Letter

Non-Hermitian butterfly spectra in a family of quasiperiodic lattices

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We propose a family of exactly solvable quasiperiodic lattice models with analytical complex mobility edges, which can incorporate mosaic modulations as a straightforward generalization. By sweeping a potential tuning parameter δ, we demonstrate a kind of interesting butterflylike spectra in a complex energy plane, which depicts energy-dependent extended-localized transitions sharing common exact non-Hermitian mobility edges. Applying Avila's global theory, we are able to analytically calculate the Lyapunov exponents and determine the mobility edges exactly. For the minimal model without mosaic modulation, we obtain a compactly analytic formula for the complex mobility edges, which indicates clearly mobility edges having a loop structure in the complex energy plane. Together with an analytical estimation of the range of the complex energy spectrum, we can obtain the true mobility edge. The non-Hermitian mobility edges are further verified by numerical calculations of the fractal dimension and spatial distribution of the wave functions. Tuning the parameters of non-Hermitian potentials, we also investigate the variations of the non-Hermitian mobility edges and the corresponding butterfly spectra, which exhibit a richness of spectrum structures.

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Introduction. In the past few decades, quasiperiodic lattices [1–7] have become a versatile platform to investigate disorder-induced localization transitions, which is one of the key topics of fundamental importance in the frontiers of condensed matter physics. As is well known, the topic was originally proposed by Anderson in his seminal work in the context of electronic systems with truly random disorders [8,9]. While scaling theory [10] demonstrates that for truly random systems all single-particle eigenstates are already localized in one and two dimensions even under arbitrary small but finite disorder strength and thus there is no extra space left for a localization transition to occur, quasicrystals containing so-called determinant correlated disorders [1–7] have been proven to be able to host various extended-localized transitions in low-dimensional systems. Moreover, intriguing energy-dependent localization transitions have also been revealed in various low-dimensional quasiperiodic systems, for which under the same set of parameters extended and localized single-particle eigenstates can coexist and be separated by a critical energy E_c, known as the mobility edge (ME) [11]. Inspired by the influential Aubry-André-Harper (AAH) model, various generalized AAH-like models have been proposed [12–17] and discovered to be capable of accommodating localization transitions with mobility edges [17–30]. Notably, a few of these models are exactly solvable [16,17,19,20], which is a fact highly appealing and significant for further exploration of the mobility edge physics.

In recent years, non-Hermitian physics has experienced a revival and become a renewed prosperous research field. Growing attention has been paid to the interplay of non-Hermiticity and quasiperiodicity [31–57]. Many previous research efforts [33–39,49–54] have been devoted to parity-time ($\mathcal{PT}$) symmetric [58] quasiperiodical systems, for which a well-established correspondence between the real-complex transition in eigenenergy and extended-localized transition has been revealed. However, for a general non-Hermitian quasiperiodical system, the spectrum is usually complex, and no obligate relation between the change of spectrum structure and localization transition exists. Although some recent works have demonstrated the existence of a complex mobility edge in various non-Hermitian quasiperiodical systems [48,49], analytical results of complex mobility edges are rare and thus are particularly important for the broadening of the concept of MEs from the real to the complex plane.

In this Letter, we propose a family of non-Hermitian quasiperiodic models with a compact analytical expression of complex mobility edges, which can further incorporate mosaic modulations as a straightforward generalization. Utilizing Avila's global theory, we are able to calculate the Lyapunov exponent $\Gamma(E)$ analytically and get a uniform formula of non-Hermitian mobility edge (NHME). Thus, an accurate characterization of the NHME can be implemented and its intersection with the physical spectrum produces the true mobility edge. By varying a potential parameter δ, we obtain a kind of intriguing butterflylike spectra in the complex energy plane. For a typical system, we showcase that the extended and localized states are distributed on the body and wings of butterfly, respectively, separated by the NHMEs. Varying the

