PHYSICAL REVIEW B VOLUME 12, NUMBER 3 1 AUGUST 1975

Critical behavior of amorphous magnets*

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(Received 3 March 1975)

Renormalization-group techniques are applied to a model Hamiltonian recently proposed by Harris, Plischke, and Zuckermann for the description of amorphous magnets. In this model, a single-ion uniaxial anisotropy, with a random direction of the axis of anisotropy, is introduced. Averaging over this random variable yields an (translationally invariant) effective Hamiltonian, in which the m-component spin variable is replaced by an nm-component vector, and n is finally set equal to zero. Contrary to the result for an isotropic random perturbation, the fixed-point describing the nonrandom m-component critical behavior is *unstable* with respect to terms in the Hamiltonian generated by the randomness, the appropriate crossover exponent being given exactly by $(2\phi_m - 2 + \alpha_m)$, where ϕ_m is the Fisher-Pfeuty anisotropic spin crossover exponent and α_m is the specific-heat exponent. Depending on the distribution function for the random anisotropy directions, there are seven or thirteen other fixed points. Most of these are unstable, and the recursion relations probably yield a "runaway" which is interpreted as a smeared transition. Experiments on amorphous $TbFe_2$ are discussed.

I. INTRODUCTION

The magnetic properties of quenched amorphous metals have recently drawn more and more attention.[1] Recent experiments by Rhyne *et al.*[2] on the magnetic critical behavior of amorphous $TbFe_2$ and YFe_2 showed some interesting phenomena; in comparison with the corresponding crystalline alloy, the critical temperature T_c and the spontaneous magnetization are lowered, the neutron scattering cross section shows a weak "rounded" anomaly at T_c and a further rise below T_c, and the correlation length has a maximum just below T_c.

To explain these phenomena, Harris, Plischke, and Zuckermann[3,4] recently proposed a model, based on random packing of atomic spheres. In this model, *each magnetic ion is subjected to a local anisotropy field of random orientation*. Indeed, Mössbauer absorption measurements on Fe^{57} in $HoFe_2$, $DyFe_2$, and $ErFe_2$ later turned out to agree with the proposed model.[5]

Harris *et al.*[3] used mean field theory to derive the magnetization curve below T_c, and their results were qualitatively in agreement with the measurements.[2] However, such a theory is not expected to give a correct description of the *asymptotic* critical behavior, in the region in which spin fluctuations become important. The recent development of the theory of *renormalization group*[6,7] gave some insight into the effects of these fluctuations, and yielded a useful practical tool for their study, i.e., the ϵ expansion, in which critical exponents, scaling functions, etc. are expanded in powers of $\epsilon = 4 - d$ (d is the spatial dimensionality of the system). In its standard form,[8] the ϵ expansion assumes *translational invariance* of the Hamiltonian of the system; this is not obeyed in *quenched* amorphous or disordered alloys. To overcome this difficulty in the problem of the magnet with random nonmagnetic impurities, Grinstein and Luther[9] recently derived an effective Hamiltonian, which is translationally invariant and which leads to the same free energy as their original random Hamiltonian. A similar effective Hamiltonian was also obtained by Emery,[10] using a different technique. An alternative approach, in which the renormalization group was used to derive recursion relations directly for the probability distribution for the random potentials, was proposed by Harris and Lubensky[11] and led to similar results.

In the present paper we use similar ideas to study the critical behavior of systems described by the model suggested by Harris, Plischke, and Zuckermann.[3] It turns out that the translationally invariant effective Hamiltonian derived from this model is a generalization of the ones considered previously.[9–11] It involves more parameters, yields more fixed points, and may lead to a more complicated critical behavior.

The general model and the generalization of Emery's method for obtaining an effective translationally invariant Hamiltonian are described in Sec. II. The case in which the random distribution of the directions of uniaxial anisotropy is completely isotropic in space is studied in detail in Sec. III, where the appropriate effective Hamiltonian is derived, Sec. IV, where the renormalization group recursion relations and the resulting fixed points are studied, and in Sec. V, where the possible Hamiltonian flows and the resulting types of critical behavior are discussed. Section VI is devoted to a similar analysis of a distribution which allows only anisotropies along cubic axes (appropriate for the case of nonmagnetic impurities randomly occupying sites on these axes), and

Sec. VII summarizes the conclusions.

II. MODEL

The Hamiltonian suggested by Harris *et al.*[3] has the form

$$\mathcal{H} = -\sum_{\langle ij\rangle} J_{ij}\vec{S}_i\cdot\vec{S}_j - D_0\sum_i(\hat{x}_i\cdot\vec{S}_i)^2 \quad , \tag{1}$$

where $\vec{S}_i$ is an m-component spin vector, located at the (d-dimensional) lattice site i, $\langle ij\rangle$ denotes a pair of spin sites, J_{ij} is the exchange interaction, and $\hat{x}_i$ is a unit vector which points in the local (random) direction of the uniaxial anisotropy at the site i.

The last term in (1) represents a single-ion uniaxial anisotropy in the random direction $\hat{x}_i$. For the ordered crystalline material, this anisotropy has a well defined direction along one of the coordinate axes. The model assumes that the ions in the amorphous material still feel this anisotropic single-ion field, and that the randomness of the anisotropy field is the most important characteristic of the amorphous state.[3,5]

As usual in renormalization-group calculations,[6-8] we consider a classical continuous spin model, with a weight function $\exp(-\frac{1}{2}|\vec{S}_i|^2 - v_0|\vec{S}_i|^4 - \ldots)$. Also, it is convenient to replace the summation over i by a d-dimensional integral. The exchange term, assumed to be of a short range, then gives rise to a gradient term. Finally, the partition function for the Hamiltonian (1) is replaced by

$$Z\{\hat{x}(\vec{R})\} = \int_{-\infty}^{\infty}\cdots\int\prod_{\vec{R}}[d^m S(\vec{R})]e^{\overline{\mathcal{H}}\{\hat{x}(\vec{R}),\vec{S}(\vec{R})\}} \quad , \tag{2}$$

with

$$\overline{\mathcal{H}}\{\hat{x}(\vec{R}),\vec{S}(\vec{R})\} = -\int d^dR\{\tfrac{1}{2}[r_0|\vec{S}(\vec{R})|^2 + |\vec{\nabla}\vec{S}|^2] - D(\hat{x}(\vec{R})\cdot\vec{S}(\vec{R}))^2 + v|\vec{S}(\vec{R})|^4 + \cdots\} \quad . \tag{3}$$

The dots represent higher order terms, e.g., $|\vec{S}|^6$, $|\vec{\nabla}\vec{\nabla}\vec{S}|^2$, etc. These are irrelevant, in the sense of renormalization group,[6] and therefore are ignored. The coefficients D and v are proportional to D_0 and v_0, and r_0 is linear in the temperature T (we rescaled the spins so that the coefficient of $|\vec{\nabla}\vec{S}|^2$ became unity).

