

# Uncovering gauge-dependent critical order-parameter correlations by a stochastic gauge fixing at $O(N)^*$ and Ising\* continuous transitions

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We study the  $O(N)^*$  transitions that occur in the 3D  $\mathbb{Z}_2$ -gauge  $N$ -vector model and the analogous Ising\* transitions occurring in the 3D  $\mathbb{Z}_2$ -gauge Higgs model, corresponding to the  $\mathbb{Z}_2$ -gauge  $N$ -vector model with  $N = 1$ . At these transitions, gauge-invariant correlations behave as in the usual  $N$ -vector (Ising for  $N = 1$ ) model. Instead, the non-gauge-invariant spin correlations are trivial and therefore the spin order parameter that characterizes the spontaneous breaking of the  $O(N)$  symmetry in standard  $N$ -vector (Ising) systems is apparently absent. We define a gauge fixing procedure—we name it stochastic gauge fixing—that allows us to define a gauge-dependent vector field that orders at the transition and is therefore the appropriate order parameter for the  $O(N)$  symmetry breaking. To substantiate this approach, we perform numerical simulations for  $N = 3$  and  $N = 1$ . A finite-size scaling analysis of the numerical data allows us to confirm the general scenario: the gauge-fixed spin correlation functions behave as the corresponding functions computed in the usual  $N$ -vector (Ising) model. The emergence of a critical vector order parameter in the gauge model shows the complete equivalence of the  $O(N)^*$  (Ising\*) and  $O(N)$  (Ising) universality classes.

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## I. INTRODUCTION

Gauge symmetries and Higgs phenomena are key features of theories describing collective phenomena in condensed-matter physics [1–4]. To understand these phenomena and, in particular, the major mechanisms driving phase transitions and critical phenomena in these theories, it is crucial to achieve a solid understanding of the interplay between global and gauge symmetries and, in particular, of the role that local gauge symmetries play in determining the phase structure of a model, the nature of the different phases and of the quantum and thermal transitions. Several lattice Abelian and non-Abelian gauge models have been considered, with the purpose of identifying the possible universality classes of the continuous transitions. In this paper, we focus on systems characterized by an emerging discrete gauge symmetry and, in particular, the  $\mathbb{Z}_2$  gauge group.

Lattice vector systems with  $\mathbb{Z}_2$  gauge symmetry may develop critical behaviors belonging to nonstandard  $N$ -vector universality classes, in which the fundamental vector modes cannot be identified by using gauge-invariant correlators; see, e.g., Refs. [4–22]. Such unconventional  $O(N)$  transitions occur, for example, in three-dimensional (3D)  $\mathbb{Z}_2$ -gauge  $N$ -vector models [22], i.e., in lattice  $N$ -vector models in which the global  $\mathbb{Z}_2$  symmetry is gauged, along the line that separates the spin-disordered phase from the spin-ordered one, for sufficiently small values of the gauge coupling, i.e., for large  $K$  in the phase diagram sketched in Fig. 1. These nonstandard  $O(N)$  vector universality classes, characterized by the symmetry-breaking pattern  $SO(N) \rightarrow O(N - 1)$  and by the absence of a vector order parameter, have somehow been

distinguished by adding a star, i.e., by naming them  $O(N)^*$  universality classes, see, e.g., Ref. [20]. Of course, the length-scale critical exponent  $\nu$  is the same in  $O(N)$  and  $O(N)^*$  systems.

In our paper, we mostly discuss the  $O(N)^*$  transitions developed by the  $\mathbb{Z}_2$ -gauge  $N$ -vector models, which are relatively simple, but nontrivial, representatives of statistical systems undergoing this class of continuous transitions. However, the validity of our discussion extends to generic  $O(N)^*$  transitions characterized by the absence of a local gauge-invariant vector order parameter.

In vector systems with global  $O(N)$  symmetry, continuous transitions are characterized by the spontaneous breaking of the  $O(N)$  symmetry, driven by the condensation of the  $N$ -component vector field. However, in  $\mathbb{Z}_2$ -gauge  $N$ -vector models, the correlations of the local vector operator are trivial, as a consequence of the  $\mathbb{Z}_2$ -gauge symmetry. Therefore, the spontaneous breaking of the  $O(N)$  symmetry can only be observed by considering correlations of composite gauge-invariant operators, the simplest one being an operator that transforms as a spin-two tensor under  $O(N)$  transformations. At  $O(N)^*$  transitions, this operator, as well as all gauge-invariant operators, have the same critical behavior as in the conventional  $N$ -vector model without gauge invariance. The equivalence of the gauge-invariant correlations in  $O(N)$  and  $O(N)^*$  transitions implies that gauge modes do not drive the critical behavior. As a consequence, one should be able to describe these transitions in terms of an effective Landau-Ginzburg-Wilson (LGW)  $\Phi^4$  field theory. The main issue here is the identification of the correct fundamental field  $\Phi$ . If one identifies  $\Phi$  with a coarse-grained

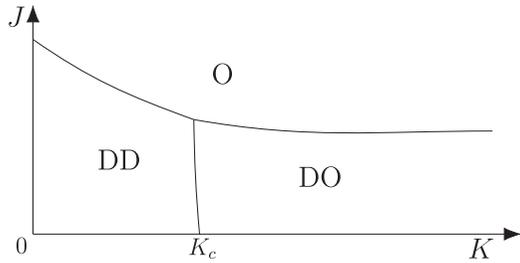


FIG. 1. Sketch of the phase diagram of the  $\mathbb{Z}_2$ -gauge  $N$ -vector model for  $N \geq 2$ , in the space of the Hamiltonian parameters  $K$  and  $J$ , cf. Eq. (1). For small  $J$ , there are two spin-disordered phases: a small- $K$  phase, in which both the spins and the gauge variables are disordered (indicated by DD), and a large- $K$  phase in which the  $\mathbb{Z}_2$ -gauge variables order (DO). In the large- $J$  phase, both spins and gauge variables are ordered (indicated by O). The  $O(N)^*$  transition line is the one that separates the DO and O phase for sufficiently large  $K$ . The three transition lines meet in a single point  $(K_*, J_*)$ ; ( $K_* \approx 0.75$ ,  $J_* \approx 0.23$ ) for small values of  $N$ ,  $N \lesssim 5$ , say; see Ref. [22] for more details.

gauge-invariant spin-2 order parameter, the LGW theory is not able to properly describe the phenomenology of the  $O(N)^*$  transitions. Indeed,  $O(N)$ -symmetric transitions driven by the condensation of a tensor spin-two field are characterized by a different symmetry-breaking pattern, thus their nature differs from that of the  $O(N)$  vector transitions, see, e.g., Ref. [22].

The critical behavior is even less conventional in the  $\mathbb{Z}_2$ -gauge Higgs model [5–7] (corresponding to a  $\mathbb{Z}_2$ -gauge  $N$ -vector model with  $N = 1$ ). In this case, there is no global  $\mathbb{Z}_2$  symmetry but, nonetheless, the transitions that occur for small gauge couplings, i.e., for large  $K$ , see the phase diagram sketched in Fig. 2, have the same universal features as Ising transitions, which are characterized by the breaking of a global  $\mathbb{Z}_2$  symmetry in standard systems. We will refer to these transitions as Ising\* transitions. Because of duality, the same Ising behavior is observed on the small- $J$  line that starts at  $J = 0$  ( $\mathbb{Z}_2$  gauge model). Also these transitions are sometimes referred to as Ising\* transitions, although the relation is obtained by the explicit use of duality; see, e.g., Refs. [23–26].

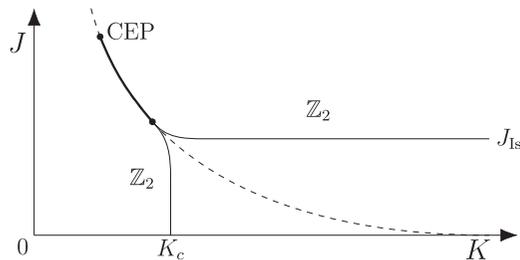


FIG. 2. Sketch of the phase diagram of the 3D  $\mathbb{Z}_2$ -gauge Higgs model. The dashed line is the self-dual line of the model, the thick line corresponds to first-order transitions along the self-dual line. The two lines labeled  $\mathbb{Z}_2$  are related by duality and correspond to Ising continuous transitions. The transition lines meet at a multicritical point along the self-dual line, at [25–28] [ $K_* = 0.7525(1)$ ,  $J_* \approx 0.22578(5)$ ], where the multicritical behavior is controlled by a multicritical  $XY$  fixed point [26,28,29]. The Ising\* transition line is the one ending at the Ising transition for  $K = \infty$  and  $J = J_{Is}$ .

A natural question concerning the  $O(N)^*$  transitions is whether it is possible to introduce a gauge fixing that allows the emergence of a vector field that orders at the transition, so it can be identified as the order parameter for the spontaneous breaking of the global  $O(N)$  symmetry, as in standard  $N$ -vector models. We can ask the same question for Ising\* transitions. In this case, the gauge fixing should turn the local  $\mathbb{Z}_2$  symmetry into a global one, which is broken at the transition by the condensation of a scalar order parameter. We mention that the search for (typically nonlocal) operators playing the role of a  $\mathbb{Z}_2$  order parameter at Ising\* transitions has recently attracted much interest, see, e.g., Refs. [25,28,30]. One possible operator is the so-called Fredenhagen-Marcu order parameter [31] that is an appropriate order parameter to characterize the Higgs phase in any lattice gauge theory. A different proposal, tailored for the  $\mathbb{Z}_2$ -gauge Higgs model, is presented in Ref. [25].

It is worth mentioning that there is another class of continuous transitions, in which critical vector modes cannot be observed by using gauge-invariant correlators. We refer to the transitions between the Coulomb and Higgs phases in noncompact lattice Abelian Higgs (AH) models, in which a complex  $N$ -vector field is minimally coupled with a noncompact  $U(1)$  gauge field, see, e.g., Refs. [32–41]. An effective description of these charged transitions is provided by the AH field theory, in which a vector and a gauge field are the fundamental variables that drive the critical behavior. However, in lattice models, the fundamental vector field does not show critical correlations because of gauge invariance. The puzzle was solved in Refs. [40,41], where it was shown that the correct order parameter for these transitions is a nonlocal gauge-invariant charged vector operator [33,34,40,41]. Equivalently, a local critical vector field is obtained by using the Lorenz gauge fixing [42].

In this paper, we show that, as in noncompact AH models, it is possible to identify a vector order parameter for  $O(N)^*$  transitions by means of an appropriate gauge-fixing procedure. However, the approach needed here—we name it stochastic gauge fixing—is more complicated than the Lorenz gauge fixing working for noncompact lattice AH models. In this approach, the gauge-fixed vector correlations show the universal critical behavior expected at transitions belonging to the standard  $O(N)$  vector universality class. Analogously, in the  $\mathbb{Z}_2$ -gauge Higgs model, we are able to observe the universal critical correlations of an emerging  $\mathbb{Z}_2$  order parameter. This shows that the  $O(N)^*$  (Ising\*) universality class is equivalent to the more standard  $O(N)$  (Ising) vector universality class, characterized by the condensation of a vector (scalar) order parameter. To validate the approach, we present numerical finite-size scaling (FSS) analyses of Monte Carlo (MC) data for  $N = 1$  and 3. Some results for  $N = 2$  were already reported in Ref. [22].

