Quantization of the hydrodynamic modes in superfluid ^{4}He with a free surface*

W. F. Saam
Department of Physics, The Ohio State University, Columbus, Ohio 43210
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Quantization of the hydrodynamic modes in superfluid ^{4}He with a free surface is carried out. A complete and orthonormal set of eigenfunctions for both ripplons and surface-reflected phonons is constructed. Applications of the formalism to the calculation of the thermal energy density near the surface, the temperature-dependent part of the surface tension, and the rate at which power is absorbed by the surface from an incident phonon beam are given.

I. INTRODUCTION

In recent years there has been considerable interest[1–6] in the properties of the elementary excitations (ripplons) associated with the presence of a free surface of superfluid ^{4}He. Of that work which has dealt with the longer-wavelength excitations, much[1,2] has followed the original idea of Atkins[6] in treating the excitations as quantized capillary waves in an incompressible fluid. One exception is the work of Edwards *et al.*,[4] who did account for the effect of phonons on the temperature-dependent part of the surface tension. Another is a calculation by the author,[3] utilizing a response-function technique of inelastic neutron scattering from the superfluid ^{4}He free surface. Nowhere, however, has there appeared the complete, unified treatment of all the hydrodynamic modes of the ^{4}He system necessary for the calculation of such quantities as transition rates arising from interactions between the modes.[7] It is the primary purpose of this paper to provide such a treatment.

In Sec. II we solve the hydrodynamic equations for an ideal compressible fluid with a planar free surface and obtain an orthonormal and complete set of modes. In Sec. III we construct a Lagrangian and Hamiltonian and then quantize the Hamiltonian in oscillator form using the complete set of oscillator modes found in Sec. II. Finally, in Sec. IV we present some applications of the formalism. We first calculate the temperature-dependent part of the energy density near the surface and from this the temperature-dependent part of the surface tension, including contributions from both ripplons and phonons. Secondly, we derive an expression for the power lost to the surface (in the form of ripplons) by phonons incident on the surface and show that the power loss might well be observable for very energetic (phonon temperature ~7 K) phonons.

II. CONSTRUCTION OF AN ORTHONORMAL AND COMPLETE SET OF MODES

The construction of the modes begins with the set of linearized hydrodynamic equations describing zero-temperature ^{4}He with a free surface. We shall assume that the surface is located a $z=0$, the bulk of the unperturbed liquid being in the half-space $z\leqslant 0$. The linearized hydrodynamic equations for the bulk are

$$\frac{\partial \delta\rho(\vec{r}t)}{\partial t}+\rho_0\vec{\nabla}\cdot\vec{v}(\vec{r}t)=0, \tag{1}$$

$$\frac{\partial \vec{v}(\vec{r}t)}{\partial t}+\vec{\nabla}\mu(\vec{r}t)=0. \tag{2}$$

Here $\delta\rho(\vec{r}t)$ is the deviation of the density from its equilibrium value ρ_0, $\vec{v}(\vec{r}t)$ is the superfluid velocity, and $\mu(\vec{r}t)$ is the chemical potential. Equations (1) and (2) must be solved subject to the boundary conditions[8]

$$\delta P(\vec{r}t)|_{z=0}=-\sigma_0\nabla_{\parallel}^2\zeta(\vec{r}_{\parallel},t), \tag{3}$$

$$\frac{\partial\zeta(\vec{r}_{\parallel},t)}{\partial t}=v_z(\vec{r},t)|_{z=0}, \tag{4}$$

where $\zeta(\vec{r}_{\parallel},t)$ is the position, relative to $z=0$, of the surface at the point $\vec{r}_{\parallel}=x\hat{\imath}+y\hat{\jmath}$, σ_0 is the (zero-temperature) surface tension, $\delta P(\vec{r}t)$ is the deviation of the pressure from its equilibrium value, and $\nabla_{\parallel}^2\equiv\partial^2/\partial x^2+\partial^2/\partial y^2$. It is convenient to introduce the velocity potential $\varphi(\vec{r}t)$ via

$$\vec{v}(\vec{r}t)=\vec{\nabla}\varphi(\vec{r}t), \tag{5}$$

and to use the relation

$$\rho_0\delta\mu(\vec{r}t)=\delta P(\vec{r}t)=s^2\delta\rho(\vec{r}t), \tag{6}$$