To obtain the free energy F of the quenched system, we average over the free energies of all possible random configurations of the directions, $\{\hat{x}(\vec{R})\}$. We assume that in the thermodynamic limit all of these configurations yield the same critical behavior.[12] Thus

$$-\beta F = \int\cdots\int\prod_{\vec{R}}[d^m\hat{x}(\vec{R})]P\{\hat{x}(\vec{R})\}\ln Z\{\hat{x}(\vec{R})\} \quad , \tag{4}$$

where $\beta = 1/k_BT$ and $P\{\hat{x}(\vec{R})\}$ is the probability of finding a given configuration of directions $\{\hat{x}(\vec{R})\}$. The integral $\int d^m\hat{x}$ is over the m-dimensional unit sphere. We now follow Emery,[10] and write

$$\ln Z\{\hat{x}(\vec{R})\} = \frac{\partial}{\partial n}[Z\{\hat{x}(\vec{R})\}]^n\Big|_{n=0} \quad . \tag{5}$$

From (2) and (3) it follows that if we define an nm-component vector

$$\vec{\sigma} \equiv (\vec{S}_1,\ldots,\vec{S}_n) \equiv (S_{11},\ldots,S_{1m},S_{21},\ldots,S_{nm}) \tag{6}$$

we can then write

$$[Z\{\hat{x}(\vec{R})\}]^n = \int\cdots\int\prod_{\vec{R}}[d^{nm}\sigma(\vec{R})]e^{\overline{\mathcal{H}}_1\{\hat{x}(\vec{R}),\vec{\sigma}(\vec{R})\}} \quad , \tag{7}$$

with

$$\overline{\mathcal{H}}_1 = -\int d^dR\Big(\tfrac{1}{2}[r_0|\vec{\sigma}(\vec{R})|^2 + |\vec{\nabla}\vec{\sigma}|^2] - D\sum_{\alpha=1}^n(\hat{x}(\vec{R})\cdot\vec{S}_\alpha(\vec{R}))^2 + v\sum_{\alpha=1}^n|\vec{S}_\alpha(\vec{R})|^4 + \cdots\Big) \quad . \tag{8}$$

We simply replace each factor Z by the expression (2) for one vector variable $\vec{S}_\alpha$. Here, $|\vec{\sigma}|^2 = \sum_{\alpha=1}^n|\vec{S}_\alpha|^2 = \sum_{\alpha=1}^n\sum_{i=1}^m S_{\alpha i}^2$. From (4), (5), (7), and (8) we now find

$$-\beta F = \frac{\partial}{\partial n}\Big(\int\cdots\int\prod_{\vec{R}}[d^{nm}\sigma(\vec{R})]e^{\overline{\mathcal{H}}_{eff}\{\vec{\sigma}(\vec{R})\}}\Big)\Big|_{n=0} \quad , \tag{9}$$

where

$$\overline{\mathcal{H}}_{eff} = -\int d^dR\Big(\tfrac{1}{2}[r_0|\vec{\sigma}(\vec{R})|^2 + |\vec{\nabla}\vec{\sigma}|^2] + v\sum_{\alpha=1}^n|\vec{S}_\alpha|^4 + \cdots\Big) - G\{\vec{\sigma}(\vec{R})\} \quad , \tag{10}$$

and

$$e^{-G\{\vec{\sigma}\}} = \int\cdots\int\prod_{\vec{R}}[d^m\hat{x}(\vec{R})]P\{\hat{x}(\vec{R})\} \times\exp\Big(D\int d^dR\sum_{\alpha=1}^n(\hat{x}(\vec{R})\cdot\vec{S}_\alpha(\vec{R}))^2\Big) \quad . \tag{11}$$

In this paper we limit ourselves to a probability distribution in which all the $\hat{x}(\vec{R})$'s are completely independent of each other. Thus we can factorize $P\{\hat{x}\}$ into

$$P\{\hat{x}(\vec{R})\} = \prod_{\vec{R}}p[\hat{x}(\vec{R})] \quad , \tag{12}$$

where $p(\hat{x})$ is the probability distribution of a single-ion anisotropy direction. With this assumption, the integral in (11) can be factorized, and we find

$$G\{\vec{\sigma}(\vec{R})\} = \int d^dR\, g[\vec{\sigma}(\vec{R})] \quad , \tag{13}$$

with

$$e^{-g(\vec{\sigma})} = \int d^m\hat{x}\, p(\hat{x}) \exp\left(D\sum_{\alpha=1}^{n} (\hat{x}\cdot\vec{S}_\alpha)^2\right) \quad . \tag{14}$$

We shall consider a few examples of this function in the following sections.

To conclude this section, we now show that the free energy per degree of freedom of the system described by $\bar{\mathcal{H}}_{eff}$ is equal, in the limit $n\to 0$, to that of the original system.[10] Let

$$Z_{eff} = \int\cdots\int \prod_{\vec{R}} [d^{nm}\sigma(R)]\, e^{\bar{\mathcal{H}}_{eff}\{\vec{\sigma}(\vec{R})\}}$$

$$= \int\cdots\int \prod_{\vec{R}} [d^m\hat{x}(\vec{R})] P\{\hat{x}(\vec{R})\}[Z\{\hat{x}(\vec{R})\}]^n \; . \tag{15}$$

In the limit $n\to 0$, $Z_{eff}=1$. Therefore Eq. (9) may be written

$$\frac{1}{m}F = -\frac{1}{m\beta}\frac{\partial}{\partial n}[\ln Z_{eff}]\Big|_{n=0} = -\lim_{n\to 0}\left[\frac{1}{nm\beta}\ln Z_{eff}\right] \; . \tag{16}$$

The left-hand side is the free energy per spin component of the original model, whereas the right-hand side is the limit of the free energy per spin component of the model with $\bar{\mathcal{H}}_{eff}$.

III. ISOTROPIC CASE–EFFECTIVE HAMILTONIAN

We are now ready to choose the probability distribution function $p(\hat{x})$. For the crystalline system,

$$p(\hat{x}) = \delta^{(m)}(\hat{x}-\hat{k}) \quad , \tag{17}$$

where $\hat{k}$ is a unit vector in the direction of anisotropy (e.g., the 1-axis). Equation (14) thus becomes

$$g(\vec{\sigma}) = -D\sum_{\alpha=1}^{n} S_{\alpha 1}^2 \quad , \tag{18}$$

and the Hamiltonian $\bar{\mathcal{H}}_{eff}$ separates into n uncoupled m-vector Hamiltonians, each of which has a uniaxial anisotropy. Thus the parameter n becomes meaningless, and we return to the usual problem of spin anisotropy.[13–15] In this case, the asymptotic critical behavior will be that of an Ising model [or an $(m-1)$-component model], and typical crossover effects from m-component Heisenberg-like behavior to Ising behavior [or $(m-1)$-component behavior] will be observed.