The paper is organized as follows. In Sec. II, we introduce the  $\mathbb{Z}_2$ -gauge  $N$ -vector models and the  $\mathbb{Z}_2$ -gauge Higgs model corresponding to  $N = 1$ , and summarize the main features of their phase diagrams. In Sec. III, we define the stochastic gauge fixing, which allows us to uncover the critical vector correlations along the  $O(N)^*$  transition lines, and we discuss the relation between the stochastic gauge-fixing scheme and random-bond Ising systems. In Sec. IV, we report a numerical

study of the  $O(3)^*$  transitions and Ising\* transitions. In particular, we show that vector correlation functions are critical, in the stochastic gauge-fixing scheme. Finally, in Sec. V we summarize and draw our conclusions.

## II. THE $\mathbb{Z}_2$ -GAUGE $N$ -VECTOR MODEL

The lattice  $\mathbb{Z}_2$ -gauge  $N$ -vector model is defined on a 3D cubic lattice. Its Hamiltonian is

$$H(J, K) = H_s(J) + H_\sigma(K), \quad (1)$$

$$H_s(J) = -JN \sum_{\mathbf{x}, \mu} \sigma_{\mathbf{x}, \mu} \mathbf{s}_{\mathbf{x}} \cdot \mathbf{s}_{\mathbf{x}+\hat{\mu}}, \quad (2)$$

$$H_\sigma(K) = -K \sum_{\mathbf{x}, \mu > \nu} \sigma_{\mathbf{x}, \mu} \sigma_{\mathbf{x}+\hat{\mu}, \nu} \sigma_{\mathbf{x}+\hat{\nu}, \mu} \sigma_{\mathbf{x}, \nu}, \quad (3)$$

where the site variables  $\mathbf{s}_{\mathbf{x}}$  are unit-length  $N$ -component real vectors, and the bond variables  $\sigma_{\mathbf{x}, \mu}$  ( $\sigma_{\mathbf{x}, \mu}$  is associated with the bond starting from site  $\mathbf{x}$  in the positive  $\mu$  direction,  $\mu = 1, 2, 3$ ) take the values  $\pm 1$ . The Hamiltonian parameter  $K$  plays the role of inverse gauge coupling, therefore the  $K \rightarrow \infty$  limit corresponds to the small gauge-coupling limit. By measuring energies in units of the temperature  $T$ , we can formally set  $T = 1$  and write the partition function as

$$Z(J, K) = \sum_{\{s, \sigma\}} e^{-H(J, K)}. \quad (4)$$

For  $N = 1$ , the spin variables take the integer values  $s_{\mathbf{x}} = \pm 1$ , and the model corresponds to the so-called  $\mathbb{Z}_2$ -gauge Higgs model [5–7].

The Hamiltonian (1) is invariant under global  $SO(N)$  transformations  $\mathbf{s}_{\mathbf{x}} \rightarrow V \mathbf{s}_{\mathbf{x}}$  with  $V \in SO(N)$ , and local  $\mathbb{Z}_2$  gauge transformations:

$$\mathbf{s}_{\mathbf{x}} \rightarrow w_{\mathbf{x}} \mathbf{s}_{\mathbf{x}}, \quad \sigma_{\mathbf{x}, \nu} \rightarrow w_{\mathbf{x}} \sigma_{\mathbf{x}, \nu} w_{\mathbf{x}+\hat{\nu}}, \quad w_{\mathbf{x}} = \pm 1. \quad (5)$$

From the point of view of the symmetries, the model can be interpreted as an  $N$ -vector model, which is  $O(N) = \mathbb{Z}_2 \otimes SO(N)$  symmetric, in which the  $\mathbb{Z}_2$  symmetry is gauged, i.e., becomes local. Due to the  $\mathbb{Z}_2$  gauge invariance, the correlation function of the vector variables  $\mathbf{s}_{\mathbf{x}}$ ,

$$G_s(\mathbf{x}, \mathbf{y}) = \langle \mathbf{s}_{\mathbf{x}} \cdot \mathbf{s}_{\mathbf{y}} \rangle, \quad (6)$$

trivially vanishes for  $\mathbf{x} \neq \mathbf{y}$  and any Hamiltonian parameter  $K$  and  $J$ . For the same reason, any correlation function of local operators defined as products of an odd number of spin variables (such as a spin-3 local operator) vanishes as well. These correlations, therefore, cannot characterize the  $O(N)^*$  transitions occurring for large  $K$ ; see Figs. 1 and 2. For  $N \geq 2$ , the spontaneous breaking of the global  $O(N)$  symmetry is instead signaled by the condensation of the gauge-invariant bilinear spin-two operator:

$$Q_{\mathbf{x}}^{ab} = s_{\mathbf{x}}^a s_{\mathbf{x}}^b - \frac{1}{N} \delta^{ab}. \quad (7)$$

The  $\mathbb{Z}_2$ -gauge  $N$ -vector model is a paradigmatic model relevant for transitions in nematic liquid crystal, see, e.g., Refs. [43,44], and for systems with fractionalized quantum numbers, see, e.g., Refs. [10,11]. It shows different phases characterized by the spontaneous breaking of the global

$SO(N)$  symmetry and by the different topological properties of the  $\mathbb{Z}_2$ -gauge correlations, see, e.g., Refs. [4,6,22]. Its phase diagram for  $N \geq 2$  is sketched in Fig. 1. Two spin-disordered phases are present for small  $J$ : a small- $K$  phase, in which both spin and  $\mathbb{Z}_2$ -gauge variables are disordered (DD), and a large- $K$  phase in which the  $\mathbb{Z}_2$ -gauge variables order (DO). For large  $J$ , there is a single phase in which both spins and gauge variables order (O). These phases are separated by three transition lines. The transitions along the DD-DO line, departing from the  $J = 0$  axis at [45,46]  $K_c(J = 0) = 0.761413292(11)$ , are continuous, at least for small enough values of  $J$ , and belong to the  $\mathbb{Z}_2$ -gauge universality class [4–6] for any  $N$ . Instead, the main features of the DD-O and DO-O transitions crucially depend on  $N$  and are discussed in Ref. [22]. Here, for the convenience of the reader, we summarize the main characteristics of the DO-O transition line, which is the one relevant for the present study. The transitions along the DD-O line are more conventional. They can be associated with an effective LGW theory in which the fundamental field is obtained by coarse-graining the order parameter defined in Eq. (7). The three transition lines meet at one point, see Fig. 1, located at  $[K_* \approx 0.75, J_* \approx 0.23]$  for sufficiently small values of  $N$ ,  $N \lesssim 5$ , say.<sup>1</sup>

The transitions along the DO-O line are expected to belong to the  $O(N)^*$  universality class. This is due to the stability of the  $O(N)$  vector transition occurring for  $K \rightarrow \infty$  against gauge fluctuations. Indeed, for  $K \rightarrow \infty$  the plaquette term in the Hamiltonian (1) converges to 1. In infinite volume, we can just set  $\sigma_{\mathbf{x}, \mu} = 1$  modulo gauge transformations, obtaining the partition function of the standard  $N$ -vector model. Therefore, for  $K \rightarrow \infty$  the model (1) undergoes a continuous transition at a finite  $J_c(K = \infty)$  belonging to the  $O(N)$  vector universality class (estimates of  $J_c(K = \infty)$ , i.e., of the critical point of the standard  $N$ -vector models, can be found in Refs. [47–53]). The  $O(N)$  transition is expected to be stable against small gauge fluctuations for sufficiently large but finite values of  $K$  [22] due to the discrete nature of the gauge variables, whose fluctuations are suppressed in the large- $K$  topologically ordered phase. Therefore, the continuous DO-O transitions are expected to belong to the  $O(N)^*$  universality class. Note that gauge fluctuations are instead relevant in models with continuous Abelian and non-Abelian gauge symmetries. In that case, gauge interactions destabilize the vector critical behavior, leading to transitions of different natures, see, e.g., Refs. [39–41,54,55]. If the DO-O transitions belong to the  $O(N)^*$  universality class, we expect the presence of critical vector modes. But, because of gauge invariance, they cannot be directly identified; for instance, the correlation function  $G_s$  defined in Eq. (6) is not critical. They, however, emerge if an appropriate gauge-fixing procedure is considered, as we discuss below.

The phase diagram for  $N = 1$  is shown in Fig. 2, see, e.g., Refs. [23,25,39]. Unlike the multicomponent  $N \geq 2$  models, only two phases are present, separated by two

<sup>1</sup>Since  $K_c(J) \approx K_c(J = 0) - NJ^4$  [22] for small  $J$  and  $J_*$  is only slightly larger than  $J_c(K = \infty) = J_{O(N)} \lesssim 0.25$  ( $J_{O(N)}$  is the critical point of the standard  $N$ -vector model), we expect  $K_* \approx 0.761 - 0.003N$  for sufficiently small values of  $N$ .

$\mathbb{Z}_2$  lines that are related by duality [56] and correspond to Ising continuous transitions. They end at [46] [ $J = J_{\text{Is}} = 0.221654626(5)$ ,  $K = \infty$ ] and at [ $J = 0$ ,  $K = K_c = 0.761413292(11)$ ] and meet in a multicritical  $XY$  point along the self-dual line, located at [ $K_* = 0.7525(1)$ ,  $J_* \approx 0.22578(5)$ ], see, e.g., Refs. [23,25,27,28,39]. The endpoint of the first-order transition line, at [ $K \approx 0.688$ ,  $J \approx 0.258$ ], is expected to be an Ising critical endpoint. Transitions along the large- $K$  transition line, running almost parallel to the  $K$  axis, belong to the Ising\* universality class. This is analogous to what occurs along the DO-O line for  $N \geq 2$ , where transitions belong to the  $O(N)^*$  universality class. However, for  $N = 1$  there are no gauge-invariant spin correlations. Critical scalar correlations emerge only after implementing the stochastic gauge fixing that we will outline in the next section.

### III. STOCHASTIC GAUGE FIXING

#### A. General considerations on the standard gauge-fixing approach

In lattice models with noncompact Abelian variables, it is relatively easy to define a consistent gauge fixing procedure, see, e.g., Refs. [40–42] and references therein. In these models, indeed, the lattice gauge variables take values in  $\mathbb{R}$ , and gauge fixing can be introduced just like in continuum theories; in particular, gauge fixing procedures defined by linear functionals (like the Lorenz gauge) are particularly straightforward to implement. In models with discrete or compact bond variables, instead, due to the intrinsic nonlinear structure of the gauge fixing, several additional problems arise, even for Abelian gauge groups. We discuss here this issue for the  $\mathbb{Z}_2$  gauge group but the discussion can easily be extended to any Abelian gauge group.

