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Letter | Editors' Suggestion

Thermal transport across a Josephson junction in a dissipative environment

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At zero temperature, a Josephson junction coupled to an ohmic environment displays a quantum phase transition between superconducting and insulating phases, depending on whether the resistance of the environment is below or above the resistance quantum. This so-called Schmid transition, representative of the effect in a broad class of quantum impurity problems, turns into a crossover at finite temperatures. We determine the conditions under which the temperature dependence of the thermal conductance, which characterizes heat flow from a hot to cold resistor across the Josephson junction, displays a universal scaling characteristic of the Schmid transition. We also discuss conditions for heat rectification to happen in the circuit. Our work can serve as a guide for identifying signatures of the Schmid transition in heat transport experiments.

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Introduction. The Schmid transition between superconducting and insulating ground states is controlled by the dimensionless ratio of the resistance of the environment "seen" by a Josephson junction to the resistance quantum, $R_Q = \pi\hbar/2e^2$ [1]. This quantum phase transition affects the charge and energy transport across the Josephson junction, but not the intensive characteristics of the entire system of which the junction is a part. This subtlety of the Schmid transition, representative of a broad class of quantum impurity systems, results in a peculiar evolution in the scaling behavior of measurable quantities across the phase transition point.

The Schmid transition is currently attracting much attention both experimentally and theoretically [2–14]. The lack of clear theoretical predictions for scaling have led to the publication of erroneous conceptual claims supported by imperfect computer simulations, and to a growing confusion with the interpretation of the experimental results on charge and heat transport. A recent heat transport experiment performed with a flux-tunable superconducting quantum interference device (SQUID) challenged the existence of the Schmid transition because the thermal conductance was still depending on the flux, despite the circuit being on the insulating side of the transition [6]. According to our study, the dependence of the heat conductance on flux is a part of the expected scaling behavior, and there is no qualitative contradiction between the observations [6] and theory. Apart from resolving the current controversy, the main goal of this Letter is to provide theoretical guidance to future experiments aimed at quantitative studies of the scaling behavior.

Heat transport mediated by microwave photons was observed in superconducting circuits operating at temperatures well below the superconducting transition. Ballistic heat transport characterized by the quantum of thermal conductance $G_q(T) = \pi k_B^2 T/6\hbar$ at temperature T [15] was shown up to distances of 50 μm [16] and 1 m [17] in circuits with matched impedances. These observations raise a prospect of using thermal conductance in a circuit containing a Josephson junction to probe the evolution of energy transport across the Schmid transition.

This type of circuits may also have a practical application as heat valves. Flux-tunable heat valves were realized with superconducting circuits by connecting the reservoirs to a SQUID formed of a loop with two Josephson junctions, and applying a magnetic field through the loop [18]; a heat valve controlled by a gate was demonstrated with a Cooper pair transistor consisting of two junctions separated by a Coulomb island [19]. The existing theory of heat propagation in such devices relies on perturbation theory in the strength of coupling realized by the controllable junction between the heat reservoirs. This limits theory applicability with respect to the range of circuit parameters and temperatures. A comprehensive theory needs to account for the Schmid transition [1].

Outline. In this Letter, we develop the scaling theory for the thermal conductance $G_{\rm th}(T)$ across the Schmid transition, as well as the nonperturbative theory of the heat valve associated with it. To derive the scaling theory, we establish a relation between the thermal conductance $G_{\rm th}(T)$ and the admittance $Y(\omega, T)$ of the superconducting circuit at finite frequency ω and temperature T. This relation accounts for arbitrarily strong inelastic scattering of the photons off the junction and, therefore, goes far beyond the linear input-output theory. (A similar approach was used in finding the thermal conductance across a Kondo impurity [20] with the help of the dynamical susceptibility studied in Refs. [21,22].) After that, we extend the results for the admittance detailed in Ref. [13] to finite temperatures and evaluate $G_{\rm th}(T)$.

