Critical properties of random and constrained dipolar magnets*

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Renormalization-group techniques are used to study a model of n-coupled m-component spin systems, in the limit of large dipole-dipole interactions. As recently shown by Emery, this model describes the critical behavior of a constrained dipolar system in the limit $n \to \infty$ and that of a dipolar system with a quenched random perturbation in the limit $n \to 0$. In both cases, the unperturbed dipolar fixed point is *unstable*, and there is a crossover to a new behavior. For the constrained system, this leads to another dipolar fixed point (if $\alpha < 0$, α being the dipolar specific-heat exponent) with the same thermodynamic critical exponents, or to one with renormalized dipolar exponents (if $\alpha > 0$). These results are different from those of previous "spherical" dipolar models. For the random case, the crossover is either to a new fixed point, with very different exponents, e.g., $2\nu \simeq 1 + 1.183\epsilon$ for $m = d = 4 - \epsilon$, or away from all the fixed points found to order ϵ. One of the new fixed points in this case has complex eigenvalues of the linearized recursion relations. This is related to the fact that the recursion-relation flow is not of a gradient type for random systems.

I. INTRODUCTION

Systems which undergo phase transitions are never ideal. There are always some impurities or some other random perturbations. There are also external constraints (e.g., on the number of the spins, the volume, etc.). These may lead to drastic changes in the critical behavior. Such changes have recently been studied, using the renormalization group technique, for systems which are described by m-component order parameters, with short-range rotationally invariant interactions.

The problem of a quenched random system was studied by Grinstein and Luther[1] and by Harris and Lubensky.[2] Both find four fixed points. One point, which describes the critical behavior of the nonrandom system, is stable against the random perturbation for $m \gtrsim 4 + O(\epsilon)$ ($\epsilon = 4 - d$, where d is the dimensionality of the system). In fact, one can show[3–6] that this stability holds when $\alpha < 0$, where α is the specific-heat exponent of the nonrandom system. This seems to be the case for $d = 3$, $m > 1$.[7] Thus, one does not have to worry about small random perturbations for *XY* or Heisenberg short-range systems in three dimensions.

The problem of one type of constraint (constant volume) was recently studied by Sak.[8] Again, there are four fixed points, and the stability of the nonconstrained one is determined by the sign of α. If $\alpha > 0$ then the stable fixed point has critical exponents which are derived from those of the nonconstrained system by a Fisher renormalization.[9] The other, unstable, fixed points, are the Gaussian one and the one describing the spherical model.

The same four fixed points are found if one considers n coupled m-vector models, $\vec{S}_1, \ldots, \vec{S}_n$, with a coupling which effectively constrains lattice sums of local combinations like $(\sum_{\alpha=1}^{n} |\vec{S}_\alpha|^2)^2$, in the limit $n \to \infty$.[10–12] Such a constraint is closely related to the one which relates the $n \to \infty$ limit of the usual n-vector model to the spherical model.[11,13]

The Hamiltonians leading to the four fixed points in both cases were recently combined by Emery,[12] who showed that the random problem is the $n \to 0$ limit and the constrained problem is the $n \to \infty$ limit of the same nm-component spin Hamiltonian. Thus, considering the general case of n coupled m-component spin systems will lead to an understanding of both the random and the constrained systems.

Another direction in which the study of critical phenomena evolved recently involves dipole-dipole interactions.[14–17] It turns out, that whenever dipole-dipole interactions exist they cause an instability of the fixed point describing the short-range ferromagnetic critical behavior, and lead to a dipolar fixed point, with a different critical behavior. Since all magnetic systems have dipole-dipole interactions, together with all the above-mentioned perturbations, we devote this paper to the study of random and constrained dipolar magnets. In addition to the practical interest (dipolar critical behavior is dominant for ferromagnets with a low transition temperature[14]), there are a few theoretical reasons which make such a study interesting. The spherical model limit of dipolar magnets was considered long ago by Lax,[18] who concluded that it yields the same critical behavior as the usual, short-range, spherical model. A study of n coupled d-component dipolar spin systems[19] indeed yielded a "spherical" fixed point, in apparent agreement with Lax's conclusion. However, it turns out that this fixed point is unstable, and a more general treatment is necessary. Another theoretical interesting feature of the present study has to do with the eigenvalues of the linearized recursion relations. We find, for the first time, a fixed point which yields *complex* eigenvalues, which thus represent an oscillating flow of the Hamiltonian in parameter space under renormalization-group iterations. Such a behavior is forbidden for the usual, finite-n, case, where the recursion relations yield a gradient flow,[20] but this is no longer true in the limit $n\rightarrow 0$.

The outline of the paper is as follows: The model of n coupled m-component dipolar spin systems and the resulting recursion relations are described in Sec. II. Sections III and IV are devoted to the constrained ($n\rightarrow\infty$) and random ($n\rightarrow 0$) cases. In each case, the fixed points, the Hamiltonian flows and the resulting critical behavior are discussed. Section V summarizes the results.