The other extreme is to assume a completely isotropic distribution, with

$$p(\hat{x}) \equiv \left[\int d^m\hat{x}\right]^{-1} = (2\pi)^m K_m^{-1} = 2\pi^{m/2}/\Gamma(m/2) \quad . \tag{19}$$

Harris *et al.*[4] estimate $p(\hat{x})$ on the basis of a random packing of atomic spheres, and conclude that the assumptions (19) and (12) are confirmed for that model. With this distribution, Eq. (14) becomes

$$e^{-g(\vec{\sigma})} = \left(\int d^m\hat{x}\right)^{-1}\int d^m\hat{x}$$

$$\times\exp\left(D\sum_{i,j=1}^{m}\hat{x}_i\hat{x}_j\sum_{\alpha=1}^{n}S_{\alpha i}S_{\alpha j}\right) \quad . \tag{20}$$

Expanding the exponent in a Taylor series, and using standard angular integrals,[16] we find

$$g(\vec{\sigma}) = -m^{-1}D|\vec{\sigma}|^2 + [m^2(m+2)]^{-1}D^2|\vec{\sigma}|^4$$

$$-[m(m+2)]^{-1}D^2\sum_{\alpha,\beta=1}^{n}\sum_{i,j=1}^{m}S_{\alpha i}S_{\alpha j}S_{\beta i}S_{\beta j}$$

$$+O(|\vec{\sigma}|^6) \; . \tag{21}$$

The first term in (21) is simply a shift in r_0 of Eq. (10) (or in T_c, r_0 is linear in the temperature), and we can replace

$$r = r_0 - 2m^{-1}D \; . \tag{22}$$

The second term in (21) is similar to the usual $|\vec{\sigma}|^4$ term, which would result from an isotropic nm-component weight function.[6] The last term is new, and may lead to new effects in the critical behavior. As in (3), we ignore here higher-order terms which become irrelevant for small ϵ.[17] Combining (10), (13), (21), and (22), our effective Hamiltonian becomes

$$\bar{\mathcal{H}}_{eff} = -\int d^dR\left(\tfrac{1}{2}[r|\vec{\sigma}|^2 + |\vec{\nabla}\vec{\sigma}|^2] + u|\vec{\sigma}|^4\right.$$

$$\left. + v\sum_{\alpha=1}^{n}|\vec{S}_\alpha|^4 + 2w\sum_{\alpha,\beta=1}^{n}\sum_{i,j=1}^{m}S_{\alpha i}S_{\alpha j}S_{\beta i}S_{\beta j}\right) \; . \tag{23}$$

Note that $v>0$ (unless the crystalline system undergoes a first-order transition), $u>0$, and $w<0$. The effective Hamiltonian considered by Grinstein and Luther[9] and by Harris and Lubensky[11] was of the same form (23), but with $w=0$ and $u<0$. In their case, $(-u)$ was proportional to the second cumulant of the random (impurity) distribution, and as such had to be always negative.[11] Similarly, our u and w are related to appropriate cumulants of the distribution function, and maintain their signs and the ratio $w/u = -m/2$ for all isotropic distributions.

For $w=0$, the Hamiltonian (23) was already analyzed using the renormalization group.[7,9,11] In the following section we summarize this previous analysis, and generalize it to include the new w term. As we shall see, this generalization leads to quite a few new phenomena.

It is also interesting to consider the case in which the system is not fully amorphous, and there still is a preferred axis $\hat{k}$ along which some of the anisotropy fields tend to align. If the probability for this is q, then we can combine (17) with (19), and write

$$p(\hat{x}) = q\,\delta^{(m)}(\hat{x} - \hat{k}) + (1-q)\,2\pi^{m/2}/\Gamma(m/2)\ . \quad (24)$$

We can now substitute this into (14), and we find

$$g(\vec{\sigma}) = -qD\sum_{\alpha=1}^{n} S_{\alpha 1}^2 - (1-q)m^{-1}D\,|\vec{\sigma}|^2 + O(|\vec{\sigma}|^4)\ . \quad (25)$$

The first term here reflects, again, a spin anisotropy of the Fisher-Pfeuty type.[13–15] Irrespective of the quartic terms, this term will cause a crossover to a critical behavior which involves only the n components $\{S_{\alpha 1}\}$ or the remaining $n(m-1)$ components. In the former case, the Hamiltonian reduces to that of n coupled Ising models, and the final asymptotic critical behavior will be that of the random Ising model.[9,11] In the latter case we shall remain with an effective Hamiltonian similar to (23), but with the summations over i and j going only from 2 to m. In any case, there is no further need to discuss this case separately from the general case.

IV. ISOTROPIC CASE–RECURSION RELATIONS AND FIXED POINTS

We can now follow the standard procedures[6,7] and analyze the Hamiltonian (23) using renormalization group. We first Fourier transform the spin variables, and we introduce a spherical Brillouin zone of radius Λ. We then perform the functional integral over the spins $\vec{\sigma}(\vec{q})$ which have wave vectors $\vec{q}$ in the range $\Lambda/b < |\vec{q}| < \Lambda$, with $b > 1$. Finally, we rescale the space variables $\vec{q}$ and the remaining spin variables $\vec{\sigma}$, so that the partition function can be rewritten in the form (15), the new $\overline{\mathcal{H}}_{\text{eff}}$ being given by (23) with new parameters r', u', v', and w'. The recursion relations for these parameters are found to be (assuming r, u, v and w are all of order ϵ)

$$r' = b^2\{r + 4K_4 I(r)[(mn+2)u + (m+2)v + 2(m+n+1)w] + O(\epsilon^2)\}\ , \quad (26)$$

$$u' = b^{\epsilon}\{u - 4K_4 \ln b[(mn+8)u^2 + 2(m+2)uv + 4(n+m+1)uw + 4vw + 12w^2] + O(\epsilon^3)\}\ , \quad (27)$$

$$v' = b^{\epsilon}\{v - 4K_4 \ln b[(m+8)v^2 + 12uv + 4(m+5)vw] + O(\epsilon^3)\}\ , \quad (28)$$

and

$$w' = b^{\epsilon}\{w - 4K_4 \ln b[2(n+m+4)w^2 + 12uw + 4vw] + O(\epsilon^3)\}\ , \quad (29)$$

where K_4 $(= 1/8\pi^2)$ was defined in (19), and

$$I(r) = \int_{\Lambda b^{-1}}^{\Lambda} \frac{q^3 dq}{r+q^2} = \frac{1}{2}\left((1-b^{-2})\Lambda^2 + r\ln\frac{r+\Lambda^2 b^{-2}}{r+\Lambda^2}\right) = \tfrac{1}{2}\Lambda^2(1-b^{-2}) - r\ln b + O(r^2)\ . \quad (30)$$

We can now solve Eqs. (26)–(29) for their fixed points. We find eight fixed points, which are summarized in Table I. The first four fixed points are the same as those discussed previously,[7,9,11,18] and there are four additional fixed points. Linearizing (26)–(29) about any of these fixed points will now yield the eigenvalues, which are related to critical exponents. Linearizing (26), and writing it in the form $\Delta r' = b^{1/\nu}\Delta r$, we find that the correlation length exponent ν is given by

$$1/2\nu = \tfrac{1}{2}\lambda_r = 1 - 2K_4[(mn+2)u^* + (m+2)v^* + 2(m+n+1)w^*] + O(\epsilon^2)\ . \quad (31)$$

Linearizing the remaining three recursion relations, and diagonalizing the resulting 3×3 matrix of coefficients, we find three eigenvalues, which can be written in the form b^{λ_i}, $i = 1, 2, 3$. The fixed point is stable only if all λ_i's are negative. Otherwise, one or more combinations of u, v, and w represents a relevant parameter, and causes a crossover. This crossover may be towards another fixed point, which is more stable, or away and out of the linear region in which (26)–(29) represent valid approximations. The latter case probably represents a smeared transition.[11,19] To order ϵ, the eigenvalues λ_1, λ_2, and λ_3 are also given in Table I.