Since our aim is to use gauge fixing to expose the criticality of non-gauge-invariant quantities (and, in particular, of the would-be order parameter), two different problems must be tackled. The first problem is the definition and consistency of the gauge-fixing procedure, which means that we must show that expectation values of gauge-invariant quantities are the same in the original theory and in the gauge fixed one. The second problem concerns the existence of a nontrivial critical behavior of the gauge-variant modes in the gauge fixed theory, which is by no means guaranteed. Examples of gauge-fixing procedures which are not useful in studying the critical properties of noncompact Abelian gauge models are discussed in Refs. [40,42].

Let us start investigating the first of these two problems. In general, a gauge fixing is defined by a local function of the gauge fields  $F_x(\sigma)$  and by the requirement that  $F_x(\sigma) = 0$  for all lattice points  $x$ . To avoid the problem of the Gribov copies [57,58], we assume that the gauge fixing is complete, i.e., that for each configuration  $\{\sigma_{x,\mu}\}$  there is a unique configuration  $\{\sigma'_{x,\mu}\}$ , related to  $\{\sigma_{x,\mu}\}$  by a gauge transformation, such that  $F_x(\sigma') = 0$ . Although the relation between  $\{\sigma_{x,\mu}\}$  and  $\{\sigma'_{x,\mu}\}$  is unique, the gauge transformation that relates the two configurations is not. However, if  $w_x^{(a)}$  and  $w_x^{(b)}$  are two gauge transformations that both relate  $\{\sigma_{x,\mu}\}$  with  $\{\sigma'_{x,\mu}\}$ , it is trivial to show that  $w_x^{(a)} = c w_x^{(b)}$ , where  $c = \pm 1$ . This result implies that the completeness of the gauge fixing does not imply a

unique definition of the gauge-fixed spin. In the gauge fixing procedure that relates  $\{\sigma_{x,\mu}\}$  with  $\{\sigma'_{x,\mu}\}$ , the new spin  $s'_x$  is only defined up to a sign (the constant  $c$  defined above). This is not a limitation, if we only consider correlation functions with an even number of spins, for instance, the two-point function. Moreover, the previous result implies

$$\sum_w \frac{1}{2} \prod_x \delta[F_x(\sigma')] = 1, \quad \sigma'_{x,\mu} = w_x \sigma_{x,\mu} w_{x+\hat{\mu}}, \quad (8)$$

independently of  $\sigma_{x,\mu}$ , which is just the formalization of the previous statement that two different lattice gauge fields always correspond to the gauge-fixed  $\{\sigma'_{x,\mu}\}$  gauge field. This is just a discrete version of the standard Faddeev-Popov procedure [59] used to properly define gauge fixing in continuum field theories, and in the most general case a weight depending on  $\sigma'_{x,\mu}$  appears instead of the constant factor  $1/2$  in Eq. (8), which is typically rewritten by using auxiliary field variables (the so-called Faddeev-Popov ghosts). By inserting the identity Eq. (8) in the sum in Eq. (4) defining the partition function, it immediately follows that correlation functions of gauge-invariant operators computed in the gauge-fixed theory and in the original theory are the same. We have thus shown that in a  $\mathbb{Z}_2$  lattice gauge theory any complete gauge fixing defined by a local functional  $F_x(\sigma)$  can be used, without having to worry about complications related to ghosts fields.

A standard way of fixing the gauge consists of setting the bond variables on a maximal lattice tree equal to the identity [60,61]. A particular case is the axial gauge, in which all bonds in a given lattice direction are set equal to the identity, paying attention to the boundary conditions and adding some additional constraints on the boundaries. Another possibility that we considered is the gauge fixing obtained by using the gauge function

$$f_x \equiv -1 + \prod_{\mu} \sigma_{x-\hat{\mu},\mu} \sigma_{x,\mu}, \quad (9)$$

which somehow generalizes the Lorenz gauge of noncompact Abelian gauge theories. If we define  $F_x = f_x$  on the whole lattice, then the gauge fixing is not complete. It is, however, possible to make it complete by changing the gauge-fixing function on a lattice boundary (we do not report details, as this approach turns out to fail).

These types of gauge fixings do not, however, allow us to identify the critical vector modes that characterize the  $O(N)$  vector universality class. For instance, the gauge-fixed correlation function  $\langle s'_x \cdot s'_y \rangle$  [the fields  $s'_x$  are those obtained imposing the gauge-fixing constraint (9)] is not critical along the  $O(N)^*$  transition line. The absence of criticality can be traced back to the fact that the gauge-fixing procedure is strictly nonlocal. As a consequence, a local change of the initial configuration  $\sigma_{x,\mu}$  may give rise to a nonlocal change of the gauge-fixed configuration  $\sigma'_{x,\mu}$ , which prevents the spins  $s_x$  from acquiring a nonvanishing polarization. This phenomenon is easy to understand in the axial gauge defined by  $\sigma'_{x,3} = 1$ , but the same occurs when using the gauge-fixing constraint (9). If the original (i.e., not gauge fixed) configuration is  $\sigma_{x,\mu} = 1$  on the whole lattice, then we also have  $\sigma'_{x,\mu} = 1$ ; the gauge-fixed spins  $s'_x$  thus behave as in the  $O(N)$  vector model and, in particular, they are ferromagnetically

ordered for large  $J$ . If, however, the original configuration has a single bond misaligned in the axial direction in the bulk of the lattice, i.e.,  $\sigma_{y,3} = -1$  for a single  $y$  value in the bulk, in the gauge fixed configuration  $\sigma'_{x,\mu} = -1$  on a number of links of order  $L^3$ . As a consequence,  $s'_x$  does not order ferromagnetically for large  $J$ . Thus, even when  $K$  is very large, typical configurations  $\{s'_x\}$  are not related with the typical configurations of the  $O(N)$  vector model. Therefore,  $(s'_x \cdot s'_y)$  does not show any ferromagnetic order for large  $J$ .

### B. A new approach

In this paper, we pursue a different approach that allows us to uncover critical gauge-dependent order-parameter correlations. The basic idea is to average nongauge invariant quantities over all possible gauge transformations with a properly chosen, nongauge invariant, weight. We introduce  $\mathbb{Z}_2$  fields  $w_x = \pm 1$  defined on the lattice sites [associated with the local gauge transformations, see Eq. (5)], and an ancillary Hamiltonian  $H_w$  that generally depends on  $w_x$ ,  $s_x$ , and  $\sigma_{x,\mu}$ . If  $A(s_x, \sigma_{x,\mu})$  is a function of the fields, we define its weighted average over the gauge transformations as

$$[A(s_x, \sigma_{x,\mu})] = \frac{\sum_{\{w\}} A(\hat{s}_x, \hat{\sigma}_{x,\mu}) e^{-H_w}}{\sum_{\{w\}} e^{-H_w}}, \quad (10)$$

where  $\hat{s}_x$  and  $\hat{\sigma}_{x,\mu}$  are defined as

$$\hat{s}_x = w_x s_x, \quad \hat{\sigma}_{x,\mu} = w_x \sigma_{x,\mu} w_{x+\hat{\mu}}, \quad (11)$$

and correspond to the fields obtained by performing a gauge transformation with gauge function  $w_x$ . The average  $[A(s_x, \sigma_{x,\mu})]$  is then averaged over the fields  $s_x$  and  $\sigma_{x,\mu}$  using the original Hamiltonian (1), i.e.,

$$\langle [A(s_x, \sigma_{x,\mu})] \rangle = \frac{\sum_{\{s,\sigma\}} [A(s_x, \sigma_{x,\mu})] e^{-H}}{\sum_{\{s,\sigma\}} e^{-H}}. \quad (12)$$

Gauge-invariant observables are invariant under the procedure. Indeed, if  $A(s_x, \sigma_{x,\mu})$  is a gauge-invariant observable, then

$$A(\hat{s}_x, \hat{\sigma}_{x,\mu}) = A(s_x, \sigma_{x,\mu}) = [A(s_x, \sigma_{x,\mu})]. \quad (13)$$

In this approach, we define a vector correlation function as

$$G_V(\mathbf{x}, \mathbf{y}) = \langle [s_x \cdot s_y] \rangle. \quad (14)$$

A crucial point in the procedure is the choice of the Hamiltonian  $H_w$ . To unveil  $O(N)$  vector correlations, we would like to work in a gauge which maximizes the number of bonds with  $\hat{\sigma}_{x,\mu} = 1$ . Indeed, this implies that the Hamiltonian for the gauge-transformed fields  $\hat{s}_x$  is almost ferromagnetic. Therefore, these fields display the same critical behavior as vector fields in the  $O(N)$  model.

With this idea in mind, we consider

$$H_w(\gamma) = -\gamma \sum_{x,\mu} \hat{\sigma}_{x,\mu} = -\gamma \sum_{x,\mu} w_x \sigma_{x,\mu} w_{x+\hat{\mu}}, \quad (15)$$

where  $\gamma$  is a positive number that should be large enough—this point will be discussed in detail below—to ensure that the minima of  $H_w(\gamma)$  dominate in the average over the gauge transformations.

This procedure, which we call stochastic gauge fixing, mimics what is done in random systems with quenched disorder, for instance, in spin glasses. The variables  $s_x$  and  $\sigma_{x,\mu}$  are the disorder variables and  $e^{-H}/Z$  represents the disorder distribution, while  $w_x$  are the system variables that are distributed with Gibbs weight  $e^{-H_w}/Z_w$  at fixed disorder. In the language of disordered systems, the average  $[\cdot]$  therefore represents the thermal average at fixed disorder, while  $\langle \cdot \rangle$  is the average over the different disorder realizations.<sup>2</sup> This analogy allows us to use the wealth of available result for quenched random systems. In particular, the present procedure is thermodynamically consistent and, when the low-temperature (large  $\gamma$ ) phase is not a spin-glass phase, it admits a local field-theory representation. Thus, we can apply the standard renormalization-group (RG) machinery to correlations computed in the gauge-fixed theory.

The resulting model with the added variables  $w_x$  is a quenched random-bond Ising model [62,63] with a particular choice of bond distribution. Quenched random-bond Ising models have various phases—disordered, ferromagnetic, and glassy phases—depending on the temperature (whose role is played here by  $1/\gamma$ ), the amount of randomness of the bond distribution, and its spatial correlations, see, e.g., Refs. [62–75]. In particular, we expect the present model to undergo a quenched transition for  $\gamma = \gamma_c(J, K)$ . The transition separates a disordered phase for  $\gamma < \gamma_c(J, K)$  from a large- $\gamma$  phase, which *a priori* can be ferromagnetic or glassy. As we shall discuss, if  $J$  and  $K$  belong to the DO-O transition line, the large- $\gamma$  phase is ferromagnetic and the minimum configurations are essentially unique modulo global symmetries. Thus, the long-distance behavior of the variables  $w_x$  is the same for all  $\gamma > \gamma_c(J, K)$ : The variables  $w_x$  simply make uncorrelated short-range fluctuations around the minimum configurations obtained for  $\gamma \rightarrow \infty$ . It is thus natural to conjecture that  $\gamma$  is an irrelevant parameter, i.e., the critical behavior of the gauge-fixed quantities is the same for any  $\gamma > \gamma_c(J, K)$  along the DO-O transition line. In RG language,  $1/\gamma$  represents an irrelevant perturbation of the  $\gamma = \infty$  fixed point.