The results are most clearly represented by the scaling function $g(t)$,

$$G_{\rm th}(T) = G_0(T)g(t), \quad G_0(T) = (4R_1R_2/R^2)G_q(T), \quad (1)$$

detailing the deviation from the ballistic transport prediction. Here, $t = T/T_\star$ is the temperature normalized by the characteristic scale $T_\star$. The latter depends both on the Josephson junction parameters and a dimensionless parameter that

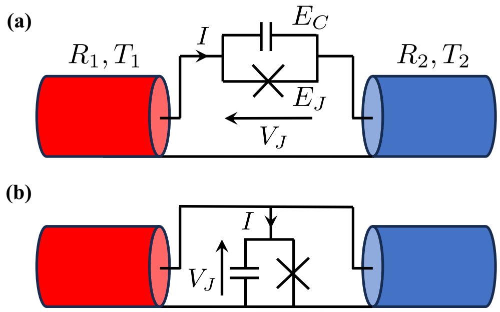

FIG. 1. Two circuits for measuring heat transport. (a) Series configuration, and (b) parallel configuration.

characterizes the wave impedance seen by the junction, $K = R_Q/2R$ with $R = R_1 + R_2$, where R_1 and R_2 are the baths' impedances in the circuit of Fig. 1(a); $G_0(T) = G_q(T)$ at matched impedances, $R_1 = R_2$.

The knowledge of the scaling function $g(t)$ allows us to find $G_{\rm th}(T)$ outside the domain of the previously used perturbation theory. The newly found nonperturbative results include the heat conductance of a high-capacitance Josephson junction (a transmon) and a Cooper pair box (a charge qubit). In a broader context, we relate the overall behavior of $g(t)$ to the Schmid transition: $g(t)$ is a monotonically increasing function of t at $K < 1/2$, its monotonicity is opposite at $K > 1/2$ (see Fig. 2). At the Schmid transition, $g(t)$ is temperature independent. We find analytically the full scaling form of $g(t)$ in the vicinity of the transition at $K = 1/2$ and at the Toulouse point ($K = 1/4$) by mapping to a free-fermion problem [23].

Relation between admittance and heat conductance. Let us start with the formula for the thermal conductance in the series configuration of Fig. 1(a), where two resistances R_1 and R_2 are held at different temperatures T and $T + \Delta T$ and connected by a Josephson junction. In linear response, $\Delta T \ll T$, the heat current from the hot to cold resistor is $P = G_{\rm th}\Delta T$, where $G_{\rm th}$ is the thermal conductance at temperature T. The latter can be related to the complex scattering phase $\delta(\omega, T) = \delta'(\omega, T) + i\delta''(\omega, T)$ off a circuit consisting of a Josephson junction in series with a resistor R, at finite frequency ω and temperature T. Equivalently, one may use the real part of the complex admittance of that circuit, $Y(\omega, T) = [1 - e^{2i\delta(\omega,T)}]/2R$, in order to express $G_{\rm th}$ as

$$G_{\rm th}(T) = \frac{R_1R_2}{\pi R^2}\int_0^\infty d\omega\omega\frac{\partial n(\omega)}{\partial T}[1 - {\rm Re}\, e^{2i\delta(\omega,T)}], \tag{2}$$

where $n(\omega)$ is the Bose function at temperature T. Hereinafter we use units with $\hbar = k_B = 1$.

To derive Eq. (2), we may use boson scattering theory. For this, we describe the resistors that appear in Fig. 1 as transmission lines held at different temperatures T_i ($i = 1, 2$). Using the quantum description of transmission lines in terms of left- and right-moving frequency-resolved quantum modes [24], we introduce incoming and outgoing bosonic modes at frequency ω in the lines, $a_i^{\rm in}(\omega)$ and $a_i^{\rm out}(\omega)$, such that $[a_i^\mu(\omega), a_j^\nu(\omega')] = 0$, $[a_i^\mu(\omega), a_j^{\nu\dagger}(\omega')] = 2\pi\delta_{\nu,\mu}\delta_{i,j}\delta(\omega - \omega')$ ($\mu, \nu = {\rm in, out}$), and the Fourier harmonics of the voltage and current at the line's end in contact with the junction are expressed as

$$V_i(\omega) = \sqrt{\omega R_i/2}\,[a_i^{\rm in}(\omega) + a_i^{\rm out}(\omega)], \tag{3a}$$

$$I_i(\omega) = \sqrt{\omega/2R_i}\,[a_i^{\rm in}(\omega) - a_i^{\rm out}(\omega)]. \tag{3b}$$

