrated in Appendix B. Thus $\mathcal{T}(p)$ can be factored as

$$\mathcal{T}(p) = \mathcal{T}_R(p)/\mathcal{T}_L(p), \quad (4.13)$$

so that

$$\ln\mathcal{T}(p) = \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} dq \frac{\ln\mathcal{T}(q)}{q-p} + \frac{1}{2\pi i}\int_{-d-i\infty}^{-d+i\infty} dq \frac{\ln\mathcal{T}(q)}{p-q} = -\ln\mathcal{T}_L(p) + \ln\mathcal{T}_R(p), \quad (4.14)$$

where again d is a positive quantity limited by the distance from the imaginary axis to any roots of $\mathcal{T}(p)$. The functions $\mathcal{T}_R(p)$ and $\mathcal{T}_L(p)$ are regular in the half-plane $\mathrm{Re}p > -d$ and in the half-plane $\mathrm{Re}p < d$, respectively.

Equation (4.11) can be rearranged to read

$$[E_z(0) - \mathfrak{M}_2(p)/2k]\mathcal{T}_L(p) = \tfrac{1}{2}[3(\omega_p/\omega)^2A\mathcal{K}_z(p) + aE_z(0)\mathcal{K}(p)]\mathcal{T}_L(p) + [\mathfrak{D}(p) + E_z(0)]\mathcal{T}_R(p). \quad (4.15)$$

The left member of Eq. (4.15) is regular in the half-plane $\mathrm{Re}p < d$. But the right member still has singularities in the half-planes on each side of the imaginary axis. To separate these singularities, define

$$\mathcal{W}_1(p) = \mathcal{K}(p)\mathcal{T}_L(p) = \frac{1}{2\pi i}\int_{d'-i\infty}^{d'+i\infty} dq \frac{\mathcal{K}(q)\mathcal{T}_L(q)}{q-p} + \frac{1}{2\pi i}\int_{-d''-i\infty}^{-d''+i\infty} dq \times \frac{\mathcal{K}(q)\mathcal{T}_L(q)}{p-q} = \mathcal{W}_{1L}(p) + \mathcal{W}_{1R}(p) \quad (4.16a)$$

and

$$\mathcal{W}_2(p) = \mathcal{K}_z(p)\mathcal{T}_L(p) = \frac{1}{2\pi i}\int_{d'-i\infty}^{d'+i\infty} dq \frac{\mathcal{K}_z(q)\mathcal{T}_L(q)}{q-p} + \frac{1}{2\pi i}\int_{-d''-i\infty}^{-d''+i\infty} \times \frac{\mathcal{K}_z(q)\mathcal{T}_L(q)}{p-q} = \mathcal{W}_{2L}(p) + \mathcal{W}_{2R}(p). \quad (4.16b)$$

The positive quantities d' and d'' satisfy $d' < d$ and $d'' < \mathrm{Re}Q$. The $\mathcal{W}_R$ functions are regular in the half-plane to the right of $-Q$; the $\mathcal{W}_L$ functions are regular in the half-plane $\mathrm{Re}p < d'$.

Equation (4.15) can now be written in the Wiener-Hopf form,

$$[E_z(0) - \mathfrak{M}_2(p)/2k]\mathcal{T}_L(p) - \tfrac{3}{2}(\omega_p/\omega)^2A\mathcal{W}_{2L}(p) - \tfrac{1}{2}aE_z(0)\mathcal{W}_{1L}(p) = \tfrac{3}{2}(\omega_p/\omega)^2A\mathcal{W}_{2R}(p) + \tfrac{1}{2}aE_z(0)\mathcal{W}_{1R}(p) + [\mathfrak{D}(p) + E_z(0)]\mathcal{T}_R(p). \quad (4.17)$$

Equation (4.17) is valid in a strip about the imaginary axis. But each member is regular in a half-plane extending in opposite directions from this strip. The two members between them define a single function of p analytic everywhere. This function must be a constant, and it is easy to see from the right member of Eq. (4.17), evaluated at large values of p, that the value of the constant is $E_z(0)$, that is,

$$\tfrac{3}{2}(\omega_p/\omega)^2A\mathcal{W}_{2R}(p) + \tfrac{1}{2}aE_z(0)\mathcal{W}_{1R}(p) + [\mathfrak{D}(p) + E_z(0)]\mathcal{T}_R(p) = E_z(0). \quad (4.18)$$

So we have, finally,

$$\mathfrak{D}(p) = \left(\frac{1 - \frac{1}{2}a\mathcal{W}_{1R}(p)}{\mathcal{T}_R(p)} - 1\right)E_z(0) - \frac{3}{2}\left(\frac{\omega_p}{\omega}\right)^2\frac{\mathcal{W}_{2R}(p)}{\mathcal{T}_R(p)}A. \quad (4.19)$$

Following some necessary preliminaries, Eqs. (4.9) and (4.19) will be used to develop an expression for the surface impedance.

A. Electric fields

From the original integral equations in the electric field components, Eqs. (2.11), uncoupled equations (3.5) and (3.8) were obtained for D, the divergence, and C, the y component of the curl. Formal solutions for these quantities were obtained above. We now discuss two closely related questions: How may the field components be obtained from D and C, and to what extent do these components satisfy the original equations (2.11)?

For the x and z components of the electric field with the spatial dependence

$$E(x, y, z) = E(z)e^{ikx},$$

and which do not become exponentially large in the limit of large z, it is easy to show that

$$E_x(z) = Be^{-kz} + \frac{1}{2}\int_0^z ds\,[C(s) - iD(s)]e^{-k(z-s)} - \frac{1}{2}\int_z^\infty ds\,[C(s) + iD(s)]e^{-k(s-z)} \quad (4.20a)$$

and

$$E_z(z) = iBe^{-kz} + \tfrac{1}{2}i\int_0^z ds\,[C(s) - iD(s)]e^{-k(z-s)} + \tfrac{1}{2}i\int_z^\infty ds\,[C(s) + iD(s)]e^{-k(s-z)}. \quad (4.20b)$$

B is a constant still to be determined.

It is useful to have expressions for the field components written directly in terms of the Laplace transforms of C and D. Because the fields have been taken to be zero for negative values of the position coordinate,[25] the integrals from 0 to z in (4.20) may be extended to $-\infty$. Thus

$$E_x(z) = Be^{-kz} + \frac{1}{4\pi i}\int_{-i\infty}^{i\infty} dp\, e^{pz}\left(\frac{\mathfrak{C}(p) - i\mathfrak{D}(p)}{k+p} - \frac{\mathfrak{C}(p) + i\mathfrak{D}(p)}{k-p}\right) \quad (4.21a)$$

and

$$E_z(z) = iBe^{-kz} + \frac{1}{4\pi}\int_{-i\infty}^{i\infty} dp\, e^{pz}\left(\frac{\mathfrak{C}(p) - i\mathfrak{D}(p)}{k+p} + \frac{\mathfrak{C}(p) + i\mathfrak{D}(p)}{k-p}\right). \quad (4.21b)$$

Equations (4.20) and (4.21) are only valid for positive z.

The results of applying Eqs. (4.20) or (4.21) for $z = 0+$ are

$$E_x(0) = B - \tfrac{1}{2}[\mathfrak{C}(k) + i\mathfrak{D}(k)] \quad (4.22a)$$

and

$$E_z(0) = i\{B + \tfrac{1}{2}[\mathfrak{C}(k) + i\mathfrak{D}(k)]\}. \quad (4.22b)$$

To show to what extent any solution of Eqs. (3.5) and (3.8) is a solution of Eqs. (2.11), and to develop some additional vital relations, we outline a procedure by which Eqs. (2.11) might be derived from Eqs. (3.5) and (3.8). The manipulations of the kernels, so essential in Sec. III, are all clearly reversible and therefore irrelevant here. The problem is one of integration.

Let $X_l(z)$ and $X_r(z)$ represent, respectively, the left and right members of Eq. (2.11a). Similarly let $Z_l(z)$ and $Z_r(z)$ represent the corresponding members of Eq. (2.11b). These functions are all defined for non-negative values of z and are bounded for large z. For each subscript, define

$$\Gamma(z) = \frac{dX(z)}{dz} - ikZ(z) \quad (4.23a)$$

and

$$\Delta(z) = ikX(z) + \frac{dZ(z)}{dz}. \quad (4.23b)$$

Integration by parts yields the following identities, suggested by Eqs. (4.20):

$$\frac{1}{2}\int_0^z ds\,[\Gamma(s) - i\Delta(s)]e^{-k(z-s)} - \frac{1}{2}\int_z^\infty ds\,[\Gamma(s) + i\Delta(s)]e^{-k(s-z)} = X(z) - \tfrac{1}{2}[X(0) - iZ(0)]e^{-kz} \quad (4.24a)$$

and

$$\tfrac{1}{2}i\int_0^z ds\,[\Gamma(s) - i\Delta(s)]e^{-k(z-s)} + \tfrac{1}{2}i\int_z^\infty ds\,[\Gamma(s) + i\Delta(s)]e^{-k(s-z)} = Z(z) - \tfrac{1}{2}i[X(0) - iZ(0)]e^{-kz}. \quad (4.24b)$$

So it follows from Eqs. (3.5), (3.8), and (4.24) that

$$X_l(z) - \tfrac{1}{2}[X_l(0) - iZ_l(0)]e^{-kz} = X_r(z) - \tfrac{1}{2}[X_r(0) - iZ_r(0)]e^{-kz} \quad (4.25a)$$

and

$$Z_l(z) - \tfrac{1}{2}i[X_l(0) - iZ_l(0)]e^{-kz} = Z_r(z) - \tfrac{1}{2}i[X_r(0) - iZ_r(0)]e^{-kz}. \quad (4.25b)$$

