Antiferromagnetic and spin spiral correlations in the doped two-dimensional Hubbard model: Gauge symmetry, Ward identities, and dynamical mean-field theory analysis

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We reconsider derivation of Ward identities for spin stiffnesses, which determine the nonlinear sigma model of magnetic degrees of freedom of interacting electrons in the presence of antiferromagnetic or incommensurate correlations. We emphasize that in the approaches, which do not break explicitly spin symmetry of the action, the spatial components of gauge kernel, which are used to obtain spin stiffnesses, remain gauge invariant even in case of spontaneous spin symmetry breaking. We derive the corrected Ward identities, which account for this gauge invariance. We find that the frequency dependence of temporal spin stiffnesses is not fixed by the obtained identities, and show that the infinitesimally small external staggered field is crucially important to obtain finite static uniform transverse susceptibility. On the other hand, the spatial spin stiffnesses are determined by the gauge kernel of the Legendre transformed theory, which is in general *different* from the gauge kernel of the original theory and obtain an explicit expression for spatial spin stiffnesses through susceptibilities and current correlation functions. We verify numerically the obtained results within dynamic mean-field theory, and obtain doping dependencies of the resulting spin stiffnesses for antiferromagnetic and incommensurate phases.

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I. INTRODUCTION

The properties of antiferromagnetic and incommensurate magnetic correlations near half-filling remain an actively discussed topic, in particular in connection with the physics of high-T_c compounds. In these compounds the commensurate long-range magnetic order is destroyed already at small doping (see, e.g., Refs. [1,2]), but the short-range magnetic order is present at higher doping. The observed short-range magnetic correlations are incommensurate and characterized by the wave vector $\mathbf{Q} = (\pi, \pi - \delta)$ [1–5]. These short-range magnetic correlations are considered to be one of the viable scenarios for pseudogap formation [6–15] .

The low-energy spin excitations of magnetic systems can be described by the nonlinear sigma model (NLSM) (see, e.g., Ref. [16]). The key parameters of this model are the temporal and spatial spin stiffnesses, which are determined from the microscopic analysis. The classical version of the O(3) nonlinear sigma model was derived in the continuum limit of the commensurate antiferromagnetic ordered classical Heisenberg model [17,18] and later generalized to the quantum case [19,20]. The nonlinear sigma model in the O(3)/O(2) manifold was considered as a continuum limit of frustrated quantum antiferromagnets with spin spiral ground state [21,22].

For itinerant (collinear) antiferromagnets the derivation of the NLSM, which also allows obtaining the respective spin stiffnesses, was proposed in Refs. [11,23–27]. This derivation uses Hubbard-Stratonovich transformation, and therefore introduces in the action the effective fluctuating field, corresponding to the order parameter, which is then fixed at its mean-field value. This approach explicitly breaks SU(2) gauge invariance of the theory, although the O(3) invariance of the resulting bosonic action remains unbroken. Breaking of the SU(2) gauge symmetry can be represented as a condensation of the Higgs field [11,27]. The derivation of the corresponding bosonic action from the microscopic models (e.g., Hubbard model), relies within this strategy on the average over the fermionic fields in the presence of the gauge field, which can be performed only in some approximate way. This approach is therefore difficult to generalize beyond the mean-field approach for fermionic (also referred as chargon) degrees of freedom.

Recently, using the SU(2)-symmetric gauge theory, which does not introduce symmetry-breaking terms or condensation of gauge fields in the action, was proposed for derivation of the nonlinear sigma model and obtaining respective spin stiffnesses [15,28,29]. In this approach the symmetry is broken only at the level of dressed single- and two-particle Green's functions via spin-asymmetric self-energy and the resulting renormalized interaction vertices. This approach is more convenient for use in combination with the many-body techniques since it does not introduce mean fields in the action. In view of that, it is also expected to preserve gauge invariance. This invariance provides, however, the difficulty since the respective spin stiffnesses, obtained by expansion of the action in gauge fields, are expected to vanish in the gauge-invariant approach. At the same time, finite spin stiffnesses were obtained in Refs. [15,28,29], and in Refs. [28,29] the Ward identities, which relate the second derivatives of the gauge kernel to the spin stiffnesses were proposed for the same SU(2) gauge symmetric action. The applicability of previously obtained results, and in particular respective Ward identities, requires therefore further investigation in view of the above discussed argument of gauge invariance of the considered theory, and the respective vanishing of second derivatives of the gauge kernel.

