Letter

No bulk thermal currents in massive Dirac fermions

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We calculate the energy current flowing in the bulk of a (2+1)-dimensional system of massive Dirac fermions and along a (1+1)-dimensional domain wall generated by flipping the sign of the particle mass. We show that, at low temperatures and in the long-wavelength limit, the system does not support a bulk thermal Hall current proportional to the temperature gradient. The only such contribution is due to states localized at the domain wall. This puts an end to a controversy existing in the literature and amends previous results obtained via first-order perturbation calculations.

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Introduction. The thermal Hall effect mirrors the Hall effect in the realm of energy transport, i.e., a transverse heat current emerges when a temperature gradient is established. Such an effect has been studied in a variety of systems, such as multiferroics [1], topological insulators [2], magnets [3–6], quantum spin liquids [7–9], and strongly correlated systems [10–16], to name a few, and it also constitutes an important diagnostics of neutral excitations. From a theoretical standpoint, thermal responses can be calculated by using Luttinger's "trick" [17], by exploiting the equivalency that exists in the linear-response regime between a nonflat metric tensor (also termed "gravitational potential" in what follows, in analogy with the electric potential) and temperature fluctuations in generating energy currents [17–19]. In this scheme, the Hamiltonian is perturbed by introducing a nontrivial metric tensor that couples to the system's energy density. The first derivative of the metric tensor then defines the gravitational field (analogous to the electric field and equivalent to the thermal gradient within linear response) that generates the longitudinal and/or Hall thermal currents [17–22].

Recently, a controversy has arisen about whether thermal Hall currents proportional to the gravitational field (i.e., to the temperature gradient) can be supported in the system's bulk [20,23,24], or whether such currents are *always* proportional to higher-order gradients of the temperature fluctuations [25–29]. References [25,26] were the first to point out a fundamental difference between the charge and thermal Hall effects. They showed that, in contrast to transverse charge currents that arise in response to uniform electric fields, within a gravitational Chern-Simons action no thermal Hall current is generated in response to a uniform gravitational field (temperature gradient) in the bulk of a (2+1)-dimensional gapped system. This result was challenged by Refs. [20,23,24]. In particular, Ref. [23] found a bulk contribution proportional to the first derivative of the gravitational potential, when calculating the thermal Hall response to first order in the metric tensor and in the long-wavelength limit. Since the gravitational Chern-Simons action does not support a bulk thermal Hall current generated by a uniform gravitational field, if the system indeed exhibits an inflow bulk current, it implies that the gravitational Chern-Simons action does not completely describe the system. There could be other effective actions, such as a torsional Chern-Simons term [28,30], that better describe the system.

More recent analytical and numerical works [27–29] are in agreement with the earlier findings of Refs. [25,26]. However, in deriving their results, Refs. [27–29] use different methods compared to Refs. [20,23,24]. Specifically, Ref. [27] employs a low-energy effective bulk theory similar to the approach in Ref. [26]. Reference [28] is based on a hydrostatic effective action. Reference [29], on the other hand, relies on numerical calculations on a lattice model. These methods are significantly different from those in Refs. [20,23,24], which are based on analytical calculations from microscopic theories and boundary theories derived from microscopic approaches. It is therefore unclear whether these latter works present shortcomings that can be remedied to get results consistent with the rest of the literature [25–29]. The scope of this Letter is thus to put an end to such controversy by showing, with an analytical calculation based on the massive Dirac fermion model, which closely follows that of Ref. [23], that it is indeed possible to obtain the correct results by including all-order contributions in the metric tensor to the system's free energy. By doing so, we show that the bulk thermal Hall response proportional to a uniform gravitational field (temperature gradient) vanishes. Thus, the thermal Hall response in the long-wavelength limit features only boundary contributions. Since the massive Dirac fermion model is the continuum low-energy theory for many topological insulator models [31–34], our results indicate that