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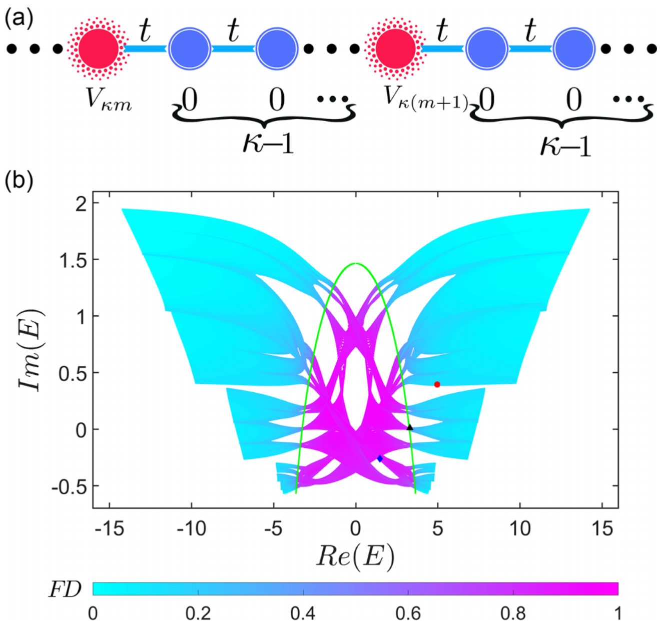

FIG. 1. (a) Sketch of the family of a one-dimensional (1D) non-Hermitian quasiperiodic lattice model. The solid blue lines denote the homogeneous hopping t. The red circles denote lattice sites with a non-Hermitian quasiperiodic potential $V_{\kappa m}$ with m being an integer, while the other circles in between denote lattice sites with zero potential. (b) The non-Hermitian butterfly spectrum of the minimal model with $\kappa = 1$. The fractal dimension (FD) of each eigenstate is encoded in the color of each energy point in the spectrum. The true non-Hermitian mobility edge is denoted by a green line which is given by a comprehensive consideration of both Eq. (3) and the actual range of the model's spectrum. Parameters: $L = 987$, $\lambda = 1$, $\alpha = 0.5$, $\gamma = \pi/2$, $\theta = 0$, $t = 1$, and the modulation parameter δ varies from -7 to 7.

non-Hermiticity parameter γ can lead to the change of the spectrum structure and NHME. We also show the deformation of the non-Hermitian butterfly spectra for systems with various parameters.

Model and non-Hermitian butterfly spectrum. We propose a family of generic non-Hermitian quasiperiodic models which are described in a unified manner by the following eigenvalue equation,

$$t(\phi_{j-1} + \phi_{j+1}) + V_j\phi_j = E\phi_j, \tag{1}$$

where j is the index of the lattice site, and t is the nearest-neighbor hopping amplitude. The core feature of the model Eq. (1) then is the non-Hermitian quasiperiodic mosaic [19] on-site potential with

$$V_j = \begin{cases} \frac{\lambda e^{i\gamma}\cos(2\pi jb+\theta)+\delta}{1-\alpha\cos(2\pi jb+\theta)}, & j = m\kappa, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

in which κ is a positive integer and $m = 1, 2, \ldots, N$. Apparently, the quasiperiodic potential occurs periodically with period κ, which is pictorially shown in Fig. 1(a). N can be seen as the number of quasicells, therefore the lattice size of the model is $L = N\kappa$. So $\kappa = 1$ is for the usual quasiperiodic lattice while each $\kappa \geqslant 2$ corresponds to a mosaic quasiperiodic lattice [19,34]. Here, the quasiperiodic on-site potential is controlled by two modulation parameters λ, δ and a deformation parameter α. The parameter b is an irrational number which is responsible for the quasiperiodicity of the on-site potential. To be concrete and without loss of generality, in this Letter we choose $b = (\sqrt{5} - 1)/2$. The parameter γ is a phase angle dictating the non-Hermitian nature of the on-site quasiperiodic potential and θ denotes a phase offset. Obviously, a non-Hermitian potential in this form does not respect parity-time ($\mathcal{PT}$) symmetry which is otherwise a key ingredient of previous works addressing non-Hermitian localization transitions [33–39,49–54]. For convenience, we shall set $t = 1$ as the energy unit in the following calculation.