From either of Eqs. (4.25), we then have

$$X_l(0) + iZ_l(0) = X_r(0) + iZ_r(0). \quad (4.26)$$

To guarantee that the fields of Eqs. (4.20) are solutions to Eqs. (2.11), it is thus sufficient to choose the integration constant B so that $Z_l(0) = Z_r(0)$. At the metal surface, J_z must vanish. So from Eq. (2.11b)

$$E_z(0) = ik\beta^{-2}C(0). \quad (4.27)$$

This equation can be combined with Eqs. (4.22), either to solve for B, as

$$2B = E_x(0) + k\beta^{-2}C(0), \quad (4.28a)$$

or to express the boundary condition in a form from which B has been eliminated, as

$$-E_x(0) + k\beta^{-2}C(0) = \mathfrak{C}(k) + i\mathfrak{D}(k). \quad (4.28b)$$

Since the surface impedance is proportional to $E_x(0)/H_y(0)$, or to $E_x(0)/C(0)$, Eq. (4.28b) is the essential equation for determining this fundamental quantity. Inserting into Eq. (4.28b) the forms of the Laplace transforms, Eqs. (4.9) and (4.19), gives the relation

$$\frac{1}{(k+N)\mathfrak{U}_R(k)} - \frac{1 - \frac{1}{2}a\mathfrak{W}_{1R}(k)}{\mathfrak{T}_R(k)}k\beta^{-2} + \frac{k+N+u}{(k+N)\mathfrak{U}_R(k)}\frac{E_x(0)}{C(0)} = i\left[\frac{3}{2}\left(\frac{\omega_p}{\omega}\right)^2\frac{\mathfrak{W}_{2R}(k)}{\mathfrak{T}_R(k)} + \left(\frac{k+N+u}{(k+N)\mathfrak{U}_R(k)} - 1\right)k\right]\frac{A}{C(0)}. \quad (4.29)$$

B. J_z ($z=0$) condition

The essential condition that has not been incorporated up to this point is the requirement that the normal current density vanishes at the metal-vacuum surface. This requirement will determine the value of the distribution-function constant A that first appeared in Eq. (2.4), and that still appears in expression (4.29).

The normal current density at the surface is obtained by setting $z = 0$ in Eq. (2.7b). In the last term in the integral, eliminate G by Eq. (3.3). The range of integration for this last term may be extended to $-\infty$, because all fields have been taken

to vanish for negative values of the position coordinate. In the other terms of the integrand of (2.7b), use the expressions (4.21) for the electric fields. The limits of integration must not be altered in these terms, because of the limited validity of Eqs. (4.21). The coefficient of A is obtained from Eq. (A12f). The resulting expression for normal current density at the surface is then

$$J_z(0)=\tfrac{3}{2}a\sigma_0\Big[\tfrac{1}{2}A+iB[\mathcal{K}_{zz+}(k)-\mathcal{K}_{xz+}(k)]+\frac{1}{4\pi}\int_{-i\infty}^{i\infty}dp\Big(\frac{\mathcal{C}(p)-i\mathcal{D}(p)}{k+p}[\mathcal{K}_{zz+}(-p)-\mathcal{K}_{xz+}(-p)]$$
$$+\frac{\mathcal{C}(p)+i\mathcal{D}(p)}{k-p}[\mathcal{K}_{zz+}(-p)+\mathcal{K}_{xz+}(-p)]\Big)-\frac{1}{3}\frac{\omega}{\omega_p^2\tau}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dp\,\mathcal{D}(p)\mathcal{K}_z(p)\Big]. \tag{4.30}$$

The subscripts + indicate one-sided Laplace transforms which are discussed in Appendix C.

We now choose A so that $J_z(0)=0$. B can be eliminated by using (4.28a), $E_z(0)$ eliminated by using (4.27), and $\mathcal{C}(p)$ and $\mathcal{D}(p)$ replaced by the expressions (4.9) and (4.19). The equation then resulting from $J_z(0)=0$ is

$$0=A\{\tfrac{1}{2}-ikI_2+i[\tfrac{3}{2}(\omega_p/\omega)^2I_4-aI_6]\}+E_x(0)\{\tfrac{1}{2}i[\mathcal{K}_{zz+}(k)-\mathcal{K}_{xz+}(k)]+I_2\}$$
$$+C(0)\{(\tfrac{1}{2}ik\beta^{-2})[\mathcal{K}_{zz+}(k)-\mathcal{K}_{xz+}(k)]+I_1+(k/\beta^2)(I_3-\tfrac{2}{3}\omega\omega_p^{-2}\tau^{-1}I_5)\}. \tag{4.31}$$

In Eq. (4.31), a new kind of integrals appears:

$$I_1=\frac{1}{4\pi}\int_{-i\infty}^{i\infty}\frac{dp}{(p+N)\mathfrak{u}_R(p)}\left(\frac{\mathcal{K}_{zz+}(-p)-\mathcal{K}_{xz+}(-p)}{k+p}+\frac{\mathcal{K}_{zz+}(-p)+\mathcal{K}_{xz+}(-p)}{k-p}\right), \tag{4.32a}$$

$$I_2=\frac{1}{4\pi}\int_{-i\infty}^{i\infty}dp\left(\frac{p+N+u}{(p+N)\mathfrak{u}_R(p)}-1\right)\left(\frac{\mathcal{K}_{zz+}(-p)-\mathcal{K}_{xz+}(-p)}{k+p}+\frac{\mathcal{K}_{zz+}(-p)+\mathcal{K}_{xz+}(-p)}{k-p}\right), \tag{4.32b}$$

$$I_3=\frac{1}{4\pi}\int_{-i\infty}^{i\infty}dp\left(\frac{1-\frac{1}{2}a\mathfrak{W}_{1R}(p)}{\mathcal{T}_R(p)}-1\right)\left(\frac{\mathcal{K}_{zz+}(-p)-\mathcal{K}_{xz+}(-p)}{k+p}-\frac{\mathcal{K}_{zz+}(-p)+\mathcal{K}_{xz+}(-p)}{k-p}\right), \tag{4.32c}$$

$$I_4=\frac{1}{4\pi}\int_{-i\infty}^{i\infty}dp\,\frac{\mathfrak{W}_{2R}(p)}{\mathcal{T}_R(p)}\left(\frac{\mathcal{K}_{zz+}(-p)-\mathcal{K}_{xz+}(-p)}{k+p}-\frac{\mathcal{K}_{zz+}(-p)+\mathcal{K}_{xz+}(-p)}{k-p}\right), \tag{4.32d}$$

$$I_5=\frac{1}{4\pi}\int_{-i\infty}^{i\infty}dp\left(\frac{1-\frac{1}{2}a\mathfrak{W}_{1R}(p)}{\mathcal{T}_R(p)}-1\right)\mathcal{K}_z(p), \tag{4.32e}$$

$$I_6=\frac{1}{4\pi}\int_{-i\infty}^{i\infty}dp\,\frac{\mathfrak{W}_{2R}(p)}{\mathcal{T}_R(p)}\mathcal{K}_z(p). \tag{4.32f}$$

All the integrands of the I integrals are analytic in the right half-plane except for poles at $p=k$ [arising from the terms $k-p$ in the denominators of I_1, I_2, I_3, I_4) and branch cuts running to the right from $p=Q$, and all fall off at least as fast as p^{-2} for large values of p. So the integrations along the imaginary axis may be replaced by residues at $p=k$ plus integrations around the branch cut. The contribution to the right-hand side of (4.31) arising from the poles at $p=k$ is

$$\tfrac{1}{2}i\left(-E_x(0)+\frac{k}{\beta^2}C(0)\right)\left[-\frac{1}{2k}+\frac{i}{3(1+ia)}\right.$$
$$\left.+\frac{(1+ia)^2}{3k^3}\left(1+\frac{iQ^3}{(1+ia)^3}\right)\right]. \tag{4.33}$$

The one-sided transforms of the kernels are expressed in Appendix C in terms of a logarithmic function L_+ defined in Eq. (C1). It is the difference in value of this function on the two sides of the cut (a difference of $2\pi i$) that prevents the integration above and below the cut from canceling. Thus the parts of the I integrals coming from integration around the branch cuts are

$$I_1'=-\tfrac{1}{2}ik\int_Q^\infty\frac{dp}{(p+N)\mathfrak{u}_R(p)}\left(\frac{(1+ia)^2}{P^5}+\frac{1}{P^3}\right), \tag{4.34a}$$

$$I_2'=-\tfrac{1}{2}ik\int_Q^\infty dp\left(\frac{p+N+u}{(p+N)\mathfrak{u}_R(p)}-1\right)\left(\frac{(1+ia)^2}{P^5}+\frac{1}{P^3}\right), \tag{4.34b}$$

$$I_3'=-i(1+ia)^2\int_Q^\infty dp\left(\frac{1-\frac{1}{2}a\mathfrak{W}_{1R}(p)}{\mathcal{T}_R(p)}-1\right)\frac{p}{P^5}, \tag{4.34c}$$

$$I_4'=-i(1+ia)^2\int_Q^\infty\left(\frac{\mathfrak{W}_{2R}(p)}{\mathcal{T}_R(p)}\right)\frac{p}{P^5}, \tag{4.34d}$$

$$I_5=\tfrac{1}{2}(1+ia)\int_Q^\infty dp\left(\frac{1-\frac{1}{2}a\mathfrak{W}_{1R}(p)}{\mathcal{T}_R(p)}-1\right)\frac{p}{P^5}, \tag{4.34e}$$

$$I_6=\tfrac{1}{2}(1+ia)\int_Q^\infty dp\left(\frac{\mathfrak{W}_{2R}(p)}{\mathcal{T}_R(p)}\right)\frac{p}{P^5}. \tag{4.34f}$$

In Eqs. (4.34), the paths of integration start at the point $p=Q$ and then proceed outward away from the

imaginary axis.

