

Sagnac effect in a rotating ring with Dirac fermions

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The observation of the Sagnac effect for massive material particles offers a significant enhancement in sensitivity when compared to optical interferometers with equal area and angular rotation velocity. For this reason, there have been suggestions to employ solid-state interferometers that rely on semiconductors and graphene. We investigate the Sagnac effect in Dirac materials governed by the relativisticlike quasiparticle dispersion law and show that the fringe shift is still determined by the mass of a free electron. This confirms that graphene is indeed a promising material for creating solid-state Sagnac interferometers. Considering monolayer graphene with its linear dispersion law and comparing it with light provides a deeper understanding of the Sagnac effect.

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Introduction. Physical phenomena associated with rotation possess a captivating allure that spans multiple levels. One of their fundamental illustrations lies in the impossibility of establishing a standard clock synchronization procedure along a closed curve in a rotating material (see Refs. [1–3]). One of the consequences of this impossibility is the renowned Sagnac effect, which refers to the phenomenon where a phase shift is observed between two coherent beams that travel along opposite paths within an interferometer situated on a rotating disk (see Refs. [4–7] for reviews). This phase shift, which was first demonstrated for light by Sagnac in [8,9], is an intrinsically relativistic effect. It is not restricted to light waves, but is observed for electron waves in vacuum [10], neutrons [11], and atoms [12] (see also the recent work in [13]). Moreover, the observation of the Sagnac phase shift for massive particles in solids, specifically superconducting Cooper pairs, dates back to as early as the work in [14].

While practical applications of the Sagnac effect currently rely on light waves, there is a compelling physical reason why massive particles are significantly more advantageous for its realization. The Sagnac fringe shift, denoted by Θ_S , with respect to the fringe position for the stationary interferometer reads [6,10,15]

$$\Theta_S = \frac{4EA\Omega}{\hbar c^2}. \quad (1)$$

This formula is applicable to waves comprising both massless and massive particles, with E being the total energy of a corresponding particle. The quantity A denotes the area of the enclosure, formed by the light or particle beams in the interferometer, orthogonally projected to the plane of rotation, Ω is the angular velocity of rotation of the interferometer

in the inertial frame, \hbar is the reduced Planck constant, and c is the vacuum speed of light. This equation neglects a small relativistic correction (discussed below). Substituting in Eq. (1) the energy $E = \hbar\omega$, where ω is either the frequency of light or the frequency of the de Broglie wave of a material particle, we recover the standard formula for the Sagnac phase shift [4–7,10]

$$\Theta_S = \frac{4\omega A\Omega}{c^2}. \quad (2)$$

When considering light in vacuum and using the dispersion relation $\omega = 2\pi c/\lambda$, where λ denotes the wavelength, we arrive at another commonly used form for the Sagnac phase shift [4–6],

$$\Theta_S = \frac{8\pi A\Omega}{c\lambda}. \quad (3)$$

In the nonrelativistic case of slow massive particles, the energy $E = mc^2$ is associated with their rest mass m and the phase fringe takes the form

$$\Theta_S = \frac{4mA\Omega}{\hbar}. \quad (4)$$

Comparing the phase shift for the matter-wave and optical interferometers, we find that, for the same (projected) area and angular velocity, the phase shift is enhanced by a factor [16] $mc^2/\hbar\omega = \omega_0/\omega$, where $\omega_0 = mc^2/\hbar$ is the de Broglie frequency mentioned above. Since the frequency ω_0 is not a commonly used quantity, it is convenient to express Eq. (4) for electrons similarly to Eq. (3), with the photon wavelength λ replaced by the Compton wavelength $\lambda_C = 2\pi\hbar/m_e c \approx 0.0243 \text{ \AA}$, where m_e is the electron mass. This representation simplifies the comparison between interferometers using light and matter and having the same area and angular velocity.