In this Letter, we shall study the general non-Hermitian case with $\kappa \geqslant 1$ and $\alpha \in (-1, 1)$ in the presence of both λ and δ terms. By applying Avila's global theory, we can derive NHMEs analytically by calculating the Lyapunov exponents for the general case with $\kappa \geqslant 1$. We shall prove that the model has exact NHMEs separating localized states and extended states. However, to facilitate our discussion we focus on the minimal model with $\kappa = 1$ and then showcase examples with $\kappa \geqslant 2$. For the minimal model, a compactly analytical formula for the non-Hermitian mobility edges (NHMEs) can be obtained,

$$[\alpha\,\mathrm{Re}(E) + \lambda\cos\gamma]^2 + \frac{[\alpha\,\mathrm{Im}(E) + \lambda\sin\gamma]^2}{1-\alpha^2} = 4t^2, \tag{3}$$

where $\mathrm{Re}(E)$ and $\mathrm{Im}(E)$ are respectively the real and imaginary parts of E. Equation (3) is our key result. For the general case with $\gamma \neq n\pi$, Eq. (3) indicates that the ME takes a complex value, which is irrelevant to the parameter δ. When $\gamma = 0$, the potential V_j is real, and the model reduces to the generalized Ganeshan-Pixley-Das Sarma (GPD) model [17,59,60]. Since E takes a real value, Eq. (3) is simplified to $\alpha\,\mathrm{Re}(E) + \lambda\cos\gamma = \pm 2t$, consistent with the results of the generalized GPD model.

Before proceeding with a rigorous proof of Eq. (3), we first conduct a numerical verification to gain an intuitive understanding. In Fig. 1(b), by implementing numerical calculations we display in the complex plane the energy spectrum of Eq. (1) with the color encoding the fractal dimension (FD) of the corresponding eigenstate. For an arbitrary normalized eigenstate ϕ, the fractal dimension is defined as $\mathrm{FD} = -\lim_{L\to\infty}\ln(\sum_j|\phi_j|^4)/\ln L$, which acts as a good indicator for distinguishing localized and extended states in that $\mathrm{FD} \to 0$ for localized states and $\mathrm{FD} \to 1$ for extended states. As the modulation parameter δ varies, we get intriguing non-Hermitian butterfly spectra displayed in Fig. 1(b), in which the localized states and extended states are well separated. Obviously, the separation between localized and extended states is energy dependent in the complex-energy plane, coinciding well with the green line plotted according to the analytical formula Eq. (3), which predicts the mobility edges forming a closed elliptic loop in the complex plane.

Representative distributions of the eigenstates corresponding to different regions of the non-Hermitian butterfly spectrum in Fig. 1(b) are shown in Fig. 2. The three eigenstates are denoted respectively by a blue diamond, black triangle, and red dot in the non-Hermitian butterfly spectrum. Clearly, Figs. 2(a) and 2(c) display a typical extended state and

