Observability of Josephson pair-quasiparticle interference in superconducting interferometers*

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Pair-quasiparticle interference current has a measurable effect on the tuning characteristics of a superconducting interferometer operating in the nonhysteretic regime: $\epsilon \equiv (2\pi/\Phi_0)L_R I_c < 1$ where $\Phi_0 = h/2e$, L_R is the inductance of the interferometer ring, and I_c is the critical current of the interferometer's weak link. In the limit $\epsilon \ll 1$ the interferometer will have a zero crossing in its magnetic flux sensitivity versus frequency characteristic that is above, on, or below resonance depending whether the coefficient of the interference current term is positive, zero, or negative, respectively. For larger ϵ, screening currents will also shift the zero crossing. Results for the magnitude of this additional shift are presented and discussed.

I. INTRODUCTION

Recently, a simple theory[1] was developed for the operation of single-junction interferometers [rf superconducting quantum interference device (SQUID)] in the nonhysteretic regime: $\epsilon < 1$.[2-5] That theory neglected the effects of quasiparticle conductance G and quasiparticle-pair interference conductance $\gamma G\cos\phi$,[6,7] and predicted that the magnetic-flux sensitivity was a perfectly antisymmetric function of frequency around the resonant frequency ω_0. That prediction (and others) have recently been confirmed by experiments[5] on toroidal point-contact SQUID operating with $\beta \equiv \Phi_0\omega_0 G/2\pi I_c \ll 1$. In this range of β the effects of quasiparticle conductance and quasiparticle-pair conductance are indeed negligible.

This paper generalizes the simple theory[1] to include both conductances. It shows that in the range of larger β the quasiparticle-pair conductance $\gamma G\cos\phi$ has a measurable effect on the tuning curve. This effect has been used by two experimental groups to determine the magnitude and sign of γ for their weak links: (i) Preliminary experiments[5] on point contacts set a bound on γ as negative and of magnitude greater than 0.5, consistent with previously reported measurements[8] by a different technique on similar point contacts. (ii) The accompanying manuscript reports the first measurements of γ in a Dayem bridge. In addition, it may prove possible to use the effect to measure, for the first time, the temperature dependence of γ.

II. THEORY

We begin by assuming that the current through the weak link shown in Fig. 1 is given by Josephson's equation[9]

$$I = I_c \sin\phi' + GV(1+\gamma\cos\phi'),\qquad(1)$$
$$\phi' = \phi - \frac{2e}{\hbar}\int_{1\ \text{through weak link}}^{2} \vec{A}\cdot d\vec{l},$$

where I_c is the weak-link's critical current, ϕ is the difference in the phase of the superconducting order parameter across it, $V = (\hbar/2e)(d\phi/dt)$ is the voltage across it, G is its quasiparticle conductance, and γ is the ratio of the magnitude of the pair-quasiparticle interference conductance to the quasiparticle conductance (σ_1/σ_0 in Josephson's notation).[9] It has been implicitly assumed that the voltage dependence of G and γ can be neglected since typical voltages $(\hbar/2e)(2\pi \times 30\ \text{MHz}) = 6\times10^{-8}$ V are small relative to energy-gap voltages $\approx 10^{-3}$ V.

The difference in the phase of the superconducting order parameter ϕ can be related to the total magnetic flux enclosed in the interferometer loop Φ_{tot} by the requirement that the order parameter must be single valued. This single valuedness requires that the total change in the phase of the order parameter in one clockwise circuit of the interferometer loop must equal $2\pi m$, where m is an integer. Thus, using the expression for the gradient of the phase inside a superconductor,[10] we can write

$$2\pi m = \phi + \frac{2e}{\hbar}\int_{2\ \text{through the superconductor}}^{1}\left(\vec{A} + \frac{m_e}{2e^2 n'}\vec{j}\right)\cdot d\vec{l}.\qquad(2)$$

For macroscopic loops the term involving the current density j can be neglected.[1] Combining Eqs. (1) and (2) and using the identity $\oint \vec{A}\cdot d\vec{l} = \Phi_{tot}$,

$$I = I_c \sin\left(-\frac{2\pi\Phi_{tot}}{\Phi_0}\right) + GV\left[1+\gamma\cos\left(-\frac{2\pi\Phi_{tot}}{\Phi_0}\right)\right],\qquad(3)$$

where $\Phi_0 = h/2e$ is one flux quantum. The negative sign on Φ_{tot} is due to the convention that positive B is into the paper in Fig. 1.[11]

