Soliton capacitors in a Cooper-pair box transmission line

Toshiyuki Fujii
Department of Physics, Asahikawa Medical University, 2-1-1-1 Midorigaoka-Higasi, Asahikawa 078-8302, Japan

Manami Nishimura
School of Education, Hiroshima University, 1-1-1 Kagamiyama, Higashihiroshima, Hiroshima 739-8524, Japan

Noriyuki Hatakenaka
Graduate School of Advanced Science and Engineering, Hiroshima University, 1-7-1 Kagamiyama, Higashihiroshima, Hiroshima 739-8521, Japan

(Received 5 February 2025; revised 13 April 2025; accepted 7 May 2025; published 19 May 2025)

We propose a superconducting circuit that supports the propagation of Korteweg–de Vries (KdV)-type voltage solitons. By employing Cooper-pair boxes as nonlinear capacitors, we realize a dual system to the previously reported current solitons based on nonlinear inductors. The proposed voltage solitons manifest as propagating capacitors, carrying a velocity-dependent induced charge. This unique characteristic facilitates local electrostatic interactions, which are dual to the magnetic interactions mediated by fluxons.

DOI: 10.1103/PhysRevB.111.174522

I. INTRODUCTION

Nonlinearity confers a fascinating property of nature when considering cases of large amplitude in many fields ranging from biology to the universe. One remarkable example is a solitary wave called a soliton. This occurs when this nonlinearity is balanced with dispersion in the nonlinear dispersive media, resulting in a stable propagation without changing its shape [1]. This is responsible for stable information communication, as demonstrated by the successful implementation of optical solitons in fiber optic communication systems [2]. Solitons also provide valuable insights into complex physical phenomena.

In electronics, nonlinear transmission lines with an LC circuit consisting of inductors L and capacitors C as basic elements serve as nonlinear dispersive media that support electrical solitons. In fact, *voltage solitons* described by the Korteweg–de Vries (KdV) equation have been experimentally confirmed in nonlinear transmission lines with *nonlinear capacitors* [3], validating fundamental soliton theory [4]. On the other hand, in the superconducting circuit, *flux (current-based) solitons* [5,6] dual to voltage solitons have been devised based on nonlinear Josephson inductance as the source of nonlinearity necessary for soliton formation. However, the realization of voltage solitons in superconducting tunnel devices has remained unexplored due to the challenge of implementing nonlinear capacitors. In a built-in tunnel junction regarded as a parallel plate capacitor, in general, its capacitance cannot be controlled as desired after it is assembled since it depends on its shape and dielectric constant. In semiconducting transmission lines, the width of the depletion layer varies with the applied voltage. This modulates the effective capacitance, thereby overcoming the challenge of implementing nonlinear capacitors in LC transmission lines [7]. However, this depletion layer mechanism is not applicable to superconducting tunnel junctions, in which capacitance and nonlinearity are governed by distinct quantum mechanical phenomena.

While current solitons have traditionally played a dominant role in signal propagation within Josephson transmission lines, recent developments in nonlinear capacitance technology [8] have suggested the feasibility of voltage solitons. These voltage solitons offer several advantages over their current-based counterparts. Specifically, they exhibit increased immunity to external magnetic fields and common-mode noise, while also enabling high-speed switching through voltage control mechanisms. Furthermore, their primary utilization of electrostatic energy significantly reduces the Joule heating losses that typically plague current solitons. The inherent properties of nonlinear capacitance contribute to maintaining waveform integrity, resulting in relatively low signal attenuation over transmission distances. These advantages are particularly relevant for applications requiring high-density circuit integration, high-speed operation, and low-power consumption.