Using the results of Appendix C and expression (4.33), Eq. (4.31) can be written in terms of the integrals (4.34) as

$$\frac{A}{C(0)}\left\{\tfrac{1}{2}-ikI_2'+i\left[\frac{3}{2}\left(\frac{\omega_p}{\omega}\right)^2 I_4'-aI_6\right]\right\}$$
$$+\frac{E_x(0)}{C(0)}\left\{i\left[\frac{1}{2k}-\frac{(1+ia)^2}{3k^3}\left(1+\frac{iQ^3}{(1+ia)^3}\right)\right]+I_2'\right\}$$
$$+\frac{k}{\beta^2}\left(-\frac{1}{3(1+ia)}+I_3'-\frac{2\omega}{3\omega_p^2\tau}I_5\right)+I_1'=0. \qquad (4.35)$$

Equations (4.35) and (4.29) can then be solved for $E_x(0)/C(0)$, which is related to the surface impedance Z_p via

$$Z_p=i\beta[E_x(0)/C(0)]. \qquad (4.36)$$

Alternatively, Eqs. (4.29) and (4.31) can be solved for $E_x(0)/C(0)$, with the final expression for the surface impedance then being

$$Z_p=-i\beta\left(\frac{S_1Q_1+S_3Q_3}{S_2Q_1+S_3Q_2}\right), \qquad (4.37)$$

where

$$S_1=\frac{1}{(k+N)\mathfrak{u}_R(k)}-\frac{1-\frac{1}{2}a\mathfrak{w}_{1R}(k)}{\mathcal{T}_R(k)}k\beta^{-2}, \qquad (4.38a)$$

$$S_2=\frac{k+N+u}{(k+N)\mathfrak{u}_R(k)}, \qquad (4.38b)$$

$$S_3=i\left(\tfrac{3}{2}\Omega^{-2}\frac{\mathfrak{w}_{2R}(k)}{\mathcal{T}_R(k)}+k(S_2-1)\right), \qquad (4.38c)$$

$$Q_1=\tfrac{1}{2}-ikI_2+i(\tfrac{3}{2}\Omega^{-2}I_4-I_6), \qquad (4.38d)$$

$$Q_2=\tfrac{1}{2}i[\mathfrak{K}_{zz+}(k)-\mathfrak{K}_{xz+}(k)]+I_2, \qquad (4.38e)$$

and

$$Q_3=\tfrac{1}{2}ik\beta^{-2}[\mathfrak{K}_{zz+}(k)-\mathfrak{K}_{xz+}(k)]+I_1+k\beta^{-2}(I_3-\tfrac{2}{3}\Omega\gamma I_5), \qquad (4.38f)$$

with

$$\Omega=\omega/\omega_p \qquad (4.39a)$$

and

$$\gamma=(\omega_p\tau)^{-1}. \qquad (4.39b)$$

The reflectance R_p is then given by

$$R_p=\left|\frac{Z_p-\cos\theta}{Z_p+\cos\theta}\right|^2 \qquad (4.40)$$

and the absorptance A_p by

$$A_p=1-R_p. \qquad (4.41)$$

V. RESULTS AND DISCUSSION

A. Dielectric functions

If the surface impedance and optical properties of an electron gas with a specularly scattering surface are calculated using the Boltzmann equation, it is found that the dynamical response of the electrons is entirely described by $\epsilon_t(\vec{q},\omega)$ and $\epsilon_l(\vec{q},\omega)$, the transverse and longitudinal dielectric functions of the bulk system. For the special case of s polarization, where the components of the electric field and current density normal to the surface are zero, the expression for the surface impedance involves only the transverse dielectric function. The absorptance in the nonlocal specular scattering theory is larger than that in a local theory, where one uses a dielectric function dependent only on the frequency, $\epsilon(\omega)=\lim_{q\to 0}\epsilon_l(\vec{q},\omega)=\lim_{q\to 0}\epsilon_t(\vec{q},\omega)$. At low frequencies ($\omega\lesssim 10^{-2}\omega_p$), the additional absorptance, appearing as the anomalous skin effect, occurs for both s and p polarization, and arises from the q dependence of $\epsilon_t(\vec{q},\omega)$. For $\omega>\omega_p$ there is additional absorptance for p-polarized light; it arises from the excitation of bulk plasmons within the metal, and is associated in the theory with the presence of the longitudinal dielectric function $\epsilon_l(\vec{q},\omega)$. Also associated with the presence of ϵ_l are absorptance contributions from single-particle excitations when $\omega\gtrsim 10^{-1}\omega_p$.

For diffuse scattering,[1,7,8] the surface impedance for s-polarized light involves only the transverse dielectric function. Using the dimensionless frequency Ω of Eq. (4.39a) and a dimensionless wave vector $\tilde{Q}=qc/\omega_p$, where q is the magnitude of the usual wave vector with dimension (length)$^{-1}$, the surface impedance can be written

$$Z_s=-\pi i\Omega\left[\int_0^\infty d\tilde{Q}_z\ln\left(1-\frac{\Omega^2}{\tilde{Q}_z^2}[\epsilon_t(\tilde{Q},\Omega)-\sin^2\theta]\right)\right]^{-1}. \qquad (5.1)$$

Here $\tilde{Q}_z$ is the z component of the wave vector $\tilde{Q}$, θ is the angle of incidence, and $\epsilon_t(\tilde{Q},\Omega)$ is the transverse dielectric function.

It will be shown that both the transverse dielectric function and a function essentially equivalent to the longitudinal dielectric function appear in the present diffuse scattering theory for p polarization. However, the theory also contains electron-gas response functions that are different from the usual dielectric functions; in this respect the theory is more complicated than either the p-polarized specular scattering theory or the s-polarized diffuse scattering theory. This occurs because of the vital role of the destruction of translational invariance in the direction normal to the surface for p polarization and diffuse scattering. For s polarization, this destruction of translational invariance does not eliminate the validity of the ordinary wave-vector concept since the current is directed parallel to the surface, a direction for which translational invariance is maintained. This, together with the fact that the incident fields induce no charge, means that the optical properties for s polarization can be completely described by $\epsilon_t(\vec{q},\omega)$, the transverse dielectric constant. For specular

scattering and p polarization, the destruction of translational invariance is effectively circumvented by the field symmetries associated with the specular scattering condition.

The transverse dielectric function can be identified by going to normal incidence, as will be discussed in Appendix D, for in this limit the theories for p and s polarization become identical. One finds that the transverse dielectric function is contained in the transformed kernel

$$\mathcal{K}_c(p) = -\tfrac{1}{2}[(1+ia)^2\mathcal{F}_2(p) + \mathcal{F}_1(p)], \tag{5.2}$$

where $\mathcal{F}_1(p)$ and $\mathcal{F}_2(p)$ are given in Eqs. (A4a) and (A4b). The dimensionless Laplace-transform variable p is related to q_z, the z component of the wave vector, by the equation $p = iq_z v_F/\omega$; here the factor i appears because one is converting from the Laplace-transform variable to a Fourier-transform variable, and v_F/ω, the unit of length in the theory, is required so that p is dimensionless. Converting q_z to the dimensionless wave-vector component $\tilde{Q}_z$ by the relation $q_z = \tilde{Q}_z\omega_p/c$, we find that $p = i\beta\tilde{Q}_z/\Omega$. Similarly the tangential component of the wave vector is $k = \beta\sin\theta = \beta\tilde{Q}_x/\Omega$, and $\tilde{Q}$, the magnitude of the wave vector, is given by $P = (p^2 - k^2)^{1/2} = i\beta\tilde{Q}/\Omega$. If we write $a = (\omega\tau)^{-1} = \gamma/\Omega$, where $\gamma = (\omega_p\tau)^{-1}$, and introduce the quantities

$$b' = \beta(\gamma - i\Omega)^{-1}, \tag{5.3}$$

$$L = \ln\left(\frac{1+ib'\tilde{Q}}{1-ib'\tilde{Q}}\right), \tag{5.4}$$

we find that

$$1 - i\frac{3}{2\Omega^2}\mathcal{K}_c(p) = \epsilon_t(\tilde{Q}, \Omega), \tag{5.5}$$

where

$$\epsilon_t(\tilde{Q}, \Omega) = 1 + \tfrac{3}{2}[(b'\tilde{Q})^3\Omega(\Omega+i\gamma)]^{-1}\{\tfrac{1}{2}i[1+(b'\tilde{Q})^2]L + b'\tilde{Q}\} \tag{5.6}$$

is the nonlocal transverse dielectric function for an infinite medium, as derived from the Boltzmann equation [see Ref. 2, Eq. (2.46)].