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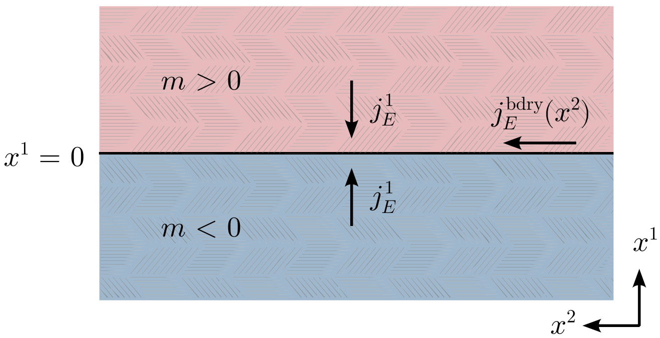

FIG. 1. The boundary located at $x^1=0$ separates two (2+1)-dimensional bulk massive Dirac fermion systems, whose masses are equal but of opposite sign: negative in the half plane $x^1<0$ and positive in the half plane $x^1>0$. A boundary current $j_E^{\rm bdry}$ flows along the edge, i.e., in the x^2 direction, and bulk currents j_E^1 flow across the boundary. These currents satisfy the continuity equation given in Eq. (1). The textured background represents a nonuniform gravitational potential, which varies in the x^2 direction. This system is equivalent to the interface between two different topological insulators, with their left and right ends connected to heat baths kept at different temperatures. The equilibrium temperature of the edge mode, as described in Eq. (2), matches that of the upstream heat bath [35,36]. Throughout this Letter, temperature T is treated as uniform in both space and time. Any spatial variation in temperature is interpreted as a variation in gravitational potential [17,18].

for these gapped systems, there is no thermal Hall bulk current proportional to the temperature gradient. In turn, these systems can be described by the gravitational Chern-Simons theory.

In this Letter we consider the system of Fig. 1, i.e., a (1+1)-dimensional boundary located at $x^1=0$ and oriented along the direction x^2 which separates two (2+1)-dimensional bulk massive Dirac fermion systems. We assume their masses in the two half spaces, $x^1>0$ and $x^1<0$, to be equal in magnitude but opposite in sign, as shown in Fig. 1. The continuity of energy currents imposes that a bulk thermal Hall current must necessarily exist to account for boundary anomalies. This is to say that, if a (1+1)-dimensional current $j_E^{\rm bdry}$ flows along the boundary, then the bulk current j_E^1 flowing *across* the boundary must satisfy the continuity equation [37]

$$j_E^1(x^1=-0)=-j_E^1(x^1=+0)=\tfrac{1}{2}\partial_2 j_E^{\rm bdry}. \tag{1}$$

This equation is central in what follows in proving that the bulk current $j_E^1(x^1)$ can only be proportional to the derivative of the gravitational field (i.e., to the second derivative of the temperature fluctuations).

On the contrary, the calculation of Ref. [23] suggests that the boundary current is directly proportional to the metric tensor. If this were true, according to Eq. (1), the bulk thermal Hall current would be proportional to the first derivative of the metric, and therefore to the temperature gradient. However, these findings are derived from lowest-order approximations of the quantities involved: Higher-order corrections could dramatically alter this conclusion.

To show that this is indeed the case, we follow the method used in Ref. [23] but we calculate contributions to the thermal current to *all* orders in the gravitational potential in the long-wavelength limit. We omit terms in the boundary current that depend on the derivatives of the metric, since such terms would correspond to contributions to the bulk thermal Hall current proportional to at least the second derivative of the gravitational potential. Our all-order calculation shows that the long-wavelength energy current flowing along a boundary located at $x^1=0$ is

$$j_E^{\rm bdry}(x^2)=\frac{\pi T^2}{12}, \tag{2}$$

i.e., it depends only on the uniform equilibrium temperature T and does not contain any term proportional to the gravitational potential itself (i.e., to the temperature fluctuations away from equilibrium). According to the thermal generalization of the Streda formula [38], we can use the result of Eq. (2) to find the thermal Hall conductivity, $\kappa_H=-{\rm sgn}(m)\pi T/12$ [26,33,38].

