## Emergent dynamics of scale-free interactions in fractional quantum Hall fluids

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We show that even with arbitrarily large cyclotron gap, Landau level (LL) mixing can be dominant with scalefree interaction in a fractional quantum Hall system, as long as the filling factor exceeds certain critical values. Such scale-free interaction with kinetic energy can serve as exact model Hamiltonians for certain composite Fermion or parton states (unlike the well-known TK Hamiltonians where the number of LLs needs to be fixed by hand), and they are natural physical Hamiltonians for 2D systems embedded in higher-dimensional space time. Even with LL mixing the null spaces of such Hamiltonians (spanned by the ground state and the quasiholes) can be analytically obtained, and we show these are the generalization of the conformal Hilbert spaces (CHS) to more than one LLs. The effective interaction between the anyons for these topological phases emerges from the kinetic energy of the "elementary particles," leading to an interesting duality between strongly and weakly interacting systems that can be easily understood via the tuning of parameters in the scale-free interaction. We also propose an experimental platform for approximately realizing such model Hamiltonians with trionlike particles that can potentially lead to very robust (non-Abelian) FQH phases from two-body Coulomb-based interaction.

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The long range bare Coulomb repulsion between electrons comes from one of the fundamental interactions between elementary particles. The inverse quadratic interaction (e.g.,  $1/r^2$ , where r is the separation between the two particles) is a universal force that is scale free; the quadratic power is fundamentally determined by the three-dimensional universe we live in, from the hydrodynamic description of the gauge Bosons mediating the interaction between charged particles. The long range nature of the interaction tends to complicate theoretical treatments, where singularities need to be properly renormalized. In Fermi liquid, such long range interaction typically can be screened by mobile charge carriers [1], leading to a Yukawa-like interaction and Friedel oscillation of electron density [2].

For strongly interacting condensed matter systems with flat bands (quenched kinetic energy), the quantum fluids of electrons are quintessentially non-Fermi liquid. In twodimensional electron gas systems with a strong perpendicular magnetic field, flat bands called Landau levels are formed. The fractional quantum Hall effect (FQHE) emerge entirely from Coulomb interaction (with no kinetic energy), when the top most Landau level is partially filled [3]. Nevertheless, extensive numerical analysis shows single-particle-like behaviors emerge from strong interaction, not for electrons, but for emergent phenomenological composite Fermions [4]. At least for finite size systems, the composite Fermion wavefunctions have large overlap with the Coulomb interaction ground states, thus capturing not only the topological properties of those quantum fluids, but also the nonuniversal quantitative behaviors quite accurately.

It is important to note that for many FQH phases, there exist model Hamiltonians and model wavefunctions (as exact eigenstates) that give ideal descriptions of such topological phases [3,5]. These include the exact quasihole counting and degeneracy, as well as the ground state topological

entanglement entropy and spectrum [6-8]. Such model Hamiltonians in the form of pseudopotentials are exact toy models for the FQH phase. For the CF wavefunctions or the exact eigenstates of the Coulomb interactions, however, these ideal properties are conjectured to be qualitatively captured but no longer quantitatively exact. In particular, the CF wavefunctions after the lowest LL (LLL) projection generally do not even have parent Hamiltonians [9,10]. It is thus interesting that the CF wavefunctions tend to quantitatively agree with the Coulomb eigenstates better than the special model wavefunctions. A more detailed exposition on the microscopic derivation of the CFs without the LLL projection has been discussed in [10,11]. The questions of why the Coulomb interaction seems to be special for the composite Fermions, and its hidden connection to the model Hamiltonians and the LLL projection, are still poorly understood.