In this paper we show explicitly that in the absence of the external magnetic fields the gauge invariance of the spatial part of the gauge kernel is preserved in the approach of Refs. [15,28,29]. We emphasize therefore that the corresponding gauge kernel in general can not be used for determination of spin stiffnesses since the corresponding contributions to the stiffnesses of the nonlinear sigma model, which are proportional to the derivatives of the gauge kernel, vanish. Instead, we show that spatial spin stiffnesses are determined by the second derivatives of the effective Legendre transformed action over gauge field, which are in general *not* identical to the gauge kernel of original theory, and derive the corresponding corrections to the Ward identities, which necessarily account for the dependence of the source terms on the gauge field at fixed order parameter. On the other hand, temporal components of the spin stiffnesses are determined by the Ward identities for the original functional W before the Legendre transformation, which yield, however, the off-diagonal components of the susceptibilities and take as an input the uniform susceptibilities in small staggered magnetic field.

We furthermore show that the obtained correction terms in Ward identities compensate contributions of Goldstone modes in spatial spin stiffnesses. Although the argument of vanishing of the contribution of Goldstone modes on the basis of vanishing of the corresponding vertices in the long-wave limit was proposed in Refs. [15,28,29], it is in general not applicable since at the same time the corresponding contributions are potentially singular due to gapless Goldstone excitations. This singularity can be avoided by introducing small external staggered magnetic field, which introduces a gap in the Goldstone excitations, which is however inconvenient for practical calculations beyond mean-field approach. The compensation of the contribution of Goldstone excitations by correction terms, which originate from the proper treatment of Legendre transform, is irrespective of the presence of the external magnetic fields and on one hand shows the universality of the resulting spin stiffnesses, but on the other hand allows performing calculations in zero external magnetic field. The obtained corrections also correspond to a certain gauge fixing in the (properly treated) Legendre transform.

Finally, we apply the developed approach to calculation of spin stiffnesses of doped two-dimensional Hubbard model with hopping between nearest- and next-nearest neighbors within the recently proposed dynamical mean-field theory (DMFT) for incommensurate long-range order [30]. We verify obtained Ward identities and determine the doping dependence of the respective spin stiffnesses, which are used for construction of the nonlinear sigma model. These stiffnesses can be further used for the analysis of the magnetic properties of the model in various temperature and doping regimes.

The plan of the paper is the following. In Sec. II we introduce the model, the respective gauge transformations, and present the modified Ward identities (the details of their derivation can be found in Appendixes A and B). We also discuss in detail the differences to the previous form of Ward identities, and identify their sources. In Sec. III we provide analytical results for the susceptibilities at the momenta close to $\mathbf{q} = \mathbf{0}, \mathbf{Q}$, obtained from the modified form of Ward identities, and reveal their important matrix structure. We show that the correct form of the transverse uniform susceptibility can be obtained only in the presence of (infinitesimally) small staggered magnetic field, without which the respective susceptibility vanishes in accordance with spin conservation. In Sec. IV we present numerical results for the doped Hubbard model within dynamic mean-field theory, which confirm our analytical results, and obtain the respective temporal and spatial spin stiffnesses. Finally, in Sec. V we present conclusions.