Equation (2) implies that $\partial_2 j_E^{\rm bdry}=0$ and therefore one can immediately conclude that there is no bulk thermal Hall current which is proportional to the gravitational field (temperature gradient). Thus, by following the same method presented in Ref. [23], our all-order calculation corrects their approximate result. Going beyond the long-wavelength approximation, one would include contributions to the boundary energy current of Eq. (2) that are proportional to the first derivative of the metric tensor. These in turn translate, via Eq. (1), into leading-order contributions to the bulk thermal Hall current that are proportional to the second derivative of the temperature fluctuations away from equilibrium. This result is in agreement with the effective theories of Refs. [25,27] and numerical results of Ref. [29].

The model. The action for $(2+1)$-dimensional Dirac fermions coupled to a gravitational field is (hereafter, we set $\hbar=1$) [39]

$$S=\int_{x,t}\sqrt{g}\bar{\psi}\left[\frac{i}{2}\left(e^{\mu}_{\ \alpha}\gamma^{\alpha}\overrightarrow{\nabla}_{\mu}-\overleftarrow{\nabla}_{\mu}\gamma^{\alpha}e^{\mu}_{\ \alpha}\right)-m\right]\psi, \tag{3}$$

where $\int_{x,t}=\int d^2x dt$ and the covariant derivative $\overrightarrow{\nabla}_\mu$ ($\overleftarrow{\nabla}_\mu$) acts on the right (left) two-component spinor field ψ. Explicitly, $\overrightarrow{\nabla}_\mu\psi=\overrightarrow{\partial}_\mu\psi+[\gamma_\alpha,\gamma_\beta]\omega_\mu^{\alpha\beta}\psi/8$, where $\overrightarrow{\partial}_\mu$ is the derivative over the temporal ($\mu=0$) and spatial ($\mu=x,y$) directions, while $\omega_\mu^{\alpha\beta}=e_\nu^{\ \alpha}e^{\nu'}_{\ \beta}\Gamma^\nu_{\mu\nu'}-e^\nu_{\ \beta}\partial_\mu e_\nu^{\ \alpha}$ is the spin connection [40]. Finally, the combinations $\gamma^0\gamma^1$, $\gamma^0\gamma^2$, and γ^0 correspond to the usual Pauli matrices σ_x, σ_y, and σ_z, respectively. In Eq. (3), we have introduced the metric $g_{\mu\nu}$, whose determinant, in modulus, is g. The factor $\sqrt{g}$ ensures invariance of the action under changes of coordinates. Throughout this Letter, we use the greek indices $\mu,\nu=0,1,2$ and $\alpha,\beta,\ldots=\hat{0},\hat{1},\hat{2}$ to denote the environment and locally flat (or internal) coordinates, respectively. In what follows, when we refer to spacelike directions only, we will use the latin letters $i,j=1,2$ for the environment coordinates, and $a,b=\hat{1},\hat{2}$ for the internal coordinates. The Minkowski metric in the locally flat space-time is taken to be $\eta_{\alpha\beta}={\rm diag}(+1,-1,-1)$. The environment and flat metrics, $g_{\mu\nu}$ and $\eta_{\alpha\beta}$, respectively, are related by a vielbein field $e_\mu^{\ \alpha}$ according to the identity [40] $g_{\mu\nu}=e_\mu^{\ \alpha}e_\nu^{\ \beta}\eta_{\alpha\beta}$. From Eq. (3), we define the energy-momentum tensor

$$\tau^{\mu}_{\nu} = e_{\nu}^{\alpha}\tau^{\mu}_{\alpha} = -e_{\nu}^{\alpha}\frac{1}{\sqrt{g}}\frac{\delta S}{\delta e_{\mu}^{\alpha}}, \tag{4}$$

and the Hamiltonian acting in an effectively locally flat space-time as

$$H = \int_x \sqrt{g}\bar{\psi}\left[\frac{i}{2}e^{0}_{\alpha}\gamma^{\alpha}\omega_0 + \frac{i}{2}\omega_0\gamma^{\alpha}e^{0}_{\alpha} - \frac{i}{2}e^{j}_{\alpha}\gamma^{\alpha}\overrightarrow{\nabla}_j + \frac{i}{2}\overleftarrow{\nabla}_j\gamma^{\alpha}e^{j}_{\alpha} + m\right]\psi. \tag{5}$$