where s is the zero-temperature sound velocity. It is now a straightforward matter to use Eqs. (1)–(6) to obtain the wave equation

$$\frac{\partial^2\varphi(\vec{r}t)}{\partial t^2} - s^2\nabla^2\varphi(\vec{r}t) = 0 \tag{7}$$

for $\varphi(\vec{r}t)$ and the boundary condition

$$\left.\frac{\partial^2\varphi(\vec{r}t)}{\partial t^2}\right|_{z=0} = \frac{\sigma_0\nabla_\parallel^2}{\rho_0}\left.\frac{\partial\varphi(\vec{r}t)}{\partial z}\right|_{z=0}, \tag{8}$$

which $\varphi(\vec{r}t)$ must obey.

Our problem now is to obtain a complete set of orthonormal eigenfunctions satisfying (7) and (8). We search for functions $\varphi(\vec{r}t)$ having the form

$$\varphi_{ql}(\vec{r}t) = \varphi_{ql}(z)e^{i\vec{q}\cdot\vec{r}_\parallel}e^{-i\omega_{ql}t}, \tag{9}$$

where $\vec{q}$ is a wave vector parallel to the surface and l is an index which will describe the z dependence. Clearly, solutions with different $\vec{q}$'s will be orthonormal. Inserting (9) into (7) and (8) yields

$$\left(\frac{\partial^2}{\partial z^2} - q^2\right)\varphi_{ql}(z) = -\frac{\omega_{ql}^2}{s^2}\varphi_{ql}(z), \tag{10}$$

$$\left.\frac{\partial\varphi_{ql}(z)}{\partial z}\right|_{z=0} = \frac{\omega_{ql}^2\rho_0}{\sigma_0 q^2}\varphi_{ql}(0). \tag{11}$$

The solutions to (10) and (11) have some important general properties. To discover these we multiply (10) by $\varphi_{ql'}^*(z)$ and integrate over z,

$$\int_{-\infty}^0 dz\left(-\varphi_{ql'}^*(z)\frac{\partial^2\varphi_{ql}(z)}{\partial z^2} + q^2\varphi_{ql'}^*(z)\varphi_{ql}(z)\right) = \frac{\omega_{ql}^2}{s^2}\int_{-\infty}^0 dz\,\varphi_{ql'}^*(z)\varphi_{ql}(z). \tag{12}$$

Integrating the first term in large parentheses on the left-hand side by parts, assuming that the contribution at $z=-\infty$ is zero, and using (11) to evaluate the contribution at $z=0$, allows one to write (12) in the form

$$\int_{-\infty}^0 dz\left(\frac{\partial\varphi_{ql'}^*(z)}{\partial z}\frac{\partial\varphi_{ql}(z)}{\partial z} + q^2\varphi_{ql'}^*(z)\varphi_{ql}(z)\right) = \frac{\omega_{ql}^2}{s^2}\left(\frac{\rho_0 s^2}{\sigma_0 q^2}\varphi_{ql'}^*(0)\varphi_{ql}(0) + \int_{-\infty}^0 dz\,\varphi_{ql'}^*(z)\varphi_{ql}(z)\right). \tag{13}$$

From this result, taken for the case $l=l'$, it is clear that ω_{ql}^2 is both real and positive, i.e.,

$$\omega_{ql}^2 = \omega_{ql}^{2*},\quad \omega_{ql}^2 \geq 0. \tag{14}$$

Next, if we take the complex conjugate of (13), interchange l and l', and subtract the result from (13), we obtain [using (14)]

$$(\omega_{ql'}^2 - \omega_{ql}^2)\frac{1}{s^2}\left(\frac{s^2\rho_0}{\sigma_0 q^2}\varphi_{ql'}^*(0)\varphi_{ql}(0) + \int_{-\infty}^0 dz\,\varphi_{ql'}^*(z)\varphi_{ql}(z)\right) = 0. \tag{15}$$