The *Gaussian fixed point* (I in Table I) is trivial, and yields $\lambda_u^{\text{I}} = \lambda_v^{\text{I}} = \lambda_w^{\text{I}} \equiv \epsilon$. Thus it is unstable for $\epsilon > 0$ or for $d < 4$. The *decoupled fixed point* (II) represents the *isotropic crystalline state*, when $D = 0$ in Eq. (1), or $g(\vec{\sigma}) = 0$. It represents the critical behavior of the usual m-component Heisenberg-like model, as studied previously.[6] We now "switch on" a small amount of random anisotropy. This introduces two perturbations, represented by the parameters u and w. Including all the $\alpha = \beta$ terms in Eq. (23) in the v term, a typical term in the u perturbation may be written

$$u\,|\vec{S}_\alpha|^2 |\vec{S}_\beta|^2\ , \quad \alpha \neq \beta\ . \quad (32)$$

The operator $|\vec{S}_\alpha|^2$ represents an energy density operator of one of the decoupled (unperturbed) m-vector models. In the vicinity of the m-vector model critical point it therefore behaves as $|\Delta T|^{1-\alpha_m} \sim \xi^{-(1-\alpha_m)/\nu_m}$, where α_m and ν_m are the specific-heat and the correlation-length exponents of the m-vector model (ξ is the correlation length). Since the two factors in (32) are totally independent (in the unperturbed system), the full operator (32) behaves as $\xi^{-2(1-\alpha_m)/\nu_m}$. Remembering the additional d-dimensional integral in (23), the exponent

TABLE I. Fixed points and exponents (to order ϵ) for isotropic case.

Fixed point	$4K_4u^*$	$4K_4v^*$	$4K_4w^*$	Exponents
I. Gaussian	0	0	0	$\lambda_u=\lambda_v=\lambda_w=\epsilon$
II. Decoupled m-component	0	$\frac{\epsilon}{m+8}$	0	$\lambda_v=-\epsilon,\ \lambda_u=\frac{4-m}{m+8}\epsilon,\ \lambda_w=\frac{m+4}{m+8}\epsilon$
III. Isotropic nm-component	$\frac{\epsilon}{nm+8}$	0	0	$\lambda_u=-\epsilon,\ \lambda_v=\lambda_w=\frac{nm-4}{nm+8}\epsilon$
IV. "Mixed"	$\frac{(4-m)\epsilon}{(m+8)nm-16(m-1)}$	$\frac{(nm-4)\epsilon}{(m+8)nm-16(m-1)}$	0	$\lambda_1=-\epsilon,\ \lambda_2=\frac{(4-m)(4-nm)}{(m+8)nm-16(m-1)}\epsilon,\ \lambda_w=\frac{(4+m)(nm-4)}{(m+8)nm-16(m-1)}\epsilon$
V. } VI. }	$\frac{x_\pm\epsilon^{a}}{2(m+n+4+6x_\pm)}$	0	$\frac{\epsilon}{2(m+n+4+6x_\pm)}$	$\lambda_1=-\epsilon,\ \lambda_2=\frac{2-m-n+(4-mn)x_\pm}{n+m+4+6x_\pm}\epsilon,\ \lambda_v=\frac{n-m-6}{n+m+4+6x_\pm}\epsilon$
VII. } VIII. }	$\frac{y_\pm\epsilon^{b}}{2(6y_\pm-4z+m+n+4)}$	$\frac{-z\epsilon}{6y_\pm-4z+m+n+4}$	$\frac{\epsilon}{2(6y_\pm-4z+m+n+4)}$	$\lambda_1=-\epsilon$; for λ_2 and λ_3 see text [Eq. (37)]

[a] $x_\pm=\{m+n-2\pm[(m+n-2)^2+12(4-mn)]^{1/2}\}/(4-mn)$.

[b] $z=(m-n+6)/(m+4)$, $y_\pm=\{(4-m)z+2n-8\pm[\{(4-m)z+2n-8\}^2+(4-mn)(12-8z)]^{1/2}\}/(4-mn)$.

with which the parameter u rescales, λ_u^{II}, is thus readily found to be[20] (using $\alpha_m=2-d\nu_m$)

$$\lambda_u^{II}=d-2(1-\alpha_m)/\nu_m=\alpha_m/\nu_m \quad . \tag{33}$$

Since series expansions[21] and experiments[22] indicate that $\alpha_m<0$ for $m>1$, we conclude that the decoupled fixed point is stable against the parameter u. Note that for $m=1$, the parameters u and w represent the same operator, and (23) reduces to the Hamiltonian discussed in Refs. 9–11. (Indeed, V ≡ I, VI ≡ III, VII ≡ II, VIII ≡ IV for $m=1$.) In this case, the present model loses its meaning, as the last term in Eq. (1) reduces to a single-ion nonrandom Ising-like interaction. We shall therefore confine ourselves in the following discussion to $m>1$.

A typical term in the w perturbation may be written

$$2w\left[\sum_{i,j=1}^{m}\left(S_{\alpha i}S_{\alpha j}-\frac{1}{m}|\vec{S}_\alpha|^2\delta_{ij}\right)\left(S_{\beta i}S_{\beta j}-\frac{1}{m}|\vec{S}_\beta|^2\delta_{ij}\right)-\frac{1}{m}|\vec{S}_\alpha|^2|\vec{S}_\beta|^2\right],\quad \alpha\neq\beta \quad . \tag{34}$$

The last term in the brackets can be included in (32). The first term is a sum over products of two single m-component model operators, $S_{\alpha i}S_{\alpha j}-m^{-1}|\vec{S}_\alpha|^2\delta_{ij}$. These operators represent spin anisotropies, of the form discussed by Fisher and Pfeuty[13,15] and by Wegner.[14] The dimension of each of these is[14] $d-\phi_m/\nu_m$, where ϕ_m is the spin-anisotropy crossover exponent from the m-component isotropic fixed point to one with a lower symmetry. Using similar arguments to the ones used above, we thus find

$$\lambda_w^{II}=d-2(d-\phi_m/\nu_m)=2\phi_m/\nu_m-d \quad . \tag{35}$$

At $d=3$, one has $\phi_2\simeq 1.175$, $\phi_3\simeq 1.25$,[15] $\alpha_2\simeq -0.02$, $\alpha_3\simeq -0.14$, $\nu_2\simeq 0.67$, and $\nu_3\simeq 0.71$.[21] Thus

$$\lambda_u^{II}\simeq -0.2\ ,\quad \lambda_w^{II}\simeq 0.52\quad (d=3,\ m=3)\ ,$$
$$\lambda_u^{II}\simeq -0.03\ ,\quad \lambda_w^{II}\simeq 0.5\quad (d=3,\ m=2)\ . \tag{36}$$

The appropriate crossover exponents are given by $\nu_m\lambda^{II}$. Note that the first order ϵ-expansion result, given in Table I, yields positive values for λ_u^{II} if $m<4$.[9,11] Thus one should be careful with drawing final conclusions from the low-order ϵ expansion if it yields a small exponent. A similar situation occurs in the cubic problem, where the sign of an exponent (our λ_v^{III}, with $m=1$, $n=3$) oscillates with the order in ϵ.[23,24]