In this Letter, we study the family of scale-free interactions between elementary Fermions (e.g., electrons) and Bosons with the electrostatic interaction  $V_{\alpha}(r) = 1/r^{\alpha}$ , confined to a two-dimensional manifold. For integer  $\alpha$  this is the Coulomb interaction within a 2D manifold in a universe with  $\alpha + 2$ spatial dimensions; though, in principle,  $\alpha$  serves as a single continuous tuning parameter for a family of Hamiltonian of the following form:

$$\hat{H}_{\alpha} = \sum_{i} \frac{1}{2m} g^{ab} \hat{\pi}_{a} \hat{\pi}_{b} + \int d^{2} r_{1} d^{2} r_{2} V_{\alpha} (|\vec{r}_{1} - \vec{r}_{2}|) \hat{\rho}_{\hat{r}_{1}} \hat{\rho}_{\hat{r}_{2}}.$$
 (1)

Here  $g^{ab}$  is the metric describing the effective mass tensor with m being the effective mass,  $\hat{\pi}_a = \hat{P}_a - e\hat{A}_a$  is the canonical momentum from the minimal coupling of the external vector potential  $\hat{A}_a$  with the uniform magnetic field  $B = \epsilon^{ab}\partial_{\hat{r}^a}\hat{A}_b$ , and the electron charge e;  $\hat{\rho}_{\hat{r}} = \sum_i \delta^2(\hat{r}_i - \vec{r})$  is the bare electron density operator.

We show even in the experimentally relevant limit of  $\hbar\omega_c \to \infty$  where  $\omega_c = eB/m$  is the cyclotron frequency, the low energy physics could naturally involve multiple LLs and the Trugman-Kivelson (TK) type interaction Hamiltonians [12] are related to the special cases of  $\hat{H}_{\alpha}$  with  $\alpha =$  $2k, k \ge 1$ . Thus, in the large  $\omega_c$  limit, strong LL mixing can still occur when the filling factor exceeds critical values, and the low energy dynamics (e.g., ground state and quasiholes) can be exactly obtained even with LL mixing. Indeed, from our understanding of the TK Hamiltonian [13,14], Eq.(1) is the exact physical model Hamiltonian with proper values of  $\alpha$ for a number of the unprojected CF and parton states with no additional assumptions. This is not the case for the TK Hamiltonian where the number of LLs needs to be manually determined [15]. The conjectured adiabatic connection to the LLL Coulomb interaction physics for these CF and parton states comes from tuning a single parameter  $\alpha$  while keeping the cyclotron energy large. Based on those theoretical results we propose experimental platforms where model Hamiltonians and non-Abelian physics from two-body interactions can be potentially realized.

It is useful to first confine the Hilbert space to the LLL, and look at the effective interaction from  $V_{\alpha}(r)$  by expanding its Fourier transform in the basis of Haldane pseudopotentials. The effective interaction in the momentum space is

$$\tilde{V}_{\alpha}(q) \sim q^{\alpha - 2} e^{-\frac{1}{2}q^2} = \sum_{i=0}^{\infty} c_{\alpha,i} V_i^{2bdy},$$
 (2)

where the two-body pseudopotentials (PP) are  $V_i^{2bdy} = L_i(q^2)e^{-\frac{1}{2}q^2}$  with  $L_i(x)$  being the *i*th Laguerre polynomial and  $c_{\alpha,i}$  being the PP weights. The Coulomb interaction is the well-known case of  $\alpha = 1$ , but Eq. (2) is only valid for  $\alpha < 2$ ; for  $\alpha \ge 2$  the Fourier transform diverges. A more careful calculation shows that as  $\alpha$  approaches 2 from below, only  $c_{\alpha\to 2,0}$  is divergent, while all other PP weights are finite. More generally, as  $\alpha \to 2k$ ,  $c_{\alpha\to 2k,i \le k-1} \to \infty$  [thus, equivalent to a divergent  $\nabla^{2k-2}\delta^2(\vec{r}_1 - \vec{r}_2)$ ], while  $c_{\alpha\to 2k,i > k-1}$  are all finite. As  $\alpha$  increases, the interaction diverges more strongly at short distances, and it is the leading PPs that become divergent first, forbidding two particles from having small relative angular momenta.