From this it follows that

$$\int_{-\infty}^0 dz\,\varphi_{ql'}^*(z)\varphi_{ql}(z) + \frac{s^2\rho_0}{\sigma_0 q^2}\varphi_{ql'}^*(0)\varphi_{ql}(0) = 0, \tag{16}$$

if $\omega_{ql}^2 \neq \omega_{ql'}^2$. This is the desired orthogonality relation between the modes. We see from (13) that this also implies

$$\int_{-\infty}^0 dz\left(\frac{\partial\varphi_{ql'}^*(z)}{\partial z}\frac{\partial\varphi_{ql}(z)}{\partial z} + q^2\varphi_{ql'}^*(z)\varphi_{ql}(z)\right) = 0;\quad \omega_{ql}^2 \neq \omega_{ql'}^2. \tag{17}$$

To proceed further, we will need explicit expressions for the eigenfunctions $\varphi_{ql}(z)$ and eigenfrequencies ω_{ql}. There are two types of solutions to (10) and (11). We will just state them; the fact that they are solutions is easily checked. The first corresponds to a capillary wave (indexed by $l=0$) and is

$$\varphi_{q0}(z) = [2\kappa b_q/(q^2+b_q^2)^{1/2}]\,e^{\kappa z}. \tag{18}$$

The quantity b_q, the inverse decay length κ, and the ripplon frequency ω_{q0} are determined from

$$b_q = \frac{\sigma_0 q^2}{2\rho_0 s^2},\quad \kappa = -b_q + (q^2+b_q^2)^{1/2},$$

$$\omega_{q0} = \left(\frac{\sigma_0\kappa q^2}{\rho_0}\right)^{1/2}. \tag{19}$$

Here $\varphi_{q0}(z)$ is normalized according to

$$\int_{-\infty}^0 dz|\varphi_{q0}(z)|^2 + \frac{s^2\rho_0}{\sigma_0 q^2}|\varphi_{q0}(0)|^2 = 1, \tag{20}$$

as is suggested by (16). The second type of solution corresponds to a phonon reflected at the surface and is

$$\varphi_{qk}(z) = e^{ikz} - R_{qk}e^{-ikz},\quad k>0. \tag{21}$$

The reflection coefficient R_{qk} and phonon frequency ω_{qk} are given by

$$R_{qk} = \frac{q^2+k^2-2ib_q k}{q^2+k^2+2ib_q k},\quad \omega_{qk} = s(q^2+k^2)^{1/2}. \tag{22}$$

The z component k of the phonon wave vector is restricted to have positive values. Otherwise, the set of eigenfunctions (18) and (21) would be overcomplete. This is easily seen if one notes that a reflected wave with wave vectors $\vec{q}$ and k is equivalent to one with wave vectors $\vec{q}$ and $-k$.

Now, it is easy to show that $\omega_{q\kappa} < sq$ for all finite q. Hence, phonon and ripplon frequencies are never equal, and the phonon and ripplon eigenfunctions are orthogonal via (16). The same holds for phonon eigenfunctions with different values of k. There remains the question of the normalization of the phonon eigenfunctions. In fact, these functions, as given in (21), obey the relation

$$\int_{-\infty}^{0} dz\, \varphi_{qk}^*(z)\varphi_{qk'}(z) + \frac{s^2\rho_0}{\sigma_0 q^2}\, \varphi_{qk}^*(0)\varphi_{qk'}(0) = 2\pi\delta(k-k'). \tag{23}$$

To prove this, note that

$$\int_{-\infty}^{0} dz\, \varphi_{qk}^*(z)\varphi_{qk'}(z) = \int_{-\infty}^{0} dz\, e^{\epsilon z}(e^{-ikz} - R_{qk}^*\, e^{ikz}) \times (e^{ik'z} - R_{qk'}\, e^{-ik'z}),$$

where ϵ is a positive infinitesimal introduced to make the integral well defined. The integral is simply done, and making use of the relation

$$1/(x+i\epsilon) = -i\pi\delta(x) + P/x, \tag{24}$$

the fact that k and k' are positive, and the definitions (19) and (22) of b_q and R_{qk} leads immediately to the verification of (23).

