Letter

Quantization of the anyonic spectral density of heat currents

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It is well known that the thermal conductance of Abelian integer and fractional quantum Hall (QH) states is quantized and is determined by the net and total chirality of downstream and upstream modes. We furthermore demonstrate that the finite frequency noise of the heat current of the Abelian particle and holelike states is quantized when the system size is smaller than the thermal equilibration length. It is shown that the finite frequency noise of the heat current is a universal and independent physical quantity, which provides information about the microscopic structure of edge excitations.

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A two dimensional electron gas at low temperature and subject to a strong perpendicular applied magnetic field exhibits the QH effect [1,2]. In the regime of the QH effect, the charge transport occurs through one dimensional chiral edge states, which are quantum analogs of skipping orbits [3]. The edge excitations in such channels strictly propagates in one direction since backscattering for a quantum Hall edge state is prohibited [4]. As a result, the QH effect has surprising peculiarities such as precise quantization of the Hall conductance [5,6]. To be specific, the Hall conductance $G = \nu G_{\mathrm{q}}$ is quantized, where the filling factor ν takes on either integer or fractional values and $G_{\mathrm{q}} = e^2/2\pi\hbar$ is the elementary quantum conductance [3]. Furthermore, excitations with fractional charge can be observed [3]. Apart from charge, the QH edge states transport heat and quantization of the heat conductance occurs. Similar to the quantization of the electrical conductance, the quantization of the heat conductance can be shown. Particularly, at filling factor $\nu = 1$ (free electrons) the thermal Hall conductance is given by the quantum heat conductance $K_{\mathrm{q}} = \pi k_{\mathrm{B}}^2 T/6\hbar$, where T is temperature. Therefore, in case of $\nu = 1$ the thermal and electrical conductances satisfy the Wiedemann-Franz law for free electrons [7,8].

It is known that hierarchical fractional QH edge states consist of multiple propagating channels, moreover some of them propagate in the opposite direction (upstream modes) of the skipping orbits (downstream modes) defined by the magnetic field [9]. In this case, in contrast to the integer QH effect, the thermal K and electrical G conductances do not satisfy the Wiedemann-Franz law [7]. Thus, the thermal conductance is a universal and independent physical quantity, which can provide additional information about the microscopic structure of edge excitations. The recent experimental progress has allowed for measurements of the thermal heat conductance for different filling factors with a high precision [10–18]. Additionally, with the help of heat transport measurements, an essential step toward the detection of the upstream modes was made [19–22]. It was confirmed that the thermal conductance of Abelian integer and fractional QH states is quantized and is determined by the net chirality of the downstream and upstream modes, $K = (N_{\mathrm{d}} - N_{\mathrm{u}})K_{\mathrm{q}}$, for a thermally equilibrated edge. In contrast, a thermally non-equilibrated edge demonstrates quantization, which is controlled by the total chirality, $K = (N_{\mathrm{d}} + N_{\mathrm{u}})K_{\mathrm{q}}$. The crossover from the equilibrated to the non-equilibrated heat transport regime is defined by the thermal equilibration length, which generally depends on temperature, scattering between modes and disorder on the edge [23].

In this Letter we make important progress in the study of thermal transport of Abelian particlelike (Laughlin-Jain series) and holelike ($1/2 < \nu < 1$) QH states. Our aim is to show that apart from the quantization of the anyonic heat flow, furthermore, the anyonic noise of the heat current, at finite frequency, is quantized when the system size is smaller than the thermal equilibration length. The quantization is completely different from the quantization of the heat conductance. This is because, in general, there is no fluctuation-dissipation theorem [24,25], which relates the spectral density and thermal conductance (or heat current) in the presence of temperature gradients. This means that the finite frequency noise of the heat current is an independent physical quantity, which can give additional information about the physics of QH edge excitations. In order to investigate the fluctuations of the heat current we consider a two-terminal setup with cold and hot reservoirs (see Fig. 1). We restrict the consideration to the case of clean edges and the ballistic nature of edge modes, namely when the size of the system is smaller than the thermal equilibration length.