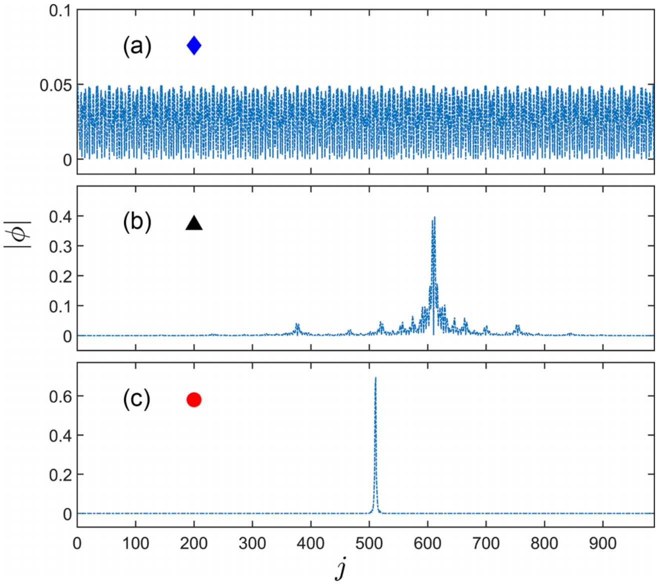

FIG. 2. Typical spatial distributions of eigenstates in different regions of the non-Hermitian butterfly spectrum. (a) Extended state corresponding to eigenenergy denoted by a blue diamond in Fig. 1. (b) Critical state corresponding to eigenenergy denoted by a black triangle in Fig. 1. (c) Localized state corresponding to eigenenergy denoted by a red dot in Fig. 1. Model parameters are the same as in Fig. 1.

localized state, respectively, while Fig. 2(b) shows a critical state with a multifractal structure.

Analytical derivation of NHME. Analytically, the non-Hermitian mobility edge of the generic non-Hermitian quasiperiodic mosaic model in Eq. (1) can be exactly derived by computing the Lyapunov exponent. For convenience, t is absorbed into λ and E in the intermediate derivation process and will be restored later. According to Avila's global theory of a one-frequency analytical $SL(2,\mathbb{C})$ cocycle [61,62], the Lyapunov exponent $\Gamma(E)$ can be calculated as

$$\Gamma(E)=\lim_{N\to\infty}\frac{1}{N\kappa}\ln\left\|\prod_{m=1}^{N}T_m\right\|, \tag{4}$$

where $\|\cdot\|$ denotes the norm of the matrix. T_m is the one-step transfer matrix of the Schrödinger operator at the mth quasicell, which can be explicitly written as

$$T_m=\begin{pmatrix}E-V_{\kappa m} & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}E & -1\\ 1 & 0\end{pmatrix}^{\kappa-1}, \tag{5}$$

with $V_{\kappa m}$ given by Eq. (2).

To ease the calculation of $\Gamma(E)$ according to Eq. (4), one can reorganize T_m as $T_m=\frac{Y_m}{X_m}$, where $X_m=1-\alpha\cos(2\pi b\kappa m+\theta)$ and

$$Y_m=\begin{pmatrix}EX_m-V_{\kappa m}X_m & -X_m\\ X_m & 0\end{pmatrix}\begin{pmatrix}a_\kappa & -a_{\kappa-1}\\ a_{\kappa-1} & -a_{\kappa-2}\end{pmatrix}.$$

Note that a convenient mathematical relation has been implemented above as

$$\begin{pmatrix}E & -1\\ 1 & 0\end{pmatrix}^{\kappa-1}=\begin{pmatrix}a_\kappa & -a_{\kappa-1}\\ a_{\kappa-1} & -a_{\kappa-2}\end{pmatrix},$$

in which

$$a_\kappa=\frac{1}{D}\left[\left(\frac{E+D}{2}\right)^{\kappa}-\left(\frac{E-D}{2}\right)^{\kappa}\right], \tag{6}$$

with $D=\sqrt{E^2-4}$.

