Letter

Einstein relation for subdiffusive relaxation in Stark chains

P. Prelovšek,[1] S. Nandy,[1,2] and M. Mierzejewski[3]
[1]*Jožef Stefan Institute, SI-1000 Ljubljana, Slovenia*
[2]*Max Planck Institute for the Physics of Complex Systems, D-01187 Dresden, Germany*
[3]*Institute of Theoretical Physics, Faculty of Fundamental Problems of Technology, Wrocław University of Science and Technology, PL-50-370 Wrocław, Poland*

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We investigate chains of interacting spinless fermions subject to a finite external field F (also called Stark chains) and focus on the regime where the charge thermalization follows the subdiffusive hydrodynamics. First, we study reduced models conserving the dipole moment and derive an explicit Einstein relation which links the subdiffusive transport coefficient with the correlations of the dipolar current. This relation explains why the decay rate Γ_q of the density modulation with wave vector q shows q^4 dependence. In the case of the Stark model, a similar Einstein relation is also derived and tested using various numerical methods. They confirm an exponential reduction of the transport coefficient with increasing F. On the other hand, our study of the Stark model indicates that upon increasing q there is a crossover from subdiffusive behavior, $\Gamma_q \propto q^4$, to the normal diffusive relaxation, $\Gamma_q \propto q^2$, at the wave vector q^* which vanishes for $F \to 0$.

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Introduction. Macroscopic systems driven by finite external forces/fields are usually described within the extended concept of local equilibrium via local thermodynamic parameters as, e.g., the temperature and chemical potential. Such a description can fail when we are dealing with systems isolated from the environment. Prominent recent examples are those of the Stark systems of interacting fermions, as realized in tilted cold-atom lattices [1–3]. Such systems reveal several novel theoretical insights and challenges, discussed mostly within one-dimensional (1D) models. Since noninteracting particles subject to a finite external field, F, exhibit Stark localization, the problem shares some similarities with the many-body localization (MBL) (for an overview, see Refs. [4–7]) and it is known as the Stark MBL [8–18]. It is also well established that effective models, derived at large F from the Stark model (usually by invoking the Schrieffer-Wolff transformations), can exhibit Hilbert-space fragmentation [19–24] that violates the eigenstate thermalization hypothesis (ETH) [25–27]. In systems with strongly fragmented Hilbert space, the latter can be linked with the emergence of additional integrals of motion [28,29].

Still at moderate F, the cold-atom experiments [1] as well as numerical simulations of 1D models [30] reveal the hydrodynamic relaxation of the inhomogeneous particle distributions towards the steady state, which corresponds to the infinite-temperature ($T \to \infty$) equilibrium. For small wave vectors $q \ll 1$ the relaxation rates follow a particular subdiffusive (SD) law $\Gamma_q \propto q^4$, rather than the normal diffusive behavior, $\Gamma_q \propto q^2$. This is well consistent with the fact that at $F \neq 0$, the dipole moment emerges in macroscopic systems as an additional conserved quantity [2,3,19,30,31] and the phenomenological description can be given in terms of the fracton hydrodynamics [31–36]. Despite the wide consensus that such a description is appropriate for isolated driven systems, so far the theoretical studies are mostly based on phenomenological hydrodynamic approaches. There are so far very few quantitative results for the subdiffusion in microscopic models [30] as well as theoretical attempts to express the SD transport coefficient D_S [1,37] in terms of the response functions.

Here, we present an analysis, focused on the particle/density relaxation and anomalous diffusion in the hydrodynamic regime. We consider the standard 1D Stark model, i.e., the chain of interacting spinless fermions subject to a finite external field F, as well as the effective models which involve extended pair-hopping (EPH) interactions and strictly conserve the dipole moment P. On the one hand, from numerical results for the dynamical density structure factor $S_q(\omega)$, we extract the relaxation rates Γ_q of the density modulation. These rates reveal subdiffusive transport for $q \to 0$ as well as the value of the corresponding SD coefficient D_S. On the other hand, employing the memory-function (MF) formalism, we derive the Einstein relation that expresses $\Gamma_{q\to 0}$ in terms of the uniform ($q = 0$) correlations of the normal current, J_N, and the dipolar current, J_D. If P is conserved, the response function is determined solely by J_D and the relaxation follows the SD relation $\Gamma_q = D_S q^4$. In the EPH model, J_D is a translationally invariant operator that governs the relaxation. In the case of the full Stark model, J_D dominates the response for large L (for $F > 0$) when one observes the emergent conservation of the dipole moment. The Einstein relation as well as the numerical results for D_S in both models are tested with alternative numerical approaches. It should be emphasized that the derived Einstein relations remain valid beyond the considered models and even beyond 1D, which we mostly discuss below. Moreover, our numerical results in Stark chains for larger $q > 0$ reveal the crossover from SD $\Gamma_q \sim D_S q^4$ to normal diffusion $\Gamma_q \sim D_N q^2$ at $q \sim q^*(F)$ with vanishing $q^*(F \to 0)$, consistent with some phenomenological theories [1,37].