II. THE MODEL AND THE RECURSION RELATIONS

The partition function for a nonrandom (nonconstrained) isotropic m-component spin dipolar system may be written[14–17]

$$Z=\int\cdots\int\prod_{\vec{R}}[d^m S(\vec{R})]\,e^{\mathcal{H}_1\{\vec{S}(\vec{R})\}},\tag{1}$$

with

$$\mathcal{H}_1\{\vec{S}(\vec{R})\}=-\int d^dR\left(\frac{1}{2}\sum_{i,j=1}^{m}V^{ij}(\vec{R})S_i(\vec{R})S_j(\vec{R})+v\,|\vec{S}(\vec{R})|^4+\cdots\right),\tag{2}$$

where the dots indicate higher-order (irrelevant) terms, and where $V^{ij}(\vec{R})$ is the Fourier transform of

$$\hat{V}^{ij}(\vec{q})=r_0+q^2+g(q_iq_j/q^2)-hq_iq_j+O(q^4).\tag{3}$$

In writing Eqs. (1)–(3) we used a continuous spin model, and we replaced the lattice sum by a continuous d-dimensional integral. For an isotropic dipolar interaction, we must take $m=d=4-\epsilon$.[14,15,17] However, if the (short-range) exchange interaction between m spin components ($m\le d$) is much stronger than that between the other components, then we can use the same expressions, (2) and (3), but include only the terms with $i,\,j=1,\ldots,\,m$.[16] In the dipolar limit $g\rightarrow\infty$, this means that the spatial radial integrals are d dimensional, but the angular integrals reduce to become m dimensional. We shall consider here this more general case.

To obtain the random case, we can now replace each $V^{ii}(\vec{R})$ by $V^{ii}(\vec{R})+\psi(\vec{R})$, where $\psi(\vec{R})$ is a random variable, representing, e.g., a random short-range exchange. Averaging over the random distribution of this variable leads to a new expression for the free energy per spin,[5,12]

$$-\beta F=\lim_{n\rightarrow 0}\,[(1/n)\ln Z_{\text{eff}}],\tag{4}$$

where $\beta=1/k_BT$, and

$$Z_{\text{eff}}=\int\cdots\int\prod_{\vec{R}}[d^{nm}\sigma(\vec{R})]\,e^{\mathcal{H}_{\text{eff}}\{\vec{\sigma}(\vec{R})\}},\tag{5}$$

with the new nm component spin variable

$$\vec{\sigma}\equiv(\vec{S}_1,\ldots,\,\vec{S}_n)\equiv(S_{11},\ldots,\,S_{1m},\,S_{21},\ldots,\,S_{nm})\tag{6}$$

and

$$\mathcal{H}_{\text{eff}}=\sum_{\alpha=1}^{n}\mathcal{H}_1\{\vec{S}_\alpha(\vec{R})\}+G\{\vec{\sigma}(\vec{R})\}.\tag{7}$$

Assuming that the random distribution of each $\psi(\vec{R})$ is independent of that of all the others, one has[5,12]

$$G\{\vec{\sigma}(\vec{R})\}=\int d^dR\,\ln\langle e^{-\psi|\vec{\sigma}(\vec{R})|^2/2}\rangle\quad,\tag{8}$$

where $\langle\cdots\rangle$ denotes the average over the random distribution of the variable ψ and

$$|\vec{\sigma}|^2=\sum_{\alpha=1}^{n}|\vec{S}_\alpha|^2=\sum_{\alpha=1}^{n}\sum_{i=1}^{m}S_{\alpha i}^2.$$

Expanding (8) in powers of $|\vec{\sigma}|^2$ we find

$$G\{\vec{\sigma}(\vec{R})\}=-\int d^dR\,[\tfrac{1}{2}\langle\psi\rangle|\vec{\sigma}|^2+u|\vec{\sigma}|^4+O(|\vec{\sigma}|^6)],\tag{9}$$

where

$$u=-\tfrac{1}{8}\langle\psi^2\rangle_c=-\tfrac{1}{8}(\langle\psi^2\rangle-\langle\psi\rangle^2).\tag{10}$$

The symbol $\langle\cdots\rangle_c$ denotes the cumulant of the random distribution.[2] Finally we thus have

$$\mathcal{H}_{\text{eff}}=-\int d^dR\left(\frac{1}{2}\sum_{\alpha=1}^{n}\sum_{i,j=1}^{m}V^{ij}(\vec{R})S_{\alpha i}(\vec{R})S_{\alpha j}(\vec{R})+v\sum_{\alpha=1}^{n}\left|\vec{S}_\alpha(\vec{R})\right|^4+u|\vec{\sigma}|^4+\cdots\right),\tag{11}$$

where now r_0 in (3) is shifted to

$$r=r_0+\langle\psi\rangle.\tag{12}$$

Taking the large n limit in (11) one can show[11,12] that it becomes equivalent to the Hamiltonian of a constrained system. We shall thus start by applying the renormalization-group transformation to the case with general n.