Finally, we prove that the ripplon and phonon eigenfunctions (18) and (21) form a complete set. The question of completeness as regards the x and y coordinates is obviously handled by the $e^{i\vec{q}\cdot\vec{r}_\parallel}$ factors in the eigenfunctions. Hence, all we really need do is show that

$$\sum_l \varphi_{ql}^*(z')\varphi_{ql}(z) = \delta(z-z'). \tag{25}$$

The phonon part of the sum in (25) is ($\epsilon = 0^+$ is a convergence factor)

$$\int_0^\infty \frac{dk}{2\pi}\, e^{-\epsilon k}(e^{-ikz'} - R_{qk}^*\, e^{ikz'})(e^{ikz} - R_{qk}e^{-ikz}) = \delta(z-z') - \int_0^\infty \frac{dk}{2\pi}(R_{qk}^* e^{ik(z+z')} + R_{qk}e^{-ik(z+z')}), \tag{26}$$

where we have used (24) and the property $|R_{qk}|^2 = 1$. Using the property $R_{q,k}^* = R_{q,-k}$, the integral on the right-hand side of (26) may be put in the form

$$J_q(z+z') \equiv \int_0^\infty \frac{dk}{2\pi}(R_{qk}^* e^{ik(z+z')} + R_{qk}e^{-ik(z+z')}) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, R_{qk}e^{-ik(z+z')} = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \frac{q^2+k^2-2ib_qk}{q^2+k^2+2ib_qk}\, e^{-ik(z+z')}. \tag{27}$$

Since $z+z' < 0$, we can evaluate the integral by closing the contour in the upper half-plane, picking up the pole at $k = i\kappa$ [κ is defined in (19)] to find

$$J_q(z+z') = [2\kappa b_q/(b_q^2+q^2)^{1/2}]\, e^{\kappa(z+z')}. \tag{28}$$

The phonon part of the completeness sum is thus

$$\delta(z-z') - [2\kappa b_q/(q^2+b_q^2)^{1/2}]\, e^{\kappa(z+z')},$$

the second term of which is canceled [see (18)] by the ripplon contribution. Hence, (25) is proven.

III. QUANTIZATION OF THE MODES

The procedure we follow here is the standard one of first constructing a Lagrangian and then a Hamiltonian for the set of field equations (7) and (8). The Hamiltonian will then be quantized in terms of creation and annihilation operators for the orthonormal and complete set of modes constructed in Sec. II.

We multiply (7) by a variation $\delta\varphi(\vec{r}t)$ and integrate over the volume of the fluid and over a time interval t_1 to t_2. Imposing the conditions $\delta\varphi(\vec{r}, t_1) = \delta\varphi(\vec{r}, t_2) = 0$, carrying out some parts integrations, and expressing the result in terms of Fourier components of $\varphi(\vec{r}t)$ defined by

$$\varphi(\vec{r}t) = \sum_{\vec{q}} e^{i\vec{q}\cdot\vec{r}_\parallel}\varphi_{\vec{q}}(zt), \quad \varphi^*_{-\vec{q}}(zt) = \varphi_{\vec{q}}(zt), \tag{29}$$

we obtain

$$\delta\int_{t_1}^{t_2} dt \sum_{\vec{q}} \int_{-\infty}^{0} dz \left(\frac{|\dot{\varphi}_{\vec{q}}(zt)|^2}{2} - \frac{s^2q^2}{2}|\varphi_{\vec{q}}(zt)|^2 - \frac{s^2}{2}\left|\frac{\partial\varphi_{\vec{q}}(zt)}{\partial z}\right|^2\right) + s^2\int_{t_1}^{t_2} dt \sum_{\vec{q}} \delta\varphi_{\vec{q}}^*(zt)\frac{\partial}{\partial z}\varphi_{\vec{q}}(zt)\Big|_{z=0} = 0. \tag{30}$$

Use of the boundary condition (8) in the last term of (30), doing a parts integration in time, and multiplying by a factor ρ_0/s^2 leads to the principle of least action for the system,

$$\delta\int_{t_1}^{t_2} L\, dt = 0, \tag{31}$$

where the Lagrangian L is given by

$$L = \sum_{\vec{q}} \int_{-\infty}^{0} dz \left(\frac{\rho_0}{2s^2}|\dot{\varphi}_{\vec{q}}(zt)|^2 - \frac{\rho_0 q^2}{2}|\varphi_{\vec{q}}(zt)|^2 - \frac{\rho_0}{2}\left|\frac{\partial\varphi_{\vec{q}}(zt)}{\partial z}\right|^2\right) + \sum_{\vec{q}} \frac{\rho_0^2}{2\sigma_0 q^2}|\dot{\varphi}_{\vec{q}}(0,t)|^2. \tag{32}$$