Dynamical density-modulation relaxation. In the following, we study two 1D lattice models of interacting spinless fermions, as they emerge in the presence of the finite external field F, whereby the chain has L sites and open boundary conditions (OBCs). The coupling to the field enters the Hamiltonian via $H' = FP$ where P is the dipole moment, $P = \sum_l (l - L/2) n_l$, and n_l is the particle number operator at site l.

Isolated macroscopic Stark systems at finite $F > 0$ develop (heat up) towards a homogeneous steady state $\langle n_l \rangle = \bar{n} = N/L$ (N representing the total particle number), corresponding to $T \to \infty$ equilibrium. Further on, we analyze the dynamics of the periodic density modulation $n_q = \sum_l e^{iql} \tilde{n}_l / \sqrt{L}$, $\tilde{n}_l = n_l - \bar{n}$, and related correlation function $\phi_q(\omega)$,

$$\phi_q(\omega) = \frac{\chi_q(\omega) - \chi_q^0}{\omega} = \frac{-\chi_q^0}{\omega + M_q(\omega)},$$
$$\chi_q(\omega) = \frac{i}{\beta} \int_0^\infty dt e^{i\omega t} \langle [n_{-q}(t), n_q] \rangle, \tag{1}$$

whereby we define dynamical susceptibilities $\chi_q(\omega)$ and $\chi_q^0 = \chi_q(\omega = 0)$ that remain nonzero even at $\beta = 1/T \to 0$. In this limit, $\phi_q(\omega)$ is related with the standard dynamical structure factor $S_q(\omega) = \mathrm{Im}\,\phi_q(\omega)/\pi$. In general, $\phi_q(\omega)$ can be represented in terms of the memory function (MF), $M_q(\omega)$, that determines the relaxation rate of the density modulation, $\Gamma_q = \mathrm{Im}\, M_q(\omega = 0)$, which we later on extract also from numerical results for $\phi_q(\omega)$.

Einstein relation. An analytical step towards $M_q(\omega)$ can be made using the MF formalism [38], discussed in the textbook [39] as well as in Refs. [40,41]. It introduces the scalar product of two operators, $(A|B)$, and the Liouville operator $\mathcal{L}A = [H, A]$. In the case of $\beta \to 0$, the latter scalar product reduces to thermodynamic average, i.e., $(A|B) \sim \langle A^\dagger B \rangle$. Within this formalism one can express the correlation function as $\phi_q(\omega) = (n_q|(\mathcal{L} - \omega)^{-1}|n_q)$. The memory function can be written in the hydrodynamic regime $q \to 0$ [39] (in analogy to the perturbation theory [40]) as

$$M_q(\omega) = (\mathcal{L} n_q|(\mathcal{L} - \omega)^{-1}|\mathcal{L} n_q)/\chi_q^0. \tag{2}$$

Expanding n_q in powers of q, one obtains

$$\mathcal{L} n_q \simeq \frac{1}{\sqrt{L}} \left(iq\mathcal{L}P + iq^2 \frac{i}{2} \mathcal{L}Q \right), \quad Q = \sum_l l^2 \tilde{n}_l, \tag{3}$$

where we assumed conservation of the particle number, $\mathcal{L}N = 0$ with $N = \sum_l n_l$. The first term in Eq. (3) represents the normal (uniform) current, $J_N = i\mathcal{L}P$. It determines the hydrodynamic relaxation ($q \to 0$) in generic systems that do not conserve the dipole moment, $\mathcal{L}P \neq 0$. Namely, one obtains from Eq. (2) the standard Einstein relation [39,42–44] $\Gamma_q = q^2\, \mathrm{Im}\,\phi_N(0)/\chi_0^0 = D_N q^2$, which links the diffusion constant D_N with the current correlation function $\phi_N(\omega) = (J_N|(\mathcal{L} - \omega)^{-1}|J_N)/L$.

If the dipole moment is conserved, $\mathcal{L}P = 0$, then the hydrodynamic relaxation is determined by the second term in the expansion in Eq. (3), which can be interpreted as the dipolar current $J_D = \frac{i}{2}\mathcal{L}Q$. Similarly, one then obtains the Einstein relation $\Gamma_q = D_S q^4$ with $D_S = \mathrm{Im}\,\phi_D(0)/\chi_0^0$, however, involving the dipolar currents $\phi_D(\omega) = (J_D|(\mathcal{L} - \omega)^{-1}|J_D)/L$. To conclude this part, we note that the MF formalism straightforwardly explains the origin of SD in systems with dipole-moment conservation, $\Gamma_q = D_S q^4$. Similarly to the phenomenological fracton hydrodynamics [31–36], the only assumption we made is the conservation of the dipole moment.