For $g=h=0$, Eq. (3) reduces to that appropriate for the nondipolar case. This was treated previously.[1,2,4] For any finite g, the recursion relation for g is[15]

$$g'=b^{2-\eta}g,\tag{13}$$

so that under iteration $g\rightarrow\infty$ ($b>1$ is the spatial rescaling factor). This limit yields the dipolar fixed point in the nonrandom (nonconstrained) case. From now on we shall assume that we are already close to this ($g\rightarrow\infty$) limit. Ignoring terms of order $1/g$, the propagator for the diagrammatic expansion of the partition function (5) becomes[16]

$$G^{\alpha i,\beta j}(\vec{q}) = \delta_{\alpha\beta}[\hat{V}^{-1}(\vec{q})]^{ij} = \frac{\delta_{\alpha\beta}}{r+q^2}\left(\delta_{ij} - \frac{q_i q_j}{Q^2}\right), \quad (14)$$

where

$$Q^2 = \sum_{i=1}^{m} q_i^2 \; . \quad (15)$$

Clearly, this propagator is meaningful only if $d \geq m > 1$. The case $m = 1$ deserves special attention.[21]

It is very important to note that in the dipolar case there no longer exists a separate rotational invariance in the m-dimensional spin space. This is due to the coupling between spin and space vectors, arising from the dipolar interaction. Thus, one should not be surprised that new terms may be generated in the Hamiltonian after a renormalization group iteration, which reflect this lower symmetry. Indeed, a direct iteration with (11) immediately generates an additional term, i.e.,

$$\mathcal{H}_{\text{eff}}^{(1)} = -2w\int d^dR \sum_{\alpha,\beta=1}^{n}{}' (\vec{S}_\alpha \cdot \vec{S}_\beta)^2, \quad (16)$$

where

$$(\vec{S}_\alpha \cdot \vec{S}_\beta) = \sum_{i=1}^{m} S_{\alpha i} S_{\beta i}.$$

This is a very important term, as all the previously studied fixed points are unstable with respect to it. A similar term arises in the problem of amorphous magnets, when one introduces a uniaxial anisotropy with a random direction,[5] and in the problem of competing ferromagnetic and antiferromagnetic interactions.[4]

We are now ready to study the recursion relations for the sum of (11) and (16). Using the standard integral[15,16]

$$\int \frac{d^dq}{(2\pi)^d} G^{\alpha i,\beta j}(\vec{q}) G^{\gamma k,\delta l}(\vec{q}) = \delta_{\alpha\beta}\delta_{\gamma\delta} K_d \int q^{d-1} dq \times [A\delta_{ij}\delta_{kl} + B(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})], \quad (17)$$

where $K_d^{-1} = 2^{d-1}\pi^{d/2}\Gamma(d/2)(= 8\pi^2$ for $d = 4)$ and

$$A = \frac{m^2 - 3}{m(m+2)}, \quad B = \frac{1}{m(m+2)}, \quad (18)$$

the recursion relations for r, u, v, and w become[4,22] (to order ϵ)

$$r' = b^2\{r + 4K_4\, I(r)[(mn+2)u + (m+2)v + 2(m+n+1)w][(m-1)/m]\}, \quad (19)$$

$$u' = b^\epsilon\{u - 4K_4 \ln b[A_1u^2 + A_2uv + A_3uw + A_4vw + A_5w^2]\}, \quad (20)$$

$$v' = b^\epsilon\{v - 4K_4 \ln b[B_1v^2 + B_2uv + B_3vw]\}, \quad (21)$$

$$w' = b^\epsilon\{w - 4K_4 \ln b[C_1w^2 + C_2uw + C_3vw + C_4u^2]\}, \quad (22)$$

where b is the space rescaling factor, and

$$I(r) = \int_{\Lambda b^{-1}}^{\Lambda} \frac{q^3 dq}{r + q^2} = \frac{\Lambda^2}{2}(1 - b^{-2}) - r\ln b + O(r^2) \quad (23)$$

(Λ is the momentum cutoff). The coefficients A_i, B_i, and C_i are given in Table I. Note that the coefficient C_4 is absent in the isotropic short range case, and therefore w is *not* generated by u in that case. (An easy way to rederive the isotropic short-range case is to put $A = 1$, $B = 0$.[5])

For $u = w = 0$, the Hamiltonian reduces to a sum of n *decoupled* m-component dipolar Hamiltonians. Each of these yields the Gaussian dipolar fixed point, which is highly unstable, and the original *dipolar fixed point* with $u^* = w^* = 0$ and $4K_4v^* = \tilde{v}\epsilon + O(\epsilon^2)$, where[16]

$$\tilde{v} = \frac{m(m+2)}{(m-1)(m^2 + 10m + 12)} = \frac{m}{(m-1)\{m + 8 - [4/(m+2)]\}} \; . \quad (24)$$