To study the QH edge modes we use a low-energy effective field theory. According to the effective theory, the edge states are the collective fluctuations of one-dimensional densities [7,26]. The Hamiltonian of a QH edge is given by

$$\hat{H} = \frac{1}{4\pi}\int dx \sum_{ij} \partial_x \hat{\Phi}_i U_{ij} \partial_x \hat{\Phi}_j, \tag{1}$$

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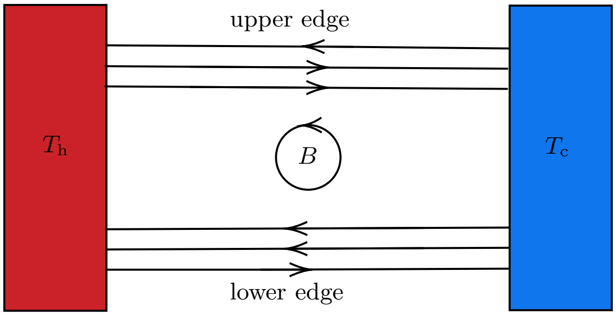

FIG. 1. Schematic of a two terminal setup. As an example the case of $\nu = 3/5$ with $N_\mathrm{d} = 1$ chiral downstream modes and $N_\mathrm{u} = 2$ chiral upstream modes on each (upper and lower) edge is shown. The direction of the downstream modes is defined by the magnetic field B. The upstream modes propagate in the opposite direction of the downstream modes. The hot edge modes are emanated from the left reservoir with temperature T_h, while the cold edge modes have a temperature of the right reservoir T_c.

where $\partial_x\hat{\Phi}_i$ are one-dimensional densities, and U_{ij} are nonuniversal interactions at the edge. These densities satisfy the Kac-Moody commutation relations

$$[\hat{\Phi}_i(x), \hat{\Phi}_j(y)] = i\pi K^{-1}_{ij}\mathrm{sgn}(x-y), \tag{2}$$

where the quadratic matrix K_{ij} stores information about the topological order of the fractional QH state, and $i, j = 1, 2, \dots, n$ refer to the edge modes. The electric charge carried by each edge mode is characterized by a charge vector t_i, such that the total edge charge density is defined by the sum $\hat{\rho} \propto \sum_i t_i\partial_x\hat{\Phi}_i$. The filling factor is determined by K_{ij} and t_i via $\nu = \sum_{ij} t_i K^{-1}_{ij} t_j$. The specific representations of the matrix K_{ij} and basis t_i can be found in Ref. [26]. It is known that the electrical conductance in this model is quantized [7]. The constant bias, V, couples to the total charge in the Hamiltonian, $\hat{H}_\mathrm{ext} = -V\int dx\hat{\rho}(x)$ and the minimization of the Hamiltonian with respect to $\hat{\rho}$ results in a quantized conductance. The average current is given by $j = \nu G_\mathrm{q} V$.

In order to study heat transport, it is convenient to rewrite Eqs. (1) and (2) on a diagonal form, with the help of the linear transformation $\partial_x\hat{\Phi}_i = \sum_j \Lambda_{ij}\partial_x\hat{\phi}_j$. The matrix Λ can be chosen such that both K_{ij} and U_{ij} are simultaneously diagonalized, i.e., $(\Lambda^T K\Lambda)_{ij} = \eta_i\delta_{ij}$ and $(\Lambda^T U\Lambda)_{ij} = v_i\delta_{ij}$. In the new bosonic basis ϕ_i the Hamiltonian takes the diagonal form

$$\hat{H} = \frac{1}{4\pi}\int dx\sum_i v_i(\partial_x\hat{\phi}_i)^2, \tag{3}$$

where the new charge densities obey the Kac-Moody commutation algebra with a diagonal prefactor on the right hand side

$$[\hat{\phi}_i(x), \hat{\phi}_j(y)] = i\pi\eta_i\delta_{ij}\mathrm{sgn}(x-y). \tag{4}$$