For atoms, the matter-wave interferometer is significantly more sensitive to rotation, with the enhancement factor

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relative to light reaching the value of 10^{10} . While the existing optical gyroscopes necessitate the utilization of either several kilometers of optical fiber or a substantial area to reach the necessary sensitivity, its enhancement for matter waves has led to proposals to utilize cold-atom interferometers in the search for smaller signals beyond the Earth's rotation [13].

For free electrons, the enhancement factor reaches the value of 10^6 , and, as mentioned earlier, the Sagnac effect was observed using an electron interferometer in vacuum [10]. The possibility of realizing the Sagnac effect in the solid state by employing a serial array of mesoscopic ring-shaped electron interferometers was discussed in Refs. [17–20] (see also Ref. [21] for discussion of the Sagnac interferometry in carbon nanotubes).

As highlighted in [19,20], graphene emerges as a promising material for electron interferometry, owing to its extraordinary electronic properties. Recent experiments on the Aharonov-Bohm oscillations in ring-shaped gated bilayer graphene [22] and magic-angle twisted bilayer graphene [23] provide further confirmation of this potential (see also the review in [24]).

However, monolayer graphene belongs to a new class of Dirac materials with zero effective carrier mass and a linear dispersion relation similar to that of light. This unique behavior of charged carriers in graphene often prompts a reassessment of many fundamental principles of physics, leading to their deeper understanding. Thus, at first glance, due to the assertion discussed in Ref. [6] that the phase shift remains independent of the wave's phase velocity, the Sagnac effect in graphene might appear analogous to the case of light. Another issue is that the results discussed above pertain to massive particles in vacuum rather than inside solids.

The objective of the present work is to examine the Sagnac effect in graphene. More broadly, the goal is to investigate the Sagnac effect in relativistic Dirac materials and to clarify its distinctions from the previously known cases.

Model. We consider the Dirac materials characterized by the dispersion relation

$$E(\mathbf{k}) = \pm\sqrt{\hbar^2 v^2 k^2 + \Delta^2} + \mathcal{E}_D, \quad (5)$$

where \mathbf{k} represents the wave vector counted from the Dirac point, which can be either two- or three-dimensional, and the signs correspond to energy bands above and below this point. Below we will be examining a narrow ring of graphene on the plane geometry; in this case, the z component of \mathbf{k} can effectively be disregarded by setting it to zero. The quantity v in Eq. (5) denotes the Fermi velocity, and Δ is the gap in the quasiparticle spectrum.

The energy \mathcal{E}_D determines the energy position of the Dirac point, $E(\mathbf{k} = 0)$, with respect to a chosen reference point. For example, when considering the work function, the reference point is taken to be the local vacuum energy (see, e.g., Fig. 1 in [25]). As we will see below, in the case of the Sagnac effect, it is crucial to include the electron rest energy $m_e c^2$ in \mathcal{E}_D .

The gap term Δ is present, in particular, in the Hamiltonian derived by Wolf (see Ref. [26] for a review) for Bi and in similar effective Hamiltonians describing other three-dimensional (3D) Dirac materials. It can also be induced in a graphene monolayer by placing it on top of hexagonal boron nitride. As

already mentioned, the quasirelativistic spectrum (5) follows, for example, from the Wolf Hamiltonian as well as from other effective low-energy 2D or 3D Dirac Hamiltonians used to describe Dirac materials (for graphene, see, e.g., the review in [27]).

To focus on the Sagnac effect for quasiparticles with a relativisticlike dispersion, we restrict ourselves to considering the squared Dirac Hamiltonians neglecting the coupling between pseudospin and spin degrees of freedom with the rotation of the frame [28]. Thus, we assume that a free quasiparticle in the electron subsystem of the Dirac material in the inertial rest frame (t' , \mathbf{r}') of the material obeys the following wave equation:

$$\left(\hbar \frac{\partial}{\partial t'} + i\mathcal{E}_D\right)^2 \psi - v^2 \hbar^2 \nabla_{\mathbf{r}'}^2 \psi + \Delta^2 \psi = 0. \quad (6)$$

Here $\psi(t', \mathbf{r}')$ is the electron envelope wave function, which can be treated as a scalar. Seeking a solution of Eq. (6) in the form $\psi \sim \exp(-iEt'/\hbar + i\mathbf{k}\mathbf{r}')$, we reproduce the spectrum (5). After relating the energy gap Δ to the mass, $m = \Delta/v^2$, and setting $v = c$ and $\mathcal{E}_D = 0$, Eq. (6) reduces to the usual relativistic Klein-Gordon-Fock (KGF) equation.