It is easy to verify that variation of the action with respect to $\varphi_{\vec{q}}^*(zt)$ leads to the Fourier-transformed version of (7), while variation with respect to $\varphi_q^*(0,t)$ leads to the Fourier-transformed equation (8). Thus $\varphi_q(zt)$ and $\varphi_q(0,t)$ are, in fact, independent variables coupled by the constraint of continuity at $z=0$. The momentum conjugate to $\varphi_{\vec q}(zt)$ is

$$\pi_{\vec q}(zt)=\frac{\delta L}{\delta\dot\varphi_{\vec q}(zt)}=\frac{\rho_0}{s^2}\,\dot\varphi_{\vec q}^*(zt)=\frac{\rho_0}{s^2}\,\varphi_{-\vec q}(zt), \tag{33}$$

while that conjugate to $\varphi_{\vec q}(0,t)$ is

$$X_{\vec q}(0,t)=\frac{\partial L}{\partial\dot\varphi_q(0,t)}=\frac{\rho_0^2}{\sigma_0q^2}\,\dot\varphi_{\vec q}^*(0,t)=\frac{\rho_0^2}{\sigma_0q^2}\,\varphi_{-\vec q}(0,t). \tag{34}$$

Consequently, the Hamiltonian for our system is

$$H=\sum_{\vec q}\int_{-\infty}^0 dz\left(\frac{\rho_0}{2s^2}\,|\dot\varphi_{\vec q}(zt)|^2+\frac{\rho_0q^2}{2}\,|\varphi_{\vec q}(zt)|^2+\frac{\rho_0}{2}\left|\frac{\partial\varphi_{\vec q}(zt)}{\partial z}\right|^2\right)+\sum_{\vec q}\frac{\rho_0^2}{2\sigma_0q^2}\,|\dot\varphi_{\vec q}(0,t)|^2. \tag{35}$$

Recalling that φ_q is a velocity potential, we see that the second two terms in square brackets in (35) are just what we would write down for the kinetic energy of the fluid, thus justifying our above multiplication by ρ_0/s^2.

Introducing the transformation

$$\varphi_{\vec q}(zt)=\sum_l i\left(\frac{\hbar s^2}{2\rho_0\omega_{\vec q l}}\right)^{1/2}\left[C_{-\vec q,l}^\dagger\,\varphi_{-\vec q,l}^*(z)e^{i\omega_{ql}t}-C_{\vec q,l}\varphi_{\vec q,l}(z)e^{-i\omega_{ql}t}\right], \tag{36}$$

and imposing the canonical commutation relations,

$$\begin{aligned}&[\varphi_{\vec q}(zt),\pi_{\vec q'}(z't)]=i\hbar\,\delta_{\vec q,\vec q'}\delta(z-z'),\\&[\varphi_{\vec q}(zt),\varphi_{\vec q'}(z't)]=[\pi_{\vec q}(zt),\pi_{\vec q'}(z't)]=0,\end{aligned} \tag{37}$$

leads, via (33), (35) and the orthonormality results of Sec. II, to

$$H=\sum_{\vec q,l}\hbar\omega_{ql}\,(C_{\vec q,l}^\dagger C_{\vec q,l}+\tfrac12), \tag{38}$$

and the fact that the annihilation and creation operators $C_{\vec q,l}$ and $C_{\vec q,l}^\dagger$ obey the usual Bose commutation relations, as expected.

Operator expressions for the velocity $\vec v_{\vec q}(zt)$, the density deviation $\delta\rho_q(zt)$, and the surface displacement $\zeta_q(zt)$ will be useful in subsequent calculations. Utilization of (36), (5), (2), (6), and (4) leads to

$$\begin{aligned}\vec v_{\vec q}(zt)&=\sum_l i\left(\frac{\hbar s^2}{2\rho_0\omega_{ql}}\right)^{1/2}\left(i\vec q+e_z\frac{\partial}{\partial z}\right)\left[C_{\vec q,l}^\dagger\,\varphi_{-\vec q,l}(z)e^{i\omega_{ql}t}-C_{\vec q l}\,\varphi_{\vec q,l}(z)e^{-i\omega_{ql}t}\right],\\ \delta\rho_{\vec q}(zt)&=+\sum_l\left(\frac{\hbar\omega_{ql}\rho_0}{2s^2}\right)^{1/2}\left[C_{-\vec q,l}^\dagger\varphi_{-\vec q,l}^*(z)e^{i\omega_{ql}t}+C_{\vec q,l}\,\varphi_{\vec q,l}(z)e^{-i\omega_{ql}t}\right],\\ \zeta_{\vec q}(t)&=\sum_l\left(\frac{\hbar s^2\rho_0\omega_{ql}}{2\sigma_0^2q^4}\right)^{1/2}\left[C_{\vec q,l}^\dagger\,\varphi_{-\vec q,l}^*(0)e^{i\omega_{ql}t}+C_{\vec q,l}\,\varphi_{\vec q,l}(0)e^{-i\omega_{ql}t}\right].\end{aligned} \tag{39}$$