The results (36) clearly indicate that the decoupled, or nonrandom, fixed point is *unstable* with respect to the perturbation w. This fixed point was *stable* for the nonmagnetic impurity case, $w=0$, discussed previously.[7,9,11] We must therefore study the other fixed points, and the possible Hamiltonian flows. For all the other fixed points one cannot find general expressions, analogous to (33) and (35). We thus must rely on the ϵ expansion. The results are summarized in Table I, except for the last two fixed points. For these, λ_1, λ_2, and λ_3 are the eigenvalues of the matrix

$$\frac{\epsilon}{6y_\pm - 4z + m + n + 4}\begin{pmatrix} 2mz - (nm+2)y_\pm - n - m + 2 & -(m+2)y_\pm - 2 & -2(m+n+1)y_\pm + 4z - 12 \\ 12z & n - m - 6 + 2(m+6)z & 4(m+5)z \\ -6 & -2 & -(n+m+4) \end{pmatrix} \tag{37}$$

($y_\pm$ and z are defined in Table I), which can be diagonalized numerically. As we noted already, the order $-\epsilon$ results may be misleading in some cases. It is thus useful to appeal to higher order in ϵ, and to independent arguments. The analysis becomes quite lengthy for the general case. We shall thus limit ourselves from now on to the case of particular interest here, namely $n \to 0$. The results in Table I immediately show that the $n=0$ isotropic fixed point III is stable. This is borne out by calculations to order ϵ^3 near the isotropic n-component fixed point.[24] The fixed points II and IV are unstable, at least with respect to w. The magnitude of λ_w and the relation (35) for the fixed point II lead us to believe that this instability will persist (at $\epsilon = 1$) even when higher orders in ϵ are included. To order ϵ, $\lambda_u^{\mathrm{II}} > 0$ and $\lambda_2^{\mathrm{IV}} < 0$ for $d=3$, $m<4$. From (33) and (36) we saw that the first inequality holds only for $m<2$. It is reasonable to assume that the same will hold for the second inequality. For the last four fixed points, we rely on numerical results. At $n=0$, $m=3$, $x_\pm = (1 \pm 7)/4$, and thus

$$\lambda_2^{\mathrm{V}} = \tfrac{7}{19}\epsilon\,, \quad \lambda_v^{\mathrm{V}} = -\tfrac{9}{19}\epsilon\,, \quad \lambda_2^{\mathrm{VI}} = \tfrac{7}{2}\epsilon\,, \quad \lambda_v^{\mathrm{VI}} = \tfrac{9}{2}\epsilon\,, \tag{38}$$

$$\begin{aligned} &\lambda_2^{\mathrm{VII}} \simeq 3.47\epsilon\,, \quad \lambda_3^{\mathrm{VII}} \simeq 2.78\epsilon\,, \\ &\lambda_2^{\mathrm{VIII}} \simeq 0.379\epsilon\,, \quad \lambda_3^{\mathrm{VIII}} \simeq -0.473\epsilon\,. \end{aligned} \tag{39}$$

Therefore none of these is stable. The fixed points V, VII, and VIII turn out to be unstable, to order ϵ, for all values of m. However, both λ_2^{VI} and λ_v^{VI} change signs at $m \simeq 3.9$. In fact, both approach $+\infty$ as $m \to 3.9^+$ and $-\infty$ as $m \to 3.9^-$. Probably, this 3.9 must be replaced by an ϵ expansion. All we can say at this stage is that for m larger than $m_c(\epsilon) \simeq 3.9 + O(\epsilon)$, the fixed point VI becomes stable. Since 3.9 is not too far from 3, it is quite possible that higher-order terms in ϵ will show that this fixed point is actually stable at $m=3$. In fact, such situations have been previously found both in the cubic case[23,24] and in the case of the random impurity problem (the fixed point II, for $w \equiv 0$).[7] Note however an important difference: In those cases, two fixed points coincided at some critical value of m, and then interchanged their stability properties. Here, we have a fixed point which, as function of m, "goes to infinity and returns." Since infinity is probably a legitimate fixed point of our problem (although outside of the linear range), some analogy may still be drawn. Note that the "mixed" fixed point IV also goes to infinity as $n \to 0$, $m \to 1$. In that case, the borderline value $m=1$ persists to second order in ϵ.[9] Thus one cannot go beyond speculation until second-order ϵ expansion is carried out.

In summary, we have at most two stable fixed points, namely, the $n=0$ isotropic one, II, and maybe VI.

V. ISOTROPIC CASE–HAMILTONIAN FLOWS AND CRITICAL BEHAVIOR

To conclude our analysis of the isotropic case we must now start with our particular initial Hamiltonian, given by Eqs. (10) and (21) (including all irrelevant operators), and follow its flow to (or away from) some of the fixed points. The scaling behavior in the vicinity of any fixed point will be described by the critical exponents related to that fixed point.

We start by considering the simpler case, in which the irrelevant variables are ignored. Even in this case, it is difficult to draw the three-dimensional u-v-w space (we assume that r is at its critical value r_{0c}, so that $T=T_c$). In Fig. 1 we show schematically the fixed points in the u-w plane, $v \equiv 0$, based on the order $-\epsilon$ results of Table I. The arrows show the directions of the Hamiltonian flows, as governed by Eqs. (27)–(29). Typical values of the initial values of u and w [see Eq. (21)] are indicated by the point $\times$. Ignoring irrelevant variables, one sees from Eqs. (26)–(29) that the Hamiltonian flow can never lead from this initial point to the stable, $n=0$, fixed point (II). A recent study of the random isotropic impurity problem[19] shows that even if all the irrelevant operators are added to the recursion relations in that problem the new values of the parameters still maintain their meaning as cumulants of the random distribution function. As such, they never change their signs under iterations. If the same holds here, and the new distribution function (after an iteration) is still related to an isotropic distribution like (19),

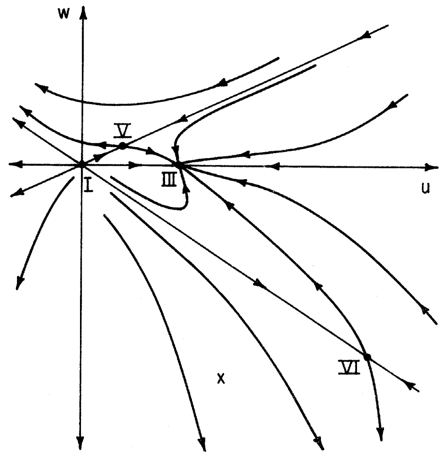

FIG. 1. Schematic flow diagram and fixed points in $u-w$ plane ($v=0$) for $\epsilon=1$, $m=3$, based on order-ϵ results. The fixed points are identified in Table I. The × indicates typical initial values of u and w.

then we may expect that the flow from the point × will not cross the line I-VI even if the irrelevant variables are included. However, this remains to be checked.