Sagnac effect from the KGF equation in the rotating frame.

Before proceeding with the general analysis, it is instructive, for an introspection, to recapitulate a simple derivation of Eq. (1) or (2) and Eq. (4) presented in Ref. [15]. Essentially, it can be regarded as an extension of the explanation of the Sagnac effect proposed by Langevin in the framework of general relativity in [29] (see also Refs. [4,7] for reviews as well as [1,2]) to the case of relativistic wave equations.

The transformation from the primed rest frame to a non-primed frame rotating about the z axis with angular velocity Ω , in cylindrical coordinates, reads

$$t' = t, \quad r' = r, \quad \phi' = \phi + \Omega t. \quad (7)$$

The KGF equation in the rotating frame was obtained in [15] and it amounts to the replacement $\partial_{t'} \rightarrow \partial_t - \mathbf{V} \cdot \nabla_{\mathbf{r}}$, where $\mathbf{V} = \boldsymbol{\Omega} \times \mathbf{r}$ is the local rotation velocity.

We consider wave signals, labeled by \pm , propagating in the counterclockwise (in the angular direction ϕ) and clockwise directions, respectively, around a circle of radius R . These signals are described by solutions of the KGF equation in the rotating frame, $\psi_{\pm}(t, \phi) = \exp[-iS_{\pm}(t, \phi)]$, with phases

$$S_{\pm}(t, \phi) = \omega t / \gamma - \gamma(\Omega \omega R / c^2 \pm k) R \phi, \quad (8)$$

where the Lorentz factor $\gamma = (1 - \Omega^2 R^2 / c^2)^{-1/2}$. The frequency ω at the source in the rotating frame and the wave number k in Eq. (8) are related by the KGF dispersion relation $\omega^2 = c^2 k^2 + m^2 c^4 / \hbar^2$. Let us assume that the two counter-propagating signals originate in phase from the same source at $\phi = 0$. Subsequently, these signals interfere after completing a full round-trip, with the counterclockwise signal at $\phi = 2\pi$ and the clockwise one at $\phi = -2\pi$. The phase difference between the two signals is therefore

$$\Theta_S = S_-(t, -2\pi) - S_+(t, 2\pi) = \frac{4\pi\gamma\omega\Omega R^2}{c^2}. \quad (9)$$

Considering that the circular interferometer has area $A = \pi R^2$, we find that the last expression reproduces Eq. (2) up

to the relativistic factor γ . Nevertheless, this distinction is not crucial for our discussion, as $\Omega R \ll c$ by several orders of magnitude, rendering the corrections to the fringe shift arising from γ (which are of the order $\Omega^2 R^2/c^2$) practically indiscernible in experimental observations.

In the nonrelativistic limit, the phase fringe shift described by Eq. (4) emerges as the KGF equation reduces to the Schrödinger equation. This can be done by transforming away the quickly oscillating rest-energy dependence $\psi = \chi \exp(-imc^2t/\hbar)$, deriving the Schrödinger equation for the slow factor χ in the rotating frame:

$$i\hbar \frac{\partial \chi}{\partial t} = \frac{1}{2m} (-i\hbar \nabla_r - m\mathbf{V})^2 \chi - \frac{1}{2} m\mathbf{V}^2 \chi. \quad (10)$$

Obtaining the solution of Eq. (10) for the circularly propagating waves [15], we then find the phases of the waves χ_{\pm} propagating in opposite directions:

$$S_{\pm}(t, \phi) = \left(\frac{\hbar k^2}{2m} - \frac{m\Omega^2 R^2}{2\hbar} \right) t - \left(\frac{m\Omega R}{\hbar} \pm k \right) R\phi. \quad (11)$$

This equation can also be derived directly by substituting the nonrelativistic limit of the KGF dispersion relation, given by $\omega \approx mc^2/\hbar + \hbar k^2/2m$, into (8) and subtracting the relativistic contribution mc^2/\hbar . It is easy to see that Eq. (11) results in the phase fringe (4) (see also the discussion in Ref. [30]).