In this way, the Lyapunov exponent can be rewritten as

$$\Gamma(E)=\lim_{N\to\infty}\frac{1}{N\kappa}\left[\ln\left\|\prod_{m=1}^{N}Y_m\right\|-\sum_{m=1}^{N}\ln|X_m|\right], \tag{7}$$

in which

$$\lim_{N\to\infty}\frac{1}{N\kappa}\sum_{m=1}^{N}\ln|X_m|=\frac{1}{2\pi\kappa}\int_0^{2\pi}\ln(1-\alpha\cos\varphi)d\varphi$$
$$=\frac{1}{\kappa}\ln\frac{1+\sqrt{1-\alpha^2}}{2}.$$

With the above preparations, we can now focus on tackling the remaining part of Eq. (7). Avila's global theory can also be generalized to investigate non-Hermitian systems [34,35,61,62]. The first step is to perform an analytical continuation of the phase in Y_m, i.e., $\theta\to\theta+i\epsilon$. Considering a large-$\epsilon$ limit, a straightforward derivation leads to

$$Y_m(\epsilon)=\frac{1}{2}e^{-i(2\pi b\kappa m+\theta)}e^{\epsilon}\times\begin{pmatrix}-\chi a_\kappa+\alpha a_{\kappa-1} & \chi a_{\kappa-1}-\alpha a_{\kappa-2}\\ -\alpha a_\kappa & \alpha a_{\kappa-1}\end{pmatrix}+o(1),$$

where $\chi=\alpha E+\lambda e^{i\gamma}$. This accordingly leads to

$$\lim_{N\to\infty}\frac{1}{N\kappa}\ln\left\|\prod_{m=1}^{N}Y_m\right\|=\frac{1}{\kappa}\epsilon+\frac{1}{\kappa}\ln f,$$

in which

$$f=\max\left\{\left|\frac{2\alpha a_{\kappa-1}-\chi a_\kappa\pm G}{4}\right|\right\}, \tag{8}$$

with

$$G=\sqrt{\chi^2a_\kappa^2-4\alpha\chi a_\kappa a_{\kappa-1}+4\alpha^2a_\kappa a_{\kappa-2}}. \tag{9}$$

Thus, we have $\kappa\Gamma_\epsilon(E)=\epsilon+\ln\frac{2f}{1+\sqrt{1-\alpha^2}}$. Avila's global theory [61] shows that $\kappa\Gamma_\epsilon(E)$ is a convex, piecewise linear function of ϵ with integer slopes. This implies that $\kappa\Gamma_\epsilon(E)=\max\{\epsilon+\ln\frac{2f}{1+\sqrt{1-\alpha^2}},\kappa\Gamma_0(E)\}$. Furthermore, Avila's global theory proves that E does not belong to the spectrum, if and only if $\Gamma_0(E)>0$, and $\Gamma_\epsilon(E)$ is an affine function in a neighborhood of $\epsilon=0$. Therefore, for any E lying in the spectrum, we have

$$\Gamma(E)=\frac{1}{\kappa}\max\left\{\ln\frac{2f}{1+\sqrt{1-\alpha^2}},0\right\}. \tag{10}$$

Then NHMEs for general κ can be determined exactly by letting $\Gamma(E)=0$. Notably, a compact formula Eq. (3) of NHMEs for the simplest case $\kappa=1$ can be obtained in this way. NHMEs for other cases with $\kappa\geqslant2$ could be found in the Supplemental Material [63].

Variations of the non-Hermitian butterfly spectrum. The richness and exact solvability of the proposed generic non-Hermitian quasiperiodic model given in Eq. (1) grants us plenty of tangible freedom to introduce variations to the

