


Lieb-Schultz-Mattis theorem with long-range interactions

 Ruochen Ma 
Kadanoff Center for Theoretical Physics, University of Chicago, Chicago, Illinois 60637, USA


(Received 18 June 2024; revised 20 August 2024; accepted 21 August 2024; published 5 September 2024)

We prove the Lieb-Schultz-Mattis theorem in d -dimensional spin systems exhibiting $SO(3)$ spin rotation and lattice translation symmetries in the presence of k -local interactions decaying as $\sim 1/r^\alpha$ with distance r . Two types of Hamiltonians are considered: Type I comprises long-range spin-spin couplings, while type II features long-range couplings between $SO(3)$ symmetric local operators. For spin- $\frac{1}{2}$ systems, it is shown that type I cannot have a unique symmetric ground state with a nonzero excitation gap when the interaction decays sufficiently fast, i.e., when $\alpha > \max(3d, 4d - 2)$. For type II, the condition becomes $\alpha > \max(3d - 1, 4d - 3)$. In $1d$, this ingappability condition is improved to $\alpha > 2$ for type I and $\alpha > 0$ for type II by examining the energy of a state with a uniform 2π twist. Notably, in $2d$, a type II Hamiltonian with van der Waals interaction is subject to the constraint of the theorem.

 DOI: [10.1103/PhysRevB.110.104412](https://doi.org/10.1103/PhysRevB.110.104412)

I. INTRODUCTION

The spectral gap is a central quantity of a quantum many-body system, which determines the correlation and entanglement properties in the ground state [1–5]. Regarding the spectral gap, a kinematic constraint is the Lieb-Schultz-Mattis (LSM) theorem [6]. It states that for a translationally invariant spin chain with $SO(3)$ spin rotation symmetry and a spin- $\frac{1}{2}$ moment per site, a unique gapped symmetric ground state is forbidden. The theorem, together with its topological argument [7,8] and higher-dimensional generalizations [3,9,10], features the fundamental role played by symmetry and topology in quantum many-body physics.

However, most discussions of the LSM theorem in the current literature focus on systems with only finite-range interactions (with a few exceptions, see Ref. [11]). Recently, it was realized that long-range interactions (i.e., power-law decay of $\sim 1/r^\alpha$ in distance r), which are ubiquitous in many-body systems, can lead to qualitatively new physics [12–18]. Thus, it is important to ask whether and to what extent the LSM constraint holds in the presence of such interactions. In this work, we address this issue, and the main results are summarized in Theorems 1 and 2. We first define the system in $1d$ and demonstrate the existence of a low-lying state with vanishing energy above the ground state following the original proof [6,19]. Then, to generalize the discussion to higher dimensions, we utilize the powerful machinery of quasiadiabatic evolution [2,3,20] to construct a low-energy state with a twisted boundary condition, which is then shown to be orthogonal to the original ground state due to the fact that each site has a spin- $\frac{1}{2}$ moment. In this reasoning, a crucial ingredient is the recent development of the Lieb-Robinson (LR) bound in power-law interacting systems [21–27].

II. SYSTEMS IN $1d$

Consider a $1d$ chain with length L and periodic boundary conditions. In each unit cell, there is a spin- $\frac{1}{2}$ moment. The

Hamiltonian is invariant under lattice translation and $SO(3)$ global spin rotation symmetry. Throughout this work, we consider two types of Hamiltonians. Type I consists of two-body antiferromagnetic interactions, with strengths decaying according to a power α in the distance,

$$H_I = \sum_{\langle ij \rangle} h_{ij}, \quad h_{ij} = \frac{\mathbf{S}_i \cdot \mathbf{S}_j}{(|i - j| + 1)^\alpha}, \quad (1)$$

where $\{ij\}$ includes all pairs of lattice sites. Type II is defined as

$$H_{II} = \sum_{\langle ij \rangle} h_{ij}, \quad h_{ij} = \frac{O_i O_j}{(|i - j| + 1)^\alpha}, \quad (2)$$

where O_i and O_j are local operators acting near i and j , respectively, and both are individually $SO(3)$ symmetric, e.g., $O_i = \mathbf{S}_i \cdot \mathbf{S}_{i+1}$. After the Jordan-Wigner transformation, H_I becomes a fermionic Hamiltonian with long-range particle hopping terms, while H_{II} has only short-range hopping but long-range density-density couplings. Later, we will also discuss generalizations to generic k -local Hamiltonians. We are interested in the spectrum of the Hamiltonians, particularly whether a unique symmetric ground state with a nonzero excitation gap Δ persists in the $L \rightarrow \infty$ limit. We present the following result.

Theorem 1. In $1d$, (1) the type I Hamiltonian cannot have a unique gapped symmetric ground state for $\alpha > 2$, and (2) the type II Hamiltonian cannot have a unique gapped symmetric ground state for $\alpha > 0$.

Proof. We follow the original proof of the LSM theorem [6]. The idea is to consider the state after a uniform 2π flux insertion (twist). We demonstrate that when $\alpha > 2$, this twisted state has a vanishing excitation energy relative to H_I .

We define the twist operator implementing the flux insertion as

$$U = \exp \left(-i \sum_{j=0}^{L-1} \frac{2\pi}{L} j S_j^z \right). \quad (3)$$

Let the ground state of H be denoted as $|\psi_0\rangle$. The energy difference ΔE between the twisted state $U|\psi_0\rangle$ and $|\psi_0\rangle$ can be expressed as

$$\begin{aligned} \langle \psi_0 | U^\dagger H U | \psi_0 \rangle - \langle \psi_0 | H | \psi_0 \rangle &= \langle \psi_0 | U^\dagger [H, U] | \psi_0 \rangle \\ &\leq \langle \psi_0 | [U^\dagger, [H, U]] | \psi_0 \rangle. \end{aligned} \quad (4)$$

We then utilize the explicit expression of U to compute the energy difference and thus bound the excitation gap as

$$\begin{aligned} \Delta E &\leq 2 \sum_{(ij)} \langle \psi_0 | \frac{S_i^x S_j^x + S_i^y S_j^y}{(|i-j|+1)^\alpha} \left[\cos \frac{2\pi(i-j)}{L} - 1 \right] | \psi_0 \rangle \\ &\leq \sum_{(ij)} \frac{2S^2}{(|i-j|+1)^\alpha} \left(\frac{2\pi(i-j)}{L} \right)^2 \\ &\leq \max[O(L^{-1}), O(L^{2-\alpha})], \end{aligned} \quad (5)$$

where S is the spin quantum number of a single site and we use $0 \leq 1 - \cos \theta \leq \frac{\theta^2}{2}$. Here, we bound the summation at large distances as

$$\sum_{j=1}^L \frac{1}{(j+1)^{\alpha-2}} \leq \int_{O(1)}^L dx \frac{1}{x^{\alpha-2}} = \max[O(1), O(L^{3-\alpha})]. \quad (6)$$

Finally, observe that

$$T U T^\dagger U^\dagger = e^{-i2\pi S_0^z} e^{i\frac{2\pi}{L} \sum_{j=0}^{L-1} S_j^z}, \quad (7)$$

where T represents lattice translation, i.e., $T S_j^z T^\dagger = S_{j+1}^z$. As $|\psi_0\rangle$ is $SO(3)$ invariant and $e^{-i2\pi S_0^z} = -1$ for $S = \frac{1}{2}$, we deduce that $U|\psi_0\rangle$ and $|\psi_0\rangle$ have different lattice momenta and therefore are orthogonal. This, along with a similar argument for H_{II} , establishes the proof of Theorem 1. ■

Let us note that our conclusion extends to more general k -local Hamiltonians (i.e., each term in the Hamiltonian acts on at most k spins, although these k spins can be far apart), which encompass many physical systems, such as van der Waals interactions, ring-exchange interactions [28,29], and multibody interactions emerging in periodically driven systems [30] and trapped ions [31,32], provided that an appropriate condition on convergence is imposed. Consider the k -local Hamiltonian

$$H = \sum_Z H_Z, \quad (8)$$

where Z is a subset of sites with cardinality $|Z| \leq k$. Assuming H has a unique symmetric ground state $|\psi_0\rangle$, following the same derivation that leads to Eq. (5), the excitation gap of H can be bounded by

$$\begin{aligned} \Delta E &\leq \langle \psi_0 | [U^\dagger, [H, U]] | \psi_0 \rangle \\ &\leq \sum_Z \|U^\dagger h_Z U + U h_Z U^\dagger - 2h_Z\|. \end{aligned} \quad (9)$$

Given the $SO(3)$ invariance of each term, it follows that [33]

$$U h_Z U^\dagger = \exp\left(-i\frac{2\pi}{L} Q_Z\right) h_Z \exp\left(i\frac{2\pi}{L} Q_Z\right), \quad (10)$$

where

$$Q_Z = \sum_x (x - x_L) S_x^z, \quad (11)$$

with the sum running over all sites on which h_Z acts nontrivially and x_L is the coordinate of the leftmost site among them (starting from zero). The eigenvalue μ of Q_Z is bounded by $-kR_Z S \leq \mu \leq kR_Z S$, where R_Z is the largest distance between any two sites in Z . The operator h_Z can be decomposed as $h_Z = \sum_\mu h_Z^\mu$, such that $[Q_Z, h_Z^\mu] = \mu h_Z^\mu$. Consequently, we have

$$\begin{aligned} \Delta E &\leq \sum_Z \|U^\dagger h_Z U + U h_Z U^\dagger - 2h_Z\| \\ &\leq \sum_Z \sum_\mu 2 \left[1 - \cos\left(\frac{2\pi}{L} \mu\right) \right] \|h_Z^\mu\| \\ &\leq \sum_Z \frac{4\pi^2 k^2 R_Z^2 S^2}{L^2} \|h_Z\| \leq \frac{\text{const}}{L} \sum_{Z \ni z} \|h_Z\| R_Z^2, \end{aligned} \quad (12)$$

where we note that $\|h_Z^\mu\| \leq \|h_Z\|$ and use translation symmetry to arrive at the last line, where the sum includes all terms that act nontrivially on a single site z . Therefore, for a generic k -local Hamiltonian, if an $\epsilon > 0$ exists such that

$$\sum_{Z \ni z} \|h_Z\| R_Z^2 < L^{1-\epsilon}, \quad (13)$$

the excitation gap vanishes algebraically for large L . The type I Hamiltonian in Eq. (1) with $\alpha > 2$ is a special case that satisfies this convergence condition.