II. THE MODEL AND WARD IDENTITIES

A. The model and gauge transformation

We consider the Hubbard model

$$H = -\sum_{i,j,\sigma} t_{ij} \hat{c}^{\dagger}_{i\sigma} \hat{c}_{j\sigma} + U \sum_i \hat{n}_{i\uparrow} \hat{n}_{i\downarrow}, \tag{1}$$

where $\hat{c}^{\dagger}_{i\sigma}$ and $\hat{c}_{i\sigma}$ are creation and destruction operators of electron at site i, spin σ, $\hat{n}_{i\sigma} = \hat{c}^{\dagger}_{i\sigma}\hat{c}_{i\sigma}$. To describe magnetic excitations, we introduce non-Abelian SU(2) gauge field by performing rotations of fermionic operators $\hat{c}_i \to \mathcal{R}_i \hat{c}_i$, $\mathcal{R}_i$ is the coordinate- and time-dependent SU(2) rotation matrix. The generating functional of the model in the rotated reference frame reads as

$$W[\mathcal{R}] = -\ln\left[\int \mathcal{D}[c, c^+] \exp(-S[c, c^+, \mathcal{R}])\right], \tag{2}$$

where c, c^+ are Grassmann variables, $S[c, c^+, \mathcal{R}]$ is the fermionic action in the rotated reference frame. We note that the fields c_i, obtained after the rotation of the reference frame, are sometimes referred as "chargons," while the spin fields, corresponding to the rotation matrices $\mathcal{R}$, are called "spinons" (see, e.g., Ref. [15]). This separation should be supplemented, however, by the gauge fixing, which will be discussed below in Sec. II C. The long-range magnetic order, which is present in the chargon sector (i.e., related to the fields c, c^+), does not necessarily imply long-range order of the spinon sector after considering the fluctuations of the metric $\mathcal{R}$.

The dependence of the action on the spinon fields can be reduced to the dependence on four SU(2) gauge fields A_μ, which are defined by

$$A_{\mu i} = \mathrm{i}\mathcal{R}^+_i \partial_\mu \mathcal{R}_i, \tag{3}$$

where we use the 4-derivative $\partial_\mu = (\partial_\tau, \nabla)$. The corresponding action takes the form

$$S[c, c^+, \mathcal{R}] = \sum_{ij} \int_0^\beta d\tau\, c^+_i \left[\left(\frac{\partial}{\partial\tau} - \mu - \mathrm{i}A_{0i}\right)\delta_{ij} - t_{ij} \exp(-\vec{r}_{ji}(\nabla - \mathrm{i}\vec{A}_i))\right] c_i + U \sum_i \int_0^\beta d\tau\, n_{i\uparrow} n_{i\downarrow}, \tag{4}$$

where μ is the chemical potential and β is inverse temperature (in energy units). We note that the external (nonuniform) magnetic field can be absorbed into the A_{0i} component of the gauge field, as we assume in the following. The fields $A_{\mu i}(\tau)$ can be expanded in Pauli matrices

$$A_{\mu i} = \frac{1}{2} \sum_{a=0,x,y,z} A^a_{\mu i} \sigma^a, \tag{5}$$

such that $A^a_\mu = \mathrm{Tr}[\sigma^a A_\mu]$. The corresponding order parameter can be written as a vector

$$m^a_i = i \frac{\delta W}{\delta A^a_{0,i}(0)} = \frac{1}{2} \langle c^+_i \sigma^a c_i \rangle. \tag{6}$$

We consider the gauge transformations $c_i \to \mathcal{V}_i c_i$, $c^+_i \to c^+_i \mathcal{V}^+_i$ which imply $\mathcal{R}_i \to \mathcal{R}_i \mathcal{V}^+_i$, $\mathcal{R}^+_i \to \mathcal{V}_i \mathcal{R}^+_i$. Expanding the field $\mathcal{V}_i(\tau)$ into components

$$\mathcal{V}_i = \exp\left(-\mathrm{i} \sum_{a=x,y,z} \mathcal{V}^a_i \frac{\sigma^a}{2}\right) \tag{7}$$

yields the following rules of infinitesimal transformation:

$$A^0_\mu \to A^0_\mu + o(\vec{\mathcal{V}}),$$
$$A^a_\mu \to A^a_\mu - \partial_\mu \mathcal{V}^a - \varepsilon_{abc} A^b_\mu \mathcal{V}^c + o(\vec{\mathcal{V}}) \tag{8}$$

(here and hereafter the summation over repeated indices is assumed). Thus, under infinitesimal spin gauge transformations, the zeroth (charge) components of the fields $A^0_{\mu i}(\tau)$ are not transformed.