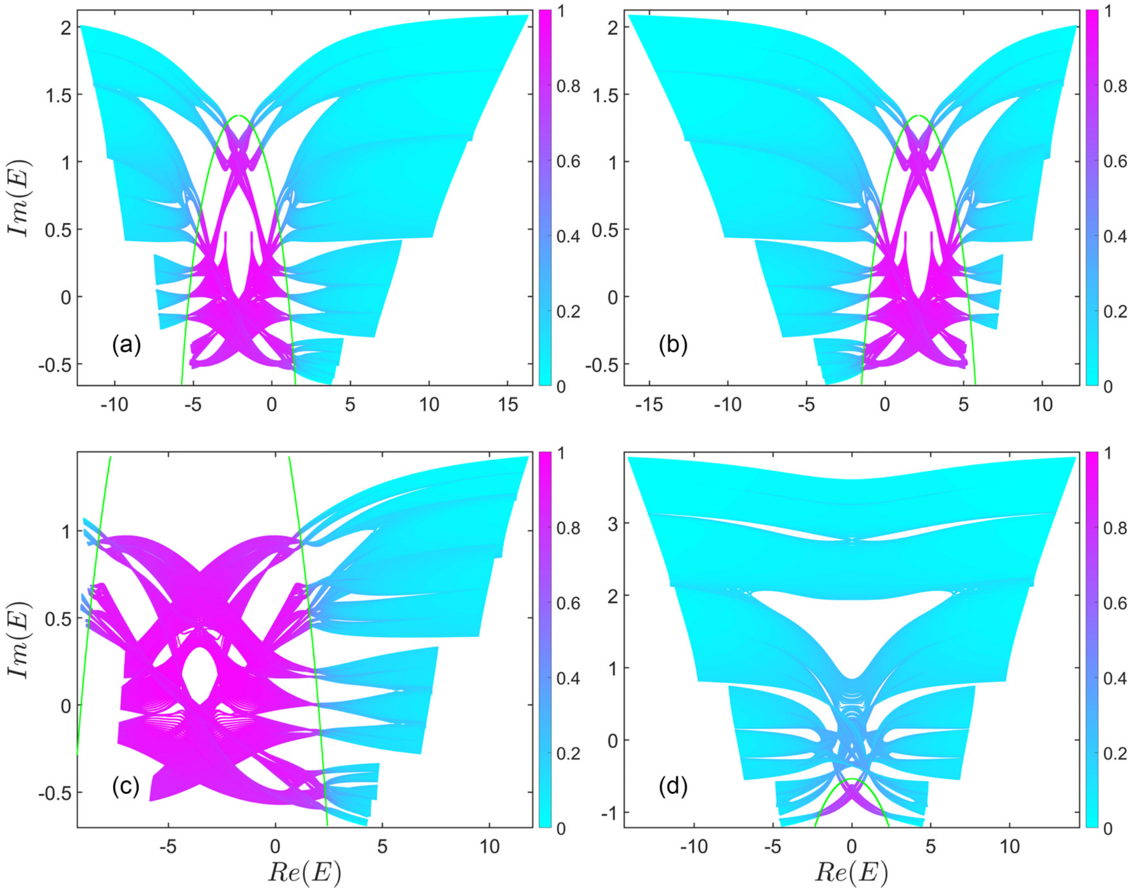

FIG. 3. Variations of the non-Hermitian butterfly spectrum for the minimal model with $\kappa = 1$ by tuning model parameters. (a) $\gamma = \pi/4$, $\lambda = 1.5$, and $\alpha = 0.5$. (b) $\gamma = 3\pi/4$, $\lambda = 1.5$, and $\alpha = 0.5$. (c) $\gamma = \pi/4$, $\lambda = 1.5$, and $\alpha = 0.3$. (d) $\gamma = \pi/2$, $\lambda = 2$, and $\alpha = 0.5$. Fractal dimension (FD) of each eigenstate is denoted by the color of each energy point in the spectrum. The true non-Hermitian mobility edge is denoted by a green line which is given by a comprehensive consideration of both Eq. (3) and the actual range of the model's spectrum. Other parameters are as follows: $L = 987$, $t = 1$, $\theta = 0$, and the modulation parameter δ varies from -7 to 7.

interesting butterfly spectrum and the exact non-Hermitian mobility edge within the spectrum.