Density dependent LL mixing. The simplest example is the Bosonic system with  $\alpha = 2$ , so within the LLL,  $c_{2,0}$ is infinity, leading to a hardcore interaction equivalent to the artificial model Hamiltonian (with infinite strength) for the Bosonic Laughlin state at filling factor  $\nu = 1/2$ . With large cyclotron energy, for  $\nu \leq 1/2$  (or more specifically for the number of magnetic fluxes  $2Q \geq 2N_b - 2$  on the spherical geometry with  $N_b$  the number of Bosons) the ground state(s) are completely within the LLL, which we can set as zero energy states. Thus, the null space of Eq. (1) is spanned by the model ground state and the quasiholes of the Laughlin  $\nu = 1/2$  phase [19], with the highest density states given by  $2Q = 2N_b - 2$ . At low temperature, no LL mixing is induced by the interaction in the limit of large cyclotron energy [see Fig. 1(a)].

Adding another Boson to the highest density quantum fluid, however, is not possible within the LLL, since the interaction will send the energy to infinity [20]. At  $2Q < 2N_b - 2$ ,



FIG. 1. An illustrative example for  $\alpha = 2$ , in (a)–(c) the circles are orbitals in three different LLs, the solid ones are occupied by Bosons. (a) All Bosons can go into the LLL when  $\nu \leq 1/2$ ; (b) An additional Boson added to the half-filled LLL has to go to the second LL to avoid infinite energy; (c) An additional Boson added to filling factor  $\nu = 2/3$  has to go to the third LL; (d) A schematic illustration of different CHS in the parameter space of  $\alpha$  and  $\nu$ . Here  $L_3 \in L_2 \in$  $L_1$  are CHS within the LLL, and  $S_2 \in S_1$  are CHS within two lowest LLs, while  $T_1$  is the CHS within the lowest three LLs [21].

strong LL mixing is present no matter how large the cyclotron gap is. Indeed, for  $\alpha = 2$ , the second term of Eq. (1) is equivalent to an infinite bosonic TK interaction. Only the second LL will be occupied for  $2Q \ge 3N_b/2 - 3$  [see Fig. 1(b)]. These are known as the exact unprojected Bosonic Jain state at  $\nu = 2/3$  and its quasiholes [13]. Further increasing the density with  $2Q > 3N_b/2 - 3$  leads to the consecutive mixing of higher LLs [see Fig. 1(c)] [24], giving a series of conjectured non-Abelian FQH states that are well studied in the context of the TK Hamiltonians [14,17], but without artificially picking a few lowest LLs by hand. A schematic illustration of different topological phases at larger  $\alpha$  is shown in Fig. 1(d), which we will also explain in more details later.

Model Hamiltonians for the composite Fermion and parton theory. The family of TK Hamiltonians involving multiple LLs give exact unprojected Jain states and parton states in many cases, but it is not obvious how they are relevant to experiments when one cannot just select just a few LLs arbitrarily. The family of Hamiltonians with scale-free interactions given in Eq. (1), however, serves as natural exact model Hamiltonians for many Jain states and parton states. It also offers transparent insights on why such theories are successful for the Coulomb interaction (with  $\alpha = 1$ ) [4], and why LLL projection in such theories could be conceptually redundant [10] and may only be useful for certain numerical computations.

This is because with Eq. (1) as the exact model Hamiltonian (e.g., for the Jain state at v = 2/3 or the 221 parton state at v = 1), it is natural to include more than one LLs. As we adiabatically tune  $\alpha$  to 1, all states are orthogonal, though they are mostly within the LLL in the presence of the large cyclotron gap: it is not necessary to project the CF wavefunctions, leading to the spurious nonorthogonality.



FIG. 2. (a) Energy spectrum on the sphere with 2Q = 14 and  $N_b = 8$ , with the black plot from exact diagonalization at  $\alpha = 2 - \epsilon$ , and the red plot at  $\alpha = 2$ , both from Eq. (1). *L* is the total angular momentum quantum number. (b) The normalized magnetoroton mode for different values of  $\alpha$ , with  $\alpha = 1.9$  (black),  $\alpha = 1$  (red), and  $\alpha = 0.1$  (blue). The horizontal dotted lines are the bottom of the multiroton continuum. (c) The dependence of  $c_{\alpha,0}$  on  $\alpha$ . (d) The dependence of the normalized magnetoron minimum energy on  $c_{\alpha,2}/c_{\alpha,0}$  (R-squared error of the linear fit is 0.9997).