The reason why the Sagnac effect survives in the nonrelativistic theory (10) of the free particle lies in the fact that nonrelativistic quantum mechanics contains relativistic elements: Its wave function is not transformed as a scalar under Galilean transformations [15,31]. Although the nonrelativistic phase (11) does not contain explicitly the de Broglie frequency $\omega_0 = mc^2/\hbar$ present in the phase of the relativistic wave function ψ , it is this frequency that enters the angular term of (11), responsible for the Galilean transformation of wave number and generating the phase difference between the two counterpropagating waves. In this regard, it is worth recalling that the original consideration of the matter waves by de Broglie relied on the special relativity and included the de Broglie frequency associated with particles at rest.

Before we return to investigating the Sagnac effect based on Eq. (6), the following points should be emphasized. This equation for an envelope wave function arises from the Schrödinger equation describing atomic orbitals, while the latter derives from a genuine Dirac equation describing electrons interacting with ions in solids. For the spinless case, this step corresponds to the connection between the KGF equation and Eq. (10). There is no need for us to follow these steps in detail; it is sufficient to take into account that the wave functions of electrons in solids contain the rapidly oscillating factor $\exp(-imc^2t/\hbar)$. Although it is not observable under most circumstances and therefore is usually dropped, this is not the case for the Sagnac effect, for which the presence of this phase factor is crucial. Considering then Eq. (6), we should set $\mathcal{E}_D = mc^2$ to ensure the presence of this frequency in the solutions. This equality is valid up to the binding energy of electrons in solids, which is significantly smaller than their rest energy.

Sagnac effect for the Dirac quasiparticles. With the above remarks in mind, we now turn to the Sagnac effect for the Dirac quasiparticles described by Eq. (6), where their velocity

$v < c$. It is clear that the approach of Ref. [15] described above for the KGF equation cannot be directly applied to the case of $v \neq c$. Indeed, we are dealing now with waves in a moving (rotating) medium, rather than in vacuum, and we cannot simply make the coordinate transformation (7) in Eq. (6), which describes waves in the material at rest.

Therefore, here we employ a different method, based on a covariant equation describing propagation of waves in a moving medium. To derive such an equation, we assume that the internal local structure (the crystalline lattice) of the material in motion is not sensitive to the arising small acceleration and that Eq. (6) is satisfied in any small region of material in its instantaneous inertial rest frame. Since we treat the wave function ψ as a scalar [recall that we ignore the particle (pseudo)spin], it is easy to express all elements of Eq. (6) in a covariant form in arbitrary space-time coordinates x^μ . The partial derivative with respect to t' (which is the proper time in the local inertial rest frame of an element of material) is covariantly expressed as $\partial_{t'} = u^\mu \nabla_\mu$, where u^μ is the element's four-velocity and ∇_μ is the covariant derivative (which, in particular, is just a partial derivative $\nabla_\mu = \partial/\partial x^\mu$ when acting on a scalar). The spatial Laplacian in the local inertial rest frame of the element of material is most easily expressed as

$$\nabla_r^2 \psi = \partial_{t'}^2 \psi - \square \psi, \quad (12)$$

where $\square = g^{\mu\nu} \nabla_\mu \nabla_\nu$ is the D'Alembertian. Combining these expressions, we immediately write Eq. (6) in arbitrary coordinates:

$$(u^\mu \nabla_\mu + i\mathcal{E}_D)(u^\nu \nabla_\nu + i\mathcal{E}_D)\psi - v^2(u^\mu \nabla_\mu u^\nu \nabla_\nu - g^{\mu\nu} \nabla_\mu \nabla_\nu)\psi + \Delta^2 \psi = 0. \quad (13)$$

Here and below we set the units $c = \hbar = 1$, unless stated explicitly otherwise. Since the small element of material is arbitrary, Eq. (13) is valid everywhere in the material in motion. Obviously, for $v = 1$ and $\mathcal{E}_D = 0$, Eq. (13) reduces to the KGF equation written in the covariant form. For material at rest, in its inertial rest frame, we have $u^0 = 1$, with the spatial components of u^μ being zero, and Eq. (13) reduces to Eq. (6), as required.