In this paper, we propose voltage solitons in superconducting circuits using Cooper-pair boxes acting as nonlinear capacitors, demonstrating that the propagating soliton carries an induced charge dependent on its velocity. A single Cooper-pair box consists of a small superconducting island that is connected to a bias voltage source V_b via a small capacitor C_b on one side and via a small Josephson junction with capacitance C_J and Josephson coupling energy E_J on the other side as shown in Fig. 1. In this configuration, the number of Cooper pairs on the island is quantized under specific bias voltage conditions. Therefore, the Cooper-pair box was devised initially as a qubit based on charge states dual to flux, i.e., a charge qubit [9–11]. Here, we use the Cooper-pair box

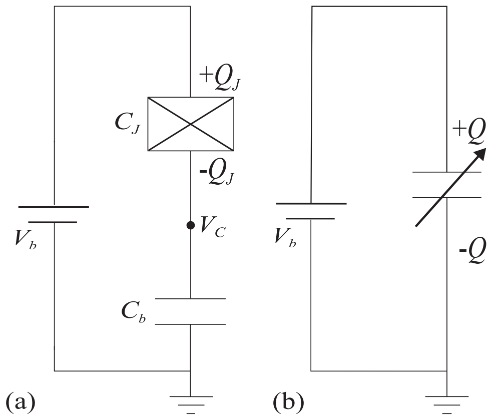

FIG. 1. (a) Cooper-pair box. C_J and C_b are the capacitance of the superconducting small junction and coupling capacitance, respectively. V_b is the bias voltage, V_c is the island potential, and Q_J is the charge stored in the junction capacitance. (b) The equivalent effective variable capacitor.

as a nonlinear capacitor [12]. Although the Cooper-pair box is a built-in device, the effective dielectric constant of the Cooper-pair box can be controlled by the bias voltage without changing its shape, which makes it possible to change the capacitance.

II. COOPER-PAIR BOX AS A NONLINEAR CAPACITOR

Let us briefly review the effective capacitance C_{eff} in the Cooper-pair box [12]. The capacitance is defined as $C_{\text{eff}} = \delta\langle Q\rangle/\delta V_b$, where $\langle Q\rangle$ is the mean value of the electric charge transferred from the battery V_b. The effective capacitance is expressed as

$$C_{\text{eff}}(V_b) = \frac{\delta\langle Q\rangle}{\delta V_b} = \frac{C_b}{C_b + C_J}\left(C_J - 2|e|\frac{\partial\langle n\rangle}{\partial V_b}\right), \quad (1)$$

where e is the elementary electric charge. The average number of excess Cooper pairs in the box $\langle n\rangle$ is given as [10]

$$\langle n\rangle = \frac{1}{2}\left[1 + \frac{\eta}{\sqrt{1+\eta^2}}\tanh\kappa\sqrt{1+\eta^2}\right], \quad (2)$$

with

$$\eta = \frac{E_c}{E_J}(2n_b - 1), \quad (3)$$

$$\kappa = \frac{\beta E_J}{2}, \quad (4)$$

where E_c and E_J are the charging energy and Josephson coupling energy, respectively. n_b is dimensionless bias voltage expressed by $n_b = C_b V_b/(2|e|)$ and $\beta = 1/k_B T$ with k_B and T being the Boltzmann constant and temperature, respectively. The average number of Cooper pairs in the box depends nonlinearly on the bias voltage (n_b). This voltage-dependent $\langle n\rangle$ gives a nonlinear voltage dependence to the C_{eff} required for soliton formation. Here, let us assume $T = 0$ for simplicity. Equation (2) reduces to

$$\langle n\rangle = \frac{1}{2}\left[1 + \frac{\eta}{\sqrt{1+\eta^2}}\right]. \quad (5)$$

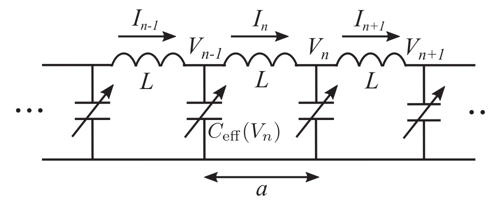

FIG. 2. Nonlinear LC transmission line consisting of constant inductance L and nonlinear capacitance depending on voltage $C_{\text{eff}}(V_n)$. The unit cell length is denoted by a. The current and the voltage on the nth unit cell are I_n and V_n, respectively.