In the series configuration of Fig. 1(a), the current flowing through the junction is $I = I_1 = -I_2$, while the voltage across the junction is $V_J = V_1 - V_2$. We then define the elastic scattering matrix at frequency ω, $S(\omega) = \{S_{ij}(\omega)\}$ ($i, j = 1, 2$), such that it relates the incoming and outgoing modes: $a_i^{\rm out} = S_{ij}a_j^{\rm in}$. Elimination of V_i and I_i in favor of I and V_J then allows us to express S in a diagonalized form,

$$S = U^T{\rm diag}(e^{2i\delta}, 1)U, \quad U = \frac{1}{\sqrt{R}}\begin{pmatrix}\sqrt{R_1} & -\sqrt{R_2}\\ \sqrt{R_2} & \sqrt{R_1}\end{pmatrix}, \tag{4}$$

where $e^{2i\delta} = (V_J - RI)/(V_J + RI)$. The unitary matrix U expresses that only one combination of the two lines' modes effectively couples to the junction. Thus, defining the admittance $Y \equiv I/V$ with $V = RI + V_J$, we recover the relation between δ and Y given just above Eq. (2). Here, let us emphasize that δ and Y must be computed under the nonequilibrium conditions fixed by the different temperatures in the leads. For reservoirs connected to a purely reactive dipole, $\delta'' = 0$ and S is unitary, such that boson scattering is purely elastic. In general, however, one should consider inelastic scattering in addition to the elastic cross section between two leads, $\sigma_{12}^{\rm el}(\omega) = (R_1R_2/R^2)|1 - e^{2i\delta(\omega)}|^2$.

To address inelastic scattering, we then introduce the partial inelastic scattering cross section $\sigma_{j|i}(\omega'|\omega)$ for a boson with frequency ω in line i to be converted into a boson with frequency $\omega' < \omega$ in line j. As the junction effectively couples to one combination of the two lines' modes only [see the discussion below Eq. (4)], we can relate $\sigma_{2|1}(\omega'|\omega) = \sigma_{1|2}(\omega'|\omega) = (R_1R_2/R^2)\sigma(\omega'|\omega)$ with the partial inelastic scattering cross section $\sigma(\omega'|\omega)$ off a Josephson junction in series with a single transmission line with resistance R. Energy conservation imposes

$$\int_0^\infty d\omega'\omega'\sigma(\omega'|\omega) = \omega\sigma^{\rm in}(\omega), \tag{5}$$

with a total inelastic cross section $\sigma^{\rm in}(\omega) = 1 - e^{-4\delta''(\omega)}$.

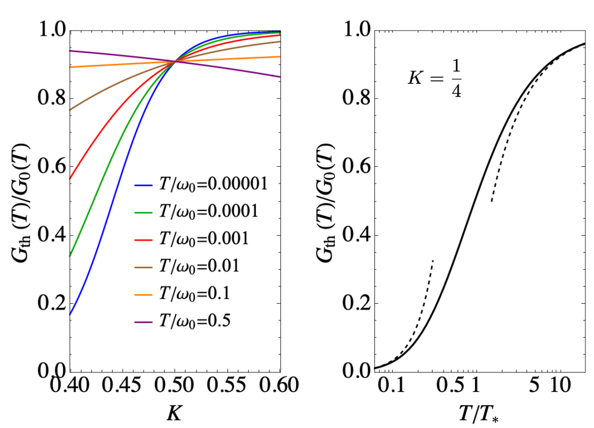

FIG. 2. Left: Thermal conductance as a function of the environment impedance for a transmon with $\lambda/\omega_0 = 0.1$ at various temperatures. Right: Thermal conductance as a function of the temperature at the Toulouse point, $K = \frac{1}{4}$. The dashed lines are the low-T and high-T asymptotes.

We may then use these relations to simplify the heat current between leads 1 and 2,

$$P \equiv \int \frac{d\omega}{2\pi}[n_1(\omega) - n_2(\omega)]\left\{\omega\sigma_{12}^{\rm el}(\omega) + \int d\omega' \sigma_{1|2}(\omega'|\omega)\right\}$$
$$= \frac{R_1 R_2}{\pi R^2}\int d\omega\omega[n_1(\omega) - n_2(\omega)][1 - \mathrm{Re}\, e^{2i\delta(\omega)}], \qquad (6)$$

where $n_i(\omega)$ are Bose functions at temperatures T_i and the admittance should be calculated under the nonequilibrium condition set by the temperature bias. Taking $T = (T_1 + T_2)/2$ and $\Delta T = T_1 - T_2$, we readily recover Eq. (2) at $\Delta T \ll T$, where the admittance is now evaluated in equilibrium at temperature T. So far, nothing was assumed about the circuit connecting the two baths. For a linear circuit with pure elastic scattering, our results match those of Refs. [25,26]. Our formalism also allows recovering the many-body results of Ref. [20] for the ohmic spin-boson description of the Kondo problem. Knowledge of the impedance entering Eq. (6) at equilibrium is also sufficient in the limit of weak coupling, but at arbitrary ΔT, similar to Refs. [20,27].