The function $\mathcal{T}(p)$, defined in Eq. (4.12), is essentially the longitudinal dielectric function. With the aid of Eqs. (4.6a) and (4.6b), Eq. (4.12) becomes

$$\mathcal{T}(p) = 1 + \frac{3}{2}\frac{a-i}{\Omega^2}\mathcal{F}_2(p) - \tfrac{1}{2}a\mathcal{F}_1(p). \tag{5.7}$$

Making substitutions similar to those used in arriving at Eq. (5.5), we find

$$\mathcal{T}(p) = 1 - \frac{1}{(\Omega+i\gamma)^2}\frac{3}{(b'\tilde{Q})^3}\left(b'\tilde{Q} - \frac{1}{2i}L\right) - \frac{1}{2}\frac{\gamma}{b'\tilde{Q}(\Omega+i\gamma)}L. \tag{5.8}$$

Equation (5.8) can be compared with the longitudinal dielectric function $\epsilon_w(\tilde{Q}, \Omega)$, obtained from the Boltzmann equation with the ordinary relaxation-time approximation, in which the electron distribution function relaxes to the unperturbed distribution function[2]:

$$\epsilon_w(\tilde{Q}, \Omega) = 1 - \frac{1}{\Omega(\Omega+i\gamma)}\frac{3}{(b'\tilde{Q})^3}\left(b'\tilde{Q} - \frac{1}{2i}L\right). \tag{5.9}$$

Equation (5.8) differs slightly from Eq. (5.9), and it is also different from the correct longitudinal dielectric function $\epsilon_l(\tilde{Q}, \Omega)$ obtained by using a scattering term in the Boltzmann equation that allows the electron distribution function to relax to the local state of charge inbalance. [An expression for $\epsilon_l(\tilde{Q}, \Omega)$ is given by Eq. (2.47) of Ref. 2.] However, all of these functions become equal when the relaxation time becomes infinite or $\gamma \to 0$:

$$\lim_{\gamma\to 0}\mathcal{T}(p) = \lim_{\gamma\to 0}\epsilon_w(\tilde{Q}, \Omega) = \lim_{\gamma\to 0}\epsilon_l(\tilde{Q}, \Omega). \tag{5.10}$$

Since, within the present theory, the relaxation of the distribution function is treated correctly by including the term $G(s)$ in Eq. (2.2) and the following equations, it is interesting that the correct longitudinal dielectric function $\epsilon_l(\tilde{Q}, \Omega)$ does not appear for a nonzero value of γ. It turns out that the correct longitudinal dielectric function can be written

$$\epsilon_l(\tilde{Q}, \Omega) = \frac{\mathcal{T}(p)}{1 - \tfrac{1}{2}a\mathcal{K}(p)}, \tag{5.11}$$

where $\mathcal{K}(p)$ is defined in Eq. (A6a); however, the particular combination of functions on the right-hand side of Eq. (5.11) does not appear in the present theory. The expression $\mathcal{T}_R(p)/[1 - \tfrac{1}{2}a\mathcal{W}_{1R}(p)]$, first appearing in Eq. (4.19) for the divergence of the electric field, is a longitudinal response function that resembles, but is not the same as, Eq. (5.11).

This situation is perhaps less surprising if one recognizes that the wave vector is a questionable concept in the diffuse scattering theory. With a wave vector specified, the distinction between transverse and longitudinal fields is immediate. When the wave vector cannot be identified, this distinction must be sought through the effects of divergence and curl operations, a considerably less transparent procedure. That the longitudinal and transverse effects could be mixed here in a nonsimple manner does not seem unrealistic.

B. Results

Using this diffuse scattering theory, the absorptance A_p was calculated as a function of frequency $\Omega = \omega/\omega_p$ for p-polarized light incident at an angle $\theta = 75°$. The Fermi velocity chosen, $\beta = v_F/c = 2.8355\times10^{-3}$, is representative of sodium. In order to show nonlocal contributions to the absorptance clearly, a small damping factor, $\gamma = 10^{-3}$, was used.

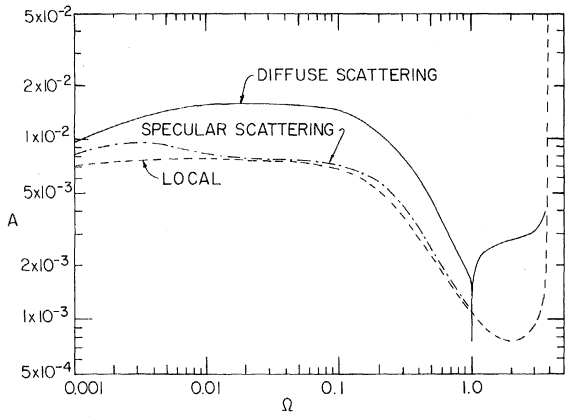

FIG. 1. Absorptance as a function of frequency $\Omega = \omega/\omega_p$ for p-polarized light incident at 75°. Solid line: diffuse scattering; dot-dashed line: specular scattering; dashed line: local theory. The calculations are for a free-electron gas representing sodium, with a damping factor $\gamma = 10^{-3}$.

The theory involves a large number of integrations along a path C at the distance d to the left of the imaginary axis of the complex p plane, and also along the imaginary axis itself. All functions are calculated at several hundred points along the imaginary axis and the integrations are performed using Weddell's rule. A function to be integrated along C is first extrapolated from the points on the imaginary axis to corresponding points on the path of integration. For accuracy, the spacing between points is decreased where the integrands vary rapidly; this occurs near the logarithmic branch points $p = \pm Q$ and also near the zeros of $\mathcal{T}(p)$, which move close to the imaginary axis when $\Omega > 1$.

Figure 1 shows the absorptance for p polarization obtained with the diffuse scattering theory, the specular scattering theory of Ref. 2, and the local theory. At low frequencies there is no essential difference between p and s polarization (see Appendix D); in fact, the surface impedance for diffuse scattering is the same for the two polarizations, the difference in absorptance arising solely from the different relations between the surface impedance and the absorptance.[26]

As the frequency increases toward the plasma frequency the p-polarized absorptance decreases for both the nonlocal and local theories, instead of remaining nearly constant as in the case of s polarization (see Fig. 7 of Ref. 8). In the frequency range $\Omega > 1.1$, the absorptance for diffuse scattering exceeds that for specular scattering by about 1×10^{-4}. This difference is so small that it cannot be shown easily on the scale of Fig. 1; the diffuse and specular scattering results therefore appear as a single solid line.

A rapid change in absorptance, not shown clearly in Fig. 1, occurs near $\Omega = 1$. The behavior near $\Omega = 1$ is shown in more detail in Fig. 2, where the absorptance difference $\delta A = A_p - A_p^{(\mathrm{spec})}$ is plotted as a function of Ω in the range $0.9 \leq \Omega \leq 1.1$, with four values of γ: $\gamma = 10^{-3}$, 3×10^{-3}, 10^{-2}, and 3×10^{-2}. As Ω increases toward $\Omega = 1$, δA at first slowly decreases, but very near $\Omega = 1$ there is a sudden drop of δA, followed by an increase to $\delta A \approx 1.5 \times 10^{-4}$ at $\Omega \approx 1.2$. For still higher values of Ω, δA decreases gradually to about 1×10^{-4}. It is evident from Fig. 2 that the change of δA near $\Omega = 1$ becomes less sharp as γ increases. For Ω slightly larger than 1, δA becomes negative; in fact, the diffuse absorptance A_p even drops below the local absorptance $A_p^{(\mathrm{loc})}$ for the two smallest values of γ. Unfortunately an accurate calculation is extremely difficult near $\Omega = 1$ if γ is small, so the size of this drop of δA for $\Omega \gtrsim 1$ is uncertain for the smallest values of γ. The curves are shown as a dotted line (for $\gamma = 3 \times 10^{-3}$) and a dot-dashed line (for $\gamma = 10^{-3}$) where the calculation may be inaccurate.

It is interesting that the absorptance at frequencies $\Omega \gtrsim 1.02$ is so nearly equal for specular and diffuse scattering. In both cases, the rise above the local absorptance, $A_p - A_p^{(\mathrm{loc})}$ and $A_p^{(\mathrm{spec})} - A_p^{(\mathrm{loc})}$, is caused largely by the excitation of bulk plasmons. In the specular scattering theory this excess absorptance can be traced mathematically to the appearance of $\epsilon_l(\tilde{Q}, \Omega)$ in the denominator of an expression which is integrated over $\tilde{Q}_z$ to give the surface impedance. The $\tilde{Q}_z$ values determined by the solution of $\epsilon_l(\tilde{Q}, \Omega) = 0$ move close to the path of integration when $\Omega > 1$, giving the additional contribution to the surface impedance and the absorptance. In the diffuse scattering theory the excess

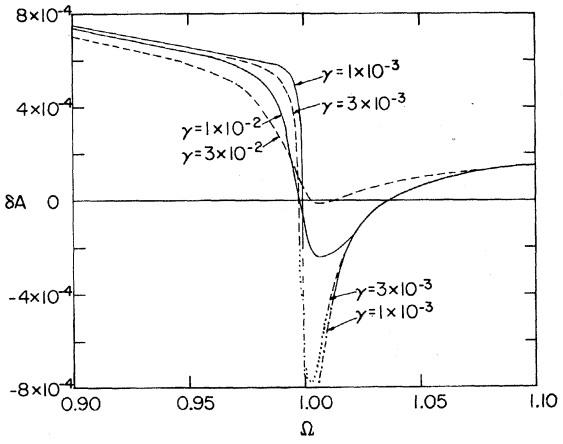

FIG. 2. Absorptance difference $\delta A = A_p - A_p^{(\mathrm{spec})}$ as a function of frequency $\Omega = \omega/\omega_p$, with four values of the damping factor: $\gamma = 10^{-3}$, 3×10^{-3}, 10^{-2}, and 3×10^{-2}. The dotted and dot-dashed portions of the curves for $\gamma = 10^{-3}$ and 3×10^{-2} indicate regions where the results shown may be somewhat inaccurate (see text).

absorptance arises from the zeros of $\mathcal{T}(p)$, which is essentially equal to $\epsilon_l(\tilde{Q}, \Omega)$. As noted above, these zeros are close to the path of integration if $\Omega > 1$, giving a contribution to the integral

$$\ln \mathcal{T}_R(p) = \frac{1}{2\pi i} \int_{-d-i\infty}^{-d+i\infty} \frac{\ln \mathcal{T}(q)}{p-q}\, dq\,, \tag{5.12}$$

and eventually to the surface impedance and the absorptance. A detailed mathematical comparison between the two cases is hard to make because of the distinctly different fashion in which the longitudinal effects appear within the two theories.