Substituting Eq. (5) into Eq. (1) and expanding as a power series around the $n_b = 0$, we obtain

$$C_{\text{eff}}(V_b) = C_0(1 - \gamma V_b) + O(V_b^2), \quad (6)$$

where

$$C_0 = \frac{C_b}{C_b + C_J}\left\{C_J - C_b\frac{\frac{E_c}{E_J}}{\left[1 + \left(\frac{E_c}{E_J}\right)^2\right]^{3/2}}\right\}, \quad (7)$$

$$\gamma = C_0^{-1}\frac{3C_b^3}{|e|(C_b + C_J)}\frac{\left(\frac{E_c}{E_J}\right)^3}{\left[1 + \left(\frac{E_c}{E_J}\right)^2\right]^{5/2}}. \quad (8)$$

Typical parameters [8] for the Cooper-pair box are $C_b \sim C_J \sim 1$ fF, $E_J \sim 6.6 \times 10^{-25}$ J, and $E_C \sim 10E_J$. Then, the order of magnitude of the capacitance C_0 is estimated to be 1 fF. The parameter γ is expected to be 100 V^{-1}. As indicated by Eq. (6), the capacitance possesses a negative nonlinear term, which stems from the second term in Eq. (1) and is caused by Cooper-pair tunneling across the junction [12]. Despite its negative nature, this term does not lead to a negative effective capacitance, $C_{\text{eff}}(V_b)$, according to (1), provided that $(E_c/E_J) \gg 1$.

III. TRANSMISSION LINES WITH NONLINEAR CAPACITORS

Let us consider the propagation of electromagnetic waves in the nonlinear transmission lines shown in Fig. 2. The fundamental elements of electric circuits are the LC circuits consisting of inductors and voltage-dependent nonlinear capacitors. The circuit equation is derived as follows. From Faraday's law, the voltage applied to the nth inductor with inductance L is derived by

$$V_n - V_{n-1} = -L\frac{dI_n}{dt}, \quad (9)$$

where V_n is the voltage applied to the nth capacitor and I_n is the current flowing through the nth inductor. We obtain

$$\frac{d}{dt}(I_{n+1} - I_n) = -\frac{1}{L}(V_{n+1} - 2V_n + V_{n-1}) \quad (10)$$

by taking the difference with the equation for the $(n+1)$th inductor. From Kirchhoff's law, we obtain

$$I_{n+1} - I_n = -\frac{d}{dt}[C_{\text{eff}}(V_n)V_n]. \quad (11)$$

By substituting Eq. (11) for Eq. (10), the circuit equation is then obtained as

$$\frac{d^2}{dt^2}[C_{\mathrm{eff}}(V_n)V_n] = \frac{1}{L}(V_{n+1} - 2V_n + V_{n-1}). \tag{12}$$

This equation is equivalent to the equation for nonlinear lattice vibration with an amplitude-dependent mass since $C_{\mathrm{eff}}(V_n)$ effectively modifies the system's inertia. This reduces to a conventional linear wave equation when $C_{\mathrm{eff}}(V_n)$ is constant.

IV. KORTEWEG–DE VRIES EQUATION

Now, let us find the nonlinear waves hidden in our circuit using the reductive perturbation method [13], which allows us to extract the stable waves balancing the nonlinearity and the dispersion from the circuit equation (12).