On the other hand, for a special subclass of k -local Hamiltonians that involve only long-range couplings among $SO(3)$ singlet local operators (or solely “density-density” couplings after the Jordan-Wigner transformation), the eigenvalue μ in the commutator $[Q_Z, h_Z^\mu] = \mu h_Z^\mu$ is bounded by a constant that does not depend on the diameter R_Z . In this case, if an $\epsilon > 0$ exists such that

$$\sum_{Z \ni z} \|h_Z\| < L^{1-\epsilon}, \quad (14)$$

then ΔE would vanish algebraically as $L \rightarrow \infty$. The type II Hamiltonian in Eq. (2) with $\alpha > 0$ is an example of such Hamiltonians.

III. HIGHER DIMENSIONS

Now let us consider a d -dimensional lattice where one specific dimension, referred to as the x direction, has a size denoted by L (chosen to be even). The Hamiltonian exhibits translational invariance and periodicity along the x direction. All other dimensions are referred to as y . The volume of the system, i.e., the total number of unit cells, is denoted as V . Like in Ref. [3], it is important to assume that the system’s width, defined as V/L , is an odd number. This assumption will be used to prove the orthogonality of the twisted state below. The thermodynamic limit is characterized by scaling the sizes of all dimensions to infinity at the same rate, specifically $V \simeq O(L^d)$. The proof in $1d$ does not directly apply, even for finite-range Hamiltonians, as the energy of $U|\psi_0\rangle$ is bounded only by $O(\frac{V}{L} \frac{1}{L})$. Nonetheless, we still have the following.

Theorem 2. In d dimensions, (1) the type I Hamiltonian cannot have a unique gapped symmetric ground state for $\alpha > \max(3d, 4d-2)$, and (2) the type II Hamiltonian cannot have a unique

gapped symmetric ground state for $\alpha > \max(3d - 1, 4d - 3)$. The theorem holds when V/L is an odd number.

Before presenting the proof, we have an additional comment. Instead of the specific form in Eqs. (1) and (2), Theorem 2 actually applies to generic k -local Hamiltonians, as long as a certain LR bound holds. Consider a Hamiltonian of the form $H = \sum_Z H_Z$, where each term H_Z is supported on a subset of sites denoted by Z . As discussed in Ref. [27], the LR bound holds when (1) the long-range interaction decays sufficiently fast, i.e., for all sites in the entire lattice Λ ,

$$\sup_{z \in \Lambda} \sum_{Z \ni z: \text{diam}(Z) \geq R} \|H_Z\| \leq \frac{\text{const}}{R^{\alpha-d}}, \quad (15)$$

with $\alpha > 2d$ and $\text{diam}(Z)$ denoting the largest distance between any two points in Z , and (2)

$$\sup_{z \in \Lambda} \sum_{y \in \Lambda} \sum_{Z \ni y, z} \|H_Z\| < \infty, \quad (16)$$

which ensures that the sum of the norms of all terms acting on any fixed site is finite.

Theorem 2 applies to generic k -local Hamiltonians that meet these two requirements. Specifically, such a Hamiltonian cannot have a unique gapped symmetric ground state when the parameter α in Eq. (15), which governs the decay of the interaction, satisfies $\alpha > \max(3d, 4d - 2)$ for generic k -local interactions or $\alpha > \max(3d - 1, 4d - 3)$ for k -local interactions among $SO(3)$ singlet local operators. One can verify that Eqs. (1) and (2) are examples of such Hamiltonians. The reason for distinguishing between type I and type II will become clear when we construct the Hamiltonian with a twist. Last, note that the two requirements above for the Hamiltonian differ from the convergence condition (13) specific to $1d$, as the variational state for which the excitation energy is estimated is constructed in a distinct way.

Proof of Theorem 2. We follow the approach described in Ref. [3], which is a proof by contradiction. We start by assuming the existence of a unique gapped ground state. Leveraging the LR bound and bounding the norm of long-range couplings, we can demonstrate the insensitivity of the energy to a twisted boundary condition. Namely, we construct a low-lying twisted state with energy algebraically small in L . Subsequently, we demonstrate that for spin- $\frac{1}{2}$ per unit cell, the 2π -twisted state has a distinct lattice momentum compared to the initial ground state and thus must be orthogonal to it. The presence of a low-energy twisted state then leads to a contradiction, thereby establishing the theorem.

We focus on a type I Hamiltonian H . To construct a state with a twisted boundary condition, we begin by constructing a twisted Hamiltonian. Unlike Hamiltonians with only finite-range interactions, where a symmetry defect can be introduced by modifying only terms in the vicinity of the twist, constructing a twisted Hamiltonian with long-range interactions is more intricate. Let us introduce a partial spin rotation operator:

$$U_R(\theta) = \exp \left[-i\theta \sum_{j \in R} S_j^z \right], \quad (17)$$

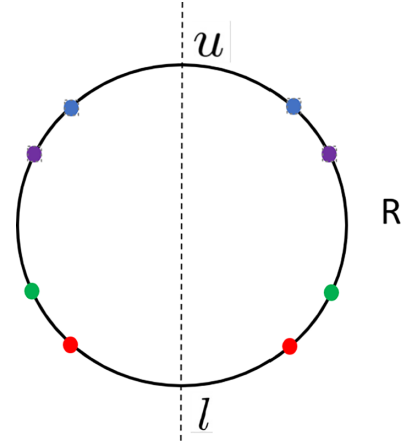


FIG. 1. Plot of the system, showing only the x direction. All sites to the right of the dashed line belong to the right half R . The lower quarter l (the shorter interval between the two red dots) contains sites with x coordinates in $[\frac{7L}{8}, \frac{L}{8} - 1]$. The upper quarter u (the interval between the two blue dots) contains sites with $x \in [\frac{3L}{8}, \frac{5L}{8} - 1]$. For later convenience, we define the region l' to be sites with $x \in [\frac{13L}{16}, \frac{3L}{16} - 1]$ (interval between the green dots) and u' to be sites with $x \in [\frac{5L}{16}, \frac{11L}{16} - 1]$ (interval between the purple dots).

where the summation extends over all spins on the right half of the lattice (see Fig 1). We then denote the Hamiltonian terms supported entirely in the lower (upper) quarter as H_l (H_u), i.e., $H_l = \sum_{i,j \in l} h_{ij}$. The twisted Hamiltonian is defined as follows:

$$H_\theta^{tw} = H_\theta^l + H - H_l, \quad (18)$$

where $H_\theta^l = U_R^\dagger(\theta) H_l U_R(\theta)$. For later convenience, we also define a Hamiltonian $H_\theta = U_R^\dagger(\theta) H U_R(\theta)$. Compared to H , in H_θ , all terms supported simultaneously on the left and right halves are modified. By definition, H_θ exhibits exactly the same spectrum for all θ , assumed to have a nonzero gap Δ .

IV. THE TWISTED STATE

We introduce the twisted state $|\psi_\theta\rangle$, with its density matrix denoted as ρ_θ , using quasiadiabatic evolution [3,20]. The density matrix of the unique ground state of H_θ [i.e., $|\phi_\theta\rangle := U_R^\dagger(\theta)|\phi_0\rangle$] is referred to as σ_θ . Without loss of generality, we choose $H_\theta \sigma_\theta = E_0 \sigma_\theta = 0$. We define the twisted state ρ_θ as

$$\partial_\theta \rho_\theta = \int_0^{L^\beta} d\tau [A^+(i\tau) - A^-(-i\tau), \rho_\theta], \quad (19)$$

with $\rho_0 = \sigma_0$. Here, β is a control parameter, and

$$A^+(i\tau) = -\frac{1}{2\pi i} e^{-(\Delta\tau)^2/2q} \int_t \frac{\partial_\theta H_\theta^{tw}(t)}{t + i\tau} e^{-(\Delta t)^2/2q}, \quad (20)$$

where $\partial_\theta H_\theta^{tw}(t) = e^{iH_\theta t} \partial_\theta H_\theta^{tw} e^{-iH_\theta t}$, $A^-(-i\tau) = [A^+(i\tau)]^\dagger$, and q is another adjustable parameter. For later convenience, we also define

$$B^+(i\tau) = -\frac{1}{2\pi i} e^{-(\Delta\tau)^2/2q} \int_t \frac{\partial_\theta H_\theta(t)}{t + i\tau} e^{-(\Delta t)^2/2q}, \quad (21)$$

where $\partial_\theta H_\theta(t) = e^{iH_\theta t} \partial_\theta H_\theta e^{-iH_\theta t}$ and $B^-(-i\tau) = [B^+(i\tau)]^\dagger$. Hereafter, unless otherwise specified, all integration regions

are $(-\infty, \infty)$. Note that ρ_θ is always a pure state for all θ since commutation with an anti-Hermitian operator generates a unitary rotation. Our goal is to bound the energy of ρ_θ relative to the Hamiltonian H_θ^{tw} , i.e., $\text{tr}(\rho_\theta H_\theta^{tw})$.

To do so, we examine the reduced density matrices in \bar{u} (the region complementary to the upper quarter), denoted as ρ_θ^1 and σ_θ^1 , respectively. Specifically, $\rho_\theta^1 = \text{tr}_u(\rho_\theta)$. We aim to demonstrate that

$$\partial_\theta \rho_\theta^1 - \partial_\theta \sigma_\theta^1 = e_\theta \quad (22)$$

has a small trace norm. Physically, this implies that in region \bar{u} , state ρ_θ closely approximates state σ_θ . We will also show that $\rho_\theta^2 = \text{tr}_l(\rho_\theta)$ closely approximates $\sigma_\theta^2 = \text{tr}_l(\sigma_\theta)$, i.e., the initial ground state.