Performing a similar calculation for the parallel configuration of Fig. 1(b), where two resistances R_1 and R_2 are connected to the same side of a junction that is grounded on its other side, yields the thermal conductance

$$\tilde{G}_{\rm th} = G_0 - G_{\rm th}. \qquad (7)$$

Furthermore, the scattering phase that appears in Eq. (2) for $G_{\rm th}$ should be evaluated for a Josephson junction shunted by an impedance R_1R_2/R, leading to $K = R_Q R/2R_1R_2$. [To derive Eq. (7) we used different relations for the current through the junction, $I = I_1 + I_2$, and the voltage, $V_J = V_1 = V_2$.]

Universal scaling. Here, we specify the results for the circuit of Fig. 1(a), where the junction has Josephson energy E_J and charging energy $E_C = e^2/2C$ with capacitance C. We first focus on a transmon ($E_J \gg E_C$). On the superconducting side of the Schmid transition ($K > \frac{1}{2}$), the phase degree of freedom at the junction is localized in a minimum of the Josephson potential at vanishing temperature and frequency. As a result, a transmon behaves as an inductive short at temperatures below the Josephson plasma frequency, $\omega_0 = \sqrt{8E_JE_C}$, such that $G_{\rm th}(T \ll \omega_0) \approx G_0(T)$.

By contrast, a strong renormalization of $G_{\rm th}(T)$ is expected on the insulating side of the transition ($K < \frac{1}{2}$), when phase slips between minima of the Josephson potential are responsible for phase delocalization. By generalizing our scaling analysis of the finite-frequency admittance [13] to finite temperatures and, then, using Eq. (2), we will show that the thermal conductance can be expressed in the scaling form of Eq. (1) in a broad range of temperatures $T \ll \omega_0$. There, the crossover temperature

$$T_\star = \frac{\omega_0}{2\pi}\left(\sqrt{\frac{2K\Gamma^2(2K)}{\Gamma(4K)}}\frac{\pi\lambda}{\omega_0}\right)^{1/(1-2K)}, \qquad (8)$$

where $\lambda = (8^5E_J^3E_C/\pi^2)^{1/4}e^{-\sqrt{8E_J/E_C}}$ is the phase slip amplitude. $T_\star$ separates a capacitivelike response, $G_{\rm th} \to 0$ at $T \to 0$, from an inductivelike one, $G_{\rm th} \approx G_0(T)$ at $T_\star \ll T \ll \omega_0$. The former regime is indicative of an insulating ground state, while in the latter one the junction is, again, essentially a superconducting short at the relevant frequencies. Below we analyze the scaling function $g(t)$ and provide its low-T and high-T asymptotes. The derivation of the admittance at finite frequency and finite temperature, necessary to find $g(t)$ and fix the prefactors in the asymptotes, is provided in the Supplemental Material [28].

Let us start with the high-T asymptote, $g(t \gg 1) = 1 - a_>(K)/t^{2-4K}$. Here, the power-law temperature scaling mirrors the bias dependence of nonlinear conductance, as well as the temperature or frequency dependence of linear conductance, across a quantum impurity in a Luttinger liquid [23]. In addition, our analysis [28] allows relating the prefactor to the line's impedance,

$$a_>(K) = \frac{3}{\pi^2}\int_0^\infty dx \frac{x/2}{\sinh(x/2)}\frac{|\Gamma(2K + ix/2\pi)|^2}{\Gamma^2(2K)}. \qquad (9)$$

The low-T asymptote takes different expressions depending on K. At $\frac{1}{4} < K < \frac{1}{2}$, the power-law temperature scaling, $g(t \ll 1) = a_<(K)t^{1/K-2}$, mirrors Kane-Fisher theory [23] for a dual description of the transmon valid in vicinity of the insulating ground state [1]. Here again, our analysis [28] allows fixing the prefactor in the asymptote,

$$a_<(K) = \frac{3b(K)}{\pi^2}\int_0^\infty dx \frac{x/2}{\sinh(x/2)}\frac{|\Gamma(1/2K + ix/2\pi)|^2}{\Gamma^2(1/2K)}, \qquad (10)$$

with $b(K) = \tilde{\tilde{c}}(1/4K)\tilde{\tilde{c}}^{1/2K}(K)$ and $\tilde{\tilde{c}}(K) = 8K^3\Gamma^4(2K)/\Gamma(4K)$.