A. Continuum approximation

To transition from the discrete equation to a continuous description, we assume that the voltage V_n varies smoothly across lattice sites. Defining the continuous field $V(x,t)$ such that $V_n(t) \approx V(x,t)$ with $x = na$ (where a is the lattice spacing), we expand the discrete terms in a Taylor series:

$$\begin{aligned} V_{n\pm1} &\approx V(x \pm a, t) \\ &= V(x,t) \pm a\frac{\partial V}{\partial x} + \frac{a^2}{2}\frac{\partial^2 V}{\partial x^2} \pm \frac{a^3}{6}\frac{\partial^3 V}{\partial x^3} + \frac{a^4}{24}\frac{\partial^4 V}{\partial x^4} + \cdots . \end{aligned} \tag{13}$$

Substituting these expansions into the discrete Laplacian $V_{n+1} - 2V_n + V_{n-1}$, we find

$$V_{n+1} - 2V_n + V_{n-1} \approx a^2\frac{\partial^2 V}{\partial x^2} + \frac{a^4}{12}\frac{\partial^4 V}{\partial x^4} + O(a^6). \tag{14}$$

The capacitance term $C_{\mathrm{eff}}(V_n)$is expanded as $C_{\mathrm{eff}}(V) = C_0(1 - \gamma V)$, where γ is a parameter describing the nonlinearity of the capacitance. Using this in the original equation and applying the continuum approximation, we obtain

$$\frac{\partial^2}{\partial t^2}[C_0(1-\gamma V)V] = \frac{a^2}{L}\frac{\partial^2 V}{\partial x^2} + \frac{a^4}{12L}\frac{\partial^4 V}{\partial x^4}. \tag{15}$$

B. Reductive perturbation method

To account for weak nonlinearity and dispersion, we introduce a small parameter $\epsilon \ll 1$ and scale the variables as follows:

$$x' = \epsilon^{1/2}(x - ct), \tag{16}$$

$$t' = \epsilon^{3/2}t, \tag{17}$$

$$V(x,t) = \epsilon V_1(x',t') + \epsilon^2 V_2(x',t') + \cdots . \tag{18}$$

Here, c is the wave speed to be determined. Substituting these expressions into the continuum equation and collecting terms at each order of ϵ, we derive equations for V_1, V_2, and higher-order terms.

1. Zeroth-order equation

At $O(\epsilon^2)$, we find the leading-order equation

$$C_0c^2\frac{\partial^2 V_1}{\partial x'^2} = \frac{a^2}{L}\frac{\partial^2 V_1}{\partial x'^2}. \tag{19}$$

Balancing terms gives the wave speed $c = a/\sqrt{LC_0}$.

2. First-order equation

At $\mathcal{O}(\epsilon^3)$, the first-order equation becomes

$$-2C_0c\frac{\partial^2 V_1}{\partial x'\partial t'} - C_0\gamma c^2\frac{\partial^2(V_1^2)}{\partial x'^2} = \frac{a^4}{12L}\frac{\partial^4 V_1}{\partial x'^4}. \tag{20}$$

Integrating once with respect to ξ and simplifying, we obtain

$$\frac{\partial V_1}{\partial t'} + \frac{\gamma c}{2}\frac{\partial(V_1^2)}{\partial x'} + \frac{a^2 c}{24}\frac{\partial^3 V_1}{\partial x'^3} = 0. \tag{21}$$

Introducing the scaled nondimensional variables $u = \alpha V_1$, $x' = \beta\xi$, $t' = \delta\tau$ with $\alpha = \gamma$, $\beta = a/2$, and $\delta = (3a/2c)$ such that the coefficients of the nonlinear and dispersive terms become unity, we arrive at the standard form of the KdV equation:

$$\frac{\partial u}{\partial \tau} + 6u\frac{\partial u}{\partial \xi} + \frac{\partial^3 u}{\partial \xi^3} = 0. \tag{22}$$

V. VOLTAGE SOLITON AS A MOVING CAPACITOR

It is important to note that voltage solitons are accompanied by an electric charge $Q = CV$ and transport this charge during their propagation. The one-soliton solution of the KdV equation (22) is given in terms of soliton velocity v by

$$u_{\mathrm{sol}} = \frac{v}{2}\mathrm{sech}^2\left(\frac{\sqrt{v}}{2}(\xi - v\tau)\right), \tag{23}$$

where v is the relative velocity normalized by the wave speed c in the circuit in $\xi - \tau$ coordinates. Here, we provide a comment on the treatment of ϵ. Reverting Eq. (23) to the original variables, the voltage propagation is expressed as

$$V = \frac{\epsilon v}{2\gamma}\mathrm{sech}^2\left\{\frac{\sqrt{\epsilon v}}{a}\left[x - c\left(1 + \frac{\epsilon v}{3}\right)t\right]\right\}. \tag{24}$$

The perturbation parameter ϵ always appears with the relative velocity parameter v in the form ϵv. Since v is arbitrary, redefining $\tilde{v} = \epsilon v$ is effectively equivalent to setting $\epsilon = 1$ [14].