To better understand why the error in Eq. (22) is small, consider the limit $q \rightarrow \infty$ and $\beta \rightarrow \infty$ and temporarily substitute $A^{+,-}$ in Eq. (19) with $B^{+,-}$. One can then verify that ρ_θ precisely matches σ_θ , the ground state with two opposite twists at $x = 0$ and $x = \frac{L}{2}$, for all θ .¹ It is found that by choosing q and L^β to be finite but large, ρ_θ can still approximate σ_θ well. When the Hamiltonian H_θ has a gap $\Delta > 0$, the LR bound enables us to demonstrate that the impact of introducing the two twists in σ_θ is primarily localized within l and u .² The motivation behind employing $A^{+,-}$ in Eq. (19) is then straightforward: Rather than σ_θ with two opposite twists, we aim to construct a state ρ_θ with a *single* twist at $x = 0$. Consequently, the LR bound suggests that the part of the system within \bar{u} resembles the state σ_θ , while ρ_θ^2 remains largely unaltered from σ_θ^2 . The reason for considering the reduced density matrices in two overlapping regions \bar{l} and \bar{u} will become apparent when we estimate the energy $\text{tr}(\rho_\theta H_\theta^{tw})$.

We now provide a more precise characterization of the error in Eq. (22). The error can be expressed as $e_\theta = e_a + e_b$, where

$$\begin{aligned} e_a &= \int_0^{L^\beta} d\tau \text{tr}_u \{ [A^+(i\tau) - A^-(-i\tau), \rho_\theta - \sigma_\theta] \}, \\ e_b &= \int_0^{L^\beta} d\tau \text{tr}_u \{ [A^+(i\tau) - A^-(-i\tau), \sigma_\theta] \} - \partial_\theta \sigma_\theta^1. \end{aligned} \quad (23)$$

Let us first analyze e_a . We observe that in $A^+(i\tau)$, and similarly for A^- , the integral over the region $|t| \geq L^\kappa$ contributes with an operator norm upper bounded by

$$\frac{\|\partial_\theta H_\theta^{tw}\|}{2\pi} \int_{|t| > L^\kappa} \frac{e^{-(\Delta t)^2/2q}}{t} \leq \frac{\|\partial_\theta H_\theta^{tw}\|}{2\pi \Delta L^\kappa} \sqrt{\frac{\pi q}{2}} e^{-(\Delta L^\kappa)^2/2q}. \quad (24)$$

Thus, if we choose $q = L^a$ and $2\kappa > a$, this piece will decay superpolynomially in L . After commuting with $\rho_\theta - \sigma_\theta$ in Eq. (23), this piece yields a term with a small trace norm in

¹This can be observed through perturbation theory, i.e., $\partial_\theta |\phi_\theta\rangle = \frac{1}{E_0 - H_\theta} \partial_\theta H_\theta |\phi_\theta\rangle = -\int_0^\infty d\tau \partial_\theta H_\theta(i\tau) |\phi_\theta\rangle$.

²The reason is that although H_θ is long range, the main contribution to $\partial_\theta H_\theta$ comes from terms in close proximity to the twists. Consequently, $B^{+,-}$, as defined in Eq. (21), is essentially supported within u and l due to the LR bound and the Gaussian that truncates the integral for large t .

e_a . Consequently, we can disregard the integral over t in the region $|t| \geq L^\kappa$.

We show that, up to a small error, e_a is only a unitary rotation of $\rho_\theta^1 - \sigma_\theta^1$, and therefore, its trace norm does not increase. To achieve this, we aim to demonstrate that $A^{+,-}$ can be approximated by an operator acting solely within \bar{u} ; i.e.,

$$\delta A^+(i\tau) = A^+(i\tau) - \frac{\text{tr}_u[A^+(i\tau)]}{\text{tr}_u \mathbb{1}_u} \mathbb{1}_u \quad (25)$$

has a small operator norm. Here, $\mathbb{1}_u$ represents the identity operator in u . The physical interpretation is that each term of $\partial_\theta H_\theta^{tw}$ acts solely within l and thus has a minimum distance of $\frac{L}{4}$ from u . When $\alpha > 2d$, $\partial_\theta H_\theta^{tw}(t)$ can be approximated by an operator within an algebraic light cone [22–24,27]. Consequently, within the time range $|t| \leq L^\kappa$ (with κ chosen to be sufficiently small), $\delta A^{+,-}$ exhibits a small operator norm. Once we have bounded the operator norm of $\delta A^{+,-}$, e_a simplifies to a unitary rotation of $\rho_\theta^1 - \sigma_\theta^1$, up to an error with a small trace norm, given by

$$\begin{aligned} e_a &= \int_0^{L^\beta} d\tau \left[\frac{\text{tr}_u[A^+(i\tau) - A^-(-i\tau)]}{\text{tr}_u \mathbb{1}_u}, \text{tr}_u(\rho_\theta - \sigma_\theta) \right] \\ &+ \text{const} \times L^{\frac{a}{2}(d+1) + \min(\beta, \frac{a}{2}) - (\alpha - 2d + 1)}. \end{aligned} \quad (26)$$

We present the detailed calculation of Eqs. (25) and (26) in the Appendix.

We now turn our attention to e_b . We claim that in the first term of e_b (the commutator), substituting $A^{+,-}$ with $B^{+,-}$ introduces only a small error. Let us evaluate the error δ resulting from this substitution,

$$\begin{aligned} \delta &= -\frac{1}{2\pi i} \int_0^{L^\beta} d\tau e^{-(\Delta \tau)^2/2q} \\ &\times \int_t \frac{[\partial_\theta H_\theta^{tw} - \partial_\theta H_\theta](t)}{t + i\tau} e^{-(\Delta t)^2/2q} - \text{H.c.} \end{aligned} \quad (27)$$

A crucial observation is that $\partial_\theta H_\theta^{tw} - \partial_\theta H_\theta$, and therefore δ , can be divided into two categories, $\delta = \delta_1 + \delta_2$: (1) δ_1 arises from terms entirely within u [see Eq. (29) below] and (2) δ_2 arises from all remaining terms in $\partial_\theta H_\theta^{tw} - \partial_\theta H_\theta$, denoted as H_m , constituting a total of $O(L^{2d})$ terms. Notably, each term in H_m acts on two spins separated by a distance of at least $\frac{L}{8}$, thus possessing an operator norm of $O(\frac{1}{L^a})$. The contribution from terms in the second category to δ is upper bounded by Shanti's bound [9]:

$$\|\delta_2\| \leq \frac{\|H_m\|}{2} \sqrt{\frac{2\pi q}{\Delta^2}} \leq \text{const} \times L^{2d - \alpha + \frac{a}{2}}. \quad (28)$$

After commuting with σ_θ in Eq. (23), δ_2 contributes a term with a trace norm bounded by the same order as specified in Eq. (28) for e_b .

We now consider δ_1 . Defining $H_\theta^u = U_R^\dagger(\theta) H_u U_R(\theta)$, one has $[\partial_\theta H_\theta^u(t) := e^{iH_\theta t} \partial_\theta H_\theta^u e^{-iH_\theta t}]$

$$\delta_1 = -\frac{1}{2\pi i} \int_0^{L^\beta} d\tau e^{-(\Delta \tau)^2/2q} \int_t \frac{\partial_\theta H_\theta^u(t)}{t + i\tau} e^{-(\Delta t)^2/2q} - \text{H.c.} \quad (29)$$

The Appendix shows that $\text{tr}_u[\delta_1, \sigma_\theta]$ has a trace norm bounded by

$$\|\text{tr}_u[\delta_1, \sigma_\theta]\|_1 \leq \max[O(L^{\frac{q}{2}(d+1)+\min(\beta, \frac{q}{2})-(\alpha-2d+1)}), \times O(L^{2d-\alpha+\frac{q}{2}})]. \quad (30)$$

Physically, δ_1 introduces a twist at $x = \frac{L}{2}$, whose effect is primarily localized in u due to the LR bound. Combining δ_1 with δ_2 , we conclude that up to small errors [whose trace norms are described by Eqs. (28) and (30)], we can substitute $A^{+, -}$ by $B^{+, -}$ in e_b .

In the Appendix, we demonstrate that

$$\int_0^{L^\beta} d\tau \text{tr}_u\{[B^+(i\tau) - B^-(-i\tau), \sigma_\theta]\} - \partial_\theta \sigma_\theta^1 \quad (31)$$

has a trace norm that decays superpolynomially in L , thereby establishing that $\|e_b\|_1$ is bounded from above by Eqs. (28) and (30). Physically, in the presence of a nonzero gap, the quadiabatic evolution continues to give a good approximation of σ_θ when q and L^β are finite but large, as mentioned earlier, with corrections that are superpolynomially small in L .

Using the same reasoning, we can show that up to a small error, the reduced density matrix in \bar{l} , ρ_θ^2 , can be approximated by σ_θ^2 . Namely, we have $\partial_\theta \rho_\theta^2 = e'_\theta$, with

$$\|e'_\theta\|_1 \leq \text{const} \times \max[L^{\frac{q}{2}(d+1)+\min(\beta, \frac{q}{2})-(\alpha-2d+1)}, L^{2d-\alpha+\frac{q}{2}}]. \quad (32)$$

In fact, since $\partial_\theta H_\theta^{tw}$ is entirely within l , e'_θ receives only contributions similar to those described in Eq. (30).

We can now determine the energy of the twisted state ρ_θ with respect to H_θ^{tw} . The first term in Eq. (18) resides entirely within \bar{u} , leading to $\text{tr}[H_I^\theta \rho_\theta] \simeq \text{tr}_{\bar{u}}[U_R^\dagger(\theta) H_I U_R(\theta) \sigma_\theta^1] = \text{tr}(H_I \sigma_0)$ up to an error of order $O(\|H_I\| \cdot \|e_\theta\|_1)$. The second part of H_θ^{tw} , $H - H_I$, can be classified into three categories: (1) For H_I , i.e., terms entirely supported in \bar{l} , we obtain $\text{tr}[H_I \rho_\theta] \simeq \text{tr}(H_I \sigma_0)$ up to an error bounded by $O(\|H_I\| \cdot \|e'_\theta\|_1)$. (2) H_u^L and H_u^R , supported in \bar{u} and acting only in the left or right half, yield $\text{tr}[(H_u^L + H_u^R) \rho_\theta] \simeq \text{tr}[(H_u^L + H_u^R) \sigma_0]$ up to an error bounded by $O(\|H_u^L\| + \|H_u^R\| \|e_\theta\|_1)$. (3) Finally, $K = H - H_I - H_{\bar{l}} - H_u^L - H_u^R$ contains $O(L^{2d})$ terms, each acting on two spins with a distance of at least $\frac{L}{8}$. Consequently, $\|K\| \leq \text{const} \times L^{2d-\alpha}$. (We analyze reduced density matrices in two overlapping regions, \bar{l} and \bar{u} , precisely to suppress the contribution from terms acting simultaneously on u and l .) Combining all results, for $\alpha > 2d$ we have

$$\text{tr}(H_\theta^{tw} \rho_\theta) \leq \text{const} \times L^d (\|e_\theta\|_1 + \|e'_\theta\|_1) + O(L^{2d-\alpha}), \quad (33)$$

where we utilized the fact that for $\alpha > d$, $\|H\| \simeq O(L^d)$. It is evident that for any $\alpha > 3d$, one can always choose a and β sufficiently small that the energy of ρ_θ (particularly, $\rho_{2\pi}$) vanishes algebraically with L . This completes the construction of a low-lying twisted state.