Extended Pair-Hopping (EPH) model. As a first example we analyze the model where P is strictly a conserved. Starting from the full Stark models, one can derive at $F \gtrsim 1$ the EPH model, either via the Schrieffer-Wolff transformation [19,20,31], or expanding the interaction in the Stark basis [29],

$$H_{\mathrm{EPH}} = \sum_l \zeta_{dr} [c_{l-r}^\dagger c_l c_{l+d+r}^\dagger c_{l+d} + \mathrm{H.c.}] + H_d. \tag{4}$$

Here, $c_l^\dagger, c_l$ refer to localized Stark states and H_d represents the Hartree-Fock diagonal term, while ζ_{dr} can be derived for given F, at least to lowest order in the interaction [29]. Here, by construction $\mathcal{L}P = 0$. One can derive explicit expression for the dipolar current $J_D = \frac{i}{2}\mathcal{L}Q = \sum_{dr} \zeta_{dr} J_D(d, r)$, where

$$J_D(d, r) = -r(r + d) \sum_j (ic_{j-r}^\dagger c_j c_{j+d+r}^\dagger c_{j+d} + \mathrm{H.c.}). \tag{5}$$

It is remarkable that explicit l dependence cancels out and J_D emerges as a translationally invariant operator. In the Supplemental Material [45] (including Ref. [46]) we show that it is a general property of translationally invariant models which conserve the particle number and the dipole moment.

Here, we do not aim to investigate closer the EPH models with realistic parameters, but rather test the existence of SD and the direct expression for the coefficient D_S for a simplified case with $H_d = 0$ and (rather arbitrarily) assuming $r = 1$ and values $\zeta_{d1} = 1, 0.75, 0.5, 0.25$ for $d = 1, \ldots, 4$. The motivation for including longer-range $d > 1$ terms is that the basic pair-hopping model with only ζ_{11} is known to exhibit strong Hilbert-space fragmentation [19,20,22,23] which invalidates the basic ETH concept, while additional terms with $\zeta_{d>1,r>1}$ are expected to suppress this effect [29]. In the following we calculate numerically $\phi_q(\omega)$, Eq. (1), using the microcanonical Lanczos method (MCLM) for finite L systems [47–49], employing a large number of Lanczos steps up to $N_L \sim 10^5$, to achieve the frequency resolution $\delta\omega \lesssim 10^{-4}$ which allows for a reliable extraction of $M_q(\omega)$ [49] even for small $q \ll 1$. The advantage of the EPH model (relative to the Stark model) is that in addition to N we use also the conservation of P (choosing the largest sector with $P = 0$) to reduce the Hilbert space and to reach $L = 32$ with $N_{st} \sim 10^7$ basis states. We expect that averaging over all sectors should give similar results, in analogy to the equivalence of canonical and microcanonical ensembles.

In Fig. 1 we present results for the density structure factor $q^4 S_q(\omega)$ as calculated via MCLM for the two lowest $q = q_m = 2m\pi/L, m = 1, 2$. Since we consider the half-filled system $\bar{n} = 1/2$ with effective $T \to \infty$, we know analytically $\chi_q^0 \sim \chi_0^0 \sim \bar{n}(1 - \bar{n}) = 1/4$. Results confirm a very sharp peak at $\omega \sim 0$, being consistent with SD hydrodynamics, i.e., $\pi S_q(\omega \sim 0) \sim \chi_q^0/\Gamma_q \sim \chi_0^0/(D_S q^4)$. Moreover, we extract also the corresponding (dynamical) SD coefficient $\tilde{D}_S(\omega) = \mathrm{Im}\, M_q(\omega)/q^4$ for the smallest $q = q_1$ and present the results in the inset, together with the numerically evaluated Einstein relation $D_S(\omega) = \mathrm{Im}\,\phi_D(\omega)/\chi_0^0$, using J_D from

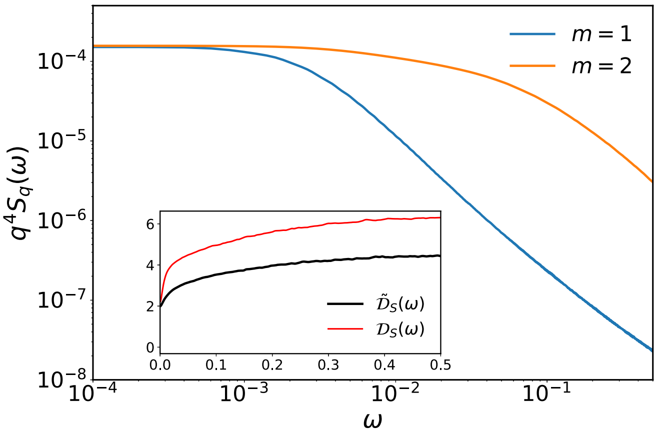

FIG. 1. Dynamical structure factor $q^4 S_q(\omega)$ for the EPH model with $L = 32$ sites, obtained via MCLM for the lowest $q = q_m = 2m\pi/L$, $m = 1, 2$. The inset shows the extracted $\tilde{D}_S(\omega) = \mathrm{Im}\, M_{q_1}(\omega)/q_1^4$, compared to the Einstein-relation result $D_S(\omega) = \mathrm{Im}\, \phi_D(\omega)/\chi_0^0$.