One can now linearize Eqs. (20)–(22) about this fixed point, and find the eigenvalues b^{λ_u} and b^{λ_w}. These are found to be

$$\lambda_u^D = -\frac{m^2 - 2m - 4}{m^2 + 10m + 12}\epsilon + O(\epsilon^2), \quad (25)$$

$$\lambda_w^D = \frac{m^3 + 5m^2 + 2m - 4}{(m-1)(m^2 + 10m + 12)}\epsilon + O(\epsilon^2). \quad (26)$$

In a previous paper,[5] we studied the scaling dimensions of the operators which multiply u and w, in the vicinity of the decoupled fixed point. We found that quite generally

$$\lambda_u^D = \alpha/\nu \quad (27)$$

and

$$\lambda_w^D = 2\phi/\nu - d, \quad (28)$$

where α, ν, and ϕ are the specific heat, correlation length, and quadratic spin anisotropy crossover[23] exponents, respectively, for the decoupled m-component system. The proof is general, and does not depend on the details of the decoupled behavior. Indeed, using the dipolar values[16,17,24]

$$1/2\nu = 1 - \frac{(m+2)^2}{2(m^2 + 10m + 12)}\epsilon + O(\epsilon^2) \quad (29)$$

and

$$\phi = 1 + \frac{m^2(m+1)}{2(m-1)(m^2 + 10m + 12)}\epsilon + O(\epsilon^2), \quad (30)$$

we find that (25) and (26) are consistent, to order ϵ, with (27) and (28).

The sign of λ_u^D may be sensitive to the order in ϵ. In any case, λ_u^D is quite small. For real Heisenberg dipolar systems, α is very probably

TABLE I. Coefficients for the u, v, w recursion relations [Eqs. (20)–(22)]; $A=(m^2-3)B$, $B=1/m(m+2)$.

General expression	$n=0$	$n\to\infty$
$A_1=(mn+8)A+(mn+4)(m+1)B$	$4(2m^2+m-5)B$	$(m-1)n+O(1)$
$A_2=2(m+2)[A+(m+1)B]=2(m+2)(m-1)/m$		
$A_3=4\{(m+n+1)A+[(m+n+1)(m+1)+4]B\}$	$4(m^3+2m^2-m+2)B$	$4n(m-1)/m+O(1)$
$A_4=4[A+(m+3)B]=4(m+1)/(m+2)$		
$A_5=4[3A+(2m+n+4)B]$	$4(m-1)(3m+5)B$	$4nB+O(1)$
$B_1=(m+8)A+(m^2+5m+12)B=(m-1)(m^2+10m+12)B$		
$B_2=4[3A+(m+5)B]=4(m-1)(3m+4)B$		
$B_3=4[(m+5)A+(m^2+4m+7)B]=4(m-1)(m^2+7m+8)B$		
$C_1=2[(n+m+4)A+(m^2+3m+n+6)B]$	$2(m-1)(m^2+6m+6)B$	$2n(m^2-2)B+O(1)$
$C_2=4[3A+(m+3)B]=4(3m^2+m-6)B$		
$C_3=4(A+B)=4(m^2-2)B$		
$C_4=4B$		

negative,[17,25] so that $\lambda_u^D<0$. However, λ_w^D is definitely positive, and is of the order of $\frac{1}{2}$ at $\epsilon=1$. Thus, the decoupled dipolar fixed point is unstable, independent of the value of n. We therefore must study the other possible fixed points, and determine the ultimate asymptotic critical behavior. At this stage we divide the discussion into two parts, and discuss separately the limits $n\to\infty$ (constrained) and $n\to 0$ (random). In both cases we find six additional nontrivial fixed points. Three of these have $v^*=0$, while the other three have $v^*\neq 0$.

III. CONSTRAINED DIPOLAR SYSTEM

We now consider Eqs. (20–(22) in the limit of large n. The last column of Table I exhibits the large-n limit of the n-dependent coefficients A_1, A_3, A_5, and C_1. All other coefficients are n-independent and of order unity. Since A_1 and C_1 are of order n, it is clear that the fixed point values of u and of w are at most of order $1/n$. Assuming this, we can, in the limit $n\to\infty$, ignore the uv and vw terms in Eq. (21), which then reduces to its form in the absence of u and w. We thus find two possible fixed point values for v, namely, $v^*=0$ or $4K_4v^*=\tilde{v}\epsilon+O(\epsilon^2)$, with $\tilde{v}$ given by Eq. (24). Linearizing Eq. (21) we find that $\lambda_v=\epsilon$ in the former case and $\lambda_v=-\epsilon+O(\epsilon^2)$ in the latter case.[17]

Turning now to Eq. (22), we can ignore, to leading order in $1/n$, the terms involving uw and w^2. The remaining terms yield $w^*=O(1/n^2)$ or

$$4K_4w^*=\frac{m(m+2)}{2(m^2-2)n}\epsilon-\frac{8K_4v^*}{n}+O\left(\frac{\epsilon^2}{n},\frac{\epsilon}{n^2}\right). \quad (31)$$

In the limit $n\to\infty$, it seems as if the first fixed point value is equivalent to setting $w^*=0$. However, this is very dangerous: the term involving u^2 *does* appear in Eq. (22) for any *finite* n, and will thus generate a *nonzero* value of w, which will then evolve to the fixed point value given by Eq. (31). It must be emphasized here, that the proper procedure is to carry the whole calculation out at a finite (although large) value of n, and let n go to infinity only at the *final* stage. Indeed, linearizing Eq. (22) about $w^*=O(1/n^2)$ yields $\lambda_w=\epsilon$ if $v^*=0$ and $\lambda_w=\lambda_w^D$ [see Eqs. (26), (28)] if $4K_4v^*\simeq\tilde{v}\epsilon$. In both cases, this fixed point will *not* be stable.