Similarly to the consideration of the KGF equation, we look for a solution of Eq. (13) in the form $\psi = \mathcal{A}e^{-iS}$, where both the amplitude \mathcal{A} and the phase S can depend on coordinates. Employing the eikonal approximation [1], i.e., neglecting the second derivatives of the amplitude, from (13) we obtain the following equation for the phase:

$$-v^2 g^{\mu\nu} \nabla_\mu S \nabla_\nu S - (1 - v^2)(u^\mu \nabla_\mu S)^2 + 2\mathcal{E}_D u^\mu \nabla_\mu S + \Delta^2 - \mathcal{E}_D^2 = 0. \quad (14)$$

The explicitly covariant form of this equation allows us to consider the problem in arbitrary space-time coordinates, for instance, in the frame rotating with the sample or in the nonrotating frame.

As in the case of the KGF equation, we consider a narrow ring of radius R rotating with angular velocity Ω . In the nonrotating frame (t', ϕ') (the radial coordinate is effectively frozen at $r' = R$), the invariant interval and the contravariant

metric tensor are

$$ds^2 = dt'^2 - R^2 d\phi'^2, \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -R^{-2} \end{pmatrix}. \quad (15)$$

Accordingly, in the rotating coordinate frame (t, ϕ) obtained by the coordinate transformation (7), the invariant interval is

$$ds^2 = \gamma^{-2} dt^2 - 2\Omega R^2 dt d\phi - R^2 d\phi^2 \quad (16)$$

and the covariant and contravariant metric tensors read

$$g_{\mu\nu} = \begin{pmatrix} \gamma^{-2} & -\Omega R^2 \\ -\Omega R^2 & -R^2 \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} 1 & -\Omega \\ -\Omega & -\gamma^{-2} R^{-2} \end{pmatrix}. \quad (17)$$

To draw parallels to the case of the KGF equation, we begin by examining Eq. (14) in the rotating frame. Since the ring is at rest in this frame, the components of the four-velocity u^μ in coordinates (t, ϕ) are $u^\mu = (\gamma, 0)$, obeying the usual normalization condition $g_{\mu\nu} u^\mu u^\nu = 1$. Again, let S_\pm be the phases of waves propagating along (+) and opposite (-) to the angular direction ϕ . For a positive-frequency solution, we then have

$$S_\pm = \omega_\pm t - k_\pm R\phi, \quad (18)$$

where ω_\pm and k_\pm are the corresponding constant frequencies and wave numbers. The source and receiver of electrons move together with the ring with the same four-velocity $u^\mu = dx^\mu(\tau)/d\tau$, where τ is the proper time, by assumption, detecting the same frequency ω for both waves:

$$\omega = \frac{d}{d\tau} S_\pm(x(\tau)) = u^\mu \nabla_\mu S_\pm = \gamma \omega_\pm. \quad (19)$$

Substituting (18) and (19) into (14) and using the relations $\nabla_t S_\pm = \omega_\pm$ and $\nabla_\phi S_\pm = -k_\pm R$, after some simple calculations, we obtain the following equation:

$$v^2 \left(\frac{k_\pm}{\gamma} - \omega \Omega R \right)^2 - (\omega - \mathcal{E}_D)^2 + \Delta^2 = 0. \quad (20)$$

Solving this equation with respect to k_\pm , we find

$$k_\pm = \gamma \omega \Omega R \pm \frac{\gamma}{v} \sqrt{(\omega - \mathcal{E}_D)^2 - \Delta^2}. \quad (21)$$