The physical interpretation of Eqs. (3) and (4) is as follows: each mode on the upper (or lower) edge describes an independent chiral density that propagates at group velocity v_i in the $\eta_i = \pm 1$ direction. Since $(\Lambda^T K\Lambda)_{ij} = \eta_i\delta_{ij}$, the number of upstream and downstream modes is a universal property of K_{ij}.

We first consider the case of filling factor $\nu = 1$ (or alternatively $\nu = 1/3$) with one edge mode at $v_1 = v$ in the two-terminal setup (see Fig. 1). The description of heat transport by a ballistic channel starts with the Hamiltonian given by Eq. (3), $\hat{H} = \int dx\hat{\mathcal{H}}(x)$, where the bosonized Hamiltonian density is $\hat{\mathcal{H}} = (v/4\pi)(\partial_x\hat{\phi})^2$. Writing the equation of motion for the Hamiltonian density $\hat{\mathcal{H}}$ in the form of the continuity equation $\partial_t\hat{\mathcal{H}} + \partial_x\hat{J}_Q = 0$, one obtains the expression for the heat current operator $\hat{J}_\mathrm{Q}(x,t) = v\hat{\mathcal{H}}(x,t)$. Taking into account the expansion of the one-dimensional density in terms of bosonic annihilation and creation operators, $\partial_x\hat{\phi}(x,t) = i\sum_{k>0}\sqrt{2\pi k/L}\hat{B}_k(x,t)$, where $\hat{B}_k(x,t) = \hat{b}_k e^{ik(x-vt)} - \hat{b}^\dagger_k e^{-ik(x-vt)}$, we obtain the following expression for heat current operator

$$\hat{J}_\mathrm{Q}(x,t) = \sum_{k,p>0}\frac{v}{2L}\sqrt{vk}\hat{B}_k(x,t)\sqrt{vp}\hat{B}^\dagger_p(x,t), \tag{5}$$

where L is the normalization length. Hereafter, in the calculations, the phases with spatial coordinate x mutually cancel each other due to momentum conservation. Averaging the above expression in the thermodynamic limit and subtracting the ground state contribution at zero temperature, we obtain the average heat current

$$J_\mathrm{Q} \equiv \langle\hat{J}_\mathrm{Q}\rangle = \frac{\pi}{12\hbar}\left(T_\mathrm{h}^2 - T_\mathrm{c}^2\right), \tag{6}$$

Setting $T_\mathrm{h/c} = T \pm \Delta T/2$ and taking the derivative with respect to $\Delta T \to 0$ we obtain the result for the quantum heat conductance $K_\mathrm{q} = \pi k_B^2 T/6\hbar$.

For the case of several edge modes, according to Eq. (3), the operator of the heat current must be provided by an additional summation with respect to i—the total number of modes on upper (or lower) edge

$$\hat{J}_\mathrm{th}(x,t) = \sum_{i;k,p>0}\frac{\eta_i v_i}{2L}\sqrt{v_i k}\hat{B}_{k,i}(x,t)\sqrt{v_i p}\hat{B}^\dagger_{p,i}(x,t), \tag{7}$$