Table II summarizes, to orders ϵ and $1/n$, the eight fixed points that we find for large n. The first four have $w^*=O(1/n^2)$ (represented by zero in this table), and the last four correspond to ω^* given by (31). The values of v^* are equal to zero or to the original dipolar fixed point value, and the values of u^* then follow readily.

The first four fixed points in Table II are very similar to the ones found in the case of the isotropic short-range case.[8,10] In addition to the Gaussian (I) and the original dipolar (II) fixed points, we find a "spherical" fixed point (III), which has the exponents of the spherical model, and a fixed point which has Fisher renormalized[9] dipolar exponents, e.g.,

$$\nu^{\mathrm{IV}}=\frac{\nu}{1-\alpha},\quad \alpha^{\mathrm{IV}}=-\frac{\alpha}{1-\alpha} \quad (32)$$

(ν and α are the original dipolar exponents of the fixed point II). It is thus probably not surprising that Eq. (27) becomes renormalized to yield

$$\lambda_u^{\mathrm{IV}}=-\alpha/\nu=-\lambda_u^D, \quad (33)$$

and similarly λ_w is unchanged [both ϕ and ν are renormalized by factors of $1/(1-\alpha)$].[26]

The "spherical" fixed point III is the one found in Ref. 19. Note that the recursion relations in this reference are correct only in the limit of large n; the terms of order unity in the coefficients have to be modified when the distinction between the parameters u and w is introduced.

As was already noted, none of these four fixed points is stable with respect to w. We thus must

TABLE II. Fixed points and eigenvalues for $n \to \infty \{O[\epsilon, (1/n)]\}$.

Fixed Point	$4K_4u^*$	$4K_4v^*$	$4K_4w^*$	Eigenvalues
I. Gaussian A	0	0	0	$\lambda_u = \lambda_v = \lambda_w = \epsilon$
II. Dipolar A	0	$\tilde{v}\epsilon$ [a]	0	$\lambda_u = \lambda_u^D,\ \lambda_v = -\epsilon,\ \lambda_w = \lambda_w^D$ [b]
III. Spherical A	$\frac{\epsilon}{n(m-1)}$	0	0	$\lambda_u = -\epsilon,\ \lambda_v = \lambda_w = \epsilon$
IV. Renormalized Dipolar A	$\tilde{u}\epsilon$ [c]	$\tilde{v}\epsilon$ [a]	0	$\lambda_u = -\lambda_u^D,\ \lambda_v = -\epsilon,\ \lambda_w = \lambda_w^D$
V. Spherical B	$x_+\tilde{w}_1\epsilon$ [d]	0	$\tilde{w}_1\epsilon$ [e]	$\lambda_u = \lambda_w = -\epsilon,\ \lambda_v = \epsilon$
VI. Gaussian B	$x_-\tilde{w}_1\epsilon$ [d]	0	$\tilde{w}_1\epsilon$ [e]	$\lambda_u = \lambda_v = \epsilon,\ \lambda_w = -\epsilon$
VII. Dipolar B	$y_1\tilde{w}_2\epsilon$ [f]	$\tilde{v}\epsilon$ [a]	$\tilde{w}_2\epsilon$ [g]	$\lambda_u = \lambda_u^D,\ \lambda_v = -\epsilon,\ \lambda_w = -\lambda_w^D$
VIII. Renormalized Dipolar B	$y_2\tilde{w}_2\epsilon$ [f]	$\tilde{v}\epsilon$ [a]	$\tilde{w}_2\epsilon$ [g]	$\lambda_u = -\lambda_u^D,\ \lambda_v = -\epsilon,\ \lambda_w = -\lambda_w^D$

[a]Equation (24).
[b]Equations (25)–(28)
[c]$\tilde{u}\epsilon = \alpha/[\nu n(m-1)]$
[d]$x_\pm = [-m^2 - 2m + 2 \pm (m^2-2)]/(m-1)m(m+2)$
[e]$\tilde{w}_1 = m(m+2)/2n(m^2-2)$
[f]$y_1 = -2/m,\ y_2 = -[2/(m+2)][(2m^3+5m^2+6m+4)/(m^3+5m^2+2m-4)]$
[g]$\tilde{w}_2\epsilon = \tilde{w}_1\lambda_w^D$

consider the remaining four fixed points, obtained using Eq. (31) with the two possible values of v^*. It turns out, that the thermodynamic exponents [e.g., ν, following Eq. (19)] of each of these fixed points are, in the limit $n \to \infty$, the same as those of one of the first four fixed points (V has spherical model exponents, VI has Gaussian model exponents; VII has dipolar exponents, and VIII has renormalized dipolar exponents). The only difference has to do with the sign of the exponent λ_w, which (to orders ϵ and $1/n$) is opposite to its counterpart at each of these fixed points. We are thus finding a new type of "exponent renormalization," similar to the change of sign of λ_u from II to IV or from I to III.