We expect the irrelevance of  $\gamma$  to be a general feature of the stochastic gauge fixing, which holds for any  $N$ —the numerical data we will present confirm this conjecture. Thus, for numerical convenience, we will always implement the gauge-fixing procedure using a finite value of  $\gamma$ . It is important not to confuse this type of gauge fixing with the more standard soft gauge fixing that is obtained by adding to the Hamiltonian a term of the form  $\lambda F_x^2$  involving a further parameter  $\lambda$  (where the equation  $F_x = 0$  represents the hard gauge fixing), which in the language of random systems represents an annealed average over the gauge fixing. For example, in the noncompact AH model,  $\lambda^{-1}$  is known to be a relevant perturbation of the Lorenz gauge-fixed theory, see Refs. [40,42].

It is worth noting that the global theory including the quenched stochastic gauge fixing is invariant under an extended set of local transformations with  $\mathbb{Z}_2$ -gauge parameter

<sup>2</sup>To avoid confusion, note that symbols  $[\cdot]$  and  $\langle \cdot \rangle$  have typically the opposite meaning in the random-system literature: The former represents the disorder average and the latter the thermal average.

$v_x = \pm 1$  given by

$$s_x \rightarrow v_x s_x, \quad \sigma_{x,\mu} \rightarrow v_x \sigma_{x,\mu} v_{x+\hat{\mu}}, \quad w_x \rightarrow v_x w_x. \quad (16)$$

Therefore, only observables that are invariant under this set of transformations are relevant, such as  $\hat{s}_x$  and  $\hat{\sigma}_{x,\mu}$ . Note that this local extended symmetry prevents the  $w_x$  variables from acquiring a nonvanishing expectation value, and thus the presence of a standard ferromagnetic phase in this random quenched model.

We remark that there is much freedom in the choice of the ancillary Hamiltonian  $H_w$ . The Hamiltonian (15) is quite appealing, since it provides a direct connection with quenched models already well investigated and is particularly simple. However, other choices may work as well. It is important to note that the critical behavior of the emerging vector critical correlators is expected to be universal, i.e., independent of the ancillary Hamiltonian  $H_w$ , provided that  $H_w$  has been properly chosen to make the spin-spin interactions ferromagnetic. This is essentially due to the fact that the critical behavior of all gauge-invariant quantities—for instance, the spin-two operator  $Q_x$  or the cumulants of the gauge-invariant energy—is independent of  $H_w$ : they all behave as in the  $N$ -vector (Ising) model. Therefore, along the DO-O line (or along the corresponding transition line for  $N = 1$ ),  $G_V(\mathbf{x} - \mathbf{y})$  should also behave as in the  $N$ -vector (Ising) model, if it is critical.

It is interesting to verify the general ideas of the approach in the limit  $K \rightarrow \infty$ . In this limit, we have

$$\sigma_{x,\mu} \sigma_{x+\hat{\mu},\nu} \sigma_{x+\hat{\nu},\mu} \sigma_{x,\nu} = 1, \quad (17)$$

which implies  $\sigma_{x,\mu} = \rho_x \rho_{x+\hat{\mu}}$ , with  $\rho_x = \pm 1$ , in the thermodynamic limit. It follows that

$$H_J = -NJ \sum_{x,\mu} (\rho_x s_x) \cdot (\rho_{x+\hat{\mu}} s_{x+\hat{\mu}}),$$

$$H_w = -\gamma \sum_{x,\mu} (\rho_x w_x) \cdot (\rho_{x+\hat{\mu}} w_{x+\hat{\mu}}). \quad (18)$$

Thus, if we redefine  $w'_x = \rho_x w_x$  and  $s'_x = \rho_x s_x$  [which are invariant under the local transformations of Eq. (16)], the partition function factorizes. Moreover, since  $\rho_x^2 = 1$ ,  $G_V(\mathbf{x})$  can be written as

$$G_V(\mathbf{x} - \mathbf{y}) = \langle s'_x \cdot s'_y \rangle_{O(N),J} \langle w'_x \cdot w'_y \rangle_{\text{Is},\gamma}, \quad (19)$$

where the two averages are performed in the standard  $O(N)$  and Ising models, respectively. For any  $\gamma > \gamma_{c,\text{Is}}$ , the Ising system is magnetized and therefore  $G_V(\mathbf{x})$  has the same critical behavior as in the  $N$ -vector model, confirming the irrelevance of  $\gamma$ . On the other hand, for  $\gamma < \gamma_{c,\text{Is}}$ ,  $G_V(\mathbf{x})$  is always disordered, irrespective of  $J$ .

Note that Eq. (17) also holds for  $J \rightarrow \infty$  for any value of  $K$ . Thus, also in this case the partition function factorizes. The gauge fields  $w_x$  behave as in the Ising model and so does the correlation function  $G_V(\mathbf{x} - \mathbf{y})$  as a consequence of Eq. (19).

We finally mention that the above ideas can be straightforwardly extended to lattice gauge models with other discrete groups or with continuous gauge  $U(1)$  variables, thereby allowing one to uncover order-parameter correlations that are not accessible using gauge-invariant operators.

## IV. NUMERICAL RESULTS

In this section we discuss the critical behavior of the  $O(3)^*$  and Ising\* transitions in the  $\mathbb{Z}_2$ -gauge model for  $N = 3$  and  $N = 1$ , respectively. In particular, we study the correlation function  $G_V(\mathbf{x})$  defined in Eq. (14) using the stochastic gauge fixing approach. The numerical analysis of the data shows that  $G_V(\mathbf{x})$  behaves as the vector correlation function in standard  $N$ -vector models. This result provides the last piece of evidence for the identification of the transitions as  $O(N)$  (or Ising) transitions driven by the condensation of a vector, but gauge-dependent, order parameter.

### A. Monte Carlo simulations

We perform simulations of the  $\mathbb{Z}_2$ -gauge model for  $N = 1$  and  $N = 3$ , applying the stochastic gauge fixing. In both cases, we perform runs at fixed  $K$  varying  $J$  around the critical point  $J = J_c$ , where the model undergoes an  $O(3)^*$  or Ising\* transition for  $N = 3$  and  $N = 1$ , respectively. We choose  $K = 1$ , which is larger than  $K_* \approx 0.75$ , the value of the meeting point of the three transition lines, see Figs. 1 and 2. The parameter  $\gamma$  of the ancillary Hamiltonian  $H_w$  should be large enough to guarantee that the ancillary quenched system is in the ferromagnetic phase, as discussed in Sec. III. For  $N = 3$ , we find  $\gamma_c(J_c, K = 1) \approx 0.23$ , and we thus fix  $\gamma = 0.3$ . Also, for  $N = 1$  the ancillary quenched system is in the ferromagnetic phase when using  $\gamma = 0.3$  at  $K = 1$  and  $J = J_c$ , so we use  $\gamma = 0.3$  for both values of  $N$ .

MC simulations are performed as in systems with quenched disorder, see, e.g., Refs. [62–75]. We simulate the model with Hamiltonian  $H$  and every  $N_s$  update sweeps we compute the quenched averages  $[A]$  over the gauge-fixing variables for fixed values of  $s_x$  and  $\sigma_{x,\mu}$  (at fixed disorder in the language of random systems). Simulations are performed by using standard local Metropolis updates. For  $N = 3$ , we also perform microcanonical updates of the  $s_x$  variables. Quenched averages are computed using  $10^4$  complete update sweeps of the  $w_x$  variables, at fixed  $\sigma_{x,\mu}$  and  $s_x$ .

Before presenting our results, we report the values of the critical exponents for the 3D  $O(3)$  (Heisenberg) and Ising universality classes that are used in our analyses. These exponents are known with great accuracy, see, e.g., Refs. [76–82] for  $N = 3$  and Refs. [46,76,79,82–87] for  $N = 1$ . For the 3D Heisenberg universality class, accurate estimates are reported in Ref. [77]:

$$\begin{aligned} \nu_H &= 0.71164(10), \\ \eta_H &= 0.03784(5), \\ \omega_H &= 0.759(2). \end{aligned} \quad (20)$$

Here  $\nu_H$  is the correlation-length exponent,  $\eta_H$  parametrizes the behavior of the critical two-point function of the vector field, and  $\omega_H$  is the leading scaling-correction exponent. In our FSS analyses, we also need the RG dimension  $Y_Q$  of the spin-two composite operator  $Q_x$  for the Heisenberg universality class [80,87–89]:

$$Y_{QH} = 1.2094(3). \quad (21)$$

For the 3D Ising universality class, we report [85]

$$\begin{aligned} \nu_l &= 0.629971(4), \\ \eta_l &= 0.036298(2), \\ \omega_l &= 0.8297(2). \end{aligned} \quad (22)$$

### B. Finite-size scaling

For  $N \geq 2$ , some relevant observables are obtained from the correlations of the gauge-invariant bilinear operator  $Q_x^{ab}$  defined in Eq. (7). Its two-point correlation function reads

$$G_Q(\mathbf{x}, \mathbf{y}) = \langle \text{Tr } Q_x Q_y \rangle, \quad (23)$$

from which one can also define the corresponding Fourier transform

$$\tilde{G}_Q(\mathbf{p}) = \frac{1}{L^3} \sum_{\mathbf{x}, \mathbf{y}} e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} G_Q(\mathbf{x}, \mathbf{y}), \quad (24)$$

susceptibility and correlation length:

$$\chi_q = \tilde{G}_Q(\mathbf{0}), \quad (25)$$

$$\xi_q^2 = \frac{1}{4 \sin^2(\pi/L)} \frac{\tilde{G}_Q(\mathbf{0}) - \tilde{G}_Q(\mathbf{p}_m)}{\tilde{G}_Q(\mathbf{p}_m)}, \quad (26)$$

where  $\mathbf{p}_m = (2\pi/L, 0, 0)$ . In the FSS limit, varying  $J$  around the critical point  $J_c$  at fixed  $K$ ,  $\chi_q$ , and  $\xi_q$  scale as

$$\chi_q(J, L) \approx L^{3-2Y_Q} \mathcal{C}_q(W), \quad (27)$$

$$\xi_q(J, L) \approx L \mathcal{R}_q(W), \quad (28)$$

where  $Y_Q$  is the RG dimension of the spin-two operator  $Q_x$  and

$$W = (J - J_c)L^{1/\nu}. \quad (29)$$

Scaling corrections decay as  $L^{-\omega}$ , where  $\omega$  is the leading correction-to-scaling exponent. The ratio

$$R_q = \frac{\xi_q}{L} \approx \mathcal{R}_q(W) \quad (30)$$

is RG invariant and can be used to determine  $J_c$ . Indeed, the data for different lattice sizes  $L$  have a crossing point that coincides with the critical point for large values of  $L$ .