Certain unprojected CF states will vanish entirely under LLL projection. This corresponds to eigenstates of Eq. (1) that lives entirely within higher LLs, and thus their energies are sent to infinity (as punished by the cyclotron energy).

Indeed, in Fig. 2 we can see in the large cyclotron gap limit, there is numerical evidence that scale-free interactions are qualitatively similar [19]. This is true for the magnetoroton modes (MR) of the Laughlin state [11,25] at v = 1/2, both at  $\alpha = 2$  where the second LL is involved [see Fig. 2(a)], and for  $0 \leq \alpha < 2$  when only the LLL is involved [see Fig. 2(b)]. Here the gap of the MR is controlled by both  $c_{\alpha,0}$  and the perturbation to  $V_0^{2bdy}$  (i.e.,  $c_{\alpha,i>0}$ ). In Fig. 2(b) the MR energies are normalized by  $c_{\alpha,0}$ , so the gap is entirely from  $c_{\alpha,i>0}$ . It is clear that  $c_{\alpha,0}$  approximately approaches unity as  $\alpha \to 0$  [see Fig. 2(c)], as do all other  $c_{\alpha,i}$ ; there is almost a linear relationship between the normalized gap and  $c_{\alpha,2}$ , as shown in Fig. 2(d) (see caption). Thus, as  $\alpha \to 0$ , the energy scales (e.g., the gap and the bandwidth of MR, etc.) approach zero, but the dynamics of the MR does not qualitatively change [26]. This is highly nontrivial, since for small  $\alpha$ , the perturbation to  $V_0^{2bdy}$  is very large (e.g.,  $c_{\alpha,i>0}/c_{\alpha,0} \rightarrow 1$ ). Note that at the limiting case of  $\alpha = 0$ , we have a constant interaction, thus all states within the LLL have the same energy. The integer quantum Hall effect (IQHE) thus can also be seen as a special case of the scale-free interaction [i.e., Eq. (1) with  $\alpha = 0$ ].

Conformal Hilbert spaces within multiple LL. The concept and hierarchy of conformal Hilbert spaces (CHS) within a single LL have been discussed extensively with various applications [10,27–30]. These are Hilbert spaces in a 2D manifold with emergent conformal symmetry, exact bulk-edge correspondence, containing a particular type of anyons/non-Abelions as "elementary particles." Thus, each CHS is the low energy manifold of a particular topological phase; for scale-free interaction, the corresponding CHS is spanned by all the finite energy states. At  $\alpha = 2$  (or  $\alpha = 4$ ), for Bosons (or Fermions) the null space for the scale-free interaction is isomorphic to the CHS of two LLs for  $\nu \leq 2/3$ (or  $\nu \leq 2/5$ ), completely agreeing with the CF description for the IQHE of composite Fermions (from each electron attached to two fluxes). For greater filling factor, the null spaces of the scale-free interaction involves more than two LLs, and become non-Abelian.

In Fig. 1(d) we show a "phase diagram" of different CHS defined by Eq. (1) with varying values of  $\alpha$  and  $\nu$ , and for simplicity we focus on  $\nu \leq 1$ . The blue region are the CHS within the LLL:  $L_1$  is the entire LLL,  $L_2$  is the null space of  $V_0^{2bdy}$  (Laughlin 1/2 ground states and quasiholes), and  $L_3$  is the null space of  $V_0^{2bdy} + V_2^{2bdy}$  (Laughlin 1/4 ground states and quasiholes). As the filling factor increases, more LLs are involved with  $\alpha \geq 2$ : both  $S_1$  and  $S_2$  are CHS involving two lowest LLs, and each is a subspace of two LLs but isomorphic to the Hilbert space of two LLs with a rescaling of the magnetic field. Similarly,  $T_1$  is a CHS involving three lowest LLs, but it is non-Abelian [14].