With all this information, we can now conclude that the *new dipolar fixed point* VII is stable if $\alpha < 0$ and that the *new renormalized dipolar fixed point* VIII is stable if $\alpha > 0$. For the usual Heisenberg dipolar system we thus conclude that *the constrained system has the same critical behavior as the unconstrained one.*

The nm-component model thus never yields a *stable* fixed point with *spherical* model exponents. This raises some questions with regards to its relation to Lax's spherical dipolar model.[18] Indeed, even in the nondipolar short-range case one finds that the "spherical" fixed point of our model is unstable. In that case, one recovers the spherical model critical behavior only in the limit $m \to \infty$. Indeed, it was *this* limit that Stanley[13] used in his discussion of the relation between the spherical model and the m-component spin model. Similarly, it is interesting to note that Eq. (29) and similar equations *do* yield spherical model exponents in the limit $m \to \infty$. Of course, this limit is quite artificial here, since m is meaningful only for $m \le d$. An alternative case in which the spherical model behavior (fixed point V) is recovered is when $v \equiv 0$, i.e., when the original dipolar case is described by a Gaussian model. Again, this is not realistic for real physical dipolar systems.

IV. RANDOM DIPOLAR SYSTEM

The $n \to 0$ limit of the n-dependent coefficients in the recursion relations (20)–(22) are shown in the second column of Table I. Again, the Gaussian and the nonrandom (decoupled) fixed points are found for $u = w = 0$. Both are unstable with respect to w, as discussed in Sec. II. In addition, we again find six fixed points, which may be divided into two groups of three. The first three all have $v^* = 0$. The fixed point values of u and w are then given (to order ϵ) by

$$4K_4w^* = \epsilon/(C_4x^2 + C_2x + C_1), \quad u^* = xw^*, \tag{34}$$

where x is one of the three real roots of

$$C_4x^3 + (C_2 - A_1)x^2 + (C_1 - A_3)x - A_5 = 0. \tag{35}$$

The other three fixed points have

$$4K_4w^* = (B_1 - C_3)\epsilon/[(B_1C_4x^2 + (B_1C_2 - B_2C_3)x + B_1C_1 - B_3C_3], \tag{36}$$

$$4K_4v^* = w[C_4x^2 + (C_2 - B_2)x + C_1 - B_3]/(B_1 - C_3),$$

$$u^* = xw^*,$$

where x is one of the roots of

$$(A_2-B_1)C_3x^3+[(A_1-B_2)(B_1-C_3)+(A_2-B_1)(C_2-B_2)+A_4C_4]x^2+[(A_3-B_3)(B_1-C_3)+(A_2-B_1)(C_1-B_3)$$
$$+A_4(C_2-B_2)]x+A_5(B_1-C_3)+A_4(C_1-B_3)=0. \tag{37}$$

The eigenvalues of the linearized Eqs. (20)–(22), λ_1, λ_2, and λ_3, are then found by diagonalizing the matrix

$$\begin{pmatrix} \epsilon-4K_4(2A_1u^*+A_2v^*+A_3w^*) & -4K_4(A_2u^*+A_4w^*) & -4K_4(A_3u^*+A_4v^*+2A_5w^*) \\ -4K_4B_2v^* & \epsilon-4K_4(2B_1v^*+B_2u^*+B_3w^*) & -4K_4B_3v^* \\ -4K_4(C_2w^*+2C_4u^*) & -4K_4C_3w^* & \epsilon-4K_4(2C_1w^*+C_2u^*+C_3v^*) \end{pmatrix}. \tag{38}$$

The results, for $m=2$, 3, 4, are given numerically in Table III. The eigenvalue λ_1 is always $-\epsilon+O(\epsilon^2)$, and was omitted from the table.