In the case of the  $\mathbb{Z}_2$ -gauge Higgs model, there are no gauge-invariant operators analogous to  $Q_x$ . The transition may be probed by studying the fluctuations of the gauge-invariant energy density, see, e.g., Refs. [21,25,26]. For this type of transition, the analysis of the critical gauge-dependent vector correlations may provide an alternative method to determine the critical point, besides confirming the Ising\* nature of the transition. It may also be numerically convenient, since numerical studies based on the energy cumulants are quite demanding.

We also analyze the gauge-dependent correlation function  $G_V(\mathbf{x}, \mathbf{y})$  in the stochastic gauge-fixing approach. If it develops a nontrivial critical behavior, then it is expected to scale as (assuming translation invariance for the sake of simplicity)

$$G_V(\mathbf{x} - \mathbf{y}, J, L) = L^{-2Y_V} [\mathcal{G}_V(X, W) + O(L^{-\omega})], \quad (31)$$

where  $\nu$  is the length-scale critical exponent and  $Y_V$  is the RG dimension of the vector operator, which is related to the

exponent  $\eta$  by

$$Y_V = \frac{d-2+\eta}{2} = \frac{1+\eta}{2}. \quad (32)$$

The function  $\mathcal{G}_V$  is expected to be universal, apart from a multiplicative factor and a normalization of the scaling variables  $W$ . It should only depend on the boundary conditions and lattice shape. We also consider the corresponding susceptibility  $\chi_v$  and second-moment correlation length  $\xi_v$ ,

$$\chi_v \equiv \tilde{G}_V(\mathbf{0}), \quad \xi_v^2 \equiv \frac{1}{4 \sin^2(\pi/L)} \frac{\tilde{G}_V(\mathbf{0}) - \tilde{G}_V(\mathbf{p}_m)}{\tilde{G}_V(\mathbf{p}_m)}, \quad (33)$$

where  $\tilde{G}_V(\mathbf{p})$  is defined as in Eq. (24). Two RG invariant quantities associated with  $G_V(\mathbf{x}, \mathbf{y})$  are the ratio

$$R_v \equiv \xi_v/L \quad (34)$$

and the Binder parameter

$$U_v = \frac{\langle [m_2^2] \rangle}{\langle [m_2] \rangle^2}, \quad m_2 = \frac{1}{L^3} \sum_{\mathbf{x}, \mathbf{y}} \hat{\mathbf{s}}_{\mathbf{x}} \cdot \hat{\mathbf{s}}_{\mathbf{y}}. \quad (35)$$

If the correlation function  $G_V$  is critical, then we expect  $R_v$  and  $U_v$  to behave as  $R_q$ , in the FSS limit. For example,

$$R_v(J, L) = \mathcal{R}_v(W) + O(L^{-\omega}). \quad (36)$$

Actually, to avoid the nonuniversal normalization of the argument  $W$ , one may rewrite the Binder parameter in terms of  $R_v$ :

$$U_v(J, L) = \hat{U}(R_v) + O(L^{-\omega}). \quad (37)$$

Finally, the vector susceptibility is expected to scale as

$$\chi_v(J, L) \approx L^{2-\eta} \hat{\mathcal{C}}_v(R_v). \quad (38)$$

We also consider observables involving the variables  $w_x$  only. They allow us to check that the value of  $\gamma$  we consider is sufficiently large for the quenched system to be in the ferromagnetic phase. To define appropriate observables, it is important to note that the gauge-fixed theory is gauge invariant under the transformations (16). Therefore, we consider the so-called replica observables, as commonly done in the analysis of random quenched systems. If  $w_x^{(1)}$  and  $w_x^{(2)}$  are two different system variables distributed with probability  $e^{-H_w}/Z_w$  with the same disorder distribution (same  $\sigma_{x,\mu}$ ,  $\mathbf{s}_x$  in the present context), we define the overlap variable as the product

$$O_x = w_x^{(1)} w_x^{(2)}. \quad (39)$$

The corresponding susceptibility is

$$\chi_o = L^{-3} \left\langle \left[ \left( \sum_x O_x \right)^2 \right] \right\rangle, \quad (40)$$

while the Binder parameter is defined as

$$U_o = \frac{\langle [n_2^2] \rangle}{\langle [n_2] \rangle^2}, \quad n_2 = L^{-3} \sum_{\mathbf{x}, \mathbf{y}} O_x O_y. \quad (41)$$

In the ferromagnetic phase [ $\gamma > \gamma_c(K, J)$ ], we expect the finite-size behavior:

$$\chi_o \sim L^3, \quad U_o = 1 + O(L^{-3}). \quad (42)$$

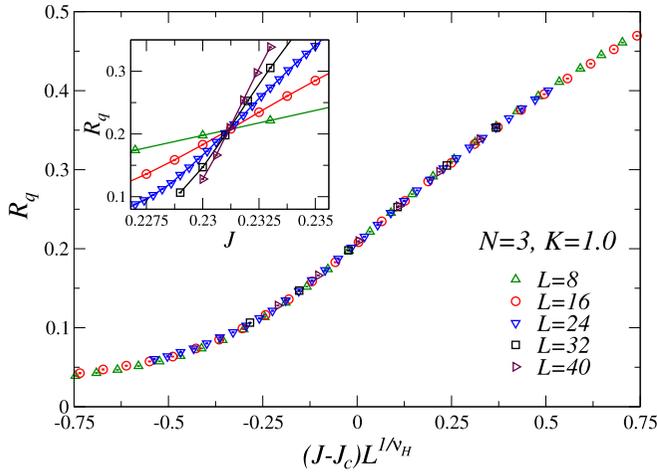


FIG. 3. Scaling plot of the ratio  $R_q$  for the  $\mathbb{Z}_2$ -gauge  $N = 3$  vector model at  $K = 1$ , with periodic boundary conditions. We plot  $R_q$  versus  $W = (J - J_c)L^{1/\nu}$ , with  $J_c = 0.23118(3)$  and  $\nu = \nu_H = 0.71164$ . In the inset, we report  $R_q$  as a function of  $J$ .

### C. Results for $\mathbb{Z}_2$ -gauge $N = 3$ vector model

We now present our numerical FSS analyses for the  $\mathbb{Z}_2$ -gauge model with  $N = 3$ . We consider periodic boundary conditions and perform simulations along the line  $K = 1$ . For  $J \approx 0.23$ , the model undergoes a continuous  $O(3)^*$  transition. This is clearly demonstrated by the FSS behavior of ratio  $R_q$  and of the susceptibility  $\chi_q$  defined in Eqs. (30) and (25). In Fig. 3, we show the results for  $R_q$ . Data have a clear crossing point and show an excellent scaling if we set  $\nu = \nu_H$ , confirming the Heisenberg nature of the transition. Fits of the data fixing  $\nu = \nu_H$  and  $\omega = \omega_H$  provide an accurate estimate of the critical point  $J_c = 0.23118(3)$ . Note that  $J_c$  is quite close to the critical value in the Heisenberg model ( $K \rightarrow \infty$ )  $J_H = 0.2310010(7)$  [48,49], indicating that the transition line is almost parallel to the  $K$  axis. The scaling behavior of  $\chi_q$ , see Fig. 4, is fully consistent with Eq. (27), using the RG

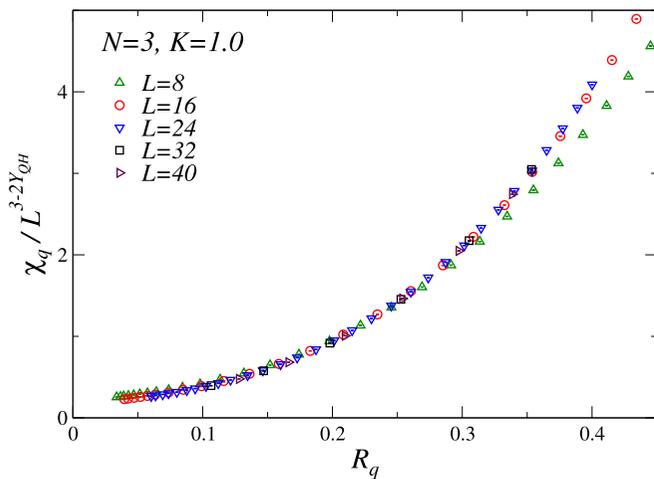


FIG. 4. Scaling plot of  $\chi_q$  for the  $\mathbb{Z}_2$ -gauge  $N = 3$  vector model at  $K = 1$ , with periodic boundary conditions. We report  $L^{-3+2Y_Q} \chi_q$  versus  $R_q$ , using the RG dimension  $Y_Q$  for the Heisenberg universality class:  $Y_Q = Y_{QH} = 1.2094$ .

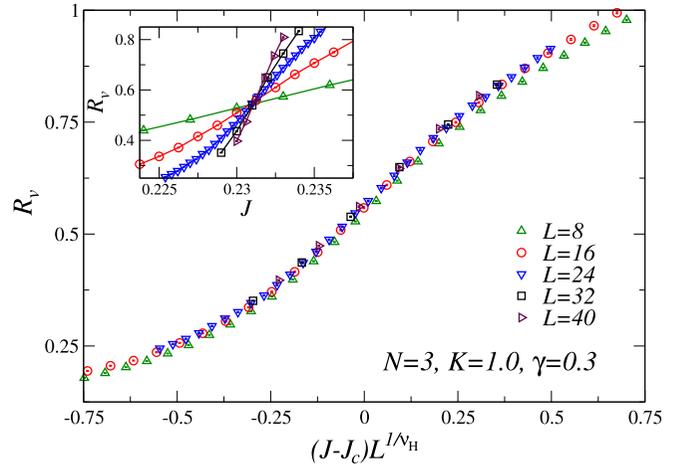


FIG. 5. Ratio  $R_v = \xi_v/L$  for the  $\mathbb{Z}_2$ -gauge  $N = 3$  vector model at  $K = 1$ , using the stochastic gauge fixing with  $\gamma = 0.3$ . Plot of  $R_v$  as a function of  $W = (J - J_c)L^{1/\nu_H}$  and as a function of  $J$  (see inset). We fix  $\nu = \nu_H = 0.71164$ .

dimension  $Y_Q = Y_{QH}$  of the spin-2 operator at the  $O(3)$  vector fixed point, again in agreement with the general scenario.