IV. APPLICATIONS

As a first example, we calculate the energy associated with the free surface at temperature T. From (35) and (39), the energy density $\epsilon(z)$ is

$$\begin{aligned}\epsilon(z)&=\sum_{\vec q}\left\langle\frac{\rho_0\vec v_{\vec q}^{\,\dagger}(z)\cdot\vec v_{\vec q}(z)}{2}+\frac{s^2}{2\rho_0}\,\delta\rho_{\vec q}^\dagger(z)\delta\rho_{\vec q}(z)+\frac{\sigma_0q^2}{2}\,\delta(z)\zeta_{\vec q}^\dagger\zeta_{\vec q}\right\rangle\\&=\sum_{\vec q l}\frac{\hbar\omega_{ql}}{2}\,\frac{1}{e^{\beta\hbar\omega_{ql}}-1}\left[\frac{s^2}{\omega_{ql}^2}\left(q^2|\varphi_{ql}(z)|^2+\left|\frac{\partial\varphi_{ql}(z)}{\partial z}\right|^2\right)+|\varphi_{ql}(z)|^2+\frac{s^2\rho_0}{\sigma_0q^2}\,|\varphi_{ql}(0)|^2\delta(z)\right],\end{aligned} \tag{40}$$

where $\beta\equiv1/k_BT$. The energy associated with the surface is just

$$\epsilon_s=\int_{-\infty}^0 dz\,[\epsilon(z)-\epsilon(-\infty)], \tag{41}$$

where

$$\epsilon(-\infty)={\sum_{\vec q l}}'\,\hbar\omega_{\vec q l}\,\frac{2}{e^{\beta\hbar\omega_{ql}}-1} \tag{42}$$

is the energy density deep in the bulk of the liquid.[9] The prime on the summation sign indicates that the ripplon mode is omitted. The surface energy is most conveniently expressed in terms of separate contributions from the phonons and ripplons ϵ_s^p and ϵ_s^r, respectively. Combination of (18), (19), and (40)–(42) yields

$$\epsilon_s^r = \sum_{\vec{q}} \frac{\hbar\omega_{q0}}{e^{\beta\hbar\omega_{q0}} - 1} . \tag{43}$$

Similarly, combination of (21), (22), and (40)–(42) leads to[10]

$$\epsilon_s^p = -\frac{1}{4}\sum_{\vec{q}} \frac{\hbar s q}{e^{\beta\hbar s q} - 1} + \sum_{\vec{q},\,\text{all } k} \frac{\hbar\omega_{qk}|1 - R_{qk}|^2}{8 b_q} \frac{1}{e^{\beta\hbar\omega_{qk}} - 1} . \tag{44}$$

While the integrals involved in (43) and (44) cannot be done analytically for all temperatures, at low temperatures an expansion in powers of the phonon thermal wave number times the length $\sigma_0/\rho_0 s^2 = 0.45$ Å proves useful. We find

$$\epsilon_s = \epsilon_s^r + \epsilon_s^p = \frac{\hbar}{3\pi}\left(\frac{\rho_0}{\sigma_0}\right)^{2/3}\left(\frac{kT}{\hbar}\right)^{7/3}\Gamma\left(\frac{7}{3}\right)\zeta\left(\frac{7}{3}\right) - \frac{1}{8\pi}\left(\frac{k_B T}{\hbar s}\right)^3 \zeta(3) + O(T^{10/3}). \tag{45}$$