Let us first consider the limit $\epsilon \equiv (2\pi/\Phi_0)I_c L_R \ll 1$. In this limit, $I_c L_R \ll \Phi_0$; thus screening flux IL_R can be neglected and $\Phi_{tot} \cong \Phi_{app}$. For

$$\Phi_{app} = \Phi_{dc} + \Phi_{ac}\sin\omega t,$$

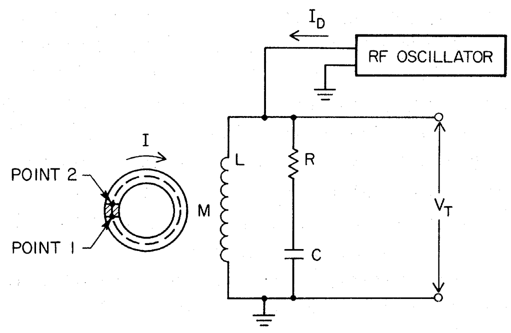

FIG. 1. Model circuit for a single-junction interferometer. A loop of superconductor that is interrupted by a Josephson weak link is magnetically coupled to a resonant tank circuit. The total voltage across the tank circuit V_T is measured as a function of the applied drive current I_D.

we can use trigonometric and Bessel-function identities to find the Fourier components of I.[12] In particular, the ω-frequency component is

$$I_\omega = -2[1+(\gamma\beta)^2]^{1/2} I_c \cos\left(\frac{2\pi\Phi_{dc}}{\Phi_0}\right) J_1\left(\frac{2\pi\Phi_{ac}}{\Phi_0}\right)$$

$$\times[\sin(\omega t+\theta')] - \omega\Phi_{ac} G \cos\omega t\,,$$

$$\theta' \equiv \arctan\gamma\beta\,, \qquad (4)$$

$$\beta \equiv \omega\Phi_0 G/2\pi I_c\,,$$

where J_1 is a Bessel function of the first kind of order 1.

The oscillating current in the interferometer loop I will induce a voltage $V_I = -M(dI/dt)$ in the tank circuit, where M is the mutual inductance. This induced voltage will result in an induced current $I_I = V_I/Z$, where

$$Z = R + i[\omega L' - (1/\omega C)]$$

is the complex impedance of the resonant circuit. Here L' is defined as the effective inductance of the resonant circuit; $L' = L - M^2/L_R$. If we choose the resonant frequency of the tank circuit $\omega_0 \equiv (L'C)^{-1/2}$ near the operating frequency ω, and if the tank circuit has a reasonably large Q ($Q \equiv \omega_0 L'/R \gtrsim 10$), then we need only consider the component of I at the fundamental frequency ω in computing I_I; the other components will be strongly attenuated by the large Z at the higher frequencies. Thus our expression for I_I becomes, using complex notation,

$$I_I = \mathrm{Re}\left\{-\frac{M}{Z}\frac{d}{dt}\left[-2I_c\cos\left(\frac{2\pi\Phi_{dc}}{\Phi_0}\right)J_1\left(\frac{2\pi\Phi_{ac}}{\Phi_0}\right)[1+(\gamma\beta)^2]^{1/2}\,\frac{\exp[i(\omega t+\theta')]}{i} - \omega\Phi_{ac}G\,e^{i\omega t}\right]\right\}$$

$$\simeq -\frac{A}{(1+\delta^2)^{1/2}}\left[[1+(\gamma\beta)^2]^{1/2}\cos\left(\frac{2\pi\Phi_{dc}}{\Phi_0}\right)J_1\left(\frac{2\pi\Phi_{ac}}{\Phi_0}\right)\sin(\omega t+\theta'-\theta)+\frac{\beta}{2}\left(\frac{2\pi\Phi_{ac}}{\Phi_0}\right)\cos(\omega t-\theta)\right], \qquad (5)$$

where $A \equiv 2(QK^2)(I_c L_R/M)$, $K^2 \equiv M^2/L'L_R$ is the coupling constant between the ring and tank circuit, $\delta \equiv 2[(\omega-\omega_0)/\omega_0]Q$ is a normalized measure of the difference between the operating frequency ω and the resonant frequency ω_0, and $\theta = \arctan\delta + \frac{1}{2}\pi$.