Note that the main contribution to the wave number contains the velocity constant v , but its correction induced by rotation does not depend on this velocity. Furthermore, one can easily see from the dispersion law (5) with $\omega = E(\mathbf{k})/\hbar$ that the second term on the right-hand side of Eq. (21) is nothing but the wave vector γk . Thus, the solution (18) with ω_\pm and k_\pm given by Eqs. (19) and (21), after restoring c and \hbar , formally coincides with the previously presented solution of the KGF equation (8). The important difference is that the supporting wave obeys the dispersion law (5). Accordingly, for the case of Dirac fermions under consideration, the Sagnac phase fringe is still determined by Eq. (9), but with the frequency $\omega = E(\mathbf{k})/\hbar$ characterized by the dispersion law (5).

It is clear that the energy \mathcal{E}_D in Eq. (5) is at least 10^5 times larger than the dispersive part. Thus we conclude that the Sagnac effect in the Dirac materials either with $\Delta = 0$ or with finite Δ is characterized by Eq. (4) with $m \approx m_e$, that

is, the wave supporting the Sagnac effect has the same nature as the waves in existing electron interferometers [22,23]. The corresponding wave numbers are equal to the Fermi wave vector k_F . We emphasize that it is incorrect to use Eq. (3) with the Fermi wavelength $\lambda_F = 2\pi/k_F$ for the Sagnac effect in this case. It is the de Broglie frequency ω_0 and the corresponding Compton wavelength λ_C that determine the phase fringe.

To illustrate the covariant character of our approach, we reproduce the results obtained above in the nonrotating coordinate frame (t', ϕ') . The four-velocity in this frame acquires the spatial (angular) component $u^\mu = (\gamma, \gamma\Omega)$, satisfying the normalization condition $g_{\mu\nu} u^\mu u^\nu = 1$ with the metric given by Eq. (15). We can look for the phases S_\pm of the waves moving in opposite directions in the same form as in Eq. (18),

$$S_\pm = \omega'_\pm t' - k'_\pm R\phi', \quad (22)$$

where ω'_\pm and k'_\pm are the corresponding constant frequencies and wave numbers in the nonrotating frame. The condition $u^\mu \nabla_\mu S_\pm = \omega$ [see Eq. (19)] in this frame reads

$$\gamma(\omega'_\pm - k'_\pm \Omega R) = \omega. \quad (23)$$

Equation (14) for S_\pm , using Eq. (23) and the condition $u^\mu \nabla_\mu S_\pm = \omega$, takes the form

$$v^2(k'_\pm{}^2 - \omega'^2_\pm) - (1 - v^2)\omega^2 + 2\mathcal{E}_D\omega + \Delta^2 - \mathcal{E}_D^2 = 0. \quad (24)$$

Equation (23) shows that the frequencies ω'_\pm are Doppler shifted relative to each other in the nonrotating frame. Expressing ω'_\pm via k'_\pm from (23) and substituting them into Eq. (24), we obtain an equation identical to Eq. (20), but with k'_\pm instead of k_\pm . This implies that $k'_\pm = k_\pm$, with the wave numbers previously determined in Eq. (21). Thus, we have found the phases S_\pm in Eq. (22). Finally, we can proceed to the rotating frame in the solution by the coordinate transformation (7). In this frame, the phases (22), using the expressions for ω'_\pm found from (23), become $S_\pm = \omega t/\gamma - k_\pm R\phi$, in full agreement with the previous result, Eqs. (18) and (19).

Conclusion. To summarize, we have shown that the phase fringe for the Sagnac effect in graphene and other Dirac materials, with the charge carrier dispersion law given by Eq. (5), is described by Eq. (4), with m being the free-electron mass (up to its binding energy in graphene). This is the same expression that was previously obtained for the Sagnac phase fringe for electrons in vacuum. The existing experiments on the Aharonov-Bohm oscillations in rings [22,23] suggest the possibility of observing the Sagnac effect in nonsuperconducting solid-state systems.

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