V. LATTICE MOMENTUM

Finally, we demonstrate that the twisted state $\rho_{2\pi}$ has a momentum distinct from the ground state σ_0 , thereby being

orthogonal to it. We define the translation in the presence of two twists [at links along the x direction with coordinates $(L-1, 0)$ and $(\frac{L}{2}-1, \frac{L}{2})$] as

$$T_{\theta, \theta'} = \exp[i\theta S_z^{(0)} + i\theta' S_z^{(\frac{L}{2})}] T, \quad (34)$$

where T represents the ordinary lattice translation along the x direction and $S_z^{(0)} = \sum_y S_{(0,y)}^z$ denotes the summation of all spin- z operators on sites with x coordinate 0, with similar notation for $S_z^{(\frac{L}{2})}$. It can be verified that (1) $T_{\theta, -\theta}$ is a symmetry of H_θ and (2) the state ϕ_θ is an eigenstate of $T_{\theta, -\theta}$ with $T_{\theta, -\theta} \phi_\theta = \phi_\theta$. Here, without loss of generality, we assume that $T \phi_0 = \phi_0$.

We now demonstrate that $\partial_\theta (T_{\theta, 0} \psi_\theta - \psi_\theta)$ has a small norm for $0 \leq \theta \leq 2\pi$, where ψ_θ is the twisted state, i.e., the state vector for ρ_θ . This would imply that $\psi_{2\pi}$ is almost an eigenvector of $T_{2\pi, 0}$ with an eigenvalue of 1. As the width of the system is odd, we have $T = -T_{2\pi, 0}$. Hence, $\psi_{2\pi}$ exhibits the opposite momentum compared to $\psi_0 = \phi_0$. The derivative can be computed as

$$\begin{aligned} \partial_\theta (T_{\theta, 0} \psi_\theta - \psi_\theta) &= O_1 \psi_\theta + \tilde{O}_1 (T_{\theta, 0} \psi_\theta - \psi_\theta), \\ O_1 &= iS_z^{(0)} + \int_0^{L^\beta} d\tau [T_{\theta, 0} A^+(i\tau) T_{\theta, 0}^\dagger \\ &\quad - A^+(i\tau) - \text{H.c.}], \end{aligned} \quad (35)$$

where \tilde{O}_1 is anti-Hermitian, inducing only a unitary rotation without altering the norm of $T_{\theta, 0} \psi_\theta - \psi_\theta$. We then bound the square norm of the first term in Eq. (35), specifically $\text{tr}(O_1^\dagger O_1 \rho_\theta)$.

We provide an overview of the argument here, leading to the conclusion that for $\alpha > 4d - 2$, $\psi_{2\pi}$ has lattice momentum π , while reserving a detailed discussion for the Appendix. Using a reasoning similar to that leading to Eq. (26), the operator O_1 can be approximated by $O_{l'}$, an operator supported solely in l' (see Fig. 1), with an error bounded by the second line of Eq. (26). Here, $O_{l'}$ is defined by truncating the effect of O_1 in \bar{l} , akin to Eq. (25). Consequently, we can approximate $\text{tr}(O_1^\dagger O_1 \rho_\theta)$ by $\text{tr}(O_{l'}^\dagger O_{l'} \rho_\theta)$, which is further approximated by $\text{tr}(O_{l'}^\dagger O_{l'} \sigma_\theta)$ since $l' \subset \bar{u}$. Finally, we claim that $\text{tr}(O_{l'}^\dagger O_{l'} \sigma_\theta)$ vanishes in the $L \rightarrow \infty$ limit. To demonstrate this, we observe that $O \phi_\theta := \partial_\theta (T_{\theta, -\theta} \phi_\theta - \phi_\theta) = 0$. Additionally, O can be approximated by $O \simeq O_{l'} + O_{u'}$, where $O_{u'}$ is an operator akin to $O_{l'}$ but supported entirely within u' . Hence, we have $0 = \text{tr}(O^\dagger O \sigma_\theta) = 2\text{tr}(O_{l'}^\dagger O_{l'} \sigma_\theta) + \text{tr}(O_{l'}^\dagger O_{u'} \sigma_\theta) + \text{tr}(O_{u'}^\dagger O_{l'} \sigma_\theta)$. It has been established that if σ_θ is a unique gapped ground state of a power-law Hamiltonian, the correlation function $\text{tr}(O_{l'}^\dagger O_{u'} \sigma_\theta) + \text{tr}(O_{u'}^\dagger O_{l'} \sigma_\theta)$ between two operators $O_{l'}$ and $O_{u'}$ (separated by at least a distance $\frac{L}{8}$) decays algebraically in L [23]. This ultimately bounds the norm of $O_1 \psi_\theta$, establishing that $\psi_{2\pi}$ possesses momentum π . This completes the proof of Theorem 2. ■

We can establish the second part of Theorem 2 following the same line of reasoning. For type II Hamiltonians [or their generic k -local generalizations, i.e., Hamiltonians that involve k -body couplings of $SO(3)$ singlet local operators], the same LR bound still applies when the two conditions (15) and (16) are satisfied. The only distinction lies in the introduction of a twist: in the construction of H_θ^l , H_θ^u and H_θ , only the terms

acting nontrivially on the twist are modified. For the simplest example, where $H = \sum_{ij} h_{ij}$, with $h_{ij} = \frac{O_i O_j}{(|i-j|+1)^\alpha}$, the twist affects only terms in which either O_i or O_j acts precisely on the twist. In contrast, for a type I Hamiltonian, in H_θ^I in Eq. (18), all terms between the left and right halves need to be twisted. This distinction results in smaller errors for type II cases, thereby extending the applicability of the theorem. We present a detailed discussion in the Appendix.

VI. SUMMARY AND OUTLOOK

In this study, we established an LSM theorem applicable to systems with long-range interactions. The proof involves the construction of a low-energy twisted state through quasiadiabatic evolution and the LR bound for power-law Hamiltonians, which was shown to be orthogonal to the ground state when each unit cell has spin $\frac{1}{2}$. Moreover, in 1d, we enhanced the theorem's applicability by explicitly constructing a low-lying state featuring a uniform 2π twist.

The result we obtained is weaker than that in Ref. [3]. In that work, the excitation gap is bounded by $\Delta \leq \text{const} \frac{\ln L}{L}$, whereas here, we conclude only that a nonzero gap $O(L^0)$ is forbidden. An interesting future direction would be to explore a similar finer bound on Δ for power-law interacting Hamiltonians, i.e., understanding how quickly Δ vanishes at large L . Moreover, whether the proof can be extended to other (e.g., discrete) on-site symmetries, with a projective representation per site, remains an open question. Another interesting question is whether the bounds given in this work are sharp, i.e., whether we can find counterexamples for which the theorems do not hold when the bounds on the decaying power are violated.

Note added. Recently, I became aware of another work that explores the LSM theorem with long-range interactions [34].

ACKNOWLEDGMENTS

I am grateful to M. Cheng, T. Grover, C. Wang, W. Ye, and, especially, M. Levin for discussions. I also thank Y. Chen for help with *Mathematica*. I am supported in part by Simons Investigator Award No. 990660 and by the Simons Collaboration on Ultra-Quantum Matter, which is a grant from the Simons Foundation (Grant No. 651440).

APPENDIX

In this Appendix, we provide technical details for several equations in the main text.

1. Equations (25) and (26)

In the definition of e_a in Eq. (23), $A^{+,-}$ involves an integral over t . We showed in Eq. (24) that the contribution from $|t| \geq L^\kappa$ is negligible. Now we argue that $A^{+,-}$ can be approximated by an operator acting only in \bar{u} , namely, Eq. (25) has a small operator norm.

We have the following expression:

$$\frac{\text{tr}_u[A^+(i\tau)]}{\text{tr}_u \mathbb{1}_u} \mathbb{1}_u = \int d\mu(U) U A^+(i\tau) U^\dagger, \quad (\text{A1})$$

where U is the unitary group acting on u and $\mu(U)$ is the Haar measure for U [35]. As a result,

$$\|\delta A^+(i\tau)\| \leq \int d\mu(U) \| [U, A^+(i\tau)] \|. \quad (\text{A2})$$

To bound the right-hand side, we compute the norm of the commutator $\| [\partial_\theta H_\theta^{tw}(t), U] \|$, where U is an operator supported entirely in the upper quarter with unit norm. A crucial observation is that $\partial_\theta H_\theta^{tw}$ contains only terms in the lower quarter l , supported simultaneously in the left and right halves. The minimum distance from each of these terms to the upper quarter is $\frac{L}{4}$. We then apply the LR bound to estimate the commutator as

$$\begin{aligned} \| [\partial_\theta H_\theta^{tw}(t), U] \| &\leq \sum_{ij} \| [\partial_\theta h_{ij}^\theta(t), U] \| \\ &\leq \text{const} \sum_{ij} \frac{C(t, r_{ij}^u)}{(|i-j|+1)^\alpha}, \end{aligned} \quad (\text{A3})$$

where (1) $h_{ij}^\theta = U_R^\dagger(\theta) h_{ij} U_R(\theta)$; (2) the pairs ij are such that $i, j \in l$ and one of them is in the left half; (3) $C(t, r)$ is the LR bound,

$$C(t, r) = \sup_{\|A\|=\|B\|=1} \| [A(t), B] \|, \quad (\text{A4})$$

with r representing the minimal distance of A and B ; and (4) r_{ij}^u denotes the minimal distance from the pair ij to the upper quarter u .