Eq. (5). The agreement is reasonable given that both numerical approaches can suffer from finite-size effects. Moreover, the considered EPH model can still exhibit some features of the Hilbert-space fragmentation [19,20,22,23], which could influence the presumed ETH.

Stark model. We turn further to the properties of the prototype Stark model, i.e., a 1D chain of interacting spinless fermions in the presence of a finite external field F,

$$H = t \sum_i (c_{l+1}^\dagger c_l + c_l^\dagger c_{l+1}) + V \sum_l \tilde{n}_{l+1} \tilde{n}_l + V' \sum_l \tilde{n}_{l+2} \tilde{n}_l + FP, \tag{6}$$

with $\tilde{n}_l = n_l - \bar{n}$, $n_l = c_l^\dagger c_l$. Fermions interact via the nearest-neighbor (V) and next-nearest-neighbor (V') repulsion. We consider half filling, i.e., $\bar{n} = N/L = 1/2$, and set $t = 1$ as the unit of energy. We introduce $V' \neq 0$ in order to suppress the integrability (and dissipationless transport) at $F \to 0$, although the latter effect appears not to be important for $F \gg 0$. In the main text, we restrict the numerical results to the case $V = V' = 1$ while in the Supplemental Material [45] we discuss also results for $V = 2$, $V' = 0$.

In the case of the Stark model, Eq. (6), we cannot apply the same analysis as for the EPH model, since $\mathcal{L}P \neq 0$ and the conservation of P emerges only in the thermodynamic limit, $L \to \infty$ [30]. Still, one can derive from the Hamiltonian (6) both contributions to $\mathcal{L}n_q$ in Eq. (3): $J_N = i\mathcal{L}P = \sum_l J_l$ and $J_D = \frac{i}{2}\mathcal{L}Q = J_N/2 + \sum_l lJ_l$, where $J_l = itc_l^\dagger c_{l+1} + \mathrm{H.c.}$ Neglecting possible off-diagonal correlations, we can then express the corresponding Einstein relation from Eq. (2),

$$M_q(\omega) \simeq [q^2 \phi_N(\omega) + q^4 \phi_D(\omega)]/\chi_0^0. \tag{7}$$

Since in general $\phi_N(\omega) \neq 0$, we can proceed by showing that for finite $F > 0$ and $L \to \infty$ one obtains $\phi_N(\omega \to 0) = 0$. The latter result is related to the emergent conservation of P [30], and is verified also numerically in the Supplemental Material [45]. Consequently, we expect that at $F > 0$ the hydrodynamic $q \to 0$ behavior will be dominated by SD with $\Gamma_q = D_S q^4$ with $D_S = \mathrm{Im}\, \phi_D(\omega \to 0)/\chi_0^0$.

We present numerical results for $\mathrm{Im}\, \phi_D(\omega)$ in Fig. 2(a). A sharp decline of $\phi_D(\omega)$ at $\omega \simeq 0$ is a finite-size effect. By construction J_D requires OBC, however, there is no strictly steady ($\omega = 0$) current in finite systems with OBC, as discussed in Ref. [50]. Still, results in Fig. 2(a), obtained with MCLM on an $L = 28$ chain, indicate that beyond $F > F_*(L) \gtrsim 0.4$, there is a well-defined value $D_S = \mathrm{Im}\, \phi_D(\omega \to 0)/\chi_0^0$, revealing already an exponential-like dependence on F. We also note that $\mathrm{Im}\, \phi_D(\omega)$ for larger ω, as in Fig. 2(a), does not have a direct relation to $M_q(\omega)$, since it neglects the $\phi_N(\omega)$ contribution in Eq. (7).