The most striking result in Table III is that two of the eigenvalues of the fixed point E are *complex*. This is the first physical case in which complex eigenvalues have been found. It must be noted, that the matrix (38) is not symmetric, and as such does not *have* to have real eigenvalues. It has been shown by Wallace and Zia,[20] that real eigenvalues will result if the recursion relations may be presented as gradient flows, i.e.,

$$\sum_B \eta^{AB}\frac{du_B}{dl}=\frac{\partial V\{u_A\}}{\partial u_A}, \tag{39}$$

where we replaced b by e^l and transformed the equations to differential form, the symbol u_A denotes u, v, or w, the function V is a scalar potential, and η^{AB} is a Reimannian metric. Indeed, for general n and m Eqs. (20)–(22) can be written in the form (39), V being a linear combination of ϵu_A^2, $\epsilon u_A u_B$, $u_A^2 u_B$, u_A^3 and uvw, and η^{AB} being given by

$$\{\eta^{AB}\}=\frac{nm}{3}\begin{pmatrix} nm+2 & m+2 & 2(n+m+1) \\ m+2 & m+2 & 2(m+2) \\ 2(n+m+1) & 2(m+2) & 2n(m+1)+2(m+3) \end{pmatrix} \tag{40}$$

[the factors of 2 in the last row and column result from the definition of w, Eq. (16)]. When $n\rightarrow 0$, $\eta_{AB}\rightarrow 0$. Thus, η^{AB} cannot be a Reimannian metric in this limit. Note that this will be the case even in the nondipolar random case, when $w=0$ and (40) reduces to a 2×2 matrix [for $m=1$, this matrix is given in Eq. (20) of Ref. 20, of which our Eq. (40) is a generalization]. We thus conclude that *the recursion relations describing random systems cannot be described as gradient flows*. This opens up a few interesting possibilities, as discussed in Ref. 20. One of these is the possibility of finding complex eigenvalues.

Of the eight fixed points, two are found to be stable: A and C in Table III. The transition will be sharp, with well-defined critical exponents, if the starting Hamiltonian is in the region of attraction of one of these fixed points. Otherwise, we shall find a "runaway" which probably represents a smeared transition (as found by Lubensky[2] for the random nondipolar Ising case).

For the nonrandom phase transition to be of second order, v in Eq. (2) must be positive. From Eq. (10) it follows that the initial value of u is negative.[2] The parameter w does not appear in the original Hamiltonian. However, the first iteration will lead, through the last term in Eq. (22), to a small negative value of w. If no higher-order terms are included, then Eq. (22) does not allow a change in the sign of w: as w approaches zero, the last term in (22) will always push it back to negative values. However, it is not at all clear that higher-order terms (e.g., the coefficient of $|\vec{\sigma}|^6$, which may be positive) will not allow w to change sign. In the nondipolar case, when w is absent, it has been recently shown[27] that even if all the irrelevant variables are included, then u is still forced to remain negative after iterations, maintaining its cumulantlike interpretation as given by Eq. (10). The variable w has no such interpretation here (although it may possibly be related to cumulants of a random distribution of anisotropic single-ion interactions with random directions, see Ref. 5). Thus, it is *a priori* possible that higher-order irrelevant variables will push w to become positive, while u will remain negative. If this happens, then the stable fixed point A will probably be reached, and the transition will be sharp. The

TABLE III. New fixed points and eigenvalues for $n=0$.

m		Fixed points $4K_4u^*$	$4K_4v^*$	$4K_4w^*$	Exponents λ_2, $\lambda_3(=\lambda_v$ if $v^*=0)$
2	A	-2.608ϵ	0	1.719ϵ	-6.184ϵ, -8.303ϵ
	B	0.1298ϵ	0	0.06325ϵ	0.6304ϵ, -0.4711ϵ
	C	1.478ϵ	0	-0.4182ϵ	-2.355ϵ, -0.9529ϵ
	D	-0.08158ϵ	-0.7988ϵ	0.3848ϵ	$(2.006\pm 0.5065i)\epsilon$
	E	1.3137ϵ	0.2640ϵ	-0.5197ϵ	0.7740ϵ, -2.231ϵ
	F	-0.08319ϵ	0.3048ϵ	0.003429ϵ	0.9345ϵ, -0.1461ϵ
3	A	-3.214ϵ	0	2.316ϵ	-16.93ϵ, -23.65ϵ
	B	0.09191ϵ	0	0.04047ϵ	0.4404ϵ, -0.4574ϵ
	C	0.3289ϵ	0	-0.03702ϵ	-0.8255ϵ, -0.5300ϵ
	D	0.00075ϵ	-0.5187ϵ	0.2231ϵ	$(2.190\pm 0.2608i)\epsilon$
	E	0.2749ϵ	0.1012ϵ	-0.07866ϵ	0.4287ϵ, -0.7783ϵ
	F	-0.008405ϵ	0.1556ϵ	$2.47\times 10^{-5}\epsilon$	0.7602ϵ, -0.02061ϵ
4	A	-0.5091ϵ	0	0.4174ϵ	-3.974ϵ, -5.778ϵ
	B	0.07944ϵ	0	0.03105ϵ	0.3695ϵ, -0.4428ϵ
	C	0.2359ϵ	0	-0.01444ϵ	-0.6874ϵ, -0.5116ϵ
	D	0.008686ϵ	-0.3658ϵ	0.1554ϵ	$(2.045\pm 0.1484i)\epsilon$
	E	0.1944ϵ	0.07656ϵ	-0.04640ϵ	0.4177ϵ, -0.6421ϵ
	F	0.01798ϵ	0.1005ϵ	$8.6\times 10^{-5}\epsilon$	0.6296ϵ, 0.05100ϵ

exponent ν, derived from linearizing Eq. (19), is then found to be

$$2\nu_A \simeq 1+1.275\epsilon+O(\epsilon^2),\quad m=2\,,$$
$$2\nu_A \simeq 1+4.034\epsilon+O(\epsilon^2),\quad m=3\,, \qquad (41)$$
$$2\nu_A \simeq 1+1.183\epsilon+O(\epsilon^2),\quad m=4.$$