It is important to recall that one of the key objectives of the reductive perturbation method is to identify appropriate timescales and spatial scales where nonlinearity and dispersion are balanced, which is achieved by selecting a suitable order for the perturbation parameter ϵ. In the resulting nonlinear evolution equations, such as the KdV equation, the effects of this small parameter are often absorbed into the scaled variables and parameters of the equation, and itself does not explicitly appear.

Note that due to the special properties of the KdV soliton, the quantities corresponding to the amplitude and wave number are expressed in terms of the soliton velocity. By applying the continuum approximation to the capacitance, $C(x) = C_{\mathrm{eff}}(V_n)a$, the charge associated with this soliton

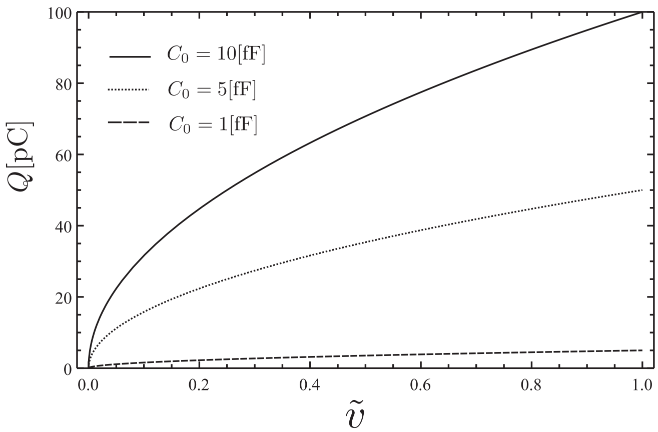

FIG. 3. Charge Q associated with voltage solitons as a function of $\tilde{v}$ at different capacitances C_0.

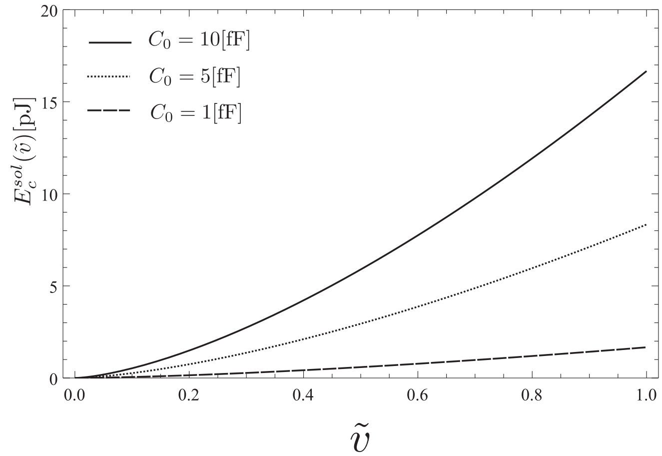

FIG. 4. Electrostatic energy of the voltage soliton $E_c^{\mathrm{sol}}(\tilde{v})$ as a function of soliton velocity $\tilde{v}$ for different capacitances C_0.

can be obtained based on the sectional quadrature method $Q = \sum_n C_{\mathrm{eff}}(V_n)V_n$ as

$$\begin{aligned} Q &= \frac{1}{a}\int C[V(x)]V(x)dx \\ &= \frac{1}{a}C_0\int V(x)dx - \frac{1}{a}C_0\gamma\int V(x)^2dx \\ &= \epsilon^{1/2}\frac{1}{\gamma}C_0\int u_{\mathrm{sol}}d\xi + O(\epsilon^{3/2}) \\ &= \epsilon^{1/2}\frac{C_0}{\gamma}\sqrt{v} + O(\epsilon^{3/2}) \\ &= \frac{C_0}{\gamma}\sqrt{\tilde{v}}, \end{aligned} \tag{25}$$

where higher-order infinitesimals associated with the order of perturbation are ignored.