where the factor η_i in front of velocity v_i is a consequence of the commutation relation in Eq. (4) and the bosonic operators satisfy the standard commutation relation $[\hat{b}_{k,i}, \hat{b}^\dagger_{p,j}] = \delta_{kp}\delta_{ij}$. The edge modes with linear spectrum $\epsilon_{k,i} = v_i k$, emanating from left and right reservoirs, are thermally populated with a Bose-Einstein distribution $f_{\mathrm{B},i}(\epsilon_{k,i}) = (e^{\epsilon_k/T} - 1)^{-1}$ at temperature $T = T_\mathrm{h/c}$. We can visualize these modes as collective bosonic excitations propagating at the edge and contributing to heat transfer. Repeating the same steps as in the case of one chiral edge mode, one arrives to the final result for the thermal conductance, $K_\mathrm{th} = K_\mathrm{q}N_+$, which is quantized and depends on the total chirality, $N_+ = N_\mathrm{d} + N_\mathrm{u}$. It is worth mentioning that in the regime of a small thermal equilibration length, compared to the system's size, the thermal conductance $K_\mathrm{th} = K_\mathrm{q}N_-$ is as well quantized and it is defined by net chirality, $N_- = N_\mathrm{d} - N_\mathrm{u}$. In particular, for filling factor $\nu = 2/3$ one needs to take into account the diffusive character of transport carefully to obtain the nonzero conductance [7,23].

The aim of this work is to demonstrate that the nonsymmetrized spectral density of heat current fluctuations,

$$S_\mathrm{th}(\omega) = \int dt e^{i\omega t}\langle\delta\hat{J}_\mathrm{th}(t)\delta\hat{J}_\mathrm{th}(0)\rangle, \tag{8}$$

is quantized, where $\delta\hat{J}_{\mathrm{th}}(t)=\hat{J}_{\mathrm{th}}(t)-\langle\hat{J}_{\mathrm{th}}(t)\rangle$. Here the spectral function is calculated at the same spatial point x [see Eq. (7)]. The integrand of the Fourier transform depends only on one time variable due to time translation invariance. The average in Eq. (8) is taken with respect to the equilibrium density matrix $\hat{\rho}_{\mathrm{h}}\otimes\hat{\rho}_{\mathrm{c}}$, with $\hat{\rho}_l\propto\exp(-\hat{H}/T_l)$ and $l=\mathrm{h},\mathrm{c}$.

In what follows, we first consider the case of filling factor $\nu=1$ with one downstream mode ($N_{\mathrm{d}}=1$) at the QH edge. Substituting the heat current operator $\hat{J}_{\mathrm{Q}}$ from Eq. (5) into Eq. (8) and applying Wick's theorem for the bosonic operators $\hat{b}_k$ and $\hat{b}_k^{\dagger}$, we obtain the final result for spectral density in the form

$$S_{\mathrm{Q}}(\omega)=\frac{\omega}{48\pi}\sum_{l=\mathrm{h},\mathrm{c}}S_{\mathrm{Q}}^{l}(\omega),$$

$$S_{\mathrm{Q}}^{l}(\omega)=[(2\pi k_BT_l)^2+(\hbar\omega)^2][1+\coth(\hbar\omega/2k_BT_l)].\quad(9)$$

First, at equilibrium $T_{\mathrm{h}}=T_{\mathrm{c}}=T$ and for zero frequency, one obtains $S_{\mathrm{Q}}(0)=2k_BT^2K_{\mathrm{q}}$. Second, because of the quadratic term $(\hbar\omega)^2$ in square brackets, the finite frequency noise at zero temperature $T_{\mathrm{h}}=T_{\mathrm{c}}=0$ does not vanish and is given by $S_{\mathrm{Q}}(\omega)=(\hbar\omega)^3\mathrm{sgn}(\omega)/24\pi\hbar$. This noise is of pure quantum nature and originates from quantum vacuum fluctuations. The nonvanishing spectral density function at zero temperature for narrow constrictions such as quantum point contact, molecular and acoustic phonon wires was already demonstrated in Refs. [25,27,28]. It is worth mentioning that for pure 1D mode the result is the same for bosonic, fermionic, and anyonic cases, in agreement with the general fact that the exchange statistics do not affect the simple thermodynamic properties of 1D systems.