We now analyze the vector correlation  $G_V$ , defined in Eq. (14). As discussed in Sec. III, the parameter  $\gamma$  of the ancillary Hamiltonian  $H_w$  must be chosen so the ancillary system is in the ferromagnetic phase. This can be easily checked by looking at the behavior of the overlap observables defined in Eqs. (40) and (41). For  $\gamma = 0.3$ , with increasing  $L$ , we find that  $\chi_o/L^3 \approx 0.6755$  and that the difference  $U_o - 1$  vanishes as  $L^{-3}$ , as expected for a ferromagnetic phase. Therefore, in the following we compute the vector observables fixing  $\gamma = 0.3$ .

Results for  $R_v = \xi_v/L$  are reported in Fig. 5. The correlation length increases as  $L$ , as expected for a critical observable, and shows a crossing point at a value of  $J$  that is very close to the critical point  $J_c$  obtained fitting the gauge-invariant observable  $R_q$ . More precisely, fits of  $R_v$  give  $J_c = 0.23128(8)$ , which is in good agreement with the estimate  $J_c = 0.23118(3)$  determined above. The  $R_v$  data also show excellent FSS behavior if we use the Heisenberg estimate of the correlation length exponent; see Fig. 5. To obtain a robust check that  $G_V$  has the same critical behavior as the Heisenberg correlation function, in Fig. 6 we report  $U_v$  versus  $R_v$  together with the universal scaling function for the analogous vector quantities computed in the Heisenberg model (the curve is taken from Ref. [90]). Also in this case the agreement is excellent. Finally, we determine the scaling behavior of  $\chi_v$ . The vector susceptibility scales very nicely according to Eq. (38) if we set  $\eta = \eta_H$ , where  $\eta_H$  is the Heisenberg exponent for vector correlations, see Fig. 7.

In conclusion, the above FSS analyses demonstrate the effectiveness of the stochastic gauge fixing outlined in Sec. III. Indeed, it allows us to identify a critical gauge-dependent vector field that orders at the transition, and therefore can be taken as the order parameter for the spontaneous breaking of the  $O(N)$  symmetry to  $O(N - 1)$ . In turn, this allows us to obtain a complete mapping of the RG operators of the  $O(N)$  and  $O(N)^*$  transitions.

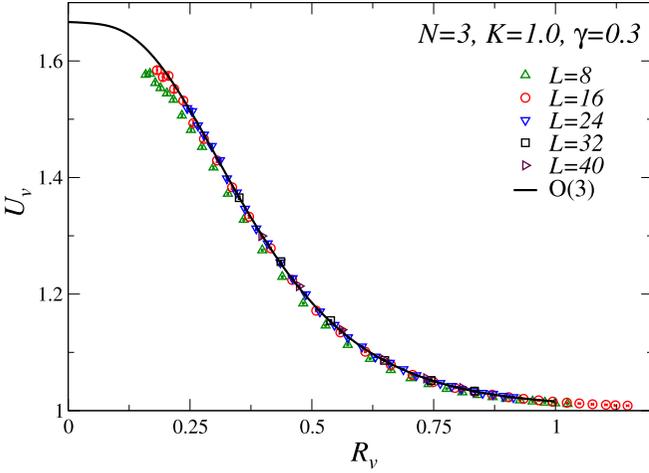


FIG. 6.  $U_v$  as a function of  $R_v$  for the  $\mathbb{Z}_2$ -gauge  $N = 3$  vector model at  $K = 1$ , using the stochastic gauge fixing with  $\gamma = 0.3$ , for different values of the lattice size  $L$  and of the coupling  $J$ . The data follow the FSS behavior (37). Moreover, the asymptotic FSS curve is consistent with the universal FSS curve for the standard O(3) vector model with periodic boundary conditions (solid line). For the O(3) curve, we use the parametrization reported in Ref. [90].

#### D. Results for the $\mathbb{Z}_2$ -gauge Higgs model

We now present results for the  $\mathbb{Z}_2$ -gauge Higgs model. We study the critical behavior at the Ising\* transition along the line  $K = 1$ , using the stochastic gauge fixing, to uncover the gauge-dependent spin critical fluctuations. We use open boundary conditions to avoid the long autocorrelation times associated with the Polyakov lines.<sup>3</sup> We fix  $\gamma = 0.3$ :  $\chi_o$  and

<sup>3</sup>Note that definitions reported in Sec. IV B are valid both for open and periodic boundary conditions.

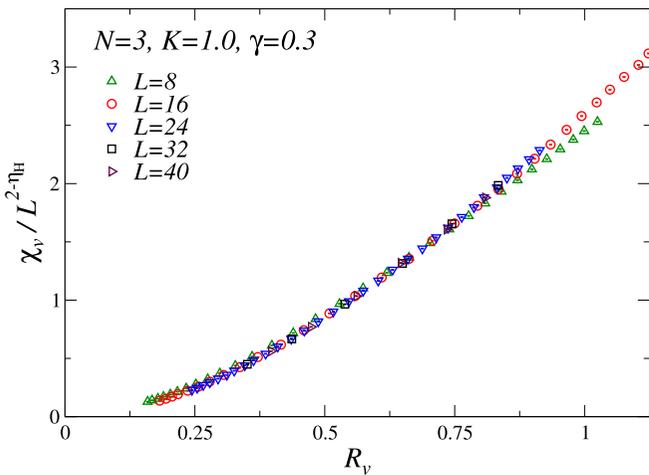


FIG. 7. Vector susceptibility  $\chi_v$  for the  $\mathbb{Z}_2$ -gauge  $N = 3$  vector model at  $K = 1$ , using the stochastic gauge fixing with  $\gamma = 0.3$ . The excellent scaling of the ratio  $\chi_v/L^{2-\eta_H}$  with  $\eta_H = 0.03784$  confirms that along the O(3)\* transition line the correlation function  $G_v$  behaves as the vector correlation function in the O(3) vector model.

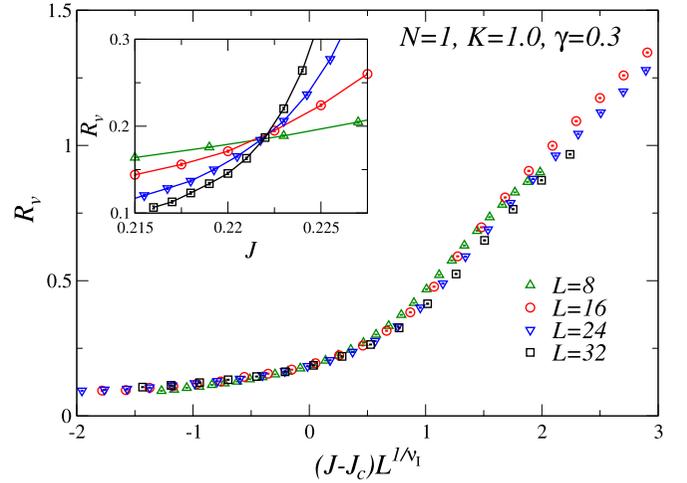


FIG. 8. Ratio  $R_v = \xi_v/L$  for the  $\mathbb{Z}_2$ -gauge Higgs model at  $K = 1$ , using the stochastic gauge fixing with  $\gamma = 0.3$ . We plot  $R_v$  versus  $W = (J - J_c)L^{1/\nu_I}$  using  $\nu = \nu_I = 0.629971$  and the estimate  $J_c = 0.22185(10)$ , obtained by biased fits of  $R_v$  using  $\nu = \nu_I$ . The inset shows the same data versus  $J$ .

$U_o$  close the transition behave as in Eq. (42), confirming that the ancillary system is ferromagnetic for this value of  $\gamma$ .

The correlation  $G_v$  behaves as the spin correlation function at the standard Ising transition. This is confirmed by the data of  $R_v = \xi_v/L$  shown in Fig. 8. The correlation length  $\xi_v$  shows an excellent scaling if one uses the Ising critical exponent  $\nu_I$ . A fit of the data provides the accurate estimate  $J_c = 0.22185(10)$ , which is again quite close to the critical value in the  $K \rightarrow \infty$  limit, i.e.,  $J_{I_S} = 0.221654626(5)$  of the standard Ising model [46]. The critical nature of the gauge-dependent correlations is further confirmed by the plot of  $U_v$  versus  $R_v$  shown in Fig. 9. Data approach an asymptotic scaling curve, as predicted by the FSS Eq. (37). Finally, the scaling behavior of the susceptibility  $\chi_v$  is definitely consistent with Eq. (38), setting  $\eta = \eta_I$ , where  $\eta_I$  is the Ising exponent. Data reported in Fig. 10 show an excellent scaling.

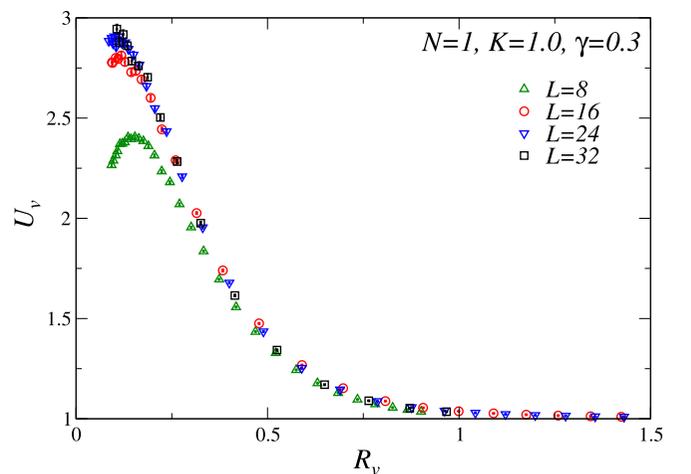


FIG. 9. Vector Binder parameter  $U_v$  as a function of  $R_v$  for the  $\mathbb{Z}_2$ -gauge Higgs model at  $K = 1$ , using the stochastic gauge fixing with  $\gamma = 0.3$ .

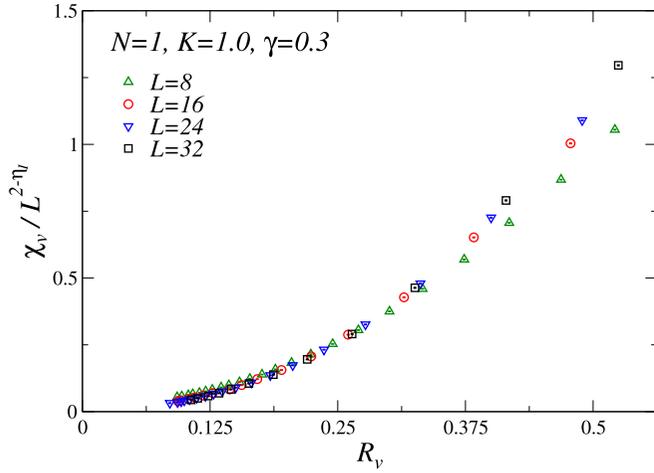


FIG. 10. Vector susceptibility  $\chi_v$  for the  $\mathbb{Z}_2$ -gauge Higgs model at  $K = 1$ , using the stochastic gauge fixing with  $\gamma = 0.3$ . We set  $\eta = \eta_l = 0.036298$ . The excellent scaling confirms that  $G_V$  along the Ising\* transition line behaves as the standard Ising spin-spin correlation function.