B. Ward identities for the gauge kernel

In the following we introduce the gauge kernel $K^{ad}_{\mu,x;\nu,x'} = \delta^2 W/(\delta A^a_{\mu,x} \delta A^d_{\nu,x'})$ $(a, d = 0, x, y, z)$, which explicit form is obtained in Appendix A. The derivation of Ward identities for the gauge kernel, obtained from the invariance of the functional W under gauge transformation (8), is presented in Appendix B 1. For the components of the kernel, which contain at least one spatial index $n > 0$, we find the identity

$$\left(\partial_{\mu,x}\delta_{ab} + \epsilon_{acb} A^c_{\mu,x}\right) K^{bd}_{\mu,x;n,x'} = 0. \tag{9}$$

For vanishing fields $A_{\mu,x}$ this is the condition of spin conservation, which is similar to charge conservation in the U(1) case (see, e.g., the discussion in Ref. [31]). The conjugated condition, obtained by interchange $\mu \leftrightarrow n$, $x \leftrightarrow x'$, $a \leftrightarrow d$ (also in the presence of the fields A_μ), corresponds to the condition of gauge invariance, which is in accordance with Eq. (8). We stress that in the presence of long-range magnetic order in the chargon sector, the spin conservation in zero external fields and gauge invariance of the spatial components of the gauge kernel remains unbroken.

For the temporal and mixed temporal-spatial components of the gauge kernel we find the identity (see Appendix B 1)

$$\partial_{\mu,x} K^{ad}_{\mu,x;0,x'} + \epsilon_{acb} A^c_{0,x} \chi^{bd}_{xx'} = i\varepsilon_{adb} m^b_x \delta_{x,x'}; \tag{10}$$

here and in the following we assume summations over repeated indices, $\chi^{bd}_{xx'} = K^{bd}_{0,x;0,x'}$ is the tensor of the nonuniform dynamic susceptibility of the chargon sector. For vanishing temporal component $A_{0,x} = 0$ (including also the external magnetic field), Eq. (10) in the uniform limit implies *absence* of time dependence of diagonal components of the uniform dynamic chargon spin susceptibility $\sum_{\mathbf{x}\mathbf{x}'} \chi^{aa}_{xx'}$, which is another consequence of spin conservation in the system. We note again that internal magnetic fields (such as staggered magnetization in the chargon sector) do not prevent spin conservation.

It was suggested in Ref. [32] that the interband contribution to the uniform susceptibility determines the spin-wave velocity. Since this contribution can be selected by considering the $\omega \to 0$ limit of the susceptibility taken *after* the uniform $\mathbf{q} \to 0$ limit (see the discussion in Ref. [29]), vanishing of the dynamic uniform susceptibility, obtained from Eq. (10), looks contradictory to the suggestion of identifying dynamic susceptibility with the interband contributions to the uniform susceptibility. As we argue below in Sec. III, the reason of this contradiction is in the necessity of introducing infinitesimally small external staggered magnetic field, which is switched off only *after* the uniform and static limits are taken. As we show below, this external staggered magnetic field, absent in previous consideration of Ward identities [28,29], is crucial to obtain *finite* interband contribution to the uniform susceptibility.

While in the theories using the Hubbard-Stratonovich transformation [11,23–27], the gauge kernel (which appears after expanding the action to the second order in gauge fields) is used for determination of spin stiffnesses, the above consideration implies that for zero external field the gauge kernel itself (without performing Hubbard-Stratonovich transformation and/or introducing mean field) does not determine spatial spin stiffnesses since the corresponding contributions to the nonlinear sigma model, which are proportional to the spatial derivatives of the gauge kernel, vanish according to Eq. (9). This difficulty occurs because of the absence of gauge-fixing contributions in the considered action, which coincides with that of Refs. [15,28,29]. Presence of the symmetry breaking in the chargon sector (e.g., spin-asymmetric self-energies of fermionic degrees of freedom) does not resolve this difficulty since the symmetry is *spontaneously* broken at the level of the particular solution (e.g., the mean-field or dynamic mean-field approximation, considered below in Sec. IV), and formally the action is still rotation invariant. As we show in Sec. II C, the spin stiffness is in fact determined by the derivatives of the Legendre transformed action, which are in general *different* from the gauge kernel.