The minimal model with $\kappa = 1$ possesses a favorable and compact analytic formula Eq. (3) for the exact NHME, which is simply an ellipse equation with properties that are familiar to us. Apparently, the center of the exact NHME lies at the point $(-\lambda\cos\gamma/\alpha, -\lambda\sin\gamma/\alpha)$ of the complex plane. As the non-Hermiticity parameter γ varies from 0 to 2π, the center of the ellipse, which is also the center of the possibly presented extended region in the butterfly spectrum, will run around a circle. In other words, the parameter γ determines the orientation of the center of the ellipse, while the ratio between λ and α controls the distance of the ellipse center from the origin of the complex plane. In Figs. 3(a) and 3(b), we show the non-Hermitian butterfly spectrum and the corresponding NHME line for the minimal model with a different non-Hermiticity parameter γ, i.e., in Fig. 3(a), $\gamma = \pi/4$, and in Fig. 3(b), $\gamma = 3\pi/4$. It is clearly shown that the orientation of the non-Hermitian mobility edge line is dependent on the non-Hermiticity parameter γ.

Moreover, it is obvious that the semimajor axis and the semiminor axis of the ellipse are $a = |2t/\alpha|$ and $b = |2t\sqrt{1-\alpha^2}/\alpha|$, respectively. As the hopping amplitude t has been set to be the energy unit throughout this Letter, the deformation parameter α is the sole parameter which can be used to monitor the size of the ellipse. Since the energy area inside the ellipse in the complex plane corresponds to extended eigenstates, changing the value of α may alter the portion of extended eigenstates in the whole spectra. This can be clearly illustrated by comparing Figs. 3(a) and 3(c). Compared to Fig. 3(a), the deformation parameter α for Fig. 3(c) decreases from 0.5 to 0.3, resulting in the obvious enlargement of the extended region in the exotic non-Hermitian butterfly spectra.

From Fig. 3(d), we see that the number of localized eigenstates increases with the increase of λ, whereas the number of extended eigenstates decreases. For Figs. 1(b) and 3(d), all model parameters are the same except λ increased from 1 to 2. As a result, the extended region in the non-Hermitian butterfly spectrum clearly shrinks. It is worth noting that in order to precisely modulate the interesting non-Hermitian butterfly spectrum and the NHME, we also need to consider the structure and the actual distribution range of the system's energy spectrum. A rough analytic estimation of the range of the model's complex energy spectrum is provided in the Supplemental Material [63].