The above FSS results show again the effectiveness of the stochastic gauge fixing outlined in Sec. III, which allows us to identify the universal gauge-dependent spin correlations which characterize standard Ising transitions, see, e.g., Ref. [76].

### E. Transitions of the ancillary quenched model

In the previous subsections, we focused on the critical behavior of the vector correlations defined by using the stochastic gauge fixing at the transition point of the original gauge-invariant lattice model. We now focus on the ancillary quenched random-bond model. For any  $J$  and  $K$ , it undergoes a quenched transition at  $\gamma_c(J, K)$ , which separates a small- $\gamma$  disordered phase from a large- $\gamma$  ordered phase. Here we wish to investigate the nature of this transition in the different phases of the gauge-invariant lattice model. We only report results for  $N = 3$ , but the general picture should be valid for any  $N$ , including the gauge-Higgs model with  $N = 1$ . We focus on the behavior along the line  $K = 1$ , considering three values of  $J$ : (i)  $J = 0.2 < J_c$  in the DO phase; (ii)  $J = J_c \approx 0.2312$ , where the gauge-invariant model is critical; and (iii)  $J = 0.26 > J_c$ , in the O phase.

For  $J = 0.2$ , in the DO phase, the overlap variables show the presence of a transition for  $\gamma \approx 0.222$ , with critical exponents that are definitely consistent with the Ising values. This is clearly confirmed by the data shown in the lower panel of Fig. 11, where we plot the overlap Binder parameter as a function of  $(\gamma - \gamma_c)L^{1/\nu_l}$ , with Ising exponent  $\nu_l$  (we use  $\gamma_c = 0.22185(3)$  as obtained from the crossing point of the overlap Binder parameter). Moreover, the data of the overlap susceptibility  $\chi_o$  at  $\gamma_c$  (not shown) are consistent with the behavior  $\chi_o \sim L^{2-\eta_{ql}}$ , where  $\eta_{ql} = 1 + 2\eta_l = 1.072596(4)$  is the Ising exponent associated with the RG dimension of the overlap variable. We expect these results to be the same in the whole DO phase. This is confirmed by the results of Ref. [75], that also observed a pure Ising critical behavior on the line  $J = 0$  in the DO phase.

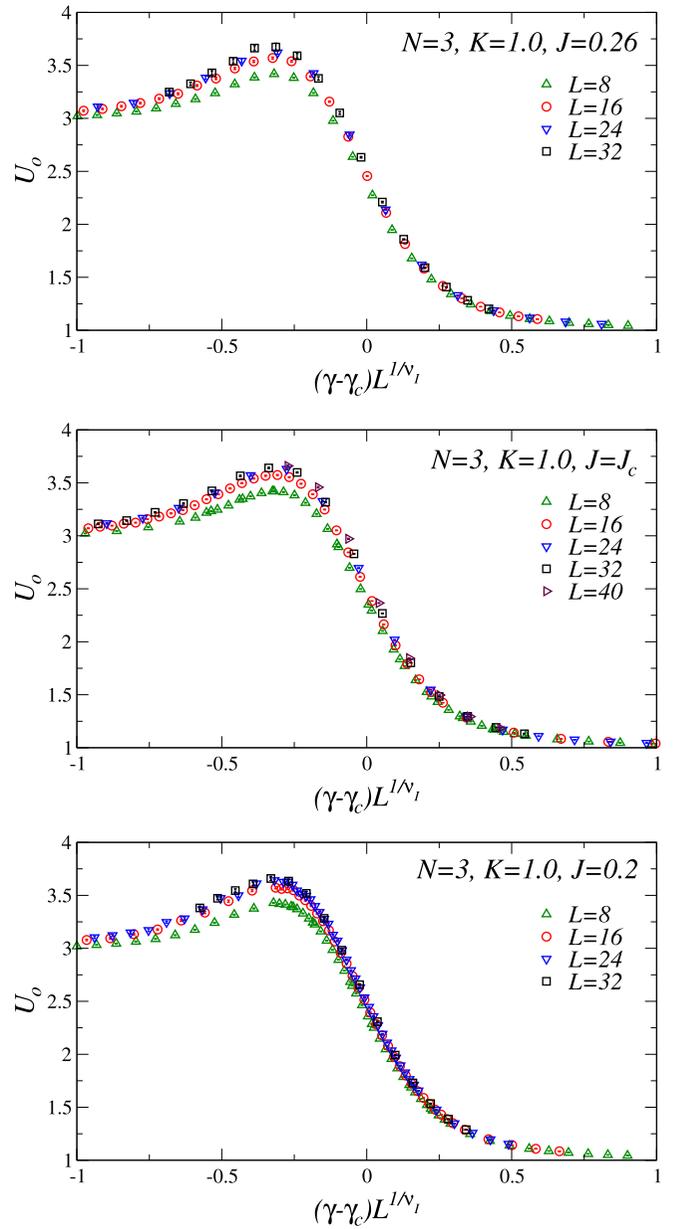


FIG. 11. Overlap Binder parameter  $U_o$  for  $N = 3$ ,  $K = 1$  and three different values of  $J$  as a function of  $(\gamma - \gamma_c)L^{1/\nu_l}$ , where  $\nu_l = 0.629971$  is the Ising critical exponent. Results for (top)  $J = 0.26 > J_c$  (O phase) with  $\gamma_c = 0.22178$ ; (middle)  $J = 0.23118 \approx J_c$  with  $\gamma_c = 0.22178$ ; and (bottom)  $J = 0.2 < J_c$  (DO phase) with  $\gamma_c = 0.22185$ .

These results indicate that the type of bond disorder that occurs in the DO phase does not destabilize the pure Ising fixed point. This is different from what occurs in generic random-bond models with spatially uncorrelated bimodal or Gaussian bond distributions. In that case, the Harris criterion [62] predicts the instability of the pure Ising fixed point against quenched disorder due to the positive value of the 3D Ising specific-heat exponent. Indeed, in the presence of spatially uncorrelated quenched disorder, one expects that the ferromagnetic critical behavior belongs to another universality class, the so-called randomly diluted Ising universality class,

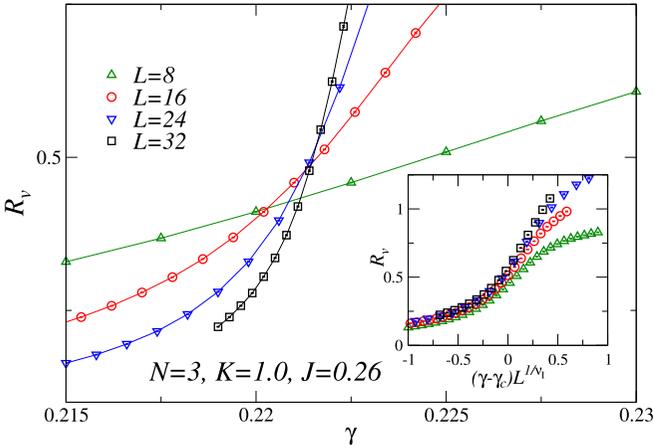


FIG. 12. The length-scale ratio  $R_v = \xi_v/L$  defined in Eq. (33) as a function of the stochastic gauge fixing parameter  $\gamma$  for  $N = 3, K = 1$ , and  $J = 0.26 > J_c$ . In the inset, we report a scaling plot using the Ising critical exponent  $\nu_I = 0.629971$ .

with critical exponents [70,76]  $\nu_{rI} = 0.683(2)$  and  $\eta_{rI} = 0.036(1)$ . The observed pure Ising critical behavior, i.e., the apparent stability of the pure Ising fixed point, is related to the different nature of the quenched disorder in the present model, which corresponds to a topologically ordered phase.

In the DO phase, for  $J < J_c$ , we do not expect the stochastic gauge fixing to lead to any ordering of the vector correlation  $G_V$  defined in Eq. (14), for any  $\gamma$ , because the condensation of the gauge-dependent vector field should be accompanied by the condensation of the gauge-invariant bilinear operator  $Q_x$ , which only occurs for  $J > J_c$  and is not affected by the stochastic gauge fixing. Numerical data confirm this general picture.

Let us now turn our attention to the O phase. For  $J \rightarrow \infty$ , the arguments reported at the end of Sec. III B indicate that the  $w_x$  variables behave as Ising variables. Therefore, for  $\gamma = \gamma_c$ , overlap variables should behave as in the Ising case. Results for  $J = 0.26 > J_c$  are fully consistent with this picture, as demonstrated by the data reported in the top panel of Fig. 11. The overlap Binder parameter shows the expected FSS if we set  $\nu = \nu_I$ , where  $\nu_I$  is the Ising critical exponent. In the plot, we use  $\gamma_c = 0.22178(6)$ , as obtained by biased fits of the data setting  $\nu = \nu_I$ . For  $J \rightarrow \infty$ ,  $G_V$  behaves as the spin-spin correlation function in the Ising model; see Sec. III B. It is natural to expect the same behavior in the whole ordered phase. The critical nature of  $G_V$  for  $\gamma = \gamma_c$  in the O phase is demonstrated by the data for  $R_v$  computed for  $J = 0.26$  reported in Fig. 12. There is a clear crossing point, indicating that  $\xi_v \sim L$  for  $\gamma = \gamma_c$ , at approximately the same value of  $\gamma_c$  obtained from the analysis of the overlap variables. However, in the ordered phase, scaling corrections are large. They are probably related to the nearby presence of the critical point  $J = J_c \approx 0.23$ , where also the disorder is critical. We have also investigated the behavior of the susceptibility  $\chi_v$ , which is expected to scale:  $\chi_v \sim L^{2-\eta_I} \sim L^{1.96}$ . Data show significant scaling corrections. Fits of  $\chi_v$  versus  $L^\kappa$  at the critical point  $\gamma = \gamma_c$  provide estimates of  $\kappa$  that increase as lower- $L$  data are discarded. If we consider only data with  $L \geq 16$ , this behavior is consistent with  $\eta = \eta_I$  and the presence of large scaling

corrections parametrized by the exponent  $\omega = \omega_I$ . These large scaling corrections are probably due to the spin modes being not fully magnetized, given the relatively small lattice sizes we consider and the small distance between  $J = 0.26$  and the critical point.