In curved space-time, the energy current is related to the energy-momentum tensor via $j^i_E = \sqrt{g}\tau^i_{\nu=0} = \sqrt{g}g_{0\mu}\tau^{i\mu}$ [26]. Because of the Lorentz invariance of massive Dirac fermions, $\tau^{\mu\nu} = \tau^{\nu\mu}$. Thus, we can rewrite $j^i_E = \sqrt{g}g_{0\mu}\tau^{\mu i} = \sqrt{g}g_{0\mu}g^{i\nu}\tau^{\mu}_{\nu}$. Assuming a perturbation of the Luttinger's type [17], the vielbein becomes [26] $e^{\hat{0}}_{\mu} = \delta^{\hat{0}}_{\mu}(1+\phi_g)$, and $e^{a}_{\mu} = \delta^{a}_{\mu}$. In this case, the expectation value of the energy current is given by [41]

$$j^2_E(x) = -[1+\phi_g(x)]^2\frac{\delta F}{\delta e^{\hat{2}}_0(x)}. \tag{6}$$

The partition function and the free energy are defined according to the usual relations as $Z = \mathrm{Tr}(e^{-\beta H})$ and $F = -\beta^{-1}\ln Z$, respectively. Here, $\mathrm{Tr}(\dots)$ denotes the trace in the Fock space [42]. To derive these equations, we used that $\delta e^0_{\alpha}/\delta e^{\hat{2}}_0 = 0$ and $\delta\sqrt{g}/\delta e^{\hat{2}}_0 = 0$, which we rigorously demonstrate to hold true for a perturbation of the Luttinger's type in the Supplemental Material [41].

Boundary fermions. Consider a boundary at $x^1 = 0$ between the gapped bulk at $x^1 < 0$ with negative mass and that at $x^1 > 0$ with positive mass (see also Fig. 1). The boundary is extended in the whole x^2 direction. The Hamiltonian for boundary fermions can be derived by employing the standard method used in Ref. [23]. Details are given in the Supplemental Material [41] for completeness. The full Hamiltonian of Eq. (5) is split it into three parts, $H = \int d^2x\psi^{\dagger}[\mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2]\psi$, where we defined $e_{\mu}^{\alpha} = \delta_{\mu}^{\alpha} + h_{\mu}^{\alpha}/2$, $e^{\mu}_{\alpha} = \delta^{\mu}_{\alpha} - h^{\mu}_{\alpha}/2$, and $h/2 = \sqrt{g} - 1$,

$$\mathcal{H}_0 = -\frac{i}{2}\gamma^{\hat{0}}\delta^j_{\alpha}\gamma^{\alpha}\overrightarrow{\partial}_j + \frac{i}{2}\overleftarrow{\partial}_j\gamma^{\hat{0}}\gamma^{\alpha}\delta^j_{\alpha} + m,$$

$$\mathcal{H}_1 = \frac{h}{2}\mathcal{H}_0 + \left(1+\frac{h}{2}\right)(h^j_{\alpha}/2)\big(i\gamma^{\hat{0}}\gamma^{\alpha}\overrightarrow{\partial}_j - i\overleftarrow{\partial}_j\gamma^{\hat{0}}\gamma^{\alpha}\big),$$

$$\mathcal{H}_2 = \left(1+\frac{h}{2}\right)\frac{i}{2}(\delta^{\mu}_{\alpha} - h^{\mu}_{\alpha}/2)\gamma^{\hat{0}}\big(\gamma^{\alpha}\omega_{\mu} + \omega_{\mu}\gamma^{\alpha}\big). \tag{7}$$