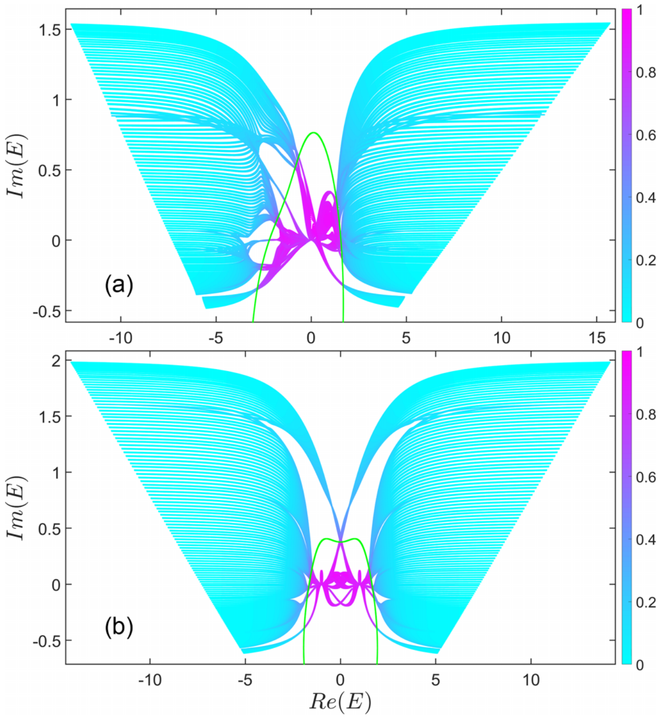

FIG. 4. The non-Hermitian butterfly spectrum of the generic non-Hermitian mosaic quasiperiodic model with (a) $\kappa = 2$ and (b) $\kappa = 3$. Fractal dimension (FD) of each eigenstate is denoted by the color of each energy point in the spectrum. The exact non-Hermitian mobility edge is denoted by a green line which is obtained by numerically solving the exact relation Eq. (11) and considering the actual range of the model's spectrum at the same time. Parameters: (a) $L = 610$, $\lambda = 1.1$, $\alpha = 0.5$, $\gamma = \pi/4$, $\theta = 0$, $t = 1$; (b) $L = 987$, $\lambda = 1$, $\alpha = 0.5$, $\gamma = \pi/2$, $\theta = 0$, $t = 1$. The modulation parameter δ varies from -7 to 7.

Next, we showcase the spectra and NHMEs for the cases with $\kappa = 2$ and $\kappa = 3$. The exact NHME for general κ can be obtained by solving the following exact relation [63],

$$\frac{2f}{1 + \sqrt{1 - \alpha^2}} = 1, \tag{11}$$

with f given by Eq. (8). As shown in Figs. 4(a) and 4(b), both spectra display intriguing butterflylike structures and the numerical results agree well with curves of NHMEs plotted according to solving Eq. (11) with $\kappa = 2$ and $\kappa = 3$, respectively.

Summary and discussions. In summary, we have proposed a family of exactly solvable 1D quasiperiodic lattice models with complex MEs. With the help of Avila's global theory, we derived a compactly analytical formula of NHMEs, which indicates clearly how the complex mobility edges form and are affected by modulation parameters. Our models exhibit intriguing butterflylike spectra in the complex energy plane with extended and localized states separated by NHMEs. Tuning the parameters of non-Hermitian potentials leads to the change of the NHMEs and deformation of the butterfly spectra, which exhibit rich structures. Our models can be directly extended to cases incorporating mosaic modulations. Our analytical results provide a firm ground for the broadening of the concept of MEs from the real to the complex plane.

For Hermitian systems, localization implies the absence of particle transport, while delocalization signifies particle motion. From the perspective of wave functions, the meaning of localized or extended for an eigenstate of a non-Hermitian system is similar. A localized eigenstate indicates that the distribution of the wave function is highly confined to a few lattice sites, while an extended state implies that the distribution spreads across almost all lattice sites. However, there may exist losses and gains in a non-Hermitian system. Thus, the amplitude of the initial wave shall increase or decrease exponentially due to the presence of gain or loss. If we normalize the wave function during the evolution process [64], the expansion dynamics is very similar to the Hermitian quasiperiodic systems. To gain some intuitive physical insights, a numerical implementation of single-particle quantum walks in non-Hermitian quasiperiodic lattices can be found in the Supplemental Material [63]. The expansion dynamics of the particle is clearly affected by the localization properties of eigenstates.

The general non-Hermitian quasiperiodic model in this Letter may be experimentally implemented on several state-of-the-art experimental platforms. For example, Hermitian quasiperiodic models have already been realized in photonic lattices [7,65–67]. Non-Hermiticity can be further incorporated [68–73] through a delicate design of the complex refractive index or through the direct introduction of gain and loss elements. The complex on-site potential Eq. (2) could be hopefully realized along this line. Alternatively, the model could also be implemented in phononic crystals [74,75] provided that the characteristic frequency, gain, and loss in each active unit are precisely and independently controlled by adjusting the cavity height, the loudspeaker, and the microphone, respectively. In addition, the ultracold atom platform [6,21,76] is also a good candidate to be further generalized to realize this class of general non-Hermitian quasiperiodic models with mosaic modulations.

Note added. Recently, we became aware of a work which also demonstrates complex mobility edges in a different non-Hermitian system [77]. Their work examines the issue mainly from the perspective of the specific structure of the complex mobility edges. In this Letter, in addition to revealing the structure by deriving an analytical formula for the genuine complex mobility edges exactly, we further study the physical spectra and other related properties of a family of non-Hermitian quasiperiodic models.

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