The fact that the diffuse and specular p-polarization theories give such different results for $\Omega < 1$ and nearly identical results for $\Omega > 1$ suggests strongly that when the bulk plasmon is sharply defined, the excitation of plasmons, which is the dominant nonlocal-absorption mechanism for $\Omega > 1$, is but little affected by the details of the individual-electron scattering events, not a surprising result in view of the many-electron character of the plasmon. When, on the other hand, the dominant longitudinal excitations are single particle in character and spread over a wide range of wave vectors, as they are for $\Omega < 1$, the individual-electron scattering behavior is important and diffuse scattering, providing an additional scattering process, enhances the absorptance associated with the single-particle excitations. In the frequency range $\Omega > 1.02$, the slight excess of the diffuse scattering absorptance over the specular scattering absorptance can be attributed to the residual single-particle excitations.

The anomalous behavior of the absorptance very near $\Omega = 1$, as shown in Fig. 2, is not completely understood. For Ω just above 1, the optically excited bulk plasmon has a small wave-vector component normal to the surface, comparable in magnitude to the wave vector associated with the dominant transverse effects. It is reasonable, then, that cancellation effects could occur between the transverse (local) and longitudinal (bulk-plasmon) contributions when evaluating the quadratic products which yield the absorptance. At higher frequencies the plasmon wave vector becomes larger, the corresponding wavelength becomes much smaller than the penetration depth of the transverse currents, and such cancellation is no longer possible. This point of view is also suggested by the fact that the anomaly extends downward in frequency to about $\Omega \approx 1 - \gamma$ and the plasmon half-width is roughly γ.

This argument should be viewed with caution since, as presented, it would also be applicable to the case of specular scattering, where the anomaly does not occur. As noted above, however, longitudinal and transverse effects are much less readily identifiable in the diffuse scattering theory and the mixing of the two in this case may well account for the anomalous behavior. This point is now being investigated in connection with photoemission studies for which the details of the spatial distribution of the fields are needed.[27]

If one uses the longitudinal dielectric function as obtained from the Boltzmann equation, the plasmon is a well-defined excitation for all $\Omega > 1$; that is, there is no Landau damping. The plasmon frequency for large q is asymptotic to the line $\omega = qv_F$, which is the low-q edge of the single-particle excitation region. Thus, there is no opportunity for the single-particle contributions to dominate the longitudinal effects for high frequencies, where the plasmon is Landau damped, as does occur when the self-consistent-field (SCF) (Lindhard) longitudinal dielectric function is used. If this could happen, it is likely that the diffuse and specular surface impedances, while essentially equal for $\Omega \gtrsim 1$, would be quite different for higher frequencies, where the plasmon is Landau damped. This point cannot be checked with the present Boltzmann-equation-based theory but is probably not of particular consequence within the purely optical realm. In connection with recent suggestions concerning nonlocal effects in photoemission,[27] it could be very important.

The role of the constant A, which is defined in Eq. (2.4) and is adjusted so that the normal component of the current density at the surface is zero, was investigated by repeating the calculation of the absorptance with $A = 0$. Using Eq. (4.29) and setting $A = 0$, we find that the surface impedance is

$$Z_p^{(A=0)} = -i\beta S_1/S_2\,, \tag{5.13}$$

where S_1 and S_2 are defined in Eqs. (4.38a) and (4.38b).

A further instructional simplification of the theory was made by neglecting all longitudinal effects. This was done by taking $A = 0$ and also setting equal to zero the second term in S_1, which contains the longitudinal response function $\mathcal{T}_R(k)$. The surface impedance then becomes

$$Z_p^{(T)} = \frac{-i\beta}{k+N+u}\,. \tag{5.14}$$

Using

$$u = \frac{1}{2\pi i} \int_{-d-i\infty}^{-d+i\infty} dq \ln \mathfrak{u}(q)\,, \tag{5.15}$$

together with Eqs. (4.4) and (5.5), introducing the variables Ω and $\tilde{Q}$, and letting $d \to 0$ and $N \to 0$, we find

$$Z_p^{(T)} = \frac{-\pi i\Omega}{\pi\Omega \sin\theta + \int_0^\infty d\tilde{Q}_z \ln\{1 - (\Omega^2/\tilde{Q}_z^2)[\epsilon_t(\tilde{Q}, \Omega) - \sin^2\theta]\}}\,. \tag{5.16}$$

If $\Omega \ll 1$, the first term in the denominator can be neglected, and $Z_p^{(T)}$ becomes identical to the surface impedance for s polarization, Eq. (5.1).

In Table I the absorptance obtained from the ap-

TABLE I. Absorptance for p-polarized light incident at $\theta = 75°$, for selected values of the frequency. The absorptance values A_p, $A_p^{(A=0)}$, and $A_p^{(T)}$ are obtained, respectively, from the correct surface impedance Z_p and the modified forms $Z_p^{(A=0)}$ and $Z_p^{(T)}$. The absorptance $A_p^{(\mathrm{loc})}$ is found from the surface impedance $Z_p^{(\mathrm{loc})} = [\epsilon(\Omega) - \sin^2\theta]^{1/2}/\epsilon(\Omega)$, with the local dielectric function $\epsilon(\Omega) = 1 - [\Omega(\Omega + i\gamma)]^{-1}$; $A_p^{(\mathrm{spec})}$ is the absorptance with specular scattering. The Fermi velocity $\beta = v_F/c = 2.8355\times10^{-3}$ and the damping factor $\gamma = (\omega_p\tau)^{-1} = 10^{-3}$.

Ω	A_p	$A_p^{(A=0)}$	$A_p^{(T)}$	$A_p^{(\mathrm{loc})}$	$A_p^{(\mathrm{spec})}$
0.01	1.54×10^{-2}	1.52×10^{-1}	1.51×10^{-2}	7.68×10^{-3}	8.28×10^{-3}
0.1	1.42×10^{-2}	7.48×10^{-1}	1.17×10^{-2}	6.88×10^{-3}	7.14×10^{-2}
0.8	2.33×10^{-3}	5.14×10^{-1}	1.28×10^{-3}	1.32×10^{-3}	1.38×10^{-3}
1.2	2.34×10^{-3}	3.27×10^{-3}	6.30×10^{-4}	9.31×10^{-4}	2.20×10^{-3}
1.5	2.61×10^{-3}	2.66×10^{-3}	4.24×10^{-4}	8.21×10^{-4}	2.54×10^{-3}

proximate expressions for the surface impedance, $Z_p^{(A=0)}$ and $Z_p^{(T)}$, is compared with the correct absorptance. The local absorptance $A_p^{(\mathrm{loc})}$ and the absorptance with specular scattering, $A_p^{(\mathrm{spec})}$, are also given. The Fermi velocity and the angle of incidence are the same as used previously. A similar comparison of the surface-impedance values is shown in Table II.

The absorptance $A_p^{(A=0)}$ is much too large for $\Omega < 1$, but is nearly correct for $\Omega > 1$. The erroneous values of $A_p^{(A=0)}$ and $Z_p^{(A=0)}$ at low frequencies show the importance of the condition that the normal component of the current density must vanish at the surface. It is evident that $Z_p^{(T)}$ and $A_p^{(T)}$ approach the correct values Z_p and A_p at low frequencies, where the longitudinal effects vanish. At the higher frequencies, however, this approximation fails completely: The absorptance $A_p^{(T)}$ drops below the local absorptance $A_p^{(\mathrm{loc})}$, and at $\Omega = 1.5$ the imaginary part of $A_p^{(T)}$ even has the wrong sign. The approximation $A = 0$, which fails at low frequencies, is somewhat better when $\Omega > 1$. However, the surface impedance $Z_p^{(A=0)}$ still differs significantly from Z_p, even though the absorptances $A_p^{(A=0)}$ and A_p are nearly the same. A comparison of the diffuse and local results shows that the imaginary parts of the surface impedances Z_p and $Z_p^{(\mathrm{loc})}$ are almost equal. The additional absorptance for diffuse scattering, as compared with the local absorptance, is due essentially to an increase in the resistive part of the surface impedance.

ACKNOWLEDGMENTS

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APPENDIX A: PROPERTIES OF THE KERNELS

The kernels appearing in Eqs. (2.6) and (2.7) are a set of closely related functions. Because their properties and the relations between them are so critical in uncoupling the integral equations, their properties are derived in some detail in this section.[28]

Defining relations

The kernels are defined in Ref. 2 (with different names) in terms of integrals over an angle θ.[29] We have found it convenient to introduce as an alternative variable of integration $x = \sec\theta$. With

$$y = k(x^2 - 1)^{1/2}|z|\,, \qquad K(z) = \int_1^\infty dx\, x^{-1} e^{(i-a)x|z|} J_0(y)\,. \tag{A1a}$$

The other kernels, along with the names given them

TABLE II. Surface impedance for p-polarized light incident at $\theta = 75°$, for selected values of the frequency. Z_p is the correct surface impedance for diffuse scattering, whereas $Z_p^{(A=0)}$ and $Z_p^{(T)}$ are simplified forms defined by Eqs. (5.13) and (5.14). $Z_p^{(\mathrm{loc})}$ is the local surface impedance as defined in the caption for Table I, and $Z_p^{(\mathrm{spec})}$ is the surface impedance with specular scattering. The Fermi velocity and γ are given in the caption for Table I.