Results for the transport coefficient $D_S(F)$ are summarized in Fig. 2(b). Besides the results from the Einstein relation evaluated via $\phi_D(\omega \to 0)$ [see Fig. 2(a)], we include also results for $D_S = \Gamma_q/q^4$ obtained from two alternative approaches applied for the $L = 24$ chain and the smallest $q_1 = 2\pi/L$. Namely, we extract Γ_q directly from the MCLM results for $S(q, \omega \to 0)$, as well as from the decay of the inhomogeneous density profile where F is introduced via the time-dependent flux [30,51]. The latter method evaluates the relaxation rate Γ_q (see Ref. [52] for the details), with the advantage of periodic boundary conditions and consequently resolving also very small D_S, i.e., reaching larger $F \simeq 2$. Results in Fig. 2(b) are

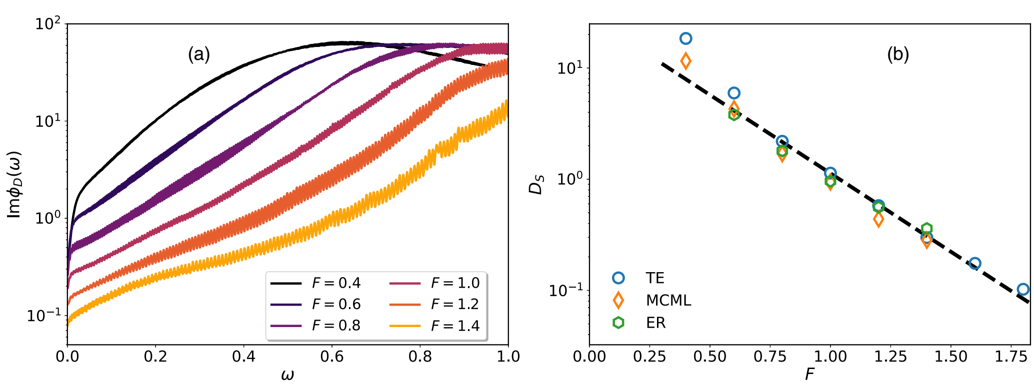

FIG. 2. (a) Dipolar-current correlations $\mathrm{Im}\, \phi_D(\omega)$ (in log scale) as calculated with MLCM for the Stark model on $L = 28$ sites for different fields F. (b) Subdiffusion coefficient $D_S(F)$, calculated from the rates Γ_{q_1}, obtained via time evolution (TE) of the density profile and via MCLM on an $L = 24$ chain, and directly from the Einstein relation (ER), i.e., from $\phi_D(\omega)$.

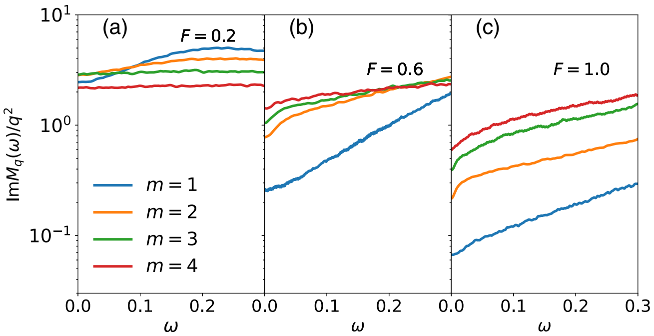

FIG. 3. Dynamical relaxation rates $\Gamma_q(\omega)/q^2$ (in log scale), as extracted from MCLM results for $\phi_q(\omega)$ on an $L = 24$ chain and $q_m = 2m\pi/L$, $m = 1$–4, for three different regimes of F.

quantitatively consistent in the broad regime of $F > F_*(L) \sim 0.4$, confirming the validity of the Einstein relation for SD transport as well as the exponential dependence of $D_S(F)$. In the Supplemental Material [45] we study also boundary-driven open systems [30,52] allowing for the analysis of considerably larger $L \leqslant 50$.

Crossover to normal diffusion. Let us finally examine closer $M_q(\omega)$ at larger q as extracted numerically from $\phi_q(\omega)$, again for parameters $V = V' = 1$. Figure 3 shows MCLM results for $\mathrm{Im}\, M_q(\omega)/q^2$ obtained for various $q = q_m = 2m\pi/L$ with $1 \leqslant m \leqslant 4$ and $L = 24$ so that $q \leqslant \pi/3$. One can resolve three regimes. Despite finite-size limitations, results at small $F = 0.2 < F_*(L)$, shown in Fig. 3(a), are approximately consistent with normal diffusion for all presented q_m, i.e., $\mathrm{Im}\, M_q(\omega) \propto q^2$. At intermediate $F_*(L) < F < F_c \sim 1$, as in Fig. 3(b), only the smallest $q = q_1$ evidently deviates, the latter being the signature of the SD transport $\mathrm{Im}\, M_q(\omega \sim 0) \propto q^4$. Still, the relaxation functions $\mathrm{Im}\, M_q(\omega)/q^2$ nearly overlap for larger q which can be interpreted as an effective normal diffusion $\mathrm{Im}\, M_q(\omega) \propto q^2$. Finally, for large $F \gtrsim F_c$, as in Fig. 3(c), an anomalous SD-like relaxation appears for all $q < 1$.