If the tank circuit alone is used to apply the ac magnetic flux to the interferometer ring, then we have the simple relationship $I_T = (\Phi_{ac}/M)\sin\omega t$, where I_T is the total ac current in the tank circuit. The difference between the total current I_T and the induced current I_I is the current flowing in the tank circuit as a result of the drive current I_D. This component will have a magnitude of approximately $[Q/(1+\delta^2)^{1/2}]I_D$. Thus we can write

$$I_D = [(1+\delta^2)^{1/2}/Q][I_T - I_I]\,, \qquad (6)$$

$$V_T = \omega L' I_T\,,$$

where V_T is the total ac voltage across the tank circuit, and I_I is given by Eq. (5). Figure 2 is a plot of V_T vs I_D generated from these equations for a particular set of parameters. Figure 3 shows

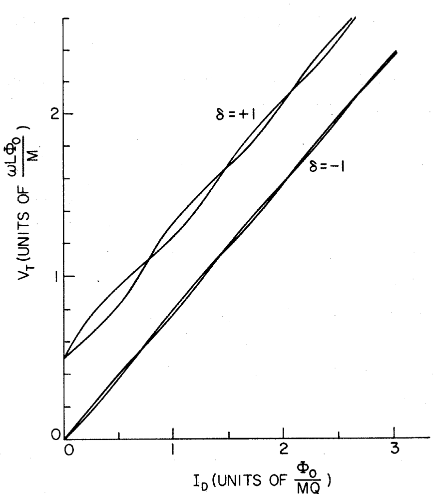

FIG. 2. Voltage across the tank circuit V_T vs the drive current I_D above $\{\delta = [2(\omega-\omega_0)/\omega_0]Q = +1\}$ and below ($\delta = -1$) resonance. These curves are plotted assuming that the coefficient of the $\cos\phi$ term γ is equal to -1 for $\beta = 1$, $(QK^2)(I_c L_R/\Phi_0) = (1/4\pi)$, and $\cos(2\pi\,\Phi_{dc}/\Phi_0) = \pm 1$. Note that the modulation amplitude is larger above resonance. The curves for $\delta = +1$ are offset for clarity.

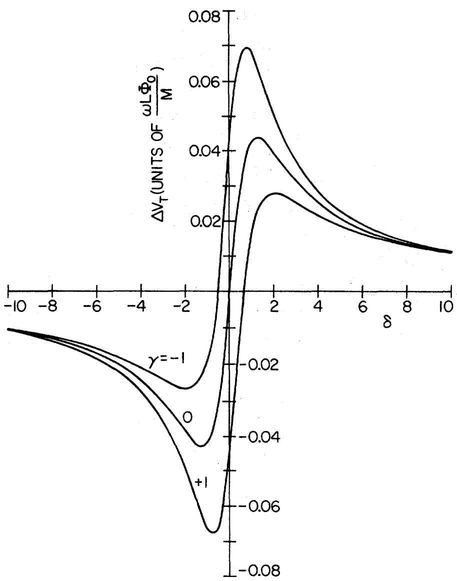

FIG. 3. Modulation amplitude ΔV_T at its second maximum as a function of the tuning parameter δ. Note that for the coefficient of the $\cos\phi$ term γ equal to $+1, 0$, or -1 there is a zero crossing above, on, or below resonance, respectively. The curves are plotted for $\beta=\frac{1}{2}$ and $(QK^2)(I_cL_R/\Phi_0)=(1/4\pi)$.

the magnitude of the modulation amplitude,

$$\Delta V_T \equiv V_T(\Phi_{dc}=0) - V_T(\Phi_{dc}=+\tfrac{1}{2}\Phi_0)$$

at its second maximum as a function of the tuning parameter $\delta \equiv 2(\omega-\omega_0)Q/\omega_0$. Note that for $\gamma=0$ this curve has a zero crossing at exactly $\delta=0$: The interferometer tunes antisymmetrically around resonance as previously described.[1] If $\gamma\neq 0$, the zero crossing is shifted from $\delta=0$: The zero crossing is above resonance for γ positive and below resonance for γ negative.