Additionally, we assume the deviations h, h^{α}_{μ}, and h^{μ}_{α} in Eq. (7) to be small. Therefore, in the following calculations we treat $\mathcal{H}_0$ as the unperturbed Hamiltonian, while $\mathcal{H}_1$ and $\mathcal{H}_2$ are treated perturbatively. We assume that the metric depends only on x^2 near the boundary. Thus, in Luttinger's case [17], $\phi_g(x) = \phi_g(x^2)$. Then the two directions x^1 and x^2 are completely decoupled in the boundary Hamiltonian. The wave function of the boundary mode obtained from the Hamiltonian $\mathcal{H}_0$ then factorizes into the product of a plane wave in the x^2 direction, $\psi_2(x^2)$, and of a two-components evanescent spinor wave function in the x^1 direction, $\psi_1(x^1)$: $\psi(x^1,x^2) = \psi_2(x^2)\psi_1(x^1)$. The formal solution of the evanescent spinor is given by [23]

$$\psi_1(x^1) = \exp\left[i\gamma^1\int_0^{x^1}dx'^1 m(x'^1)\right]|s\rangle. \tag{8}$$

The two-component spinor $|s\rangle$ corresponding to the boundary state satisfies $i\gamma^1|s\rangle = \mathrm{sgn}(m)|s\rangle$, where $\mathrm{sgn}(m)$ indicates the sign of the mass in the half space $x^1 < 0$. The other eigenstate of $i\gamma^1$ corresponds to a state that cannot be normalized [23]. Therefore, the boundary Hamiltonian obtained from the unperturbed bulk Hamiltonian $\mathcal{H}_0$ is $\tilde{\mathcal{H}}_0 = i\,\mathrm{sgn}(m)\partial_2$.

The derivation of the interaction terms $\mathcal{H}_1$ and $\mathcal{H}_2$ term is more involved. We therefore relegate it to the Supplemental Material [41] and quote only the final result, i.e., $\tilde{\mathcal{H}}_1 = \zeta(x^2)[-\frac{i}{2}(\overrightarrow{\partial}_2 - \overleftarrow{\partial}_2)]$ and $\tilde{\mathcal{H}}_2 = 0$, with

$$\zeta(x^2) = \tfrac{1}{2}\big(h - h^2_{\hat{2}} - h^2_{\hat{0}}\big) - \tfrac{1}{4}h\big(h^2_{\hat{2}} + h^2_{\hat{0}}\big), \tag{9}$$

where, according to our choice of mass signs, $\mathrm{sgn}(m) = -1$, i.e., the same as in Ref. [23].

We are now in the position to derive the effective boundary free energy at finite temperature, and from it the boundary energy current. To do so, we use the Hamiltonian (7) to write the partition function as $Z = \int\mathcal{D}\psi^*\mathcal{D}\psi\exp(-S^{\mathrm{bdry}}[\psi^*,\psi,\zeta])$ with imaginary-time boundary action [42]

$$S^{\mathrm{bdry}} = \int_{x^2,\tau}\psi^*(x^2,\tau)(\partial_\tau + \tilde{\mathcal{H}}_0 + \tilde{\mathcal{H}}_1)\psi(x^2,\tau), \tag{10}$$

where $\int_{x,\tau} = \int_0^{\beta}d\tau\int dx$. Performing the integration over the fermionic fields, the effective free-energy functional of the gravitational field is obtained as $F^{\mathrm{bdry}}[\zeta] = \beta^{-1}S^{\mathrm{bdry}}[\zeta]$. The effective action can be expressed as $S^{\mathrm{bdry}}[\zeta] = \sum_{l=1}^{\infty}\mathrm{Tr}[(G_0\Sigma)^l]/l$, where the trace is to be taken over real space x^2 and imaginary time τ [41], up to a constant which is independent of ζ. The inverse Green's function and self-energy in momentum space are defined as $G_0^{-1}(k,\tau;k',\tau') = -\delta_{k,k'}(\partial_\tau + k)\delta(\tau,\tau')$ and $\Sigma(k,\tau;k',\tau') = [\zeta(k-k')(k+k')/2]\delta(\tau,\tau')$.

At low temperature, the Fermi distribution function $f(p)$ can be approximated using the Sommerfeld expansion as $f(p) \simeq \theta(-p) - (\pi^2T^2/6)d\delta(p)/dp$. Based on the method provided in Refs. [23,43], after some lengthy algebra [41], we obtain the following complete expression for the boundary free energy up to order T^2 in the long-wavelength limit [44],

$$F^{\mathrm{bdry}}[\zeta] = \frac{\pi T^2}{12}\int_{-\infty}^{\infty}dx^2\frac{\zeta(x^2)}{1+\zeta(x^2)}. \tag{11}$$

This equation is one of the central results of our Letter. In what follows we will use it to derive the boundary energy current and show that it is independent of ϕ_g under the long-wavelength approximation.