C. Legendre transformation and Ward identities for the modified gauge kernel

To extract spatial spin stiffnesses, we perform Legendre transformation, following Refs. [28,29]. To this end we explicitly pick out the part of the external magnetic field, which regulates the order parameter, by performing the shift $A_0 \to A_0 + iJ$, and introduce the functional

$$\Gamma[\phi, A] = W[J, A] - \phi^a_x J^a_x,$$
$$J^a_x = -\frac{\delta \Gamma}{\delta \phi^a_x}, \quad \phi^a_x = \frac{\delta W}{\delta J^a_x} = i \frac{\delta W}{\delta A^a_{0,x}}, \tag{11}$$

which is identical to that considered in Refs. [28,29]. The second derivatives of the transformed functional $\kappa^{ab}_{xx'} = \delta^2 \Gamma/(\delta \phi^a_x \delta \phi^b_{x'})$ determine the inverse susceptibilities of the chargon sector according to the standard relation

$$\kappa^{ac}_{xx'} \frac{\delta^2 W}{\delta J^c_{x'} \delta J^b_{x''}} = -\kappa^{ac}_{xx'} \frac{\delta^2 W}{\delta A^c_{0,x'} \delta A^b_{0,x''}} = -\delta_{a,b} \delta_{xx''}. \tag{12}$$

One can then obtain the following relations between the functional derivatives:

$$\frac{\delta \Gamma}{\delta A^a_m} = \frac{\delta W}{\delta A^a_m}, \tag{13}$$

$$\frac{\delta \Gamma}{\delta A^a_{0,x}} = \frac{\delta W}{\delta A^a_{0,x}} = -i\phi^a_x, \tag{14}$$

$$\frac{\delta^2 \Gamma}{\delta A^a_{\mu,x}\delta A^b_{0,x'}} = 0. \tag{15}$$

Equation (15) implies that second derivatives of the functional Γ over the gauge fields, involving at least one A_0 component, vanish, and Γ depends only *linearly* on the field A_0. This linear dependence directly follows from Eq. (11) at fixed ϕ since the change of A_0 necessarily yields the opposite change of iJ to keep ϕ fixed. Notably, although our functional Γ coincides with that considered in Refs. [28,29], the second derivative $\delta^2\Gamma/\delta A_0^2$ was suggested in these studies to determine the temporal components of spin stiffness (i.e., the transverse susceptibilities). Our derivation of temporal components of spin stiffness is presented below.

First, we emphasize that the components of the modified gauge kernel $M^{ab}_{\mu,x;\nu,x'} \equiv \delta^2\Gamma/(\delta A^a_\mu \delta A^b_\nu)$ are in general different from the gauge kernel $K^{ab}_{\mu,x;\nu,x'}$. It is crucial that at fixed fields ϕ the sources J depend implicitly on the gauge fields A. We have

$$\frac{\delta J^a_x}{\delta A^b_{\mu,x'}} = -\frac{\delta^2 \Gamma}{\delta \phi^a_x \delta A^b_{\mu,x'}} = -\frac{\delta}{\delta \phi^a_x}\frac{\delta W}{\delta A^b_{\mu,x'}}$$
$$= -\frac{\delta J^c_{x''}}{\delta \phi^a_x}\frac{\delta^2 W}{\delta J^c_{x''}\delta A^b_{\mu,x'}} = \kappa^{ac}_{xx''}\frac{\delta^2 W}{\delta J^c_{x''}\delta A^b_{\mu,x'}}. \tag{16}$$

Therefore, we obtain

$$M^{ab}_{\mu,x;\nu,x'} = \frac{\delta^2 W}{\delta A^a_{\mu,x}\delta A^b_{\nu,x'}} + \frac{\delta^2 W}{\delta A^a_{\mu,x}\delta J^c_{x''}}\frac{\delta J^c_{x''}}{\delta A^b_{\nu,x'}}$$
$$= K^{ab}_{\mu,x;\nu,x'} - K^{ac}_{\mu,x;0,x''}\kappa^{cd}_{x''x'''}K^{db}_{0,x''';\nu,x'}. \tag{17}$$