Consequently, the temperature-dependent part of the surface tension $\delta\sigma$ (Helmholtz free energy per unit area for the surface) is

$$\delta\sigma = -\frac{\hbar}{4\pi}\left(\frac{\rho_0}{\sigma_0}\right)^{2/3}\left(\frac{kT}{\hbar}\right)^{7/3}\Gamma\left(\frac{7}{3}\right)\zeta\left(\frac{7}{3}\right) + \frac{1}{16\pi}\left(\frac{k_B T}{\hbar s}\right)^3 \zeta(3) + O(T^{10/3}). \tag{46}$$

A more interesting application of the results of Secs. II and III involves the calculation of transition rates. To the harmonic Hamiltonian which we have derived we add perturbations whose form is determined via standard quantum hydrodynamic arguments. The Hamiltonian, valid to third order in small quantities, is

$$\begin{aligned} H &= H_0 + H_1 + H_2, \\ H_0 &= \int_{-\infty}^{\infty} dx\,dy \int_{-\infty}^{0} dz \left(\frac{\rho_0}{2}\vec{v}^2(\vec{r}) + \frac{s^2}{2\rho_0}[\delta\rho(\vec{r})]^2\right) + \frac{1}{2}\int_{-\infty}^{\infty} dx\,dx\,\sigma_0[\vec{\nabla}_\parallel \zeta(x,y)]^2, \\ H_1 &= \int_{-\infty}^{\infty} dx\,dy \int_{-\infty}^{0} dz \left[\frac{\vec{v}(\vec{r})\cdot\delta\rho(\vec{r})\vec{v}(\vec{r})}{2} + \frac{1}{6}\left(\frac{\partial}{\partial\rho}\frac{s^2}{\rho}\right)[\delta\rho(r)]^3\right], \\ H_2 &= \int_{-\infty}^{\infty} dx\,dy \left(\frac{\rho_0}{2}\vec{v}(\vec{r})\cdot\zeta(x,y)\vec{v}(\vec{r}) + \frac{1}{2}\frac{\partial\sigma_0}{\partial C}\delta C(\vec{x}y)\right)_{z=0} [\vec{\nabla}_\parallel\zeta(xy)]^2. \end{aligned} \tag{47}$$

Here H_0 is the harmonic Hamiltonian, H_1 is the third-order perturbation commonly used for uniform systems, and H_2 contains the perturbations localized at the surface. The first part of H_2 has been used previously by the author[2] to calculate ripplon damping in the approximation that the liquid is incompressible. The second part gives the contribution arising from curvature dependence of the surface tension. The deviation $\delta C(x,y)$ in the curvature C is related to the surface displacement by

$$\delta C(x,y) = -\nabla_\parallel^2 \zeta(x,y). \tag{48}$$

Here and in (47), $\vec{\nabla}_\parallel$ is the gradient operator in the two-dimensional space parallel to the surface. The quantity $\partial\sigma_0/\partial C$ has not been measured for ^{4}He, but an estimate based on a model calculation[11] gives $\partial\sigma_0/\partial C = 3.7\times10^{-2}$ erg/cm. The smallness of this result can easily be shown to justify the neglect of the curvature term in the above-mentioned ripplon damping calculation for ripplon wavelengths larger than a few interatomic spacings.

As an example, we now use (47) to obtain the fractional power lost to the surface (via ripplon creation) by a beam of phonons incident on the surface. It is not difficult to show that for incoming phonon wavelengths large compared to the characteristic length $\sigma_0/\rho_0 s^2 = 0.45$ Å, the only important process involved is one where the phonon breaks up into a ripplon plus another phonon and that the dominant matrix element is governed by the first part of H_2 in (47). If we characterize the incoming phonon by wave vectors $\vec{q}$ and k, the outgoing phonon by wave vectors $\vec{q}''$ and k'', and the ripplon by the single wave vector $\vec{q}'$, the relevant matrix element becomes, using (39) and (47),

$$\langle \vec{q}'; \vec{q}'', k | H_1 + H_2 | \vec{q}, k\rangle \approx -\sqrt{2}\left(\frac{\hbar s^2}{\rho_0}\right)^{3/2} \rho_0 k k'' \sqrt{q'} \times (\omega_{qk}\omega_{q''k''}\omega_{q'0})^{-1/2}\delta_{\vec{q},\vec{q}''+\vec{q}'} . \tag{49}$$