Let us now estimate the magnitude of Eq. (A3). First, as previously discussed, r_{ij}^u is at least $\frac{L}{4}$. It has been demonstrated that in the presence of power-law interactions, when $\alpha > 2d + 1$, the system still exhibits a linear light cone [23],

$$C(t, r) \leq \text{const} \times |A| \frac{t^{d+1} \ln^{2d} r}{(r - vt)^{\alpha-d}}, \quad (\text{A5})$$

where $|A|$ represents the number of sites acted upon by A and v is a finite velocity. The discussion also applies to $\alpha \in (2d, 2d + 1]$, where, in this range, a nonlinear LR light cone is present [22], given by

$$C(t, r) \leq \text{const} \frac{t^{(\alpha-d)/(\alpha-2d)}}{r^{\alpha-d}} \quad (\text{A6})$$

for $t \leq \text{const} \times r^{\alpha-2d}$. $C(t, r)$ is an algebraic function of t and r , decaying as r increases. Thus, we can replace r_{ij}^u with $\frac{L}{4}$ in Eq. (A3) without altering its order. Second, a direct computation indicates that for $\alpha > d + 1$, the summation in Eq. (A3) is dominated by pairs ij where i and j are in close proximity and therefore in the vicinity of the twist. In other words, the summation is on the order of

$$\sum_{\substack{i \in l \cap L \\ j \in l \cap R}} \frac{1}{(|i-j|+1)^\alpha} \simeq O(L^{d-1}), \quad (\text{A7})$$

which corresponds to the area of the twist.

From the discussion in the previous paragraph, we can bound the commutator as follows:

$$\begin{aligned} & \left\| \left[\int_0^{L^\beta} d\tau \delta A^+(i\tau), U \right] \right\| \\ & \leq \text{const} \int_0^{L^\beta} d\tau e^{-(\Delta\tau)^2/2q} \int_{|t| < L^\kappa} dt \frac{t^{d+1} e^{-(\Delta t)^2/2q}}{L^{\alpha-2d+1} |t + i\tau|}. \end{aligned} \quad (\text{A8})$$

Here, we chose $\kappa < 1$, which will not lead to any contradictions below, and ignored the logarithmic correction. Two observations are useful for bounding the integral in Eq. (A8): (1) For $L^\kappa \gg q^{\frac{1}{2}} = L^{\frac{q}{2}}$ (i.e., $\kappa > \frac{q}{2}$), the integration over t is dominated by $t \simeq O(q^{\frac{1}{2}}/\Delta)$; therefore, the right-hand side of Eq. (A8) has the same order in L as $\int_0^{L^\beta} d\tau e^{-(\Delta\tau)^2/2q} \frac{q^{\frac{d+2}{2}}}{L^{\alpha-2d+1} |(q^{\frac{1}{2}}/\Delta) + i\tau|}$. (2) Doing the integration over τ leads to the final result,

$$\left\| \left[\int_0^{L^\beta} d\tau \delta A^+(i\tau), U \right] \right\| \leq \text{const} \frac{L^{\frac{a(d+1)}{2}} L^{\min(\beta, \frac{q}{2})}}{L^{\alpha-2d+1}}, \quad (\text{A9})$$

where the constant factor does not scale with L .³ The same analysis can be applied to A^- . We deduce that, up to an error whose trace norm is bounded by Eq. (A9), upon commuting with $\rho_\theta - \sigma_\theta$, e_a is simply a unitary rotation of $\rho_\theta^1 - \sigma_\theta^1$ and thus does not increase its trace norm. Namely,

$$\begin{aligned} e_a &= \int_0^{L^\beta} d\tau [\text{tr}_u \{A^+(i\tau) - A^-(-i\tau)\}, \text{tr}_u(\rho_\theta - \sigma_\theta)] \\ &+ \text{const} \times L^{\frac{a}{2}(d+1) + \min(\beta, \frac{q}{2}) - (\alpha-2d+1)}. \end{aligned} \quad (\text{A10})$$

This is precisely Eq. (26).

Finally, we note that the analysis above actually applies to generic k -local Hamiltonians satisfying Eqs. (15) and (16). To demonstrate this, we require an LR bound for generic Hamiltonians satisfying the two properties mentioned above [27],

$$C(t, r) \leq 2|A|e^{vt-r^{1-\sigma}} + \text{const} \frac{f(vt)}{r^{\sigma\alpha}}, \quad (\text{A11})$$

where σ can take any value within $(\frac{d+1}{\alpha+1}, 1)$ and $f(vt)$ is upper bounded by

$$f(x) \leq \text{const}(x + x^{1+\frac{d}{1-\sigma}})|A|^{n^*+2}, \quad (\text{A12})$$

with n^* being the smallest integer greater than or equal to $\frac{\sigma d}{\sigma\alpha-d}$. For k -local Hamiltonians, $|A|$ is bounded by a constant and is therefore not relevant for the discussion below. We conjecture that all the derivations below, and consequently Theorem 2, also apply to Hamiltonians in which the number of spins acted on by each term is unbounded, provided, for instance, that the norm decays superpolynomially with the number of spins it acts on. A thorough investigation of more general Hamiltonians is left for future work.

³Explicitly integrating the expression in Eq. (A8) results in a special function which shares the same order in L as the expression in Eq. (A9), accompanied by a Gamma function coefficient.

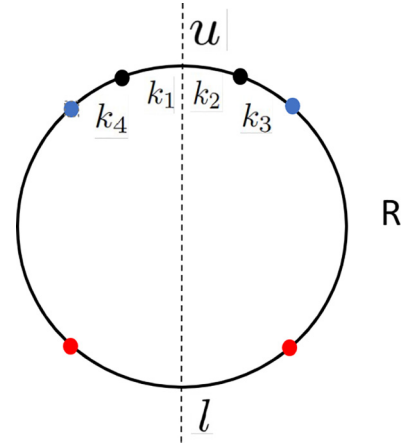


FIG. 2. The region k_1 includes sites with $x \in [\frac{L}{2}, \frac{9L}{16} - 1]$. Similarly, k_2 consists of sites with $x \in [\frac{7L}{16}, \frac{L}{2} - 1]$; k_3 comprises sites with $x \in [\frac{3L}{8}, \frac{7L}{16} - 1]$, and k_4 covers sites with $x \in [\frac{9L}{16}, \frac{5L}{8} - 1]$.

Furthermore, we can demonstrate that Eq. (A7) still holds for generic Hamiltonians satisfying Eq. (15), i.e., $\|\partial_\theta H_\theta^{tw}\|$ is bounded by $O(L^{d-1})$. Repeating the analysis leading to Eqs. (A9) and (A10), we find that for generic k -local Hamiltonians satisfying Eqs. (15) and (16), we have

$$\begin{aligned} & \left\| \left[\int_0^{L^\beta} d\tau \delta A^+(i\tau), U \right] \right\| \\ & \times \leq \text{const} \frac{L^{\frac{a}{2}(\frac{d}{1-\sigma}+1)} L^{\min(\beta, \frac{q}{2}) + (1-\sigma)(\alpha-d)}}{L^{\alpha-2d+1}} \end{aligned} \quad (\text{A13})$$

and

$$\begin{aligned} e_a &= \int_0^{L^\beta} d\tau [\text{tr}_u \{A^+(i\tau) - A^-(-i\tau)\}, \text{tr}_u(\rho_\theta - \sigma_\theta)] \\ &+ \text{const} \times L^{\frac{a}{2}(\frac{d}{1-\sigma}+1) + \min(\beta, \frac{q}{2}) + (1-\sigma)(\alpha-d) - (\alpha-2d+1)}. \end{aligned} \quad (\text{A14})$$

As we will see below, although considering generic k -local Hamiltonians complicates the calculation, it does not alter the final results: The essential point is that for any $\alpha > 2d - 1$, one can always find σ arbitrarily close to 1 and a and β sufficiently small that the error in Eq. (A13) vanishes in the $L \rightarrow \infty$ limit.

2. Equation (30)

Let us now consider the error δ_1 defined in Eq. (29). In particular, we demonstrate Eq. (30), i.e., $\text{tr}_u[\delta_1, \sigma_\theta]$, has a small trace norm. To show this, we consider the trace of this operator with an operator O with unit norm. O must be supported within \bar{u} . We have

$$\text{tr}_{\bar{u}}(O \text{tr}_u[\delta_1, \sigma_\theta]) = \text{tr}([O, \delta_1] \sigma_\theta). \quad (\text{A15})$$

We now bound the operator norm of $[O, \delta_1]$. Note that $\partial_\theta H_\theta^u$ can be further divided into two parts, $\partial_\theta H_\theta^u = K_1 + K_2$: (1) The terms in K_1 act on two spins in the region, k_1 and k_2 (as depicted in Fig. 2). Note that the minimal distance from each of these terms to O is $\frac{L}{16}$. Following exactly the same argument

leading to Eq. (A9), the LR bound constrains the commutator

$$\left\| \left[-\frac{1}{2\pi i} \int_0^{L^\beta} d\tau e^{-(\Delta\tau)^2/2q} \int_t \frac{K_1(t)}{t+i\tau} e^{-(\Delta t)^2/2q} - \text{H.c.}, O \right] \right\| \leq \text{const} \times L^{\frac{a}{2}(d+1)+\min(\beta, \frac{a}{2})-(\alpha-2d+1)}. \quad (\text{A16})$$

For a generic k -local Hamiltonian with properties (15) and (16), this commutator is similarly bounded by Eq. (A13).

(2) K_2 includes all other terms in K , namely, terms that act on either the region (k_1, k_3) , (k_2, k_4) , or (k_3, k_4) . The norm of each term in K_2 is upper bounded by $(\frac{16}{L})^\alpha$, and K_2 contains $3 \times (\frac{L^d}{16})^2$ terms in total. As a result, we have (Shanti's bound)

$$\left\| \left[-\frac{1}{2\pi i} \int_0^{L^\beta} d\tau e^{-(\Delta\tau)^2/2q} \int_t \frac{K_2(t)}{t+i\tau} e^{-(\Delta t)^2/2q} - \text{H.c.}, O \right] \right\| \leq \text{const} \times L^{2d-\alpha+\frac{a}{2}}. \quad (\text{A17})$$

For general k -local Hamiltonians, the property (15) ensures that the same order holds. Combining Eqs. (A16) and (A17), we arrive at Eq. (30). This is because we have bounded the trace in Eq. (A15) with any arbitrary O with unit norm, thereby bounding the trace norm of $\text{tr}_u[\delta_1, \sigma_\theta]$.