Ω	Z_p	$Z_p^{(A=0)}$	$Z_p^{(T)}$	$Z_p^{(\mathrm{loc})}$	$Z_p^{(\mathrm{spec})}$
0.01	1.004×10^{-3}	1.070×10^{-2}	9.867×10^{-4}	4.995×10^{-4}	5.391×10^{-4}
	$-1.008\times10^{-2}i$	$-8.546\times10^{-3}i$	$-1.001\times10^{-2}i$	$-1.001\times10^{-2}i$	$-9.952\times10^{-3}i$
0.1	1.006×10^{-3}	9.780×10^{-2}	8.583×10^{-4}	5.147×10^{-4}	5.337×10^{-4}
	$-1.007\times10^{-1}i$	$-1.896\times10^{-2}i$	$-9.123\times10^{-2}i$	$-1.010\times10^{-1}i$	$-1.008\times10^{-1}i$
0.8	1.073×10^{-2}	1.284	3.401×10^{-4}	6.129×10^{-3}	6.358×10^{-3}
	$-2.166\,i$	$-4.516\times10^{-1}i$	$-4.569\times10^{-1}i$	$-2.174\,i$	$-2.166\,i$
1.2	1.538×10^{-2}	4.611×10^{-3}	2.378×10^{-4}	6.105×10^{-3}	1.441×10^{-2}
	$+2.592\,i$	$+1.179\,i$	$-5.688\times10^{-1}i$	$+2.592\,i$	$+2.592\,i$
1.5	3.251×10^{-3}	1.303×10^{-3}	1.915×10^{-4}	1.024×10^{-3}	3.175×10^{-3}
	$+1.106\,i$	$+6.631\times10^{-1}i$	$-6.328\times10^{-1}i$	$+1.106\,i$	$+1.105\,i$

in Ref. 2, are

$$K_x(z)=\overline{K}'(z)=\int_1^\infty dx\, x^{-2}(x^2-1)^{1/2}\, e^{(1-a)x|z|}\, J_0'(y), \tag{A1b}$$

$$K_z(z)=\overline{K}(z)=-\operatorname{sgn}(z)\int_1^\infty dx\, x^{-2}\, e^{(i-a)x|z|}\, J_0(y), \tag{A1c}$$

$$K_{xx}(z)=K''(z)=\int_1^\infty dx\, x^{-3}(x^2-1)\, e^{(i-a)x|z|}\, J_0''(y), \tag{A1d}$$

$$K_{xz}(z)=K'(z)=\operatorname{sgn}(z)\int_1^\infty dx\, x^{-3}(x^2-1)^{1/2} \times e^{(i-a)x|z|}\, J_0'(y), \tag{A1e}$$

and

$$K_{zz}(z)=\tilde{K}(z)=\int_1^\infty dx\, x^{-3}\, e^{(i-a)|z|}\, J_0(y). \tag{A1f}$$

J_0 is the Bessel function of zero order; a prime indicates differentiation with respect to the argument.

A closely related representation, particularly useful for small values of $|z|$, is obtained by using as the variable of integration y, the argument of the Bessel function. The argument of the exponential becomes

$$\nu=ik^{-1}(1+ia)(y^2+k^2z^2)^{1/2},$$

so the kernels can be written

$$K(z)=\int_0^\infty dy\, y(y^2+k^2z^2)^{-1}\, e^{\nu}\, J_0(y), \tag{A2a}$$

$$K_x(z)=\int_0^\infty dy\, y(y^2+k^2z^2)^{-3/2}\, e^{\nu}\, J_0'(y), \tag{A2b}$$

$$K_z(z)=-kz\int_0^\infty dy\, y(y^2+k^2z^2)^{-3/2}\, e^{\nu}\, J_0(y), \tag{A2c}$$

$$K_{xx}(z)=\int_0^\infty dy\, y^3(y^2+k^2z^2)^{-2}\, e^{\nu}\, J_0''(y), \tag{A2d}$$

$$K_{xz}(z)=kz\int_0^\infty dy\, y^2(y^2+k^2z^2)^{-2}\, e^{\nu}\, J_0'(y), \tag{A2e}$$

and

$$K_{zz}(z)=k^2z^2\int_0^\infty dy\, y(y^2+k^2z^2)^{-2}\, e^{\nu}\, J_0(y). \tag{A2f}$$

The kernels K_z and K_{xz} are odd functions of their arguments; the other four kernels are even functions.

Laplace transforms and another integral representation

The well-known two-sided Laplace transforms of the kernels are simply expressible in terms of three related functions. Using script letters to denote Laplace transforms, a transform variable p, and the quantities

$$P=(p^2-k^2)^{1/2} \tag{A3a}$$

and

$$L=\ln\left(\frac{1+i(a+P)}{1+i(a-P)}\right), \tag{A3b}$$

these three functions are

$$\mathcal{F}_1(p)=P^{-1}L, \tag{A4a}$$

$$\mathcal{F}_2(p)=P^{-3}[L-2iP/(1+ia)], \tag{A4b}$$

and

$$\mathcal{F}_3(p)=P^{-5}[L-2iP/(1+ia)-2(iP)^3/3(1+ia)^3]. \tag{A4c}$$

The choice of phase of P is irrelevant, so long as one does not change in midformula. The functions $\mathcal{F}_n(p)$ are all analytic at $p=\pm k$ and throughout the region $-1\leq \mathrm{Re}p\leq 1$. Their only singularities are logarithmic branch points at $p=\pm Q$, where

$$Q=[k^2-(1+ia)^2]^{1/2} \tag{A5}$$

is a complex number in the fourth quadrant. The branch lines extend to infinity away from the imaginary axis. As p approaches infinity, $\mathcal{F}_1(p)$ vanishes as p^{-1} and $\mathcal{F}_2(p)$ and $\mathcal{F}_3(p)$ vanish as p^{-2}. L is defined to be zero where P is zero.

In terms of the functions $\mathcal{F}_n(p)$, the transforms of the kernels are

$$\mathcal{K}(p)=\mathcal{F}_1(p), \tag{A6a}$$

$$\mathcal{K}_x(p)=i(1+ia)k\mathcal{F}_2(p), \tag{A6b}$$

$$\mathcal{K}_z(p)=-i(1+ia)p\mathcal{F}_2(p), \tag{A6c}$$

$$\mathcal{K}_{xx}(p)=-\tfrac{3}{2}(1+ia)^2k^2\mathcal{F}_3(p)-\tfrac{1}{2}[(1+ia)^2+k^2]\mathcal{F}_2(p) -\tfrac{1}{2}\mathcal{F}_1(p), \tag{A6d}$$

$$\mathcal{K}_{xz}(p)=-\tfrac{1}{2}kp[3(1+ia)^2\mathcal{F}_3(p)+\mathcal{F}_2(p)], \tag{A6e}$$

and

$$\mathcal{K}_{zz}(p)=-\tfrac{3}{2}(1+ia)^2k^2\mathcal{F}_3(p)-[(1+ia)^2+\tfrac{1}{2}k^2]\mathcal{F}_2(p). \tag{A6f}$$

Of course, the original functions of z can be recovered from the Laplace transforms by the inverse transformation

$$F(z)=(2\pi i)^{-1}\int_{-i\infty}^{i\infty} dp\, e^{pz}\mathcal{F}(p). \tag{A7}$$

In applying this process to the kernels, or preferably to the functions $\mathcal{F}_n$, a particularly simple form is obtained for $z>0$ ($z<0$) by deforming the contour of integration around the left (right) branch cut. The difference in the value of L on the two sides of the cut contributes a factor $2\pi i$. The resulting inverse Laplace transforms of the $\mathcal{F}_n$ functions are

$$F_1(z)=\int_Q^\infty dp\, P^{-1}\, e^{-|z|p}, \tag{A8a}$$

$$F_2(z)=\int_Q^\infty dp\, P^{-3}\, e^{-|z|p}, \tag{A8b}$$

and

$$F_3(z) = \int_Q^\infty dp\, P^{-5} e^{-|z|p}. \tag{A8c}$$

In these expressions, the phase of P is no longer arbitrary. Considered as a function of p, P has branch cuts beginning at the points $p = \pm k$ and extending outward indefinitely in a direction away from the imaginary axis. Along the real axis between the branch points, P is given the value $-i(k^2 - p^2)^{1/2}$, so that in the fourth quadrant ReP is positive. Paths of integration in expressions like those of Eqs. (A8) are to be understood as extending away from the imaginary axis from Q, which is always in the fourth quadrant.