Consequently, the transition rate for the process phonon → ripplon + phonon, where we sum over final states of the outgoing phonon is, assuming the liquid temperature to be sufficiently low that the normal fluid may be ignored,

$$R_{\vec{q},k\to\vec{q}'} = \frac{2\pi}{\hbar^2}\sum_{\vec{q}'',k''>0} |\langle \vec{q}'; \vec{q}'', k | H_1 + H_2 | \vec{q}, k\rangle|^2 \times \delta(\omega_{qk} - \omega_{q'k''} - \omega_{q'0}) \approx 2\frac{\hbar s^2}{\rho_0}\frac{k^2 q' k''}{\omega_{qk}\omega_{q'0}} . \tag{50}$$

Here, $k'' > 0$ is determined by

$$k''^2 = \left[\frac{\omega_{qk}}{s} - \left(\frac{\sigma_0}{\rho_0 s^2} \right)^{1/2} q'^{\,3/2} \right]^2 - (\vec{q} - \vec{q}')^2 . \qquad (51)$$

Since we are assuming that the incoming phonon wave vector is small compared to $\rho_0 s^2/\sigma_0$, (51) may be replaced by

$$k'' \cong [\omega_{qk}^2/s^2 - (\vec{q} - \vec{q}')^2]^{1/2} . \qquad (52)$$

From (50) and (52), the differential power dP being transmitted to the surface per unit ripplon solid angle is

$$\frac{dP}{d\Omega_{\vec{q}'}} = P_0 A \frac{\hbar k}{2\pi^2 \rho_0 \omega_{qk}} \times \int_0^{q_0} q'^2 \left(\frac{\omega_{qk}^2}{s^2} - (\vec{q} - \vec{q}')^2 \right)^{1/2} dq' , \qquad (53)$$

where $P_0 = \hbar\omega_{qk} s (sk/\omega_{qk})$ is the incoming power per unit area, A is the surface area, and q_0 is determined by the condition that the quantity in large parentheses in (53) be zero. While the integral in (53) may be done analytically, the resulting expression is complicated. For the case where the incoming phonon wave vector makes an angle of 45° (i.e., $q = k$) with the surface and $\vec{q}'$ is parallel to $\vec{q}$, we obtain

$$\frac{1}{P_0 A} \frac{dP}{d\Omega_{\vec{q}'}} \bigg|_{\vec{q} \parallel \vec{q}',\, q = k} = \frac{0.17 \hbar q^4}{\rho_0 s} . \qquad (54)$$

Consequently, we obtain as an estimate of the fraction of the incoming energy deposited on the surface just 2π times the right-hand side of (54). For one-degree phonons (i.e., $\sqrt{2}\, q = k_B/\hbar s$) this fraction is 0.72×10^{-4}, so that very little energy is transferred to the surface, a result consistent with experimental observations.[12] However, since the fractional energy transferred is proportional to the fourth power of the phonon temperature, the effect might well be observable for, say, 7 °K phonons. At higher energies competing processes involving desorption of a helium atom from the surface or roton creation at the surface may become important. Of course, neither of these processes can be dealt with using the theory presented here.

An interesting related problem would be to develop quantum hydrodynamics, along the lines presented here, for the case of thin films of ^{4}He (in which case the modes become discrete[13]) in order to see if the theory would yield the attenuation seen by Anderson and Sabisky.[14]

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[2]W. F. Saam, Phys. Rev. A 8, 1918 (1973).

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[7]A microscopic treatment of a similar problem for a simple-cubic crystal has been given by A. A. Maradudin and D. L. Mills [Phys. Rev. 173, 881 (1968)]. These authors calculate the damping of a Rayleigh surface mode due to interactions with bulk phonons. Another calculation for solids, more closely related to the one presented here, is given by H. Ezawa [Ann. Phys. (N.Y.) 67, 438 (1971)] who quantizes (using the continuum approximation) the elastic waves in a semi-infinite solid.

[8]See, e.g., I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity* (Benjamin, New York, 1965), Chap. 15.

[9]Oscillatory factors $e^{\pm 2ikz}$ in the phonon parts of $\epsilon(z)$ give vanishing contributions to $\epsilon(-\infty)$.

[10]Equations (43) and (44) may also be derived from the results of Ref. 4. However, the approach taken there does not afford one the possibility of actually calculating the energy density.

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[12]D. O. Edwards (private communication) has observed no measurable attentuation of 1–2-K phonons at the surface for liquid temperatures of the order of 30 mK.

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