3. Equation (31)

We now demonstrate that the expression in Eq. (31) has a trace norm that decays superpolynomially in L . To see this, let us examine the matrix element of the operator $B^+(i\tau)$,

$$[B^+(i\tau)]_{ab} = \sqrt{\frac{q}{2\pi\Delta^2}} (\partial_\theta H_\theta)_{ab} \int_\omega \Theta[\omega - (E_a - E_b)] \times e^{(E_a - E_b)\tau - (\sqrt{\frac{q}{2\Delta^2}}\omega + \sqrt{\frac{\Delta^2}{2q}}\tau)^2}, \quad (\text{A18})$$

where a and b label eigenstates of H_θ and we used the convolution theorem. When acting on the gapped ground state σ_θ , we need to consider only elements with $|E_a - E_b| \geq \Delta$. For sufficiently large q ($q = L^a \gg L^\beta$, i.e., $a > \beta$), (1) when $E_a < E_b$, the matrix element $[B^+(i\tau)]_{ab}$ becomes $e^{(E_a - E_b)\tau} [\partial_\theta H_\theta]_{ab}$, and (2) for $E_a > E_b$, $B^+(i\tau)$ goes to zero, both with corrections of order $O(e^{-\frac{q}{2\Delta^2}(E_a - E_b)^2})$, decaying superpolynomially in L . The same argument applies to B^- . Thus, in the presence of a nonzero gap Δ , up to superpolynomial corrections, the first term in Eq. (31) can be approximated as

$$-\int_0^{L^\beta} d\tau \text{tr}_u[\partial_\theta H_\theta(i\tau)\sigma_\theta + \sigma_\theta \partial_\theta H_\theta(-i\tau)]. \quad (\text{A19})$$

Subsequently, Eq. (31) transforms into an integral over the region $\tau > L^\beta$, whose trace norm once again vanishes superpolynomially in the presence of a nonzero Δ . This is evident from a direct calculation; i.e., Eq. (31) becomes

$$-\sum_{a \neq 0} \text{tr}_u \left[\sigma_\theta \frac{\partial_\theta H_\theta |a\rangle \langle a|}{E_0 - E_a} e^{L^\beta(E_0 - E_a)} + \text{H.c.} \right], \quad (\text{A20})$$

whose trace norm decays superpolynomially in L when $\beta > 0$.

4. Lattice momentum

We now bound the norm square of the first term in Eq. (35), namely, $\text{tr}(O_1^\dagger O_1 \rho_\theta)$. If this quantity vanishes in the $L \rightarrow \infty$ limit, we will have $\text{tr}(T_{0,0}\rho_0) = \text{tr}(T_{2\pi,0}\rho_{2\pi}) = -\text{tr}(T_{0,0}\rho_{2\pi})$, indicating that the twisted state $\psi_{2\pi}$ is orthogonal to the original ground state ϕ_0 .

As discussed in Eq. (25), the operator $\int_0^{L^\beta} A^+(i\tau)$ [and thus the translated version $T_{\theta,0} \int_0^{L^\beta} A^+(i\tau) T_{\theta,0}^\dagger$] can be approximated by an operator supported solely in \bar{u} , with an error bounded by Eq. (A9). In fact, this argument allows us to approximate it even further as an operator acting in a smaller region l' , with an error of the same order. Therefore, we can express O_1 as $O_{l'} + \delta O_1$, where $O_{l'}$ is entirely within l' . Consequently, we approximate $\text{tr}(O_1^\dagger O_1 \rho_\theta)$ by $\text{tr}(O_{l'}^\dagger O_{l'} \rho_\theta)$, with an error bound given by

$$|\text{tr}[O_1 \delta O_1 \rho_\theta]| \leq \|O_1\| \cdot \|\delta O_1\| \leq \text{const} \times L^{d-1+\frac{a}{2}} L^{\frac{a}{2}(d+1)+\min(\beta, \frac{a}{2})-(\alpha-2d+1)}. \quad (\text{A21})$$

Here, we utilize the facts that (1) for $\alpha > d+1$, $\|\int_0^{L^\beta} d\tau A(i\tau) - \text{H.c.}\| \leq \text{const} \times L^{d-1+\frac{a}{2}}$, as $\|\partial_\theta H_\theta^{tw}\| \simeq O(L^{d-1})$ is dominated by terms near the twist [see Eq. (A7)], and (2) $\|\delta O_1\|$ is bounded by Eq. (A9). For $\alpha > 3d-2$, we can choose a and β sufficiently small that this error vanishes in the $L \rightarrow \infty$ limit. Similarly, for general k -local Hamiltonians, due to Eq. (A13), we have

$$|\text{tr}[O_1 \delta O_1 \rho_\theta]| \times L^{d-1+\frac{a}{2}} L^{\frac{a}{2}(\frac{d}{1-\sigma}+1)+\min(\beta, \frac{a}{2})+(1-\sigma)(\alpha-d)-(\alpha-2d+1)}. \quad (\text{A22})$$

Again for $\alpha > 3d-2$, we can choose σ sufficiently close to 1 and a and β sufficiently small that this error vanishes in the $L \rightarrow \infty$ limit.

Additionally, we have $\text{tr}(O_{l'}^\dagger O_{l'} \rho_\theta) = \text{tr}_{\bar{u}}(O_{l'}^\dagger O_{l'} \rho_\theta^1) = \text{tr}_{\bar{u}}(O_{l'}^\dagger O_{l'} \sigma_\theta^1)$, with an error bounded by (Hölder's inequality)

$$2\pi \|O_{l'}\|^2 \|e_\theta\|_1 \leq \text{const} \|O_1\|^2 \|e_\theta\|_1 \leq \text{const} \times L^{4d-2-\alpha+\frac{3a}{2}}. \quad (\text{A23})$$

For $\alpha > 4d-2$, we can always choose a and β such that this error vanishes as L grows large.

We then claim that $\text{tr}_{\bar{u}}(O_{l'}^\dagger O_{l'} \sigma_\theta^1) = \text{tr}(O_{l'}^\dagger O_{l'} \sigma_\theta)$ vanishes in the thermodynamic limit. To see this, note that $\partial_\theta(T_{\theta,-\theta}\phi_\theta - \phi_\theta) = O\phi_\theta = 0$, where

$$O = i(S_z^{(0)} - S_z^{(\frac{L}{2})}) - \int_0^\infty d\tau [T_{\theta,-\theta} \partial_\theta H_\theta(i\tau) T_{\theta,-\theta}^\dagger - \partial_\theta H_\theta(i\tau)]. \quad (\text{A24})$$

When H_θ has a unique gapped ground state, within an error with an operator norm that decays superpolynomially in L , we can approximate the second line in Eq. (A24) as follows:

$$\int_0^{L^\beta} d\tau \{T_{\theta,-\theta} [B^+(i\tau) - B^-(-i\tau)] T_{\theta,-\theta}^\dagger - [B^+(i\tau) - B^-(-i\tau)]\}. \quad (\text{A25})$$

This approximation is justified by the same reasoning elucidated below Eq. (A18). Furthermore, up to a small error, $B^{+,-}$

in Eq. (A25) can be approximated by $\tilde{B}^{+,-}$, which is defined as

$$\tilde{B}^+(i\tau) = -\frac{1}{2\pi i} e^{-(\Delta\tau)^2/2q} \int_t \frac{\partial_\theta \tilde{H}_\theta(t)}{t + i\tau} e^{-(\Delta\tau)^2/2q}, \quad (\text{A26})$$

where $\tilde{H}_\theta := H_\theta^l + H_\theta^u$. This approximation is valid because each term in $\partial_\theta H_\theta - \partial_\theta \tilde{H}_\theta$ operates on two spins separated by a distance of at least $\frac{L}{8}$ and therefore has a small operator norm [the error resulting from this substitution in Eq. (A25) is of the same order as that in Eq. (28)]. Consequently, we can express O as the sum of O'_1 , O'_2 , and δO , with $\|\delta O\|$ bounded by Eq. (28). Here, O'_1 is approximately the operator O_1 defined in Eq. (35), but with $T_{\theta,0}$ replaced by $T_{\theta,-\theta}$:

$$O'_1 = iS_z^{(0)} + \int_0^{L^\beta} d\tau [T_{\theta,-\theta} A^+(i\tau) T_{\theta,-\theta}^\dagger - A^+(i\tau) - \text{H.c.}]. \quad (\text{A27})$$

Additionally,

$$O'_2 = -iS_z^{(\frac{L}{2})} + [T_{\theta,-\theta} \delta_1 T_{\theta,-\theta}^\dagger - \delta_1], \quad (\text{A28})$$

where δ_1 is defined in Eq. (29).

Since $O\phi_\theta = (O'_1 + O'_2 + \delta O)\phi_\theta = 0$, we find

$$\begin{aligned} |\text{tr}[(O'_1 + O'_2)^\dagger (O'_1 + O'_2) \sigma_\theta]| &= 2|\text{tr}[(O'_1 + O'_2)^\dagger \delta O \sigma_\theta]| \\ &\leq \text{const} \times L^{3d-1-\alpha+a}. \end{aligned} \quad (\text{A29})$$

For $\alpha > 3d - 1$, we can always choose a and β such that this expression vanishes for large L . Hence,

$$F := 2\text{tr}(O_1^\dagger O'_1 \sigma_\theta) + \text{tr}(O_1^\dagger O'_2 \sigma_\theta) + \text{tr}(O_2^\dagger O'_1 \sigma_\theta) = 0, \quad (\text{A30})$$

where we utilize $\text{tr}(O_1^\dagger O'_1 \sigma_\theta) = \text{tr}(O_2^\dagger O'_2 \sigma_\theta)$, up to a vanishing error again for $\alpha > 3d - 1$. Due to the LR bound, $\int_0^{L^\beta} d\tau A(i\tau) - \text{H.c.}$ and δ_1 can be approximated by an operator acting within l' and u' , respectively. Employing this approximation gives an error of the order specified in Eq. (A21) in the quantity F [or Eq. (A22) for generic k -local Hamiltonians], which vanishes for large L when $\alpha > 3d - 2$. Consequently, we can substitute $T_{\theta,-\theta}$ by $T_{\theta,0}$ in O'_1 since now the operator conjugated by $T_{\theta,-\theta}$ acts as an identity near coordinate $x = \frac{L}{2}$ (we can similarly substitute $T_{\theta,-\theta}$ by $T_{0,-\theta}$ in O'_2). Thus, we deduce that

$$F = 2\text{tr}(O_{l'}^\dagger O_{l'} \sigma_\theta) + \text{tr}(O_{l'}^\dagger O_{u'} \sigma_\theta) + \text{tr}(O_{u'}^\dagger O_{l'} \sigma_\theta) = 0, \quad (\text{A31})$$

where $O_{u'}$ is the restriction of O'_2 in u' .