As a numerical example, for $\gamma\beta \ll 1$, the null occurs at roughly $\theta=\theta'$ which implies $\gamma\beta\approx\Delta\delta = 2\Delta f Q/f_0$, where Δf is the difference in frequency between the null and the resonant frequency f_0. Substituting for β this gives $\Delta f \approx \gamma f_0^2 G\Phi_0/2I_cQ$. For $|\gamma|\approx 1$, $f_0=30$ MHz, $G=0.1$ mho, $I_c=0.1$ μA, and $Q=25$ we have $\Delta f\approx 36$ kHz, an easily resolvable frequency shift. The ratio of the maximum flux sensitivity above the null to the maximum flux sensitivity below the null is an even more sensitive measure of γ. A simple formula for this ratio has not, however, been found and complete V_T-vs-δ curves must be generated to fit it. (Note: for the above example, $\beta=0.06$ so $\gamma\beta\ll 1$ is satisfied; if a low inductance, e.g., torroidal geometry, is used so that $L_R\approx 10^{-10}$ H, then $\epsilon\approx 0.03$ and screening corrections are small.[13])

For $\epsilon \gtrsim 0.1$, the screening flux cannot be neglected and we must write $\Phi_{tot}=\Phi_{app}+IL_R$. The equation for I which results when this expression for Φ_{tot} is substituted into Eq. (3) has no known analytic solution. However, a series expansion in the parameter $\epsilon\equiv(2\pi/\Phi_0)L'I_c$ can be made. The details of that series expansion in $(2\pi/\Phi_0)L'I_c$ will be included together with computer-generated theoretical curves and experimental data in a joint paper[14] with Deaver's group at the University of Virginia and will not be duplicated here. Some results of that work, however, are relevant to this paper and are summarized in Appendix B.

III. SUMMARY

Quasiparticle-pair interference conductance $G\gamma\cos\phi$ produces a measurable change in the tuning characteristics of rf SQUID operating in the nonhysteretic regime:

$$\epsilon\equiv(2\pi/\Phi_0)L_RI_c<1.$$

The null in the tuning characteristic shifts from the resonant frequency linearly with γ in the limit $\beta\gamma\ll 1$. The shifts are of order 10 kHz for reasonable SQUID parameters. In addition, the maximum magnetic-flux sensitivities above and below the null become different. The ratio of these maxima is also a sensitive function of γ.

Measurements based on this theory have been reported on point contacts and microbridges. It may prove possible in the future to do measurements on other types of weak links and to determine experimentally, for the first time, the temperature dependence of γ.

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APPENDIX A

A simple argument gives insight into the source of the effect. Figure 4(a) shows a vector representation of the current flowing in the tank circuit I_T. Equation (4) shows that in response to the resultant ac magnetic flux MI_T there will be an ω frequency component of current in the interferometer, I_ω, with two parts: (i) one which is sensitive to the sign of the dc magnetic flux and leads I_T by a phase angle θ', and (ii) one which is insensitive to the sign of the dc magnetic flux. For the moment we will neglect the second term which can be shown merely to complicate things both analytically and graphically while not changing the quali-

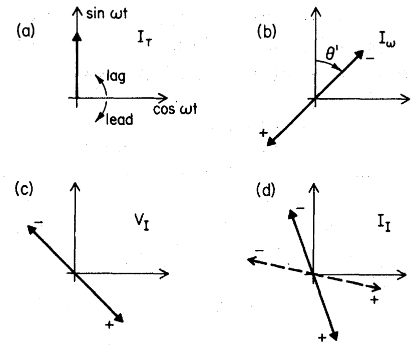

FIG. 4. Vector representations of (a) the total current in the tank circuit I_T; (b) the resultant current at frequency ω in the interferometer ring I_ω for $\cos(2\pi\Phi_{dc}/\Phi_0) = \pm 1$; (c) the induced voltage in the tank circuit V_I; and finally (d) the resultant induced signal current I_I above (shown broken) and below (shown solid) resonance. It has been assumed that α is positive; note that I_I is becoming perpendicular to I_T above resonance.

tative results we are after. Figure 4(b) shows a vector representation of the first term for $\cos(2\pi\Phi_{dc}/\Phi_0) = \pm 1$ assuming $\omega\Phi_0 G/2\pi I_c = 1$, $\gamma = +1$, and $\Phi_{ac} \approx 0.3\Phi_0$ (the first maximum of J_1). This current in the interferometer loop will induce a voltage $V_I = -M(dI/dt)$, which lags it by 90°, in the tank circuit as shown in Fig. 4(c). This voltage will result in an induced signal current, $I_I = V_I/Z$, which lags the voltage above resonance and leads it below resonance as shown in Fig. 4(d). The important point to note is that above resonance a null is being approached: The magnitudes of I_T, I_I, and $(I_T - I_I) \alpha I_D$ are the same for $\Phi_{dc} = +\frac{1}{2}\Phi_0$ as for $\Phi_{dc} = -\frac{1}{2}\Phi_0$. Thus for $\gamma = +1$ the null will be above resonance. A similar argument can be made to show that for $\gamma = -1$ the null will be below resonance.