The second term in the right-hand side of Eq. (17) is crucial to fulfill the relation (15) and obtain the correct form of the Ward identities for the functional Γ. In particular, for $\mu = 0$ or $\nu = 0$ it yields $M_{\mu\nu} = 0$ in agreement with Eq. (15). In Sec. III C we show that the second term in the right-hand side of Eq. (17) cancels contribution of Goldstone modes in the first term. This is physically justifiable since Goldstone modes, being the low-energy excitations, should not give contribution to the spin stiffness of the nonlinear sigma model. On the other hand, taking into account the dependence $J[A]$ at fixed ϕ, represented by Eq. (16), implies gauge fixing in the Legendre transform (11) at fixed equilibrium ϕ, which is characterized by $J[0] = 0$.

The respective Ward identity for the second derivatives $M^{ab}_{\mu,x;\nu,x'}$ takes the form (see Appendix B 2)

$$\partial_{m,x}\partial_{n,x'}M^{ab}_{m,x;n,x'} = -i\varepsilon_{adb}\phi^d_x\partial_{0,x'}\delta_{x,x'}$$
$$+\varepsilon_{b\tilde{d}\tilde{c}}\phi^{\tilde{d}}_{x'}\left(\varepsilon_{adc}\phi^d_x\kappa^{c\tilde{c}}_{xx'} + i\varepsilon_{a\tilde{c}c}A^c_{0,x}\delta_{xx'}\right). \tag{18}$$

The first term in the right-hand side of Eq. (18) vanishes for almost uniform components of $M^{ab}_{m,x;nx'}$ since it contains an order-parameter vector, oscillating in space for nonzero wave vector $\mathbf{Q}$. Equation (18) is *different* from that derived in Refs. [28,29] by the the derivatives in the left-hand side acting on the space components only. We also note that if the gauge kernel K^{ab} would be equal to M^{ab}, as it was assumed in Refs. [15,28,29], the left-hand side of Eq. (18) would vanish in the zero external field according to Eqs. (9) and (15), while the right-hand side is finite.

For diagonal components we obtain the identity, which allows us to evaluate the spatial components of the spin stiffness,

$$\partial_{m,x}\partial_{n,x'}M^{aa}_{m,x;n,x'} = \varepsilon_{a\tilde{d}\tilde{c}}\phi^{\tilde{d}}_{x'}\left(\varepsilon_{adc}\phi^d_x\kappa^{c\tilde{c}}_{xx'} + i\varepsilon_{a\tilde{c}c}A^c_{0,x}\delta_{xx'}\right). \tag{19}$$

One can see that the corresponding spin stiffnesses are proportional to the derivatives M^{aa}_{mn} of the functional Γ over gauge fields, which are in general *different* from the gauge kernel [see Eq. (17)]. Since the temporal parts $M_{\mu\nu}$ with either $\mu = 0$ or $\nu = 0$ vanish according to Eq. (15), we can formally add the respective time derivatives to the left-hand sides of Eq. (19), but they do not provide actual contribution, in contrast to Refs. [28,29]. This implies that the respective (diagonal) components of the inverse susceptibilities κ are frequency independent. At the same time, the temporal components of the spin stiffnesses are determined by the off-diagonal components of inverse susceptibility κ (see explicit calculation in Sec. III).