Finally, let us consider the *connected* correlation function

$$C_{l'u'} = \text{tr}(O_{l'}^\dagger O_{u'} \sigma_\theta) - \text{tr}(O_{l'}^\dagger \sigma_\theta) \text{tr}(O_{u'} \sigma_\theta). \quad (\text{A32})$$

Since O'_1 and O'_2 are supported within l' and u' , respectively, their distance r is at least $\frac{L}{8}$. It has been demonstrated that if σ_θ is the unique gapped ground state of a power-law Hamiltonian, the connected correlation between two operators (with unit norm) separated by r is upper bounded by a function $G(r)$ decaying in r at the same rate as the LR bound in Eq. (A5) [23], i.e., $G(r) \simeq \frac{1}{r^{\alpha-d}}$. In the expression for $G(r)$, the volumes of the regions u' and l' do not appear

due to $S_z^{(0)}$, $S_z^{(\frac{L}{2})}$, $\partial_\theta H_\theta^{lw}$, and $\partial_\theta H_\theta^{uu}$ being sums of operators acting on a finite number of sites. Specifically, up to a vanishing error quantified by Eq. (A21), $C_{l'u'} \simeq \langle O_1^\dagger O_2 \rangle_c = \text{tr}(O_1^\dagger O_2 \sigma_\theta) - \text{tr}(O_1^\dagger \sigma_\theta) \text{tr}(O_2 \sigma_\theta)$. Let us consider the contribution $\langle \int_0^{L^\beta} d\tau [A^+(i\tau) - \text{H.c.}] \delta_1 \rangle_c$ to $\langle O_1^\dagger O_2 \rangle_c$, while other terms can be analyzed similarly. We have

$$\begin{aligned} &\left\langle \int_0^{L^\beta} d\tau [A^+(i\tau) - \text{H.c.}] \delta_1 \right\rangle_c \\ &= -\frac{1}{\pi^2} \int_0^{L^\beta} d\tau_1 d\tau_2 e^{-\frac{\Delta^2(\tau_1^2 + \tau_2^2)}{2q}} \int_{t_1, t_2} \frac{t_1 t_2 e^{-\frac{\Delta^2(t_1^2 + t_2^2)}{2q}}}{(t_1^2 + \tau_1^2)(t_2^2 + \tau_2^2)} \\ &\quad \times \langle \partial_\theta H_\theta^{lw}(t_1 - t_2) \partial_\theta H_\theta^{uu} \rangle_c. \end{aligned} \quad (\text{A33})$$

As $\partial_\theta H_\theta^{uu}$ is a sum of terms, each acting on *two* spins within u , and $H_\theta^{lw}(t_1 - t_2)$ can be approximated by an operator within l' , we thus have $\langle \partial_\theta H_\theta^{lw}(t_1 - t_2) \partial_\theta H_\theta^{uu} \rangle_c \leq \frac{\text{const}}{L^{\alpha-d}} \|\partial_\theta H_\theta^{lw}\| \|\partial_\theta H_\theta^{uu}\|$. This immediately leads to Eq. (A34). We therefore conclude that

$$C_{l'u'} \leq \text{const} \times L^{2(d-1+\frac{\alpha}{2})-\alpha+d}. \quad (\text{A34})$$

Here, in the calculation we use the fact that, for $\alpha > d + 1$, the dominant contribution to $\int_0^{L^\beta} d\tau [A^{+,-}(i\tau) - \text{H.c.}]$ and δ_1 again comes from spin-spin couplings in the vicinity of the two twists, at links $(L - 1, 0)$ and $(\frac{L}{2} - 1, \frac{L}{2})$, respectively, i.e., Eq. (A7). For $\alpha > 3d - 2$, we can choose a and β such that this expression vanishes for large L .

Last, we claim that

$$\text{tr}(O_{l'}^\dagger \sigma_\theta) = \text{tr}(O_{u'} \sigma_\theta) = 0. \quad (\text{A35})$$

This can be easily seen by noting that $\text{tr}(O_{l'}^\dagger \sigma_\theta)$ and $\text{tr}(O_{u'} \sigma_\theta)$ vanish. This is because S_z , $\partial_\theta H_\theta^{uu}|_{\theta=0}$, and $\partial_\theta H_\theta^{lw}|_{\theta=0}$ are odd under a spin rotation of π along the S_x direction and the assumption that $|\phi_0\rangle$ is $SO(3)$ symmetric. Given that $O_{l'}$ and $O_{u'}$ are spatial truncations of O'_1 and O'_2 , respectively, Eq. (A35) holds up to an error of the order specified in Eq. (A9). Consequently, we have demonstrated that $\text{tr}(O_{l'}^\dagger O_{l'} \sigma_\theta)$ vanishes for large L , thus concluding the proof of the theorem. For generic k -local Hamiltonians, Theorem 2 holds when the parameter α governing the decay of the interaction in Eq. (15) satisfies $\alpha > \max(3d, 4d - 2)$.

5. Type II Hamiltonians

We conclude by discussing type II Hamiltonians. We construct the twisted Hamiltonian in a similar manner. However, a crucial distinction arises. In a type I Hamiltonian $H_I = \sum_{ij} h_{ij}$, following the conjugation $H_\theta = U_R^\dagger(\theta) H_I U_R(\theta)$, all terms supported simultaneously on the left and right halves undergo modification. The number of modified terms amounts to approximately L^{2d} . In contrast, for a type II Hamiltonian $H_{II} = \sum_{ij} \frac{O_i O_j}{(|i-j|+1)^\alpha}$, only those terms such that either O_i or O_j acts exactly on the twist, i.e., across the links $(L - 1, 0)$ or $(\frac{L}{2} - 1, \frac{L}{2})$, are modified, totaling approximately L^{2d-1} . This observation similarly extends to H_θ^{uu} and H_θ^{lw} , as well as to any generic k -local Hamiltonians.

Following a similar line of reasoning, we construct a twisted state as in Eq. (19) and examine the reduced density matrices in \bar{u} (the region complementary to the upper quarter), denoted as ρ_θ^1 . Again, we aim to demonstrate that

$$\partial_\theta \rho_\theta^1 - \partial_\theta \sigma_\theta^1 = e_\theta = e_a + e_b \quad (\text{A36})$$

has a small trace norm, where the two errors e_a and e_b are defined in Eq. (23). Repeating the derivation for type I Hamiltonians, we can demonstrate that the magnitude of e_a remains unchanged, as indicated in Eq. (A14). This is because for all generic k -local Hamiltonians satisfying the two requirements in Eqs. (15) and (16), the LR bound in Eq. (A11) holds, and this bound straightforwardly applies to type II Hamiltonians, which are a special subclass of k -local Hamiltonians.

On the other hand, the estimation for e_b is slightly different. Repeating the derivation, we again claim that in the first term of e_b (the commutator), substituting $A^{+,-}$ with $B^{+,-}$ introduces only a small error. Let us evaluate the error δ resulting from this substitution:

$$\begin{aligned} \delta = & -\frac{1}{2\pi i} \int_0^{L^\beta} d\tau e^{-(\Delta\tau)^2/2q} \\ & \times \int_t \frac{[\partial_\theta H_\theta^{tw} - \partial_\theta H_\theta](t)}{t + i\tau} e^{-(\Delta t)^2/2q} - \text{H.c.} \end{aligned} \quad (\text{A37})$$

A crucial observation is that $\partial_\theta H_\theta^{tw} - \partial_\theta H_\theta$, and therefore δ , can be divided into two categories, $\delta = \delta_1 + \delta_2$: (1) δ_1 arises from terms supported entirely within u , as defined in Eq. (29); (2) δ_2 comes from all remaining terms in $\partial_\theta H_\theta^{tw} - \partial_\theta H_\theta$, denoted as H_m . Notably, each term in H_m has a diameter of at least $\frac{L}{8}$. Crucially, for a generic k -local type II Hamiltonian, H_m contains only terms acting nontrivially on the twist. Using Eq. (15), we find that $\|H_m\|$ is bounded by $O(\frac{1}{L^{\alpha-2d+1}})$. The contribution from terms in the second category to δ is then upper bounded by Shanti's bound:

$$\|\delta_2\| \leq \frac{\|H_m\|}{2} \sqrt{\frac{2\pi q}{\Delta^2}} \leq \text{const} \times L^{2d-1-\alpha+\frac{q}{2}}. \quad (\text{A38})$$

Comparing Eq. (A38) with Eq. (28), we find that for the special subclass of type II Hamiltonians, δ_2 becomes smaller. This explains why the applicability of the theorem is extended.

We now consider δ_1 . Following the derivation leading to Eq. (30), we need to bound the operator norm of $[O, \delta_1]$, where O is an operator supported within \bar{u} with unit norm. Note that $\partial_\theta H_\theta^u$ can be further divided into two parts, $\partial_\theta H_\theta^u = K_1 + K_2$: (1) The terms in K_1 are supported solely in the regions $k_1 \cup k_2$ (as depicted in Fig. 2). Note that the minimal distance from each of these terms to O is $\frac{L}{16}$. Following exactly the same argument leading to Eq. (A13), for a generic k -local Hamiltonian with properties (15) and (16), the LR bound

constrains the commutator

$$\begin{aligned} & \left\| \left[-\frac{1}{2\pi i} \int_0^{L^\beta} d\tau e^{-(\Delta\tau)^2/2q} \int_t \frac{K_1(t)}{t + i\tau} e^{-(\Delta t)^2/2q} - \text{H.c.}, O \right] \right\| \\ & \leq \text{const} \times L^{\frac{q}{2}(\frac{d}{1-\sigma}+1)+\min(\beta, \frac{q}{2})+(1-\sigma)(\alpha-d)-(\alpha-2d+1)}. \end{aligned} \quad (\text{A39})$$

(2) K_2 includes all other terms in K , namely, terms that act nontrivially in region k_3 or k_4 or both. Note that the diameter of each term in K_2 is at least $\frac{L}{16}$, and therefore $\|K_2\| \leq O(L^{2d-1-\alpha})$. As a result, we have (Shanti's bound)

$$\begin{aligned} & \left\| \left[-\frac{1}{2\pi i} \int_0^{L^\beta} d\tau e^{-(\Delta\tau)^2/2q} \int_t \frac{K_2(t)}{t + i\tau} e^{-(\Delta t)^2/2q} - \text{H.c.}, O \right] \right\| \\ & \leq \text{const} \times L^{2d-1-\alpha+\frac{q}{2}}. \end{aligned} \quad (\text{A40})$$

Compared with Eq. (A17), for the special class of type II Hamiltonians, this error becomes smaller due to the fact that they contain only couplings among $SO(3)$ singlet local operators, and therefore, K_2 contains only terms that act nontrivially on the twist.