APPENDIX B

Screening current must be considered if $\epsilon \ll 1$ is not satisfied. Its effect is introduced between Eqs. (3) and (4) of the text. Specifically, the approximation $\Phi_{tot} \approx \Phi_{app}$ breaks down and we must use the complete expression $\Phi_{tot} = \Phi_{app} + IL_R$. Substituting this into Eq. (3) and introducing dimensionless parameters,

$$i/\epsilon = \sin(-a - b\sin\omega t - i) - \beta\left(b\cos\omega t + \frac{1}{\omega}\frac{di}{dt}\right)[1 + \alpha\cos(-a - b\sin\omega t)] , \quad (7)$$

where

$$i \equiv \epsilon I/I_c$$

$$a = 2\pi(\Phi_{dc}/\Phi_0) ,$$

$$b = 2\pi(\Phi_{ac}/\Phi_0) .$$

The calculation then proceeds by expanding the trigonometric functions, successively approximating i on the right-hand side, identifying terms to various orders in ϵ, and using Bessel-function identities to identify the ω-frequency components. Details of the calculation will be included in a joint theoretical and experimental manuscript[13] with Deaver's group. To first order the main effect is to

$$\cos(a)\, J_1(b) - \tfrac{1}{2}\epsilon\cos(2a)\, J_1(2b)$$

and to replace $\gamma\beta$ by $(\gamma - 2\epsilon)\beta$.

The first replacement changes the shape of the V_T-vs-I_c and the V_T-vs-δ curves. Specifically it introduces an asymmetry in their shape above and below the null. The second replacement has more importance for the present work. It means that γ_{eff}, the γ measured by fitting the ΔV_T-vs-δ curves, will be $\gamma_{eff} = \gamma - 2\epsilon$. This correction must be taken into account if $\epsilon \ll \gamma$ is not satisfied.

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[1]P. K. Hansma, J. Appl. Phys. 44, 4191 (1973).

[2]The regime $\epsilon < 1$ is called the "nonhysteretic regime" to emphasize that Φ_{tot}, the total flux enclosed in the interferometer loop, is a nonhysteretic function of Φ_{app}, the applied flux. For $\epsilon > 1$ the hysteresis of Φ_{tot} vs Φ_{app} forms the basis for a different well-understood mode of interferometer operation. The nonhysteretic regime was first described in Ref. 3 and discussed further in Ref. 4. Reference 5 contains experimental measurements which are compared with the theoretical predictions of Ref. 1 and this paper.

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[6]A great deal of interest in quasiparticle-pair conductance was stimulated by the publication of the first experimental measurements [N. F. Pedersen, T. F. Finnegan, and D. N. Langenberg, Phys. Rev. B 6, 4151 (1972)]. Since that time a controversy has arisen because of an apparent discrepency between the experimentally measured and theoretically predicted sign of this conductance. Reference 7 contains an excellent summary of the controversy and references to recent work.

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[8]D. A. Vincent and B. S. Deaver, Jr., Phys. Rev. Lett. 32, 212 (1974).

[9]B. D. Josephson, Adv. Phys. 14, 419 (1965). There is evidence that, at least in the limit of very low critical currents, many types of weak links to have sinusoidal current-phase relationships. P. E. Gregers-Hansen,

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[10]See, for example, R. P. Feynman, *Lectures on Physics* (Addison-Wesley, Reading, Mass., 1965), Vol. 3.

[11]This is opposite of the convention used in Ref. 1. It is used here since then $V = -(d\Phi_{tot}/dt)$ in agreement with the conventions of standard texts on coupled resonant circuits.

[12]See, for example, G. N. Watson, *Theory of Bessel Functions* (Cambridge U. P., Cambridge, England, 1962), p. 22.

[13]See appendix B and Ref. 5.

[14]A. Callegari, B. Deaver, P. K. Hansma, R. Rifkin, and A. Vincent, (unpublished).