From these results, we already note that, at $K \to \frac{1}{2}$, the t dependence of the high- and low-T asymptotes for $g(t)$ weakens, and $a_>(K), a_<(K) \to 1$. Actually, at $|K - \frac{1}{2}| \ll 1$, both asymptotes are combined into a formula that describes the entire crossover at any $T \ll \omega_0$,

$$G_{\rm th}(T) = \frac{G_0(T)}{1 + \mathcal{T}(2\pi T/\omega_0)^{4\delta K}}, \quad \mathcal{T} = \left(\frac{\pi\lambda}{\omega_0}\right)^2 \ll 1, \quad (11)$$

with $\delta K = K - \frac{1}{2}$. Equation (11) is valid on either side of the Schmid transition. It mirrors the crossover for temperature dependence of conductance across an impurity in a weakly interacting electron gas found in Ref. [31]. In the left panel of Fig. 2, we use Eq. (11) to plot the K dependence of the heat conductance at various fixed temperatures, as K varies across the transition point. We observe that the normalized heat conductance increases/decreases with T on the insulating/superconducting side of the Schmid transition. As expected, $G_{\rm th}$ remains close to G_0 at $K > \frac{1}{2}$, while a full crossover from 0 to G_0 can be observed at $K < \frac{1}{2}$. At $K = \frac{1}{2}$, the ratio $G_{\rm th}(T)/G_0(T)$ is T independent.

Returning to the low-T asymptote, we find that the prediction $g(t) \propto t^{1/K-2}$ is not applicable at $K < \frac{1}{4}$, as it would have predicted a stronger suppression-in-T of heat conductance than the one of a capacitor, $G_{\rm th} \propto T^3$. In fact, the correct answer originates from a capacitive contribution to $Y(\omega)$, not captured by Schmid's duality argument [13,32]. Inserting this T-independent admittance into Eq. (2), we find [28]

$g(t) = a_<(K)t^2$ with

$$a_<(K) = \frac{1}{20\pi}\Gamma^2\left(\frac{1/2}{1-2K}\right)\Gamma^2\left(\frac{1-3K}{1-2K}\right)\left(\frac{\tilde{\tilde{c}}(K)}{4K^2}\right)^{\frac{1}{1-2K}}. \tag{12}$$

While both low-T asymptotes at $K \to \frac{1}{4} \pm 0$ yield the same T^3 dependence for $G_{\text{th}}(T)$, the prefactors given by Eqs. (10) and (12) are different. This indicates a nonanalytical dependence of G_{th} around $K = \frac{1}{4}$.

At the so-called Toulouse point ($K = \frac{1}{4}$), we may use the exact free-fermion solution [23] to find the entire crossover for the scaling function [28],

$$g(t) = 1 + \frac{3}{\pi^3 t}\int_0^\infty dx \frac{x}{\sinh^2(x/2)} \times \text{Im}\left[\psi\left(\frac{1}{2} + \frac{2}{\pi^2 t} - \frac{ix}{2\pi}\right) - \psi\left(\frac{1}{2} + \frac{2}{\pi^2 t}\right)\right]. \tag{13}$$

Equation (13) reproduces the high-T asymptote, $g(t \gg 1) = 1 - 3/4t$, at $K = \frac{1}{4}$; it also matches the sum of the low-T asymptotes at $K \to \frac{1}{4} - 0$ and $K \to \frac{1}{4} + 0$, $g(t \ll 1) = 3\pi^4 t^2/80$. The result is illustrated in the right panel of Fig. 2. The relative contributions of the low-T asymptotes at $K \to \frac{1}{4} - 0$ and $K \to \frac{1}{4} + 0$ to the result at $K = \frac{1}{4}$ were obtained in Ref. [33], where a similar issue was studied in the context of an impurity in a Luttinger liquid. Here, we also find the absolute amplitude of the effect in terms of the circuit parameters.