Combining Eqs. (A38), (A39), and (A40) and following the same derivation as for a generic type I Hamiltonian, we arrive at

$$\begin{aligned} \|e_\theta\|_1 & \simeq \|e'_\theta\|_1 \\ & \leq O(L^{\frac{q}{2}(\frac{d}{1-\sigma}+1)+\min(\beta, \frac{q}{2})+(1-\sigma)(\alpha-d)-(\alpha-2d+1)}). \end{aligned} \quad (\text{A41})$$

As a result, for $\alpha > 2d$ we have

$$\text{tr}(H_\theta^{tw} \rho_\theta) \leq \text{const} \times L^d (\|e_\theta\|_1 + \|e'_\theta\|_1) + O(L^{2d-\alpha}), \quad (\text{A42})$$

where we utilized the requirement (16), which ensures that the energy is extensive, i.e., $\|H_\theta^{tw}\| \simeq O(L^d)$. Thus, for $\alpha > 3d - 1$, it is always possible to select suitable values for σ , a , and β such that the energy of $\rho_{2\pi}$ vanishes with L algebraically. This completes the construction of a low-lying twisted state.

It can also be shown that the lattice momentum of the twisted state $\psi_{2\pi}$ is π , making it orthogonal to the original ground state ϕ_0 when $\alpha > 4d - 3$. The derivation closely resembles that of the type I Hamiltonian scenario, with the only distinction being the magnitude of Eq. (A23), now given by

$$\begin{aligned} 2\pi \|O_I\|^2 \|e_\theta\|_1 & \leq \text{const} \|O_1\|^2 \|e_\theta\|_1 \\ & \leq \text{const} \\ & \times L^{4d-3-\alpha+\min(\beta, \frac{q}{2})+\frac{q}{2}(\frac{d}{1-\sigma}+3)+(1-\sigma)(\alpha-d)} \end{aligned} \quad (\text{A43})$$

since now the magnitude of $\|e_\theta\|_1$ is given by Eq. (A41). For any $\alpha > 4d - 3$, it is possible to select σ sufficiently close to 1 and $a > \beta > 0$, both sufficiently small, ensuring the vanishing of this error in the thermodynamic limit. We conclude that in d spatial dimensions, a k -local type II Hamiltonian cannot have a unique gapped ground state for $\alpha > \max(3d - 1, 4d - 3)$.

[1] M. B. Hastings, An area law for one-dimensional quantum systems, *J. Stat. Mech.* (2007) P08024.

[2] M. B. Hastings, Locality in quantum systems, [arXiv:1008.5137](https://arxiv.org/abs/1008.5137).

- [3] M. B. Hastings, Lieb-Schultz-Mattis in higher dimensions, *Phys. Rev. B* **69**, 104431 (2004).
- [4] M. B. Hastings and T. Koma, Spectral gap and exponential decay of correlations, *Commun. Math. Phys.* **265**, 781 (2006).
- [5] T. Kuwahara and K. Saito, Area law of noncritical ground states in 1D long-range interacting systems, *Nat. Commun.* **11**, 4478 (2020).
- [6] E. Lieb, T. Schultz, and D. Mattis, Two soluble models of an antiferromagnetic chain, *Ann. Phys.* **16**, 407 (1961).
- [7] M. Oshikawa, Commensurability, excitation gap, and topology in quantum many-particle systems on a periodic lattice, *Phys. Rev. Lett.* **84**, 1535 (2000).
- [8] G. Y. Cho, C.-T. Hsieh, and S. Ryu, Anomaly manifestation of Lieb-Schultz-Mattis theorem and topological phases, *Phys. Rev. B* **96**, 195105 (2017).
- [9] B. Nachtergaele and R. Sims, A multi-dimensional Lieb-Schultz-Mattis theorem, *Commun. Math. Phys.* **276**, 437 (2007).
- [10] I. Affleck, Spin gap and symmetry breaking in CuO_2 layers and other antiferromagnets, *Phys. Rev. B* **37**, 5186 (1988).
- [11] Y. Tada, Lieb-Schultz-Mattis theorem in higher dimensions from approximate magnetic translation symmetry, *Phys. Rev. Lett.* **127**, 237204 (2021).
- [12] J. Eisert, M. van den Worm, S. R. Manmana, and M. Kastner, Breakdown of quasilocalty in long-range quantum lattice models, *Phys. Rev. Lett.* **111**, 260401 (2013).
- [13] Z. X. Gong, M. F. Maghrebi, A. Hu, M. L. Wall, M. Foss-Feig, and A. V. Gorshkov, Topological phases with long-range interactions, *Phys. Rev. B* **93**, 041102(R) (2016).
- [14] Z.-X. Gong, M. Foss-Feig, F. G. S. L. Brandão, and A. V. Gorshkov, Entanglement area laws for long-range interacting systems, *Phys. Rev. Lett.* **119**, 050501 (2017).
- [15] S. Maity, U. Bhattacharya, and A. Dutta, One-dimensional quantum many body systems with long-range interactions, *J. Phys. A* **53**, 013001 (2020).
- [16] Z. Gong, T. Guaita, and J. I. Cirac, Long-range free fermions: Lieb-Robinson bound, clustering properties, and topological phases, *Phys. Rev. Lett.* **130**, 070401 (2023).
- [17] Z.-X. Gong, M. Foss-Feig, S. Michalakis, and A. V. Gorshkov, Persistence of locality in systems with power-law interactions, *Phys. Rev. Lett.* **113**, 030602 (2014).
- [18] T. Kuwahara and K. Saito, Absence of fast scrambling in thermodynamically stable long-range interacting systems, *Phys. Rev. Lett.* **126**, 030604 (2021).
- [19] H. Tasaki, The Lieb-Schultz-Mattis theorem: A topological point of view, [arXiv:2202.06243](https://arxiv.org/abs/2202.06243).
- [20] M. B. Hastings and X.-G. Wen, Quasiadiabatic continuation of quantum states: The stability of topological ground-state degeneracy and emergent gauge invariance, *Phys. Rev. B* **72**, 045141 (2005).
- [21] C.-F. Chen and A. Lucas, Finite speed of quantum scrambling with long range interactions, *Phys. Rev. Lett.* **123**, 250605 (2019).
- [22] M. C. Tran, A. Y. Guo, C. L. Baldwin, A. Ehrenberg, A. V. Gorshkov, and A. Lucas, Lieb-Robinson light cone for power-law interactions, *Phys. Rev. Lett.* **127**, 160401 (2021).
- [23] M. C. Tran, C.-F. Chen, A. Ehrenberg, A. Y. Guo, A. Deshpande, Y. Hong, Z.-X. Gong, A. V. Gorshkov, and A. Lucas, Hierarchy of linear light cones with long-range interactions, *Phys. Rev. X* **10**, 031009 (2020).
- [24] M. Foss-Feig, Z.-X. Gong, C. W. Clark, and A. V. Gorshkov, Nearly linear light cones in long-range interacting quantum systems, *Phys. Rev. Lett.* **114**, 157201 (2015).
- [25] T. Kuwahara and K. Saito, Strictly linear light cones in long-range interacting systems of arbitrary dimensions, *Phys. Rev. X* **10**, 031010 (2020).
- [26] C.-F. (A.) Chen, A. Lucas, and C. Yin, Speed limits and locality in many-body quantum dynamics, *Rep. Prog. Phys.* **86**, 116001 (2023).
- [27] D. V. Else, F. Machado, C. Nayak, and N. Y. Yao, Improved Lieb-Robinson bound for many-body Hamiltonians with power-law interactions, *Phys. Rev. A* **101**, 022333 (2020).
- [28] G. Misguich, C. Lhuillier, B. Bernu, and C. Waldtmann, Spin-liquid phase of the multiple-spin exchange Hamiltonian on the triangular lattice, *Phys. Rev. B* **60**, 1064 (1999).
- [29] A. Paramekanti, L. Balents, and M. P. A. Fisher, Ring exchange, the exciton Bose liquid, and bosonization in two dimensions, *Phys. Rev. B* **66**, 054526 (2002).
- [30] M. Bukov, L. D'Alessio, and A. Polkovnikov, Universal high-frequency behavior of periodically driven systems: From dynamical stabilization to Floquet engineering, *Adv. Phys.* **64**, 139 (2015).
- [31] R. Blatt and C. F. Roos, Quantum simulations with trapped ions, *Nat. Phys.* **8**, 277 (2012).
- [32] A. Kyprianidis, F. Machado, W. Morong, P. Becker, K. S. Collins, D. V. Else, L. Feng, P. W. Hess, C. Nayak, G. Pagano, N. Y. Yao, and C. Monroe, Observation of a prethermal discrete time crystal, *Science* **372**, 1192 (2021).
- [33] H. Tasaki, *Physics and Mathematics of Quantum Many-Body Systems*, Graduate Texts in Physics (Springer, Cham, 2020), Vol. 66.
- [34] R. Liu, J. Yi, S. Zhou, and L. Zou, Lieb-Schultz-Mattis theorems and generalizations in long-range interacting systems, [arXiv:2405.14929](https://arxiv.org/abs/2405.14929).
- [35] S. Bravyi, M. B. Hastings, and F. Verstraete, Lieb-Robinson bounds and the generation of correlations and topological quantum order, *Phys. Rev. Lett.* **97**, 050401 (2006).