The induced charge is proportional to the square root of the soliton velocity. This arises from the voltage being proportional to the soliton area, which is the product of its height ($\tilde{v}$) and its width ($1/\sqrt{\tilde{v}}$), both determined by the soliton velocity. Consequently, the amount of charge associated with a soliton can be controlled by manipulating its velocity (see Fig. 3). This unique characteristic establishes the voltage soliton as another form of a *moving capacitor*, offering a different avenue for charge transport.

The electrostatic energy when the soliton is considered as a capacitor is given as

$$\begin{aligned} E_c^{\mathrm{sol}}(\tilde{v}) &= \frac{C_0}{2a}\int_{-\infty}^{\infty} V(x)^2dx \\ &= \frac{C_0}{4\gamma^2}\int_{-\infty}^{\infty} u_{\mathrm{sol}}(\xi,\tau)^2d\xi \\ &= \frac{C_0}{6\gamma^2}\tilde{v}^{3/2} \equiv \frac{E_c^{\mathrm{sol}}}{6}\tilde{v}^{3/2}. \end{aligned} \tag{26}$$

Figure 4 shows the electrostatic energy of the soliton $E_c^{\mathrm{sol}}(\tilde{v})$ as a function of $\tilde{v}$ for various representative values of C_0.

VI. QUANTIZATION OF SOLITON CHARGE

Finally, let us consider a ring-shaped transmission line and investigate the quantization of effective charge due to the quantization of soliton velocity. This quantization originates from the quantization of phase difference across the junction. Owing to the fact that the transmission line is made of a superconductor, the voltage and phase difference are linked by Josephson's relation. Therefore, the phase difference across the junction is described as

$$\begin{aligned} \phi &= \frac{2|e|}{\hbar}\int V(x,t)dt \\ &= \frac{3a|e|}{\gamma c\hbar}\int u_{\mathrm{sol}}(\xi,\tau)d\tau \\ &= -\frac{3E_c^{\mathrm{sol}}}{\hbar\omega}\frac{|e|}{Q}\tanh\left(\frac{\sqrt{v}}{2}(\xi - \tilde{v}\tau)\right), \end{aligned} \tag{27}$$

where we introduce the characteristic frequency defined as $\omega \equiv c/a = (C_0L)^{-1/2}$. In a ring transmission line, this phase change should be an integer multiple of 2π,

$$\begin{aligned} \oint \frac{\partial\phi}{\partial x}dx &= \oint \frac{\partial\phi}{\partial\xi}d\xi \\ &= \phi(\xi = l/2,\tau) - \phi(\xi = -l/2,\tau) \\ &\simeq \phi(\xi = \infty,\tau) - \phi(\xi = -\infty,\tau) \\ &= \frac{6E_c^{\mathrm{sol}}}{\hbar\omega}\frac{|e|}{Q} = 2\pi n, \end{aligned} \tag{28}$$

where l is the nondimensional perimeter. When this quantization is satisfied, the charge carried by the soliton is given by

$$Q_n = \frac{3}{\pi}\frac{E_c^{\mathrm{sol}}}{\hbar\omega}\frac{|e|}{n}. \tag{29}$$

Moreover, the soliton velocity at that time is given by

$$\tilde{v} = \left(\frac{3}{\pi\gamma\hbar\omega}\frac{|e|}{n}\right)^2 = \left(\frac{\gamma}{C_0}Q_n\right)^2. \tag{30}$$

One potential application lies in exploiting the mobile local electrostatic interactions associated with soliton motion for local manipulations of quantum devices. This approach provides quantum manipulation that is dual to the use of local magnetic interaction of the fluxon for device switching.