In Fig. 4(a) we collect results for an effective normal diffusion coefficient $\tilde{D}_N = \mathrm{Im}\, M_q(\omega = 0)/q^2$ at largest $q = q_4 = \pi/3$ as in Fig 3. Taking results for D_S from Fig. 2(b) and $\tilde{D}_N$ from Fig. 4(a) one may estimate that the crossover between SD and diffusive relaxations is expected at the wave vector q^* fulfilling the relation $D_S(q^*)^4 = \tilde{D}_N(q^*)^2$. Numerical results from such an estimate are shown in Fig. 4(b), representing a rough phase diagram of normal-SD transport, relevant at least for moderate $F < F_c$. While we expect a continuous vanishing of q^* for $F \to 0$, it is hard to numerically determine the dependence $q^*(F \ll 1)$ due to finite-size limitations. Still, the general trend of $q^*(F)$ is well consistent with phenomenological approaches [1,37]. We should also note that the explicit MF expression, Eq. (7), is restricted only to the hydrodynamic regime $q \to 0$ when $M_q(\omega)$ is small and $\phi_N(\omega \to 0)$ strictly vanishes. Therefore, it cannot be extended to the discussion of the normal/SD crossover at larger q.

Conclusions. We have analyzed the Stark models using the memory-function approach to hydrodynamics. It enables a direct consideration of the subdiffusion and the corresponding transport coefficient D_S. In models with a strictly conserved dipole moment P, the derivation yields a subdiffusive relaxation rate $\Gamma_q = D_S q^4$ and an explicit Einstein relation which links D_S with correlations of the uniform ($q = 0$) dipolar current J_D. In the full Stark problem, where P is conserved only in the thermodynamic limit, the analogous treatment is valid only in a more restricted hydrodynamic regime $q \to 0$. However, the correlations of the dipolar current are still linked with D_S via the Einstein relation. Moreover, the obtained numerical results agree with alternative numerical approaches, at least in the range of F where the finite-size limitations allow for reliable numerical studies. As a general observation, we note that the coefficient D_S reveals a strong exponential-like reduction with F, which for large $F \gg 1$ can effectively appear as the Stark MBL concept [8–18], as well as the Hilbert-space fragmentation [19–24]. It should be stressed that the presented Einstein relations are not specific to the considered model and even not to one-dimensional systems. They can be easily generalized to models which are more relevant for experiments as, e.g., the tilted Fermi-Hubbard model.

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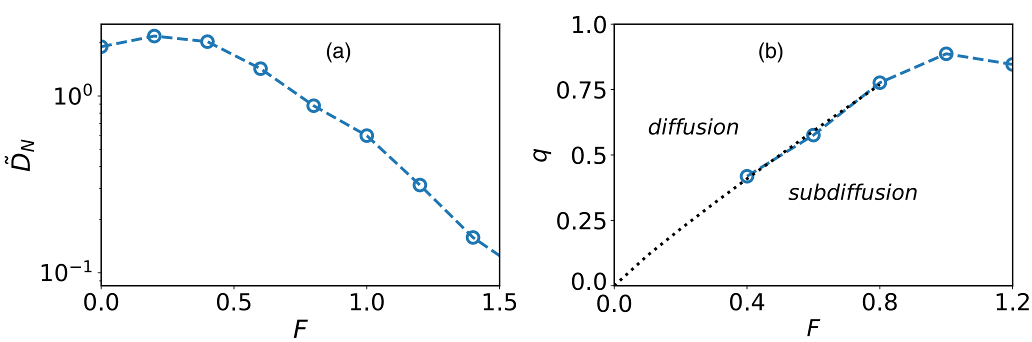

FIG. 4. (a) Effective normal diffusion coefficient $\tilde{D}_N = \Gamma_q/q^2$ for $q = \pi/3$ vs F, evaluated at $q = \pi/3$ via MCLM on $L = 24$ sites. (b) Effective phase diagram $q(F)$ with the SD/normal diffusion crossover at $q^*(F)$, with the dotted line representing the qualitative guess for small F.

The numerical calculations were partly carried out at the facilities of the Wroclaw Centre for Networking and Supercomputing. Our TEBD code was written using the ITENSORS library in JULIA [53].

[1] E. Guardado-Sanchez, A. Morningstar, B. M. Spar, P. T. Brown, D. A. Huse, and W. S. Bakr, Subdiffusion and heat transport in a tilted two-dimensional Fermi-Hubbard system, Phys. Rev. X **10**, 011042 (2020).

[2] S. Scherg, T. Kohlert, P. Sala, F. Pollmann, B. Hebbe Madhusudhana, I. Bloch, and M. Aidelsburger, Observing non-ergodicity due to kinetic constraints in tilted Fermi-Hubbard chains, Nat. Commun. **12**, 4490 (2021).