Let us compare the obtained relations to the Ward identity of Refs. [28,29] in zero external fields:

$$\partial_{\mu,x}\partial_{\nu,x'}K^{ab}_{\mu,x;\nu,x'} = \partial_{\mu,x}\partial_{\nu,x'}M^{ab}_{\mu,x;\nu,x'} = \varepsilon_{adc}\phi^c\varepsilon_{bpl}\phi^l\kappa^{dp}_{xx'}. \tag{20}$$

As we discuss above, the spatial derivatives in the left-hand side of Eq. (20) for zero external field in fact yield identically zero according to the gauge-invariance condition (9). While the temporal part $M^{ab}_{0,x;0,x'}$ vanishes too, the spatial part $M^{ab}_{m,x;n,x'}$ (and its derivatives) do not vanish, which readily shows that the left equality in the relation (20) can not be correct. This contradiction is resolved in our approach by account for the additional term in Eq. (17). The second equality in Eq. (20) is fulfilled in our approach as well, with the important difference that the contribution of the time components in the summation over μ, ν in the middle part of Eq. (20) vanishes, implying vanishing of the respective staggered components of the inverse susceptibility $\kappa_{xx'}$ in the absence of the external magnetic field, while these components, together with $M^{ab}_{0,x;0,x'}$ (and its time derivatives), were considered finite in Refs. [28,29]. In our approach this contradiction is resolved by including off-diagonal spin components of κ^{ab} and performing its inversion as a matrix, as discussed in Sec. III.

III. ANALYTICAL RESULTS FOR THE CHARGON SUSCEPTIBILITIES NEAR q = 0, Q

To calculate the chargon susceptibilities we introduce the Fourier components in the global reference frame (we consider only diagonal in momentum components of the gauge kernel)

$$M^{ab}_{q;\mu\nu} = \int d\tau \sum_{\mathbf{xx'}} e^{-i\mathbf{q}(\mathbf{x}-\mathbf{x'})+i\omega(\tau-\tau')}M^{ab}_{\mu,x;\nu,x'}, \tag{21}$$

$$\kappa^{ab}_{\mathbf{qq}',\omega} = \int d\tau \sum_{\mathbf{xx}'} e^{-i\mathbf{qx}+i\mathbf{q}'\mathbf{x}'+i\omega(\tau-\tau')}\kappa^{ab}_{xx'}, \tag{22}$$

and $\chi_{\mathbf{qq}',\omega}$ defined similarly to $\kappa_{\mathbf{qq}',\omega}$.

A. Commensurate antiferromagnetic order in chargon sector

In the commensurate antiferromagnetic (AFM) case we assume the spatial dependence of the order parameter $\phi_x = m(0, 0, \cos(\mathbf{Qx}))$ with $\mathbf{Q} = (\pi, \pi)$ and introduce also staggered field $A_0^a = ih\phi_x^a/m$. The general form of the susceptibility matrix $\chi_{\mathbf{qq}',\omega}$ at $\mathbf{q}, \mathbf{q}' = \mathbf{0}, \mathbf{Q}$, allowed by Ward identity (10), is presented in Appendix B 1. Since the Ward identities do not fix the $\chi^{xx,yy}_{\mathbf{Q},\omega}$ components of the susceptibility, we parametrize these components by the temporal spin stiffness χ_ω (which is in general frequency dependent) as

$$\chi^{xx,yy}_{\mathbf{Q},\omega} = \frac{m^2}{hm + \chi_\omega\omega^2}, \tag{23}$$

and assume $\chi^{xy,yx}_{\mathbf{Q},\omega} = 0$. For $\mathbf{q}, \mathbf{q}' = \mathbf{0}, \mathbf{Q}$ we obtain with account of Eq. (23)

$$\chi^{ab}_{\mathbf{qq}',\omega} = d_\omega \left(\begin{array}{cc:cc} h\chi_\omega & 0 & 0 & \chi_\omega\omega \\ 0 & h\chi_\omega & -\chi_\omega\omega & 0 \\ \hdashline 0 & \chi_\omega\omega & m & 0 \\ -\chi_\omega\omega & 0 & 0 & m \end{array}\right), \tag{24}$$