We now "switch on" the parameter v. As noted above, this parameter is assumed to be positive (or else the "nonrandom" transition is of first order). Both the fixed points I and VI are strongly unstable with respect to v. Thus, if we start with a small value of v, we expect the flow to go towards larger positive values. This flow will cross three planes which are parallel to the u-w plane and which include the fixed points VIII ($4K_4u^*\simeq 0.0915\epsilon$, $4K_4v^*\simeq 0.0676\epsilon$, $4K_4w^*\simeq -0.0263\epsilon$ at $m=3$), II ($u^*=w^*=0$, $4K_4v^*=\frac{1}{11}\epsilon$, and IV ($4K_4u^*=-\frac{1}{32}\epsilon$, $4K_4v^*\simeq\frac{1}{8}\epsilon$, $w^*=0$). The remaining fixed point VII ($4K_4u^*\simeq 0.0237\epsilon$, $4K_4v^*\simeq -0.495\epsilon$, $4K_4w^*\simeq 0.193\epsilon$) is below the u-w plane, and will thus not be "seen." None of these fixed points is stable, and all of them have values of u^* and w^* which are inconsistent with the ratio indicated by Eq. (21). We thus conclude that the final flow is very probably away from all these fixed points, and out of the linear region. The meaning of this "runaway" is still open to discussion, but it is quite probable that it represents a *smeared* transition.[11,19] This may be the explanation for the anomalous behavior of the critical scattering function observed in $TbFe_2$.[2]

If the initial values of u and w are small (small D), then the flow starts in the vicinity of the "decoupled" fixed point II, and one observes the usual m-component critical exponents until, close enough to T_c, the crossover due to the parameter w takes over, and "smeared" effects prevail. The boundary of this crossover region may be estimated from

$$|w|\,\xi^{\lambda_w^{II}}\sim 1 \quad . \tag{40}$$

Estimating w from (21) [$w\sim D^2/2m(m+2)$ in the appropriate rescaled units], we can thus conclude that it is quite reasonable to expect ξ to grow larger only until it reaches a maximum of the order $|w|^{-1/\lambda_w^{II}}\sim[2m(m+2)/D^2]^2$. A numerical comparison with measurements on $TbFe_2$[2] will be quite interesting.

If the flow starts closer to one of the fixed points VI or VIII (V and IV are excluded by the signs of w and of u), one may observe the exponents of these fixed points prior to the runaway. Thus near VIII one has [see (31)]

$$1/2\nu^{\rm VIII}\simeq 1-0.155\epsilon,\quad m=3,\quad n=0\,, \tag{41}$$

or $\nu^{\rm VIII}\simeq 0.59$ at $\epsilon=1$. Similarly, the order-ϵ result near VI yields

$$1/2\nu^{\rm VI}=1+\tfrac{5}{8}\epsilon,\quad m=3,\quad n=0\,, \tag{42}$$

or $\nu^{\rm VI}\simeq 0.31$ at $\epsilon=1$. A value of ν which is less than $\frac{1}{2}$ is very unusual, and should be considered with caution. This value is very probably an artifact of the divergence of all exponents of the fixed point VI at $m\simeq 3.9$.

To conclude, we mention two additional possibilities. First, $m_c(1)$ may be less than 3. In this case, both the fixed point values $u^{\rm VI}$ and $w^{\rm VI}$ will have signs opposite to those indicated in Fig. 1. This may change the flow diagrams, and yield some flows which lead from the initial point to the $n=0$ stable fixed point III. (The stable fixed point VI will not be reached.) Second, even if this is not the case, our conjectures as regards the cumulant nature of the parameters may be special to the isotropic impurity case, and may not apply here. Thus irrelevant variables may yield flows which will go to the $n=0$ fixed point. If this is the case, then we expect the transition to be *sharp*, with a divergent correlation length and with critical exponents given by the $n\to 0$ analytic continuation of the usual n-component exponents. For example,[6] to order ϵ^2,

$$2\nu^{\rm III}=1+\tfrac{1}{8}\epsilon+\tfrac{15}{256}\epsilon^2\,,\quad \eta^{\rm III}=\tfrac{1}{64}\epsilon^2\,, \tag{43}$$

etc. The observed "smeared" transition may be an indication against this possibility.

VI. CUBIC CASE

As another example of the application of the model, we now consider briefly a distribution function $p(\hat{x})$ which allows $\hat{x}$ to point only along one of the $2m$ axis-directions of a cubic lattice,

$$p(\hat{x})=\frac{1}{2m}\sum_{i=1}^{m}\left[\delta^{(m)}(\hat{x}-\hat{k}_i)+\delta^{(m)}(\hat{x}+\hat{k}_i)\right]\,, \tag{44}$$

where $\hat{k}_1,\ldots,\hat{k}_m$ are unit vectors along the axes. Although probably not appropriate for a true amorphous system, this distribution function may describe the situation in which the magnetic ions are located on the sites of a cubic lattice, and some nonmagnetic impurities are randomly occupying sites along the cubic axes, thus causing a local uniaxial single-ion interaction in the directions of these axes. Substituting in (14), we find

$$e^{-g(\vec{\sigma})} = \frac{1}{m}\sum_{i=1}^{m} \exp\left(D\sum_{\alpha=1}^{n} S_{\alpha i}^2\right) \quad . \tag{45}$$

Expanding in powers of $\vec{\sigma}$, this yields

$$g(\vec{\sigma}) = -m^{-1} D|\vec{\sigma}|^2 + \frac{D^2}{2m^2}|\vec{\sigma}|^4 - \frac{D^2}{2m}\sum_{i=1}^{m}\sum_{\alpha,\beta=1}^{n} S_{\alpha i}^2 S_{\beta i}^2 + O(|\vec{\sigma}|^6) \quad . \tag{46}$$

The first term leads to a shift in r_0, as in (22), and the second term is similar to the u term considered earlier. However, the third term now has *cubic* symmetry in the m-dimensional subspace of each vector $\vec{S}_\alpha$. Combining (46) with (10), we thus write

$$\overline{\mathcal{H}}_{\text{eff}} = -\int d^dR\left(\tfrac{1}{2}[r|\vec{\sigma}|^2 + |\vec{\nabla}\vec{\sigma}|^2] + u|\vec{\sigma}|^4 + v\sum_{\alpha=1}^{n}|\vec{S}_\alpha|^4 + w\sum_{i=1}^{m}\sum_{\alpha,\beta=1}^{n} S_{\alpha i}^2 S_{\beta i}^2 + y\sum_{i=1}^{m}\sum_{\alpha=1}^{n} S_{\alpha i}^4 + \cdots\right). \tag{47}$$

The last term here, $\sum_{i,\alpha} S_{\alpha i}^4$, was added because it combines the symmetries of the v term and the w term in the n-dimensional and m-dimensional subspaces. (Note the symmetry $v \leftrightarrow w$, $m \leftrightarrow n$.) This term must be included, because it is generated by the renormalization-group iterations. Note that w is not the same as in the previous sections.

Again, the Hamiltonian (47) reduces to that of the *random nonmagnetic impurity* for $w = y = 0$. For $u = w = 0$ it reduces to n decoupled *cubic Hamiltonians*, as treated in Refs. 23 and 24. For $w = 0$ it reduces to the effective Hamiltonian of *random nonmagnetic impurities* (introducing an *isotropic* random shift in T_c) *in a lattice of cubic symmetry*: In this case, the term $v|\vec{S}(\vec{R})|^4$ in Eq. (3) must be replaced by[23] $v|\vec{S}(\vec{R})|^4 + y\sum_{i=1}^{m} S_i^4$, and the rest of the arguments remain as before.[9,11] (This is another reason why y should be introduced for cubic symmetric cases). In our Hamiltonian $v > 0$, $u > 0$, and $w < 0$. The cubic parameter y may be of either sign.[23]