[3] T. Kohlert, S. Scherg, P. Sala, F. Pollmann, B. Hebbe Madhusudhana, I. Bloch, and M. Aidelsburger, Exploring the regime of fragmentation in strongly tilted Fermi-Hubbard chains, Phys. Rev. Lett. **130**, 010201 (2023).

[4] R. Nandkishore and D. A. Huse, Many-body localization and thermalization in quantum statistical mechanics, Annu. Rev. Condens. Matter Phys. **6**, 15 (2015).

[5] F. Alet and N. Laflorencie, Many-body localization: An introduction and selected topics, C. R. Phys. **19**, 498 (2018).

[6] D. A. Abanin, E. Altman, I. Bloch, and M. Serbyn, *Colloquium:* Many-body localization, thermalization, and entanglement, Rev. Mod. Phys. **91**, 021001 (2019).

[7] P. Sierant, M. Lewenstein, A. Scardicchio, L. Vidmar, and J. Zakrzewski, Many-body localization in the age of classical computing, arXiv:2403.07111.

[8] M. Schulz, C. A. Hooley, R. Moessner, and F. Pollmann, Stark many-body localization, Phys. Rev. Lett. **122**, 040606 (2019).

[9] E. van Nieuwenburg, Y. Baum, and G. Rafael, From Bloch oscillations to many-body localization in clean interacting systems, Proc. Natl. Acad. Sci. USA **116**, 9269 (2019).

[10] S. R. Taylor, M. Schulz, F. Pollmann, and R. Moessner, Experimental probes of Stark many-body localization, Phys. Rev. B **102**, 054206 (2020).

[11] D. S. Bhakuni, R. Nehra, and A. Sharma, Drive-induced many-body localization and coherent destruction of Stark many-body localization, Phys. Rev. B **102**, 024201 (2020).

[12] L. Zhang, Y. Ke, W. Liu, and C. Lee, Mobility edge of Stark many-body localization, Phys. Rev. A **103**, 023323 (2021).

[13] W. Morong, F. Liu, P. Becker, K. S. Collins, L. Feng, A. Kyprianidis, G. Pagano, T. You, A. V. Gorshkov, and C. Monroe, Observation of Stark many-body localization without disorder, Nature (London) **599**, 393 (2021).

[14] Y.-Y. Wang, Z.-H. Sun, and H. Fan, Stark many-body localization transitions in superconducting circuits, Phys. Rev. B **104**, 205122 (2021).

[15] E. V. H. Doggen, I. V. Gornyi, and D. G. Polyakov, Stark many-body localization: Evidence for Hilbert-space shattering, Phys. Rev. B **103**, L100202 (2021).

[16] T. M. Gunawardana and B. Buča, Dynamical l-bits and persistent oscillations in Stark many-body localization, Phys. Rev. B **106**, L161111 (2022).

[17] G. Zisling, D. M. Kennes, and Y. B. Lev, Transport in Stark many-body localized systems, Phys. Rev. B **105**, L140201 (2022).

[18] X. Wei, X. Gao, and W. Zhu, Static and dynamical Stark many-body localization transition in a linear potential, Phys. Rev. B **106**, 134207 (2022).

[19] P. Sala, T. Rakovszky, R. Verresen, M. Knap, and F. Pollmann, Ergodicity breaking arising from Hilbert space fragmentation in dipole-conserving Hamiltonians, Phys. Rev. X **10**, 011047 (2020).

[20] V. Khemani, M. Hermele, and R. Nandkishore, Localization from Hilbert space shattering: From theory to physical realizations, Phys. Rev. B **101**, 174204 (2020).

[21] Z.-C. Yang, F. Liu, A. V. Gorshkov, and T. Iadecola, Hilbert-space fragmentation from strict confinement, Phys. Rev. Lett. **124**, 207602 (2020).

[22] S. Moudgalya and O. I. Motrunich, Hilbert space fragmentation and commutant algebras, Phys. Rev. X **12**, 011050 (2022).

[23] S. Moudgalya, B. A. Bernevig, and N. Regnault, Quantum many-body scars and Hilbert space fragmentation: A review of exact results, Rep. Prog. Phys. **85**, 086501 (2022).

[24] X. Feng and B. Skinner, Hilbert space fragmentation produces an effective attraction between fractons, Phys. Rev. Res. **4**, 013053 (2022).

[25] J. M. Deutsch, Quantum statistical mechanics in a closed system, Phys. Rev. A **43**, 2046 (1991).

[26] M. Srednicki, Chaos and quantum thermalization, Phys. Rev. E **50**, 888 (1994).

[27] L. D'Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol, From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics, Adv. Phys. **65**, 239 (2016).

[28] T. Rakovszky, P. Sala, R. Verresen, M. Knap, and F. Pollmann, Statistical localization: From strong fragmentation to strong edge modes, Phys. Rev. B **101**, 125126 (2020).