At $K \to 0$, the transmon is almost disconnected and behaves as a capacitor with capacitance $C_\star = e^2/\pi^2\lambda$ [13], such that $Y^{-1}(\omega) = R + i/\omega C_\star$ at $T \ll T_\star = \lambda/\sqrt{2}$ [28]. Inserting the admittance into Eq. (2) we find that the crossover in G_{th} from 0 to G_0 actually occurs on the temperature scale $KT_\star \ll T_\star$, with asymptotes $g(t \ll K) = t^2/40K^2$, in agreement with Eq. (12) at $K \to 0$, and $g(K \ll t \ll 1) = 1 - 3\sqrt{8}K/t$. At higher temperature, $T \sim T_\star$, the already small correction to $g = 1$ changes to decay even faster, $g(t \gg 1) = 1 - 6K/t^2$ [cf. Eq. (9) at $K \to 0$].

Duality. By duality [1], for a charge qubit ($E_J \ll E_C$) on the superconducting side of the Schmid transition, $K > \frac{1}{2}$,

$$G_{\text{th}}(T) = G_0[1 - g(T/\Theta_\star, 1/4K)], \quad T \ll \Gamma, \tag{14}$$

with the same function $g(t)$ as in the transmon case studied above, and with another crossover temperature $\Theta_\star$ obtained from $T_\star$ of Eq. (8) after substitutions: $\lambda \to E_J$, $\omega_0 \to 2e^\gamma\Gamma$ ($\gamma \approx 0.58$ is Euler's constant), and $K \to 1/4K$, with plasma resonance linewidth $2\Gamma = 1/RC$. The ground state at $K > \frac{1}{2}$ is superconducting, leading to an inductivelike response, $G_{\text{th}}(T \to 0) = G_0$.

On the insulating side of the transition, $G_{\text{th}}(T)$ is strongly suppressed with respect to $G_0(T)$ at any $T \ll \Gamma$.

Rectification. Phase δ in Eq. (6) depends on the nonequilibrium distribution $n(\omega)$ set by the temperature bias and, thus, on temperatures of both baths. This naturally leads to heat rectification [27], a difference in heat currents at opposite signs of the temperature bias, provided the device is asymmetric ($R_1 \neq R_2$). Heat rectification was recently measured in a superconducting circuit [34]. To quantify this effect in the circuit of Fig. 1(a) connecting a hot bath at temperature T and cold one at $T = 0$, we assume $R_1 \gg R_2$. In this case $n(\omega)$ is an equilibrium distribution at temperature T or at $T = 0$, depending on the sign of the temperature bias. These assumptions allow us to characterize heat rectification with the ratio

$$\mathcal{R} = \frac{\int_0^\infty d\omega\omega[1 - \text{Re}\, e^{2i\delta(\omega,T)}]/(e^{\omega/T} - 1)}{\int_0^\infty d\omega\omega[1 - \text{Re}\, e^{2i\delta(\omega,0)}]/(e^{\omega/T} - 1)}. \tag{15}$$

Strong rectification requires a strong nonlinearity of the circuit. Thus, rectification remains small ($\mathcal{R} \approx 1$) at $K > \frac{1}{2}$ or $T \gg T_\star$, when the junction essentially behaves as an inductive shunt. Rectification is also small at $T \ll T_\star$ and $K < \frac{1}{4}$, when the junction behaves as a pure capacitor. Actually, a strong rectification is achieved at $T \lesssim T_\star$ and $\frac{1}{4} < K < \frac{1}{2}$; rectification reaches the maximal value of $\mathcal{R} = 11$ at $T \to 0$ and $K \to \frac{1}{4} + 0$ [28].

Discussion. In this Letter we used the scattering theory for interacting bosons to find a compact formula that relates heat conductance to the finite-frequency admittance of a Josephson junction in series with resistors at different temperatures. We analyzed the emergent scaling behavior of the thermal response of the circuit, and elucidated the manifestation of the Schmid transition in the thermal conductance as a crossing point at $K = \frac{1}{2}$ of the scaled thermal conductances measured at different temperatures. We hope our work provides a guide for identifying clear signatures that would confirm the existence of the Schmid transition in future heat transport experiments [35].

In light of our theory, the flux dependence of the heat transport across a nominally insulating circuit containing a SQUID [6] is not surprising. Indeed, the characteristic temperature scale entering the thermal response depends on the effective Josephson energy of the SQUID, which in turn depends on the flux. A quantitative comparison is far more challenging. The limited data of Ref. [6] allow us to infer that the dependence of G_{th} on T is stronger than linear. This is consistent with the device being in the insulating state, but more data taken on a suitable device is needed for a quantitative comparison with our theory.

Note added. Recently we became aware of a study [40] of the charge qubit limit on the insulating side of the transition, which is complementary to the results presented in this Letter.

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