Finally, let us consider the behavior for  $J = J_c$ . In this case, we are dealing with a multicritical point, where both the gauge-invariant observables and the ancillary variables  $w_x$  are critical. As we already discussed, both in the DO and O phases, the variables  $w_x$  show Ising criticality. Apparently, the ordering of the spin degrees of freedom has little influence on the behavior of the gauge variables  $w_x$ . This is also supported by the estimates of the critical point:  $\gamma_c = 0.22185(3)$  for  $J = 0.2$ ,  $\gamma_c = 0.22178(3)$  at  $J = J_c$ , and  $\gamma_c = 0.22178(6)$  at  $J = 0.26$ . We thus expect a pure Ising behavior of the overlap correlations along the whole line  $K = 1$ , from  $J = 0$  [75] to  $J = \infty$ , including the critical point  $J = J_c$ . More generally, Ising behavior should occur in the DO and O phases, including the DO-O transition line. This is in agreement with the exact results for  $K = \infty$  reported at the end of Sec. III B. Indeed, in this limit the overlap variables behave as Ising variables for any value of  $J$ , because of the factorization of the partition function. Numerical results are fully consistent with this picture, as demonstrated by the data reported in the middle panel of Fig. 11.

Let us finally discuss the behavior of  $G_V$  at the critical point. In the limit  $K \rightarrow \infty$ , the correlation function factorizes as indicated in Eq. (19). In infinite volume at fixed  $J = J_c$ , we can rewrite it as

$$G_V(\mathbf{x}) = \frac{Z}{r^{1+\eta_H}} G_{\text{Is}}(\mathbf{x}), \quad (43)$$

where  $G_{\text{Is}}(\mathbf{x})$  is the Ising correlation function and  $r = |\mathbf{x}|$ . We thus predict that  $\xi_v$  behaves as an Ising correlation length, while the susceptibility  $\chi_v$  is expected to scale as  $L^{2-\eta_{HI}}$ , with  $\eta_{HI} = 1 + \eta_H + \eta_I = 1.07414(10)$ . We expect these results to hold on the whole DO-O transition line. This is in fully supported by the numerical data. For instance, see the lower panel of Fig. 13, where we observe an excellent scaling if we plot  $R_v$  in terms of  $W = (\gamma - \gamma_c)L^{1/\nu_I}$ , using the Ising critical exponent. Analogously, the susceptibility  $\chi_v$  scale as  $L^{2-\eta_{HI}}$  with  $\eta_{HI} = 1.07414(10)$ ; see the upper panel of Fig. 13.

It is interesting to observe that, since the point  $J = J_c$ ,  $\gamma = \gamma_c(J_c)$  is a multicritical, it is characterized by two different length scales with exponents  $\nu_I$  and  $\nu_H$ , respectively. The Heisenberg length scale can be identified, for instance, by varying  $J$  and correspondingly fixing  $\gamma = \gamma_c(J)$ . Along this line, vector observables computed from  $G_V(\mathbf{x})$  would scale in terms of  $W = (J - J_c)L^{1/\nu_H}$ , i.e., with the Heisenberg length-scale exponent.

## V. CONCLUSIONS

In this paper, we address the nature of the so-called  $O(N)^*$  and Ising\* universality classes that differ from the standard 3D  $O(N)$  vector and Ising universality classes because of the absence of critical vector correlations. These transitions occur, for example, in generalized lattice  $\mathbb{Z}_2$ -gauge models, i.e., the 3D  $\mathbb{Z}_2$ -gauge  $N$ -vector models, where all gauge-invariant quantities develop critical behaviors analogous to

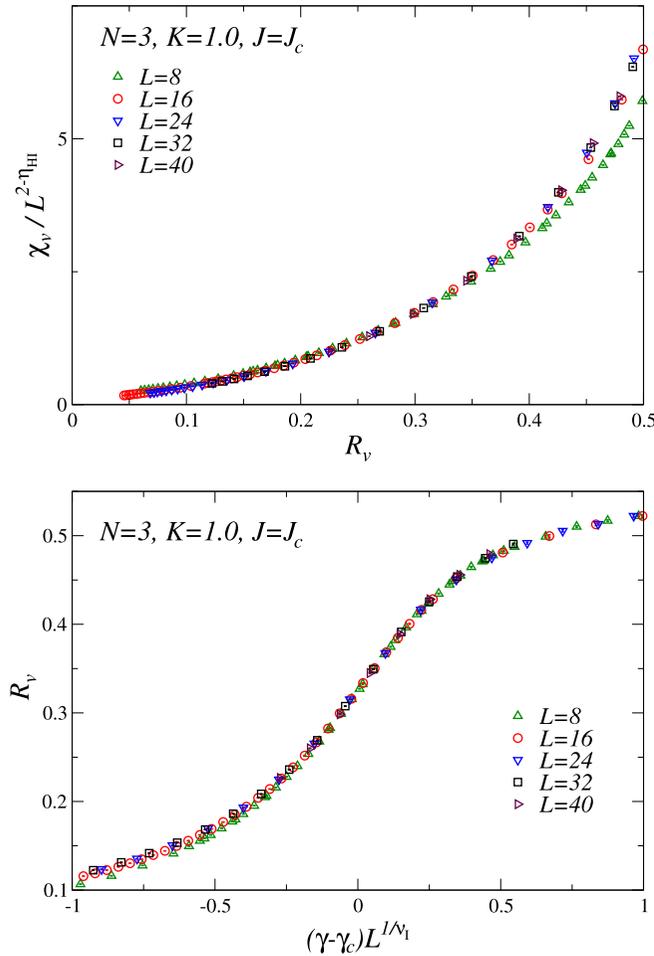


FIG. 13. Scaling of the  $R_v = \xi_v/L$  and of the susceptibility  $\chi_v$ , see Eq. (33), as a function of the stochastic gauge fixing parameter  $\gamma$ . Results for  $N = 3$ ,  $K = 1$ , and  $J = 0.2312 \approx J_c$ . We plot  $R_v$  versus  $(\gamma - \gamma_c)L^{1/n_v}$  with  $\gamma_c = 0.22178$  (bottom), and  $\chi_v/L^{2-\eta_H}$  versus  $R_v$  with  $\eta_H = 1 + \eta_l + \eta_H = 1.07414$  (top).

those of the standard  $N$ -vector or Ising (for  $N = 1$ ) model, without exposing critical order-parameter vector correlations. The fundamental spin field is not gauge invariant and therefore its correlation functions are trivial. This apparently precludes interpreting these transitions as the result of the condensation of a vector order parameter that breaks the  $O(N)$  symmetry [in the gauge model, the global symmetry group is  $SO(N)$ ] down to  $O(N - 1)$ , as in the standard  $N$  vector model.  $O(N)^*$  transitions occur in 3D  $\mathbb{Z}_2$ -gauge  $N$ -vector models with Hamiltonian (1), along the line between the spin-disordered and spin-ordered phase for sufficiently large values of the inverse gauge coupling  $K$ ; see Fig. 1. Analogous Ising\* transitions, see Fig. 2, occur in the one-component model ( $N = 1$ ), which is also known as the 3D  $\mathbb{Z}_2$ -gauge Higgs model.

We extend the characterization of the  $O(N)^*$  transitions, showing that one can define a proper gauge-fixing procedure, which does not change gauge-invariant correlations but allows one to define vector correlations that behave as in the  $N$ -vector model. For this purpose, we propose a gauge-fixing procedure, which we name stochastic gauge fixing, that is

quite different from the usual gauge-fixing procedures that are used in field theory. Here, gauge-dependent quantities are averaged over gauge transformations weighted by an ancillary gauge-dependent Hamiltonian. This leads to an extended quenched model, which can be interpreted as a random-bond Ising model, in which the bonds are distributed according to the Gibbs weight associated with the original gauge-invariant Hamiltonian. The random-bond Ising model we consider differs from those typically studied in the random-system literature. Indeed, one typically considers spatially uncorrelated bond distributions, see, e.g., Refs. [62–74]. Instead,  $O(N)^*$  (Ising\*) transitions separate two phases characterized by the fact that the  $\mathbb{Z}_2$ -gauge variables show a topological order.

It is important to remark that, if the ancillary Hamiltonian is chosen such that the gauge-fixed vector correlations are critical, then their critical behavior should necessarily coincide with that of vector correlations in the standard  $O(N)$  universality class (spin-spin correlations for  $N = 1$ ), given that all gauge-invariant observables—for instance, the spin-two operator  $Q_x^{ab}$  defined in Eq. (7) or the cumulants of the gauge-invariant energy—behave as in the  $N$ -vector model. By using the stochastic gauge fixing, we are thus able to identify the missing critical vector fields, obtaining a full correspondence between  $O(N)^*/\text{Ising}^*$  and standard  $O(N)/\text{Ising}$  universality classes.

In this work, we define a simple ancillary system that makes gauge-fixed vector correlations critical. Explicit numerical results are obtained for  $N = 3$  and  $N = 1$ . We show that the stochastic gauge fixing procedure allows us to observe the missing vector field which orders at the transition, characterizing the breaking of the  $O(N)$  global symmetry. Analogously, in the  $\mathbb{Z}_2$ -gauge model, we identify the scalar field that breaks the global  $\mathbb{Z}_2$  symmetry, which is absent in the gauge-invariant model (it is gauged) but is restored by the gauge-fixing procedure.

Note that the use of the stochastic gauge fixing in the  $\mathbb{Z}_2$ -gauge Higgs model may also turn out to be convenient from a purely numerical point of view, since it simplifies (and likely makes more accurate) the analyses required to investigate the critical behavior of the model: In the gauge invariant model, only cumulants of the energy can be studied, while in the gauge fixed model we have access to all the standard observables commonly used to study magnetic transitions.

The results presented in this paper show that the  $O(N)^*$  transitions in  $\mathbb{Z}_2$ -gauge  $N$ -vector models can still be described by an effective  $O(N)$ -symmetric LGW  $\Phi^4$  theory, without gauge fields. However, the fundamental field in the effective theory is not gauge invariant. We remark that this behavior shows some similarities and also significant differences with respect to the one emerging at the Coulomb-Higgs transitions in the lattice AH models with noncompact  $U(1)$  gauge variables, which are associated with a stable charged fixed point of the AH field theory, see, e.g., Refs. [32,39]. At variance with  $O(N)^*$  transitions, at Coulomb-Higgs transitions gauge modes play a fundamental role and therefore cannot be integrated out. On the other hand, as in  $O(N)^*$  transitions, the fundamental vector field is not gauge invariant and thus critical vector correlations are only observed when the Lorenz gauge fixing is used or, equivalently, if one considers nonlocal gauge-invariant charged operators [35,40,41].

It is finally important to remark that the stochastic gauge-fixing method introduced in this paper to study the  $O(N)^*$  transitions of the  $\mathbb{Z}_2$ -gauge  $N$ -vector model can be easily generalized and extended to generic statistical models undergoing  $O(N)^*$  transitions in the presence of an emerging local discrete gauge symmetry. Actually, this approach can be straightforwardly extended to continuous gauge groups, such as the  $U(1)$  group. However, its utility within this extended context of continuous gauge groups must be checked to understand whether it allows us to expose further features arising from gauge-dependent modes. This may provide a convenient way of studying observables whose form is very

complicated (and, in general, nonlocal) when written in a manifestly gauge-invariant form, going beyond the techniques which use standard gauge-fixing approaches [33,34,40–42].

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