From these numbers, and from the eigenvalues given in Table III, it seems that some divergence probably occurs in the vicinity of $m=3$.[28] Therefore, it is very difficult to deduce a reliable number from the order-ϵ result (41). It is, however, clear that the fixed point A has quite a large value of the exponent ν, and it would be very interesting to try to verify this.

The second stable fixed point C has a positive value of u and a relatively small negative value of w. By previous experience,[27] it seems unlikely that this fixed point will ever be reached.

Similarly, the fixed point D, which has complex eigenvalues, has a negative value of v and therefore will probably never be reached. It is also quite unstable, since the real part of these eigenvalues is positive and of the order 2ϵ. Similar arguments exclude a flow to the vicinity of B and E. The last fixed point F turns out to be very close to the dipolar nonrandom fixed point.

If our conjecture as regards the Hamiltonian flow towards the fixed point A is wrong, then the flow will have to be away from all the fixed points. For a small concentration of impurities, this flow will be described by the original nonrandom exponents, e.g., Eq. (29). A crossover to a new, random, behavior will be felt when the temperature will be close enough to T_c so that

$$w\xi^{\lambda_w^D}\sim 1\,, \qquad (42)$$

$\xi\sim(T-T_c)^{-\nu}$ being the correlation length. Since w starts as $u^2\sim p^2$, where p is the concentration of impurities,[1] one may have to go quite close to T_c to see these effects. The same criterion (42) applies to the crossover to the behavior described by the fixed point A.

It should be emphasized, that our discussion is based on results to lowest order in ϵ. It is quite possible that higher-order terms will change the signs of some of the exponents, and change the conclusions. In the nondipolar random case one also finds two stable fixed points, and one of them seems to be reachable by iterations if $1<m<4+O(\epsilon)$.[1,2] However, higher-order terms and general arguments show that at $d=3$ this happens only for $1<m<2$, namely, for no real physical case.[4] In contrast to that situation, our order-ϵ calculations show that the fixed point A remains stable and reachable for a large range of values of m (in practice, we calculated up to $m=11$). It thus seems more safe to hope that this may remain true at higher orders in ϵ. This remains to be checked.

V. SUMMARY

Contrary to the case of short-range isotropic interactions, we find that the dipolar fixed point for XY and Heisenberg systems is *unstable* with respect to either an external constraint or a random perturbation. These perturbations generate a new term in the Hamiltonian, and this term causes the Hamiltonian to flow away from the unperturbed dipolar fixed point, with a crossover exponent

$$\phi_w = \nu\lambda_w^D = 2(\phi - 1) + \alpha. \tag{43}$$

In the constrained case, the flow is towards a new nonzero value of w. However, the asymptotic critical behavior still has the original unconstrained dipolar exponents if $\alpha < 0$ and Fisher renormalized dipolar exponents if $\alpha > 0$, just as one might expect on general grounds.[9] This model does *not* yield a *stable* fixed point with spherical model exponents, as was suggested by Lax.[18]

In the random case, we find (again in contrast to the nondipolar case) that the nonrandom fixed point is *never* stable. There is a flow away from this fixed point, with the crossover exponent (43). This flow may lead to a new, stable, fixed point, with probably a very large value of the exponent ν, or away from all fixed points, probably yielding a smeared transition.[2,27]

Dipolar critical behavior is observable in ferromagnets with a low transition temperature, e.g., EuO.[14,25] It would be very interesting to perform measurements on EuO with many nonmagnetic impurities and to determine which of the above possibilities actually occurs. It is interesting to note, in this context, the different values of the exponents measured on single crystals of EuO, e.g., $\gamma \simeq 1.29$,[29] and on powder slab samples, e.g., $\gamma \simeq 1.40$.[30] Could the difference be accounted for by the random nature of the powder?

As a theoretical byproduct, we demonstrated that random systems cannot be described by a gradient flow in Hamiltonian space. It would be very interesting to exploit the results of this statement. For example, can there be limit cycles?

Our analysis does not apply to the dipolar Ising system. To study the behavior of this system at $d=3$ it is sufficient to study the behavior of the short-range Ising system at $d=4$.[21] This will be done separately.

Note added in proof. It has recently been shown by Khmelnitzkii (report of work prior to publication) that the "runaway" in the random short-range Ising case actually flows to a fixed point of order $\sqrt{\epsilon}$, yielding a *sharp* transition. There does not seem to be a similar fixed point in our case, and thus our conclusions are probably unchanged.

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