Energy current. We begin by recalling Eq. (6) which we now specify for the boundary case $j^{\mathrm{bdry}}_E(x^2) = -2[1+\phi_g(x^2)]^2[\delta F^{\mathrm{bdry}}[\zeta]/\delta h^{\hat{2}}_0(x^2)]$. Therefore, the energy current flowing along the boundary can be read off from the boundary effective free energy in Eq. (11) as

$$j_E^{\mathrm{bdry}}(x^2) = -2\frac{\pi T^2}{12}\frac{[1+\phi_g(x^2)]^2}{[1+\zeta(x^2)]^2}\frac{\delta\zeta(x^2)}{\delta h_0^{\hat{2}}(x^2)}. \tag{12}$$

The derivation of the functional derivative of $\zeta(x^2)$ is in general a difficult task. In the Supplemental Material [41] we have carried it out for the case in which $\zeta(x^2)$ is Luttinger's gravitational potential [17], i.e., a local dilation or contraction of space. Setting $\zeta(x^2) = \phi_g(x^2)$ and we have found that

$$\left.\frac{\delta\zeta(x^2)}{\delta h_0^{\hat{2}}(x^2)}\right|_{e_\mu^{\hat{0}}=\delta_\mu^{\hat{0}}(1+\phi_g),\, e_\mu^{a}=\delta_\mu^{a}} = -\frac{1}{2}. \tag{13}$$

Combining Eqs. (12) and (13), we obtain the energy boundary current under the long-wavelength approximation given previously in Eq. (2).

Finally, we can use Eq. (2) to calculate the system's thermal Hall conductivity based on the Streda formula [23,38,45]. Because of the definition of energy magnetization M_E^z in terms of energy current [18], the boundary energy current satisfies the relation $j_E^{\mathrm{bdry}} = -[M_E^z(x^1 = +\infty) - M_E^z(x^1 = -\infty)]$ [23], we get $M_E^z = -\mathrm{sgn}(m)\pi T^2/24$. Here, we restored the sign of the mass m using the fact that the bulk energy magnetization M_E^z is odd under parity transformation, so it has opposite signs in the two half planes: $M_E^z(x^1 = -\infty) = -M_E^z(x^1 = +\infty)$ [23]. Therefore, the thermal Hall conductivity is then given by the thermal generalization of the Streda formula [23,38,45] for the quantized thermal Hall effect,

$$\kappa_H = \frac{\partial M_E^z}{\partial T} = -\mathrm{sgn}(m)\frac{\pi T}{12}. \tag{14}$$

This corresponds to a quantized thermal Hall conductivity with Chern number $C = \mathrm{sgn}(m)/2$.

Conclusion. In this Letter, we consider the boundary modes existing at a domain wall between two (2+1)-dimensional massive Dirac fermion systems of opposite masses [23]. By systematically resumming all-order contributions in powers of the metric tensor at low temperature and in the long-wavelength limit, we have obtained a rigorous expression for the boundary free energy. From this, we have derived the boundary current generated by a gravitational potential of the Luttinger's type (i.e., a local dilation or contraction of space). We find that, at least in the low-temperature region, higher-order corrections significantly alter the results of existing first-order calculations [23]. We show that there is no bulk thermal Hall current proportional to the first derivative of the gravitational potential (i.e., proportional to the temperature gradient). Only the boundary supports such contributions, in agreement with numerical simulations [29]. In other words, tidal forces (higher-order gradients) are necessary in order to induce bulk thermal Hall currents [25]. Beyond the long-wavelength approximation, this method can also be used to calculate the bulk inflow current to higher-order temperature derivatives, such as the nonlinear thermal Hall effect [22,46–48]. Finally, using the generalization of the Streda formula to the thermal Hall effect, we recover the quantized thermal Hall conductivity for (2+1)-dimensional massive Dirac fermions with Chern number equal to sgn$(m)/2$.

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