It is interesting to note that within the red and green region the dominant dynamic energy scale is the kinetic energy. In Fig. 2(a) the energy spectra at the filling factor of v = 1/2on the spherical geometry, with the magnetic monopole at the sphere center  $2Q = 2N_b - 2$  ( $N_b$  is the Boson number), can also be understood as from the two following Hamiltonians:

$$\mathcal{H}_1 = \hat{V}_0^{2\text{bdy}}, \quad \mathcal{H}_2 = \sum_i \frac{1}{2m} g^{ab} \hat{\pi}_a \hat{\pi}_b, \tag{3}$$

where  $\mathcal{H}_1$  gives  $V_0^{2bdy}$  PP interaction within  $L_1$  (the LLL), and  $\mathcal{H}_2$  is within  $S_1$  (the null space of the TK Hamiltonian for the lowest two LLs). Apparently  $\mathcal{H}_1$  is a purely interacting Hamiltonian between Bosons, while  $\mathcal{H}_2$  only involves kinetic energy of the Bosons. Naively, it would be surprising to find these two Hamiltonians having very similar spectrum, reflecting a duality between the strongly interacting and weakly or noninteracting systems. Now we understand this duality is natural:  $\mathcal{H}_1$  is the special case of Eq. (1) with  $\alpha = 2 - \epsilon$  with lim  $\epsilon \to 0$  in the limit of the large  $\omega_c$ , while  $\mathcal{H}_2$  is the special case of  $\alpha = 2$ ; the two Hamiltonians are very close in the parameter space of  $\alpha$ .

The dispersion of the MR modes at  $\alpha = 2$ ,  $\nu = 1/2$  indicates nontrivial interaction between the Laughlin quasihole and quasielectron emerging from the kinetic energy of the Bosons. This is a generic feature of the quasiparticles in the multiple LL CHS as the null space of the interaction between the elementary particles: these quasiparticles are noninteracting without the kinetic energy of the elementary particles; they gain self-energy and start to interact only when the kinetic energy is included. Such energies come from the mixing of the higher LLs for the many-body wavefunctions. The amount of mixing depends on the separation between quasiparticles, which is worth further detailed study as an unusual microscopic mechanism for quasiparticle interaction.

*Experimental platform for non-Abelian phases.* It is still very challenging to realize non-Abelian topological phases in FQH systems. There are tentative results for the Moore-Read phase at v = 5/2 [31], but the physical interaction (the



FIG. 3. (a) A schematic illustration of charged Bosons with finite dipole moments confined to a 2D manifold. The electric field is needed to align all the dipoles and it can be either pointing up or pointing down; (b) The dependence of  $V_0^{2bdy}$  PP weight on d parametrizing the size of the dipole; (c) The dependence of the ratio of the PP weights of  $V_0^{2bdy}$  and  $V_2^{2bdy}$  on *d*.

second LL Coulomb interaction) is very different from the model Hamiltonian consisting of a pure three-body interaction [32,33]. Ideally, we would like to experimentally realize model Hamiltonians for the non-Abelian phases, for example, the two-body TK Hamiltonian within the lowest three or more Landau levels [14,22]. The scale-free interaction in Eq. (1)offers exactly that path, but it seems difficult to increase  $\alpha$  in a realistic experiment in a three-dimensional universe.

Nevertheless, the key for realizing CHS involving multiple LLs with strong LL mixing is the competition between the kinetic energy and the energy scale given by individual two-body pseudopotentials. In this way, LL mixing explicitly depends on the filling factor. It is important to note that a strong magnetic field or large  $\omega_c$  is needed, because we only hope to mix, or project into, a small number of lower LLs.