The kernels may also be written in the form of Eqs. (A8). We find

$$K(z) = \int_Q^\infty dp\, P^{-1} e^{-p|z|}, \tag{A9a}$$

$$K_x(z) = i(1+ia)k \int_Q^\infty dp\, P^{-3} e^{-p|z|}, \tag{A9b}$$

$$K_z(z) = i(1+ia)\,\mathrm{sgn}(z) \int_Q^\infty dp\, pP^{-3} e^{-p|z|}, \tag{A9c}$$

$$K_{xx}(z) = -\tfrac{1}{2} \int_Q^\infty dp \{3(1+ia)^2 k^2 P^{-5} + [(1+ia)^2 + k^2]P^{-3} + P^{-1}\} e^{-p|z|}, \tag{A9d}$$

$$K_{xz}(z) = \tfrac{1}{2} k\,\mathrm{sgn}(z) \int_Q^\infty dp\, p[3(1+ia)^2 P^{-5} + P^{-3}] e^{-p|z|}, \tag{A9e}$$

and

$$K_{zz}(z) = -\frac{1}{2} \int_Q^\infty dp \{3(1+ia)^2 k^2 P^{-5} + [2(1+ia)^2 + k^2]P^{-3}\} e^{-p|z|}. \tag{A9f}$$

Limiting behavior of the kernels

For sufficiently large values of $|z|$, the entire contribution to the integrals in (A9) arises from the immediate neighborhood of the lower limit. The asymptotic behavior of the kernels is thus

$$K(z) \sim i(1+ia)^{-1} |z|^{-1} e^{-Q|z|}, \tag{A10a}$$

$$K_x(z) \sim (1+ia)^{-2} k |z|^{-1} e^{-Q|z|}, \tag{A10b}$$

$$K_z(z) \sim (1+ia)^{-2} Q z^{-1} e^{-Q|z|}, \tag{A10c}$$

$$K_{xx}(z) \sim -i(1+ia)^{-3} k^2 |z|^{-1} e^{-Q|z|}, \tag{A10d}$$

$$K_{xz}(z) \sim i(1+ia)^{-3} kQ z^{-1} e^{-Q|z|}, \tag{A10e}$$

and

$$K_{zz}(z) \sim -i(1+ia)^{-3} Q^2 |z|^{-1} e^{-Q|z|}. \tag{A10f}$$

The behavior of the kernels for small values of $|z|$ can be obtained from Eqs. (A2) or (A9). The value of $F_1(z)$ for small $|z|$ is found most easily using $q = |z|p$ as a variable of integration. Thus, from (A8a),

$$F_1(z) = \int_{Q|z|}^\infty dq\, (q^2 - k^2 z^2)^{-1/2} e^{-q},$$

which we can rewrite as

$$F_1(z) = \int_{Q|z|}^1 dq\, (q^2 - k^2 z^2)^{-1/2} - \int_{Q|z|}^1 dq\, (q^2 - k^2 z^2)^{-1/2} \times (1 - e^{-q}) + \int_1^\infty dq\, (q^2 - k^2 z^2)^{-1/2} e^{-q}.$$

The first of these integrals is elementary. For $z = 0$, the sum of the remaining integrals is the negative of Euler's constant γ (= 0.577···).

$$F_1(z) = -\ln\{\tfrac{1}{2}[Q - i(1+ia)]|z|\} - \gamma + O(z). \tag{A11a}$$

The remaining F functions are finite at $|z| = 0$. We have

$$\frac{dF_2(z)}{dz} = -i(1+ia)^{-1}\,\mathrm{sgn}(z) + O(z\ln z), \tag{A11b}$$

$$F_2(z) = k^{-2}[i(1+ia)^{-1}Q - 1] + O(z), \tag{A11c}$$

$$\frac{dF_3(z)}{dz} = \tfrac{1}{3} i(1+ia)^{-3}\,\mathrm{sgn}(z) + O(z), \tag{A11d}$$

and

$$F_3(z) = -\tfrac{1}{3} ik^{-2}(1+ia)^{-3}Q - \tfrac{2}{3} k^{-4}[i(1+ia)^{-1}Q - 1] + O(z). \tag{A11e}$$

The F functions, even functions of their argument z, have absolute values which are monotonic decreasing functions of z^2. $F_1(z)$ is logarithmically infinite at the origin, and the other two functions are finite at the origin with discontinuous first derivatives.

The behavior of the kernels for small arguments is as follows:

$$K(z) = -\ln\{\tfrac{1}{2}[Q - i(1+ia)]|z|\} - \gamma + O(z), \tag{A12a}$$

$$K_x(z) = -k^{-1}[i(1+ia) + Q] + O(z), \tag{A12b}$$

$$K_z(z) = -\,\mathrm{sgn}(z) + O(z\ln z), \tag{A12c}$$

$$K_{xx}(z) = \tfrac{1}{2}\ln\{\tfrac{1}{2}[Q - i(1+ia)]|z|\} - \gamma + O(z), \tag{A12d}$$

$$K_{xz}(z) = \tfrac{1}{2} kz \ln|z| + O(z), \tag{A12e}$$

$$K_{zz}(z) = \tfrac{1}{2} + O(z). \tag{A12f}$$

Relations between kernels

Equations (A6) express the four even kernels as linear combinations of the three F functions. The latter can be eliminated to show that

$$2(K_{zz} - K_{xx}) = K + i(1+ia)k^{-1}K_x. \tag{A13}$$

Bessel's differential equation provides another relation between the kernels as described by Eq. (A1) or (A2):

$$K_{zz}(z) - K_{xx}(z) = K(z) + (kz)^{-1}K_{xz}(z). \tag{A14}$$

Important in uncoupling the integral equations are

the derivatives of the kernels. These derivatives can be written in terms of other kernels and the Dirac δ function as

$$\frac{dK(z)}{dz} = -i(1+ia)K_z(z) - 2z^{-1}K_{zz}(z), \tag{A15a}$$

$$\frac{dK_x(z)}{dz} = -kK_z(z), \tag{A15b}$$

$$\frac{dK_z(z)}{dz} = -i(1+ia)K(z) - kK_x(z) - 2\delta(z), \tag{A15c}$$

$$\frac{dK_{xx}(z)}{kz} = z^{-1}K_{zz}(z) + kK_{xz}(z), \tag{A15d}$$

$$\frac{dK_{xz}(z)}{dz} = i(1+ia)K_x(z) + kK_{xx}(z) \tag{A15e}$$

$$= z^{-1}K_{xz}(z) + kK_{zz}(z), \tag{A15f}$$

and

$$\frac{dK_{zz}(z)}{dz} = -i(1+ia)K_z(z) + kK_{xz}(z). \tag{A15g}$$

Of these equations, (a) follows directly from (A2), (b) from (A6), and (c), (e), and (g) from (A1). Equations (A13) and (A14) together with the other of Eqs. (A15) were used to obtain (d) and (f).

Finally, by differentiating $F_2(z)$, a relation is obtained involving the exponential function explicitly:

$$\frac{dF_2(z)}{dz} = \mathrm{sgn}(z)\int_Q^\infty dp\, e^{-|z|p}\frac{d}{dp}(P^{-1})$$

$$= -i(1+ia)^{-1}\,\mathrm{sgn}(z)\,e^{-Q|z|} + zF_1(z),$$

which is equivalent to

$$-i(1+ia)zK(z) = K_z(z) + \mathrm{sgn}(z)\,e^{-Q|z|}. \tag{A16}$$

APPENDIX B: ROOTS OF $\mathcal{T}(p)$

The solution given to Eq. (3.5) depends on the fact that $\mathcal{T}(p)$, defined in Eq. (4.12), had no roots along the imaginary axis. A demonstration of this fact follows.

From Eqs. (4.12), (A4), and (A6),

$$\mathcal{T}(p) = 1 - \frac{3}{2}\left(\frac{\omega_p}{\omega}\right)^2 i(1+ia)\frac{1}{P^3}\left(L - \frac{2i}{(1+ia)}P\right) - \frac{aL}{2P}. \tag{B1}$$

Along the imaginary axis (and along the real axis for $|p| < k$) let $P = iq$. Replacing q by $-q$ does not affect the value of $\mathcal{T}(p)$, so in this section consider q as positive. From (A3b), we have

$$L = \ln\left(\frac{1-q+ia}{1+q+ia}\right)$$

$$= -\tfrac{1}{2}\ln\left(\frac{(1+q)^2+a^2}{(1-q)^2+a^2}\right) + i\tan^{-1}\left(\frac{2aq}{1-q^2+a^2}\right), \tag{B2}$$

which we now write as

$$L = -s + it, \tag{B3}$$

where it is to be understood that $0 < t < \pi$.

Conditions for the vanishing of the real and imaginary parts of $\mathcal{T}(p)$ are, respectively,

$$\left(\frac{3(\omega_p/\omega)^2}{q^2}+1\right)\left(1 - \frac{at}{2q}\right) - \frac{3(\omega_p/\omega)^2 s}{2q^3} = 0 \tag{B4}$$

and

$$\left(\frac{3(\omega_p/\omega)^2}{q^2}+1\right)\frac{as}{2q} - \frac{3(\omega_p/\omega)^2 t}{2q^3} = 0. \tag{B5}$$

These equations can be divided to yield

$$\frac{s}{2q-a} = \frac{t}{as}. \tag{B6}$$

The last equation gives as a necessary condition for a root of $\mathcal{T}(p)$ along the imaginary axis that

$$q = a\left(\frac{s^2+t^2}{2t}\right). \tag{B7}$$

On the other hand, q may be determined as a function of s and t from the above definition of s and the expression for $\sin t$,

$$\sin t = \frac{2aq}{\{[(1-q)^2+a^2][(1+q)^2+a^2]\}^{1/2}}.$$

The solution to this pair of equations is

$$q = a\left(\frac{\cosh s - \cos t}{\sin t}\right), \tag{B8}$$

as is probably most simply verified by direct substitution.

But Eqs. (B7) and (B8) are incompatible. To see this, assume the equivalence of (B7) and (B8). Then

$$\cosh s = \frac{s^2}{t}\frac{\sin t}{t} + \frac{t}{2}\sin t + \cos t. \tag{B9}$$

But

$$1 + \tfrac{1}{2}s^2 \le \cosh s,$$

and so

$$0 \le \cosh s - 1 - \tfrac{1}{2}s^2$$

$$\le \frac{s^2}{2}\left(\frac{\sin t}{t} - 1\right) + \frac{t}{2}\sin t + \cos t - 1.$$

Now for $t \ge 0$, $t^{-1}\sin t < 1$. And in the domain $0 < t < \pi$, $\frac{1}{2}t\sin t - \cos t - 1$ is a monotonic decreasing function whose maximum value is 0. Thus the right-hand expression above is in fact negative in general, a contradiction that proves the suspected incompatibility.