[29] P. Łydżba, P. Prelovšek, and M. Mierzejewski, Local integrals of motion in dipole-conserving models with Hilbert space fragmentation, Phys. Rev. Lett. **132**, 220405 (2024).

[30] S. Nandy, Z. Lenarčič, E. Ilievski, M. Mierzejewski, J. Herbrych, and P. Prelovšek, Spin diffusion in a perturbed isotropic Heisenberg spin chain, Phys. Rev. B **108**, L081115 (2023).

[31] J. Feldmeier, P. Sala, G. De Tomasi, F. Pollmann, and M. Knap, Anomalous diffusion in dipole- and higher-moment-conserving systems, Phys. Rev. Lett. **125**, 245303 (2020).

[32] R. M. Nandkishore and M. Hermele, Fractons, Annu. Rev. Condens. Matter Phys. **10**, 295 (2019).

[33] A. Gromov, A. Lucas, and R. M. Nandkishore, Fracton hydrodynamics, Phys. Rev. Res. **2**, 033124 (2020).

[34] A. G. Burchards, J. Feldmeier, A. Schuckert, and M. Knap, Coupled hydrodynamics in dipole-conserving quantum systems, Phys. Rev. B **105**, 205127 (2022).

[35] J. Guo, P. Glorioso, and A. Lucas, Fracton hydrodynamics without time-reversal symmetry, Phys. Rev. Lett. **129**, 150603 (2022).

[36] A. Głódkowski, F. Peña-Benítez, and P. Surówka, Hydrodynamics of dipole-conserving fluids, Phys. Rev. E **107**, 034142 (2023).

[37] P. Zechmann, A. Bastianello, and M. Knap, Tunable transport in the mass-imbalanced Fermi-Hubbard model, Phys. Rev. B **106**, 075115 (2022).

[38] H. Mori, Transport, collective motion, and Brownian motion, Prog. Theor. Phys. **33**, 423 (1965).

[39] D. Forster, *Hydrodynamic Fluctuations, Broken Symmetry and Correlation Functions* (Taylor & Francis/CRC Press, Boca Raton, FL, 1975).

[40] W. Götze and P. Wölfle, Homogeneous dynamical conductivity of simple metals, Phys. Rev. B **6**, 1226 (1972).

[41] P. Jung and A. Rosch, Lower bounds for the conductivities of correlated quantum systems, Phys. Rev. B **75**, 245104 (2007).

[42] L. P. Kadanoff and P. C. Martin, Hydrodynamic equations and correlation functions, Ann. Phys. **24**, 419 (1963).

[43] J. Bonča and J. Jaklič, Spin diffusion of the t-J model, Phys. Rev. B **51**, 16083 (1995).

[44] R. Steinigeweg and J. Gemmer, Density dynamics in translationally invariant spin-1/2 chains at high temperatures: A current-autocorrelation approach to finite time and length scales, Phys. Rev. B **80**, 184402 (2009).

[45] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.110.L081105 for the analysis of the relaxation of dipole moment and related current correlations, translational invariance of the dipolar current, as well as for results for other model parameters, involving also the additional numerical approach using the boundary-driven open systems.

[46] P. Prelovšek, S. Nandy, Z. Lenarčič, M. Mierzejewski, and J. Herbrych, From dissipationless to normal diffusion in the easy-axis Heisenberg spin chain, Phys. Rev. B **106**, 245104 (2022).

[47] M. W. Long, P. Prelovšek, S. El Shawish, J. Karadamoglou, and X. Zotos, Finite-temperature dynamical correlations using the microcanonical ensemble and the Lanczos algorithm, Phys. Rev. B **68**, 235106 (2003).

[48] P. Prelovšek and J. Bonča, Ground state and finite temperature Lanczos methods, in *Strongly Correlated Systems - Numerical Methods*, edited by A. Avella and F. Mancini (Springer, Berlin, 2013).

[49] J. Herbrych, R. Steinigeweg, and P. Prelovšek, Spin hydrodynamics in the $S = \frac{1}{2}$ anisotropic Heisenberg chain, Phys. Rev. B **86**, 115106 (2012).

[50] M. Rigol and B. S. Shastry, Drude weight in systems with open boundary conditions, Phys. Rev. B **77**, 161101(R) (2008).

[51] M. Mierzejewski and P. Prelovšek, Nonlinear current response of an isolated system of interacting fermions, Phys. Rev. Lett. **105**, 186405 (2010).

[52] S. Nandy, J. Herbrych, Z. Lenarčič, A. Głódkowski, P. Prelovšek, and M. Mierzejewski, Emergent dipole moment conservation and subdiffusion in tilted chains, Phys. Rev. B **109**, 115120 (2024).

[53] M. Fishman, S. R. White, and E. M. Stoudenmire, The ITensor software library for tensor network calculations, SciPost Phys. Codebases, 4 (2022).