VII. CONCLUDING REMARK

Voltage solitons in superconducting circuits with Cooper-pair boxes acting as nonlinear capacitors have been proposed. This is dual to current solitons based on nonlinear Josephson inductance in the preceding studies. The resulting voltage solitons function as mobile capacitors, with their induced charge scaling with velocity. This property allows for local electrostatic interactions, providing a dual counterpart to the magnetic interactions associated with fluxons. In addition, the capacitance-derived nonlinearity originates from the charging energy based on the Coulomb forces so it is expected to have stronger nonlinearity than the inductance-derived nonlinearity based on the coupling between superconductors. Therefore, the proposed voltage solitons can contribute to improving the performance of the Josephson traveling wave parametric amplifier, which has been the subject of much research recently, due to its locally enhanced nonlinearity. Our findings open up different avenues for exploring nonlinear wave phenomena in superconducting circuits and may have implications for quantum information processing.

ACKNOWLEDGMENTS

We thank Tatsuya Honma and Katsuhiko Inagaki for very fruitful discussions and continuous encouragement. We also thank Yukie Matsumoto and Ai Maeda for their technical support. This work was supported in part by JSPS KAKENHI Grants No. 21K03389 and No. 22K03452.

DATA AVAILABILITY

No data were created or analyzed in this study.

[1] N. J. Zabusky and M. D. Kruskal, Interaction of "solitons" in a collisionless plasma and the recurrence of initial states, Phys. Rev. Lett. **15**, 240 (1965).
[2] A. Hasegawa and F. Tappert, Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers. I. Anomalous dispersion, Appl. Phys. Lett. **23**, 142 (1973).
[3] R. Hirota and K. Suzuki, Theoretical and experimental studies of lattice solitons in nonlinear lumped networks, Proc. IEEE **61**, 1483 (1973).
[4] M. Toda, Vibration of a chain with nonlinear interaction, J. Phys. Soc. Jpn. **22**, 431 (1967).
[5] A. Matsuda and T. Kawakami, Fluxon propagation on a Josephson transmission line, Phys. Rev. Lett. **51**, 694 (1983).
[6] H. R. Mohebbi and A. H. Majedi, Shock wave generation and cut off condition in nonlinear series connected discrete Josephson transmission line, IEEE Trans. Appl. Supercond. **19**, 891 (2009).
[7] R. Hirota and K. Suzuki, Studies on lattice solitons by using electrical networks, J. Phys. Soc. Jpn. **28**, 1366 (1970).
[8] T. Duty, G. Johansson, K. Bladh, D. Gunnarsson, C. Wilson, and P. Delsing, Observation of quantum capacitance in the Cooper-pair transistor, Phys. Rev. Lett. **95**, 206807 (2005).
[9] A. Shnirman, G. Schön, and Z. Hermon, Quantum manipulations of small Josephson junctions, Phys. Rev. Lett. **79**, 2371 (1997).
[10] V. Bouchiat, D. Vion, P. Joyez, D. Esteve, and M. H. Devoret, Quantum coherence with a single Cooper pair, Phys. Scr. **T76**, 165 (1998).
[11] Y. Nakamura, Y. A. Pashkin, and J. S. Tsai, Coherent control of macroscopic quantum states in a single-Cooper-pair box, Nature (London) **398**, 786 (1999).
[12] H. Paik, F. Strauch, R. Ramos, A. Berkley, H. Xu, S. Dutta, P. Johnson, A. Dragt, J. Anderson, C. Lobb, and F. Wellstood, Cooper-pair box as a variable capacitor, IEEE Trans. Appl. Supercond. **15**, 884 (2005).
[13] T. Taniuti and N. Yajima, Perturbation method for a nonlinear wave modulation. I, J. Math. Phys. **10**, 1369 (1969).
[14] T. Kuusela, Soliton experiments in transmission lines, Chaos Solit. Fractals **5**, 2419 (1995).