Now for the case of several modes, substituting Eq. (7) into Eq. (8) and taking into account that edge modes are independent, one arrives at the following expression for the noise of the heat current

$$S_{\mathrm{th}}(\omega)=N_{+}\cdot S_{\mathrm{Q}}(\omega),\quad(10)$$

where $N_{+}=N_{\mathrm{d}}+N_{\mathrm{u}}$ is the total number of upstream and downstream modes on the upper (or lower) edge. Therefore, the noise $S_{\mathrm{th}}(\omega)$ is quantized. The symmetric noise is given by the sum $[S_{\mathrm{th}}(\omega)+S_{\mathrm{th}}(-\omega)]/2$ and is also quantized. Moreover, if one instead of noise calculates the nth moment of heat current for any integer n, one would get N_{+} times a "unit of quantization" calculated from a single free boson mode. It is worth mentioning that there is no direct relation between thermal conductance and spectral density function in the equilibrium case, since there is no fluctuation dissipation theorem which relates these two quantities. Therefore, the spectral density of the heat current can be regarded as an independent and universal quantity which provides information about edge excitations. One needs to point out that for a special form of the confining potential, edge reconstruction can occur. We assume that the effect of edge reconstruction equally influences on the number of downstream and upstream modes [10,29] and thus does not change the statement of the paper regarding the quantization of the noise of the heat current.

To summarize, we have studied the finite frequency noise of Abelian particlelike and holelike QH states using a simple picture of downstream and upstream ballistic modes. It has been shown that the finite frequency noise of the heat current is quantized, when the system size is smaller than the thermal equilibration length, and determined by the total number of modes on the QH edge. The direct measurements of spectral density of the heat current [30] together with the thermal conductance can provide additional information about the microscopic structure of the QH edge and particularly about the existence of ballistic upstream modes.

[1] K. v. Klitzing, G. Dorda, and M. Pepper, New method for high-accuracy determination of the fine-structure constant based on quantized Hall resistance, Phys. Rev. Lett. **45**, 494 (1980).
[2] D. C. Tsui, H. L. Stormer, and A. C. Gossard, Two-dimensional magnetotransport in the extreme quantum limit, Phys. Rev. Lett. **48**, 1559 (1982).
[3] Z. F. Ezawa, *Quantum Hall Effects*, 3rd ed. (World Scientific, Singapore, 2013).
[4] M. Büttiker, Absence of backscattering in the quantum Hall effect in multiprobe conductors, Phys. Rev. B **38**, 9375 (1988).
[5] R. B. Laughlin, Quantized Hall conductivity in two dimensions, Phys. Rev. B **23**, 5632(R) (1981).
[6] R. B. Laughlin, Anomalous quantum Hall effect: An incompressible quantum fluid with fractionally charged excitations, Phys. Rev. Lett. **50**, 1395 (1983).
[7] C. L. Kane and M. P. A. Fisher, Thermal transport in a luttinger liquid, Phys. Rev. Lett. **76**, 3192 (1996).
[8] C. L. Kane and M. P. A. Fisher, Quantized thermal transport in the fractional quantum Hall effect, Phys. Rev. B **55**, 15832 (1997).
[9] M. Heiblum and D. Feldman, Edge probes of topological order, Int. J. Mod. Phys. A **35**, 2030009 (2020).
[10] M. Banerjee, M. Heiblum, A. Rosenblatt, Y. Oreg, D. E. Feldman, A. Stern, and V. Umansky, Observed quantization of anyonic heat flow, Nature (London) **545**, 75 (2017).
[11] M. Banerjee, M. Heiblum, V. Umansky, D. E. Feldman, Y. Oreg, and A. Stern, Observation of half-integer thermal Hall conductance, Nature (London) **559**, 205 (2018).
[12] Steven H. Simon, Interpretation of thermal conductance of the $\nu=5/2$ edge, Phys. Rev. B **97**, 121406(R) (2018).
[13] K. K. W. Ma and D. E. Feldman, Partial equilibration of integer and fractional edge channels in the thermal quantum Hall effect, Phys. Rev. B **99**, 085309 (2019).
[14] D. F. Mross, Y. Oreg, A. Stern, G. Margalit, and M. Heiblum, Theory of disorder-induced half-integer thermal Hall conductance, Phys. Rev. Lett. **121**, 026801 (2018).
[15] C. Wang, A. Vishwanath, and B. I. Halperin, Topological order from disorder and the quantized Hall thermal metal: Possible applications to the $\nu=5/2$ state, Phys. Rev. B **98**, 045112 (2018).