While Fermionic systems are possible, we now focus on Bosonic systems, because for Coulomb based interaction,  $V_0^{2\text{bdy}}$  is more dominant against other even PPs, while  $V_1^{2\text{bdy}}$  is less dominant against other odd PPs in comparison. This is desirable, because a good approximation to the scale-free interaction at  $\alpha = 2$  is for  $||\hat{V}_0^{2bdy}|| \gg n\hbar\omega_c \gg ||\hat{V}_{i>0}^{2bdy}||$  [34]. We can enhance the interaction effect for particles with greater charge or mass. The main challenge is thus to tune the Coulomb interaction experimentally to achieve the regime where  $||\hat{V}_0^{2bdy}|| \gg ||\hat{V}_{i>0}^{2bdy}||$ . Here we propose a "simple" physical system in which the

required window for different energy scales can be readily

tuned and achieved. It offers a pathway for realizing approximate model Hamiltonians for (non-Abelian) FQH phases. We hypothesize a collection of charged Bosons with a nonzero dipole moment confined to a two-dimensional manifold subject to a strong perpendicular magnetic field [see Fig. 3(a)], and an out-of-plane electric field to align their dipole moment. They can be considered as Bosonic analogous of "trions" that have been realized in certain semiconductor systems [35-37]. Let the total dipole and charge of each Boson be  $e_1d$  and  $e_2$ , respectively. The magnetic length and the cyclotron energy are given by  $\ell_B = \sqrt{\hbar/e_2 B}$  and  $\epsilon_c = \frac{\hbar e_2 B}{m}$  (where *m* is the Boson mass). The Boson-Boson interaction is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon} (\epsilon_1(\mathbf{r}) + \epsilon_2(\mathbf{r})), \qquad (4)$$

$$\epsilon_1 = 2e_1^2 \left(\frac{1}{r} - \frac{1}{\sqrt{r^2 + d^2}}\right), \epsilon_2 = \frac{e_2^2}{r},$$
 (5)

where  $\mathbf{r} = |\vec{\mathbf{r}}|$ . Note that for  $\mathbf{r} \gg d$  we have  $\epsilon_1 \sim \mathbf{r}^{-3}$ , but  $\epsilon_1$ vanishes as  $d \rightarrow 0$ , thus we do not have scale-free interaction at short distances [see Fig. 3(b)].

Nevertheless the magnetic length  $\ell_B$  sets a "minimum length" and we can decompose V(r) into PPs within the LLL with weights  $c_i$ . It is not surprising that we find  $\lim_{d\to 0} c_{i>0}/c_0 = 0$ , so that  $V_0^{2bdy}$  dominates [see Fig. 3(c)]. For large magnetic field we have  $\epsilon_c \gg \epsilon_1$ , thus, here we ignore  $\epsilon_1$ . Assuming we can tune  $e_1$ ,  $e_2$ , and *m* of the charged Bosons, we can then get arbitrarily close to the regime with  $e_1 \gg$  $e_2/m$ . It is thus potentially feasible to engineer the 2D system to realize many interesting (non-Abelian) FQH phases (e.g., the bosonic Jain 221 state with three LLs and the Fibonacci state with four LLs).

One focus of the future studies is the anyon interactions within multiple LLs emerging from the kinetic energy. When the energy window proposed above is achieved, we can realize robust ground states that are theoretically proposed with prominent Hall plateaus at those Abelian and non-Abelian phases. However, the quasihole excitations may not be degenerate in the thermodynamic limit due to the cyclotron energy. Numerical analysis of multi-LL systems, however, is strongly constrained by the large size of the Hilbert space. Thus, better numerical and theoretical tools are needed to understand the interesting dynamics (both gapped and gapless) for these topological phases.

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Abelian) topological phases are related to those conjectured in Ref. [23].

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- [33] With Coulomb interaction it is not clear if the 1/4 charge quasiparticles are energetically favorable, or the degeneracy required for non-Abelian statistics is robust in experiments, even though we observe an incompressible ground state leading to a Hall plateau at  $\nu = 5/2$ .
- [34] Here  $|| \cdots ||$  indicates the energy scale of the PP interaction component. For a generic two-body interaction Hamiltonian we have  $\hat{V}_q = \sum_i \hat{V}_i^{2bdy}$ , which is how  $\hat{V}_i^{2bdy}$  is defined; *n* is the number of the lowest LLs that intends to be mixed.
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