There is no incompatibility if $s = t = 0$. But this can occur only if q is zero or infinite. The former is not a point on the imaginary axis. And for the latter, the value of $\mathcal{T}(p)$ is not 0, but 1.

In the limit that $a \to 0$, t is either 0 or π. The latter is incompatible with (B9) and the former corresponds to the case $s - t = 0$ noted in the previ-

ous paragraph. Thus the case $a \to 0$ is in no sense special.

APPENDIX C: ONE-SIDED LAPLACE TRANSFORMS OF THE KERNELS

It will be no surprise that these transforms turn out to be very similar in structure to the two-sided transforms, but somewhat more complicated. They are all based on the function

$$L_+ = \tfrac{1}{2}\ln\left[\left(\frac{1+i(a+P)}{1+i(a-P)}\right)\left(\frac{p+P}{p-P}\right)\left(\frac{ip(1+ia)+QP}{ip(1+ia)-QP}\right)\right] \quad \text{(C1)}$$

divided by odd powers of P. L_+ has a branch point at $p=-Q$ but not at $p=Q$. The one-sided transform of K, or F_1', is

$$\mathcal{K}_+(p) = F_1'(p) = P^{-1}L_+ . \quad \text{(C2a)}$$

In the other one-sided transforms, there are subtraction terms whose function may be thought of as removing spurious singularities at $p=k$ ($P=0$) without introducing any other singularities, even at ∞. We introduce the functions

$$F_2'(p) = P^{-3}\left\{L_+ - P\left[\frac{i}{(1+ia)} + \frac{1}{p}\left(1-\frac{iQ}{(1+ia)}\right)\right]\right\} \quad \text{(C2b)}$$

and

$$F_3'(p) = P^{-2}F_2'(p) + \frac{1}{3P^2}\left[\frac{i}{(1+ia)^3} - \frac{1}{p^3}\left(1+\frac{iQ^3}{(1+ia)^3}\right)\right]. \quad \text{(C2c)}$$

In terms of these F' functions, the one-sided transforms of the other kernels are

$$\mathcal{K}_{x+}(p) = i(1+ia)kF_2'(p) - \frac{i(1+ia)}{kp}\left(1-\frac{iQ}{1+ia}\right), \quad \text{(C3a)}$$

$$\mathcal{K}_{z+}(p) = -i(1+ia)pF_2'(p), \quad \text{(C3b)}$$

$$\mathcal{K}_{xx+}(p) = -\tfrac{3}{2}(1+ia)^2k^2F_3'(p) - \tfrac{1}{2}[(1+ia)^2+k^2]F_2'(p) - \tfrac{1}{2}F_1'(p) + \tfrac{1}{2}p^{-1} - \frac{(1+ia)^2}{2pk^2}\left(1-\frac{iQ}{1+ia}\right) + \frac{(1+ia)^2}{2p^3}\left(1+\frac{iQ^3}{(1+ia)^3}\right), \quad \text{(C3c)}$$

$$\mathcal{K}_{xz+}(p) = -\tfrac{3}{2}(1+ia)^2kpF_3'(p) - \tfrac{1}{2}kpF_2'(p) + \frac{(1+ia)^2}{2kp^2}\left(1+\frac{iQ^3}{(1+ia)^3}\right), \quad \text{(C3d)}$$

and

$$\mathcal{K}_{zz+}(p) = -\tfrac{3}{2}(1+ia)^2k^2F_3'(p) - \tfrac{1}{2}[2(1+ia)^2+k^2]F_2'(p) + \tfrac{1}{2}p^{-1} + \frac{(1+ia)^2}{2p^3}\left(1+\frac{iQ^3}{(1+ia)^3}\right). \quad \text{(C3e)}$$

APPENDIX D: LIMITING CASES

In this appendix two limiting cases are discussed. In the limit as $\theta \to 0$, it is shown that the surface impedance for p polarization approaches the surface impedance for s polarization, given by Eq. (5.1) with $\theta = 0$. Although this limit is necessary physically, it is not immediately obvious how it arises from the final expression (4.37) for the surface impedance, and it is difficult to achieve numerically. In fact, showing that the correct result is obtained as $\theta \to 0$ was an important test of the numerical calculation.

The other limit is the low-frequency limit, which is very complicated mathematically. We have shown numerically that at low frequencies ($\Omega \lesssim \gamma$), the surface impedance approaches the approximation $Z_p^{(T)}$, given by Eq. (5.14), a fact that has already been noted in the discussion of Table I.

We now examine the limit $\theta \to 0$ or $k = \beta\sin\theta \to 0$. One can solve Eqs. (4.29) and (4.32) for the ratio $A/E_x(0)$, giving

$$\frac{A}{E_x(0)} = \frac{S_2Q_3 - S_1Q_2}{S_1Q_1 + S_3Q_3} . \quad \text{(D1)}$$

We shall show that as $\theta \to 0$,

$$S_2Q_3 - S_1Q_2 \to 0 \quad \text{(D2a)}$$

or

$$A/E_x(0) \to 0 . \quad \text{(D2b)}$$

This limit is expected physically; since $j_z \to 0$ for $\theta \to 0$, the constant A, which adjusts the distribution function for electrons leaving the surface in order to make $j_z|_{z=0} = 0$, must also go to zero. With the aid of Eq. (D2b), Eq. (4.37) becomes

$$\lim_{\theta\to 0} Z_p = -i\beta \lim_{k\to 0}\frac{S_1}{S_2} = \lim_{k\to 0} Z_p^{(T)} , \quad \text{(D3)}$$

the second equality arising because the second term in S_1 goes to zero as $k \to 0$. Therefore we finally have, from Eqs. (5.1), (5.16), and (D3),

$$\lim_{\theta\to 0} Z_p = \lim_{\theta\to 0} Z_s . \quad \text{(D4)}$$

In order to establish Eq. (D2a), we note that it can be rewritten as

$$N + u = \lim_{k\to 0}\{\tfrac{1}{2}i[\mathcal{K}_{zz+}(0) - \mathcal{K}_{xz+}(0)] + I_2\}/I_1 , \quad \text{(D5)}$$

where Eqs. (4.38) have been used with $k \to 0$. It can be shown that

$$\lim_{k\to 0}\mathcal{K}_{xz+}(k) = 0 ; \quad \text{(D6)}$$

the integrals I_1 and I_2 are then of the form

$$I_l = \int_{-i\infty}^{i\infty} S_l(p)\frac{2k}{k^2-p^2}\,dp , \quad \text{(D7)}$$

where

$$S_1(p) = \frac{1}{4\pi}\frac{\mathcal{K}_{zz+}(-p)}{(p+N)\mathfrak{u}_R(p)} , \quad \text{(D8)}$$

$$S_2(p)=\frac{1}{4\pi}\left(\frac{p+N+u}{(p+N)\mathfrak{u}_R(p)}-1\right)\mathfrak{K}_{zz+}(-p). \tag{D9}$$

Letting $p=ix$ in Eq. (D7), we have

$$\lim_{k\to 0} I_l = 2i\lim_{k\to 0}\int_{-\infty}^{\infty} S_l(ix)\frac{k}{k^2+x^2}\,dx \tag{D10}$$

$$= 2\pi i\lim_{k\to 0}\int_{-\infty}^{\infty} S_l(ix)\delta(x)\,dx \tag{D11}$$

$$= 2\pi i\lim_{k\to 0} S_l(0), \tag{D12}$$

so that

$$\lim_{k\to 0} I_1 = \frac{i}{2}\lim_{k\to 0}\frac{\mathfrak{K}_{zz+}(0)}{N\mathfrak{u}_R(0)} \tag{D13}$$

and

$$\lim_{k\to 0} I_2 = \frac{i}{2}\lim_{k\to 0}\left(\frac{N+u}{N\mathfrak{u}_R(0)}-1\right)\mathfrak{K}_{zz+}(0). \tag{D14}$$

Equation (D5) can then be verified immediately by using Eqs. (D6), (D13), and (D14).

Care is needed in the numerical calculation for small angles because of the function $k/(k^2-p^2)$ in the I_l integrals; this function approaches $\pi\delta(x)$ or $\pi i\delta(p)$ as $k\to 0$.

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[17]Here G, which has the dimensions of a field, is equal to the G of Ref. 2 divided by ev.

[18]Another requirement for a physical surface is that as much energy is carried away from the surface as is brought to it. In the low-temperature approximation we are making all corrections to the velocity distribution occur in the immediate neighborhood of the Fermi energy. In this approximation, conservation of the number of electrons guarantees conservation of energy at the surface. At higher temperatures a more complex correction to the distribution of "reflected" electrons would be required.

[19]Keep in mind that z and k are here and below dimensionless.

[20]Laplace transforms will be denoted by script symbols.

[21]The sign of P is here immaterial.

[22]This conclusion is also valid in the limit $a\to 0$ or $\tau\to\infty$. When $a=0$, this means we must demonstrate that roots of the equation $(b/q^2)[(S/2q)(q^2-1)+1]+q^2-\beta^2=0$, where $P=i\rho$, as above, and $S=\ln|(1+q)/(1-q)|$, cannot occur along the imaginary axis in the p plane. This demonstration can be made without difficulty.

[23]We adopt the convention of writing denominators in integrals like those in Eq. (4.6) so that their real parts are always positive.

[24]In general, we will denote by R or L subscripts functions of p regular, respectively, in a right or left half-plane.

[25]The Laplace transforms of the fields are all understood to be one-sided transforms.

[26]For s polarization the absorptance is $A_s=1-R_s$, with $R_s=|(Z_s\cos\theta-1)/(Z_s\cos\theta+1)|^2$.

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[29]This θ should not be confused with the incident angle.