[16] B. Lian and J. Wang, Theory of the disordered $\nu = 5/2$ quantum thermal Hall state: Emergent symmetry and phase diagram, Phys. Rev. B **97**, 165124 (2018).
[17] R. A. Melcer, B. Dutta, C. Spånslätt, J. Park, A. D. Mirlin, and V. Umansky, Absent thermal equilibration on fractional quantum Hall edges over macroscopic scale, Nat. Commun. **13**, 376 (2022).
[18] S. K. Srivastav, R. Kumar, C. Spånslätt, K. Watanabe, T. Taniguchi, A. D. Mirlin, Y. Gefen, and A. Das, Vanishing thermal equilibration for hole-conjugate fractional quantum Hall states in graphene, Phys. Rev. Lett. **126**, 216803 (2021).
[19] A. Bid, N. Ofek H. Inoue, M. Heiblum, C. L. Kane, V. Y. Umansky, and D. Mahalu, Observation of neutral modes in the fractional quantum Hall regime, Nature (London) **466**, 585 (2010).
[20] Y. Gross, M. Dolev, M. Heiblum, V. Umansky, and D. Mahalu, Upstream neutral modes in the fractional quantum Hall effect regime: Heat waves or coherent dipoles, Phys. Rev. Lett. **108**, 226801 (2012).
[21] R. Bhattacharyya, M. Banerjee, M. Heiblum, D. Mahalu, and V. Umansky, Melting of interference in the fractional quantum Hall effect: Appearance of neutral modes, Phys. Rev. Lett. **122**, 246801 (2019).
[22] R. Kumar, S. K. Srivastav, C. Spånslätt, K. Watanabe, T. Taniguchi, Y. Gefen, A. D. Mirlin, and A. Das, Observation of ballistic upstream modes at fractional quantum Hall edges of graphene, Nat. Commun. **13**, 213 (2022).
[23] I. V. Protopopov, Y. Gefen, and A. D. Mirlin, Transport in a disordered $\nu = 2/3$ fractional quantum Hall junction, Ann. Phys. **385**, 287 (2017).
[24] E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics, Part 2: (Landau and Lifshits Course of Theoretical Physics, Vol.9)* (Butterworth-Heinemann Oxford, 1980).
[25] D. V. Averin and J. P. Pekola, Violation of the fluctuation-dissipation theorem in time-dependent mesoscopic heat transport, Phys. Rev. Lett. **104**, 220601 (2010).
[26] X. G. Wen, Chiral Luttinger liquid and the edge excitations in the fractional quantum Hall states, Phys. Rev. B **41**, 12838 (1990).
[27] D. Sergi, Energy transport and fluctuations in small conductors, Phys. Rev. B **83**, 033401 (2011).
[28] F. Zhan, S. Denisov, and P. Hänggi, Electronic heat transport across a molecular wire: Power spectrum of heat fluctuations, Phys. Rev. B **84**, 195117 (2011).
[29] R. Sabo, I. Gurman, A. Rosenblatt, F. Lafont, D. Banitt, J. Park, M. Heiblum, Y. Gefen, V. Umansky, and D. Mahalu, Edge reconstruction in fractional quantum Hall states, Nat. Phys. **13**, 491 (2017).
[30] J. P. Pekola and B. Karimi, *Colloquium:* Quantum heat transport in condensed matter systems, Rev. Mod. Phys. **93**, 041001 (2021).