The recursion relations now become

$$r' = b^2\{r - 4K_4 I(r)[(mn+2)u + (m+2)v + (n+2)w + 3y] + \cdots\} \quad , \tag{48}$$

$$u' = b^\epsilon\{u - 4K_4 \ln b[(mn+8)u^2 + 2(m+2)uv + 2(n+2)uw + 6uy + 2vw] + \cdots\} \quad , \tag{49}$$

$$v' = b^\epsilon\{v - 4K_4 \ln b[(m+8)v^2 + 6vy + 12uv + 4vw] + \cdots\} \quad , \tag{50}$$

$$w' = b^\epsilon\{w - 4K_4 \ln b[(n+8)w^2 + 12uw + 4vw + 6wy] + \cdots\} \quad , \tag{51}$$

and

$$y' = b^\epsilon\{y - 4K_4 \ln b[9y^2 + 12uy + 12vy + 12wy + 8vw] + \cdots\} \quad . \tag{52}$$

For $n = 0$, there are fourteen fixed points, as summarized in Table II. In the general case there will be a few additional fixed points, e.g., decoupled n-component cubic and nm-component cubic (these go to infinity for $n \to 0$). The fixed points I, II, III, and VI (all in the u-v plane) are the same as the first four in Table I, and as those considered previously.[9,11] The fixed points I, II, V, and VIII (all in the v-y plane) are the same as those found for the pure cubic case.[23] By the symmetry $v \leftrightarrow w$, $n \leftrightarrow m$, the fixed points IV, VII, and X are the $m \to 0$ limits of the fixed points II, VI, and IX, respectively.

Also included in Table II are the order-ϵ exponents λ_1, λ_2, λ_3, and λ_4 resulting from the diagonalization of the linearized recursion relations (49)–(52). Again, the Gaussian fixed point is unstable. The stability of the decoupled fixed point II representing the nonrandom noncubic case can be studied similarly as the previous case: one easily shows that

$$\lambda_u^{\text{II}} \equiv \alpha_m/\nu_m \,, \quad \lambda_w^{\text{II}} \equiv 2\phi_m/\nu_m - d \quad . \tag{53}$$

The exponent λ_y^{II} relates to the cubic instability, and was calculated to order ϵ^3 by Ketley and Wallace.[24] It turns out, that $\lambda_y^{\text{II}} < 0$ and $\lambda_2^{\text{VIII}} > 0$ for $m < n_c(\epsilon)$, with[7]

$$n_c(\epsilon) = 4 - 2\epsilon + \frac{5}{12}[6\zeta(3) - 1]\epsilon^2 + O(\epsilon^3) \simeq \frac{4 + 3.176\epsilon}{1 + 1.294\epsilon} \,, \tag{54}$$

or $n_c(1) \simeq 3.13$. Thus, if $w \equiv 0$, then the fixed point II is stable and the fixed point VIII is unstable for $m = 2, 3$, $d = 3$. This will be the case for a *random cubic problem* (with an *isotropic* random interaction). However, $\lambda_w^{\text{II}} > 0$ for $m = 2, 3$, $d = 3$, and the w instability will lead away from this fixed point.

Similar general arguments may be applied to a few other fixed points:

$$\lambda_u^{\text{IV}} = \alpha_0/\nu_0 \,,$$
$$\lambda_v^{\text{IV}} = 2\phi_0/\nu_0 - d = 2/\nu_0 - d = \alpha_0/\nu_0 \,, \tag{55}$$
$$\lambda_y^{\text{IV}} = \lambda_y^{\text{II}}(m = 0) > 0$$

($\phi_0 = 1$ to all orders in ϵ^{25}),

$$\lambda_u^{\text{V}} = \lambda_v^{\text{V}} = \lambda_w^{\text{V}} = \alpha_1/\nu_1 \quad , \tag{56}$$

and

TABLE II. Fixed points and exponents (to order ϵ) for random cubic case, $n=0$.

	Fixed point	$4K_4u^*$	$4K_4v^*$	$4K_4w^*$	$4K_4y^*$	Exponents
I.	Gaussian	0	0	0	0	$\lambda_u=\lambda_v=\lambda_w=\lambda_y=\epsilon$
II.	Decoupled m-component	0	$\frac{\epsilon}{m+8}$	0	0	$\lambda_v=-\epsilon,\ \lambda_u=\frac{4-m}{m+8}\epsilon,\ \lambda_w=\frac{m+4}{m+8}\epsilon,\ \lambda_y=\frac{m-4}{m+8}\epsilon$
III.	Isotropic $n=0$	$\epsilon/8$	0	0	0	$\lambda_u=-\epsilon,\ \lambda_v=\lambda_w=\lambda_y=-\epsilon/2$
IV.	Decoupled $n=0$	0	0	$\epsilon/8$	0	$\lambda_u=\lambda_v=\epsilon/2,\ \lambda_w=-\epsilon,\ \lambda_y=-\epsilon/2$
V.	Decoupled Ising	0	0	0	$\epsilon/9$	$\lambda_u=\lambda_v=\lambda_w=\epsilon/3,\ \lambda_y=-\epsilon$
VI.	Mixed $(0,m)$	$\frac{(m-4)\epsilon}{16(m-1)}$	$\frac{\epsilon}{4(m-1)}$	0	0	$\lambda_1=-\epsilon,\ \lambda_2=\lambda_y=\frac{m-4}{4(m-1)}\epsilon,\ \lambda_w=\frac{m+4}{4(m-1)}\epsilon$
VII.	Mixed $(m,0)$	$\epsilon/4$	0	$-\epsilon/4$	0	$\lambda_1=\lambda_v=-\epsilon,\ \lambda_3=\lambda_y=\epsilon$
VIII.	Decoupled m-component cubic	0	$\frac{\epsilon}{3m}$	0	$\frac{m-4}{9m}\epsilon$	$\lambda_u=\lambda_2=\frac{4-m}{3m}\epsilon,\ \lambda_w=\frac{m+4}{3m}\epsilon,\ \lambda_4=-\epsilon$
IX.		$\frac{m-4}{24(m-2)}\epsilon$	$\frac{\epsilon}{6(m-2)}$	0	$\frac{m-4}{18(m-2)}\epsilon$	$\lambda_1=-\epsilon,\ \lambda_2=\frac{m-4}{6(m-2)}\epsilon,\ \lambda_w=\frac{m+4}{6(m-2)}\epsilon,\ \lambda_4=\frac{4-m}{6(m-2)}\epsilon$
X.		$\epsilon/12$	0	$-\epsilon/12$	$\epsilon/9$	$\lambda_1=-\epsilon,\ \lambda_v=\lambda_3=-\epsilon/3,\ \lambda_4=\epsilon/3$
XI. XII. VIII. XIV.	(α_+,β_+) (α_+,β_-) (α_-,β_+) (α_-,β_-)	$\frac{\alpha_\pm\epsilon}{2A_{\pm\pm}}$ [a]	$\frac{\epsilon}{2A_{\pm\pm}}$	$\frac{m+4}{8A_{\pm\pm}}\epsilon$	$\frac{\beta_\pm\epsilon}{2A_{\pm\pm}}$	$\lambda_1=-\epsilon$; for other exponents see text [Eq. (58)]

[a] $\alpha_\pm=[m-4\pm(m^2+48)^{1/2}]/8,\ \beta_\pm=-[m+12\pm(m^2+48)^{1/2}]/6,\ A_{\pm\pm}=6\alpha_\pm+3\beta_\pm+m+6.$

$$\lambda_u^{\mathrm{VIII}}=\alpha_m^c/\nu_m^c\ ,\quad \lambda_w^{\mathrm{VIII}}=2\phi_m^c/\nu_m^c-d\ , \tag{57}$$

where the superscript c denotes the cubic fixed point.[23,26] For the remaining fixed points we must rely on the order-ϵ results of Table II. As in the previous case, the eigenvalues of the last four fixed points were found numerically, by diagonalizing the 4×4 matrix

$$-4K_4v^*\begin{pmatrix} 4\alpha+m-4 & 2(m+2)\alpha+\frac{1}{2}(m+4) & 4\alpha+2 & 6\alpha \\ 12 & m+8 & 4 & 6 \\ 3(m+4) & m+4 & 2(m+4) & \frac{3}{2}m+6 \\ 12\beta & 12\beta+2(m+4) & 12\beta+8 & m+12+12\beta \end{pmatrix}. \tag{58}$$

One eigenvalue is $\lambda_1=-\epsilon$. The other eigenvalues, for $m=3,4$, are summarized in Table III. As before, we find interesting phenomena for m near $m_c(\epsilon)=4+O(\epsilon)$. As $m\to m_c(\epsilon)$, the fixed point XIV diverges to infinity. It is unstable for $m<m_c(\epsilon)$, and stable for $m>m_c(\epsilon)$.

Summing up all the information we have, we again conclude that at $m=2,3$ only the $n=0$ fixed point II is stable, and that the fixed point XIV may be stable at $\epsilon=1$ if $m_c(1)\lesssim3$. Note that the fixed point IX also diverges to infinity as $m\to2$.

If the $n=0$ fixed point may be reached, then a sharp transition with the exponents (43) will be observed. If not, then very probably the transition will become smeared as T_c is approached. The initial signs of u, v, and w suggest that for $m=3$ the initial "runaway" flow may probably be described by the exponents associated with the fixed points II (decoupled m-components), VII, VIII, or X. Linearizing (48), we find

$$1/2\nu=1-2K_4[(mn+2)u^*+(m+2)v^*+(n+2)w^*+3y^*]+O(\epsilon^2)\ . \tag{59}$$

Thus the exponents of VII are Gaussian, those of VIII are cubic,[23] and those of X are Ising-like. Higher-order terms in the ϵ expansion may change signs of other fixed-point values, make $m_c(1)$ smaller, etc.

VII. CONCLUSIONS

We have demonstrated that the introduction of a uniaxial anisotropy with a random direction *destabilizes* the fixed point describing the nonrandom m-component critical behavior. This result is to be contrasted with the case of an isotropic random

TABLE III. Numerical results for cubic case.

m	Fixed point	$4K_4u^*$	$4K_4v^*$	$4K_4w^*$	$4K_4y^*$	$\lambda_2, \lambda_3, \lambda_4$
3	XI	0.155ϵ	0.190ϵ	0.332ϵ	-0.712ϵ	$-1.33\epsilon, 1.43\epsilon, 1.43\epsilon$
	XII	0.0402ϵ	0.0491ϵ	0.0859ϵ	-0.0609ϵ	$-0.371\epsilon, 0.371\epsilon, -0.344\epsilon$
	XIII	0.0615ϵ	-0.0576ϵ	-0.101ϵ	0.216ϵ	$-0.435\epsilon, 0.435\epsilon, 0.403\epsilon$
	XIV	0.470ϵ	-0.440ϵ	-0.770ϵ	0.546ϵ	$3.32\epsilon, 3.32\epsilon, 3.08\epsilon$
4	XI	$\epsilon/8$	$\epsilon/8$	$\epsilon/4$	$-\epsilon/2$	$\epsilon, \epsilon, -\epsilon$
	XII	0.0417ϵ	0.0417ϵ	0.0833ϵ	-0.0556ϵ	$-\epsilon/3, -\epsilon/3, \epsilon/3$
	XIII	$\epsilon/16$	$-\epsilon/16$	$-\epsilon/8$	$\epsilon/4$	$-\epsilon/2, \epsilon/2, \epsilon/2$
	XIV	$\pm\infty$[a]	$\mp\infty$	$\pm\infty$	$\mp\infty$,	$\pm\infty, \pm\infty, \pm\infty$

[a]The signs are for $m \to 4^{\pm}$.

single-ion interaction, in which the nonrandom fixed point is *stable* for $m > 1$. In that case, a heuristic argument due to Harris[27] shows that one expects a sharp transition only if the specific heat exponent α_m is negative. One can probably construct similar arguments for the present case, with regards to the new crossover exponent $\nu_m \lambda_w^{II} = 2(\phi_m - 1) + \alpha_m$ [see (35)].

We thus expect a *crossover* from the nonrandom m-component behavior to some *new* behavior. There are two possibilities: either the Hamiltonian flow leads to the $n=0$ fixed point, and the transition is *sharp*; or it runs away, out of the range in which our approximations are valid. The experience with the isotropic case[19] leads us to believe that the second possibility is more probable. In that case, the actual asymptotic nature of the transition still remains to be investigated. However, the same experience[19] indicates that the effective Hamiltonian probably never leads to a first-order transition, and since one finds no stable fixed point to associate with a sharp transition, one concludes that the transition is "*smeared.*"[11] The exact meaning of this is still not fully understood, and is left for future interpretations. The rounded nature of the transition observed in $TbFe_2$[2] supports this conclusion.

Another interesting difference from the random isotropic case has to do with the sign of the quartic parameter u, which is now positive. An interesting generalization of the present work would involve a combination of both types of randomness; it is quite reasonable to assume that the pair distances $\{|\vec{R}_i - \vec{R}_j|\}$ are also random, so that J_{ij} in Eq. (1) should also be considered as a random variable. If the two types of randomness were independent, we would end up with the same effective Hamiltonian, except for the sign of u (which would now become a sum of a negative and a positive term and thus will possibly be of either sign). This would enlarge the possible range of Hamiltonian flows. A more realistic model will have to correlate the randomness in J_{ij} to that in D. This remains to be studied.

There are many other possible generalizations: the magnitude of D may be random, the sign of D may be random (leading to competing ferromagnetic and antiferromagnetic interactions and to multicritical points[25]), more "realistic" distribution functions $p(\hat{x})$ may be considered, long range exchange interactions can be added,[7] amorphous alloys of various metals, with different magnetic properties, may be studied, etc. All these remain for future investigations.

Note added in proof: In the case $m=1$, $n=0$, the Hamiltonian flow is not to infinity, but rather to a fixed point with u and v of order $\sqrt{\epsilon}$ [D. E. Khmelnitzkii, (report of work prior to publication)]. Thus the transition in the random Ising model is probably *sharp* (and *not smeared*). The same may happen for the fixed point VI (Table I) for $m \simeq 3.9$. However, this probably does not change our conclusions as regards the smeared nature of the transition in our case.

ACKNOWLEDGMENTS

The author thanks Alan H. Luther for raising his interest in this problem, and for several useful discussions. A. H. Luther, V. J. Emery, and T. C. Lubensky kindly supplied reports of their works prior to publication. The hospitality of P. C. Martin and H. Ehrenreich at Harvard and of S.-K. Ma at UCSD is also gratefully acknowledged.

*Work supported in part by the National Science Foundation under Grant No. GH-32774 (at Harvard) and Grant No. GP-38627X (at UCSD).

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