where $d_\omega = m/(hm + \omega^2\chi_\omega)$, the 2×2 blocks correspond to the values $\mathbf{q}, \mathbf{q}' = \mathbf{0}, \mathbf{Q}$, and $a, b = x, y$ numerates components within each 2×2 block. One can see that in agreement with spin conservation, discussed in Sec. II B, the uniform transverse susceptibilities [given by the upper left 2×2 block of Eq. (24)] vanish if the limit $h \to 0$ is taken prior to $\omega \to 0$, but they acquire a finite value $\chi_{\omega=0}$ if the limit $\omega \to 0$ is taken first. While the limit $\mathbf{q} = \mathbf{0}$ prior to taking the $\omega \to 0$ limit, as considered in Eq. (24), excludes the intraband contributions to the uniform static transverse susceptibility (cf. Ref. [29]), treatment of all interband terms (which is performed in the absence of the external staggered field h) yields zero transverse susceptibility in accordance with spin conservation (10). Therefore, finite external staggered field, switched off in the end of the calculation, is crucially important to obtain finite uniform static transverse susceptibility. Presence of small finite staggered field, which is switched off *after* taking $\mathbf{q} = 0$, $\omega \to 0$ limits, excludes interband contributions, which do not conserve momentum, making the corresponding chargon uniform susceptibility finite (see the example of mean-field calculation in Appendix C).

Inverting Eq. (24), we find

$$\kappa^{ab}_{\mathbf{qq}',\omega} = \frac{1}{m}\left(\begin{array}{cc:cc} m\chi_\omega^{-1} & 0 & 0 & -\omega \\ 0 & m\chi_\omega^{-1} & \omega & 0 \\ \hdashline 0 & -\omega & h & 0 \\ \omega & 0 & 0 & h \end{array}\right). \tag{25}$$

One can see that the diagonal terms $\kappa^{xx,yy}_{\mathbf{QQ}} = h/m$, described by the lower right 2×2 block of Eq. (25), *do not* acquire any dynamic frequency-dependent part, in agreement with Eq. (19), but in contrast to the result of Refs. [28,29], based on previous form of Ward identities. As it is discussed in Sec. II C, the latter approaches assume finiteness of the derivative $\partial_\tau^2 M^{aa}_{00}$, which in fact vanishes. It is important to stress that absence of the dynamic contributions in the diagonal $\kappa^{xx,yy}_{\mathbf{QQ}}$ terms does not contradict the form of dynamic susceptibilities (23), which acquires frequency dependence due to the *off-diagonal* terms in Eq. (25).

To obtain the momentum dependence of the susceptibilities, we use the Ward identity (19), which takes in momentum space the form

$$q_n q_l M^{yy,xx}_{q;nl} = m^2\kappa^{xx,yy}_{\mathbf{q+Q},\mathbf{q+Q},\omega} - mh. \tag{26}$$

At $\mathbf{q} = 0$ this is consistent with Eq. (25). The transverse magnetic susceptibilities near the wave vector $\mathbf{Q}$ are uniquely determined by the corresponding current correlation functions $M^{xx,yy}$. We note again that the corresponding components of the inverse susceptibilities appear to be frequency independent. Neglecting the change of the other components in Eq. (25) with $\mathbf{q}$ and inverting susceptibility matrix, we readily obtain

$$\chi^{xx,yy}_{\pm\mathbf{Q+q},\pm\mathbf{Q+q},\omega} = \frac{m^2}{hm + \chi_\omega\omega^2 + \rho q^2}, \tag{27}$$

where the spatial spin stiffness $\rho = M^{xx,yy}_{q\to 0;nn}$ (we have taken into account the diagonal form of $M^{xx,yy}_{q\to 0;nl}$ over spatial indices n, l).

B. Incommensurate spiral order in chargon sector

1. Susceptibilities at $\mathbf{q} = \mathbf{0}, \mathbf{Q}$

In the case of spiral incommensurate order we consider the order parameter $\phi_x = m(\sin(\mathbf{Qx}), 0, \cos(\mathbf{Qx}))$ and introduce external staggered field $A_0^a = ih\phi_x^a/m$. For calculations we pass to the local reference frame (see Appendix D). The susceptibilities in the local coordinate frame are diagonal with respect to momenta. The general form of the susceptibility matrix in the local reference frame, allowed by Ward identities, is presented in Appendix E. We parametrize the $\bar{\chi}^{xx}_{\mathbf{0},\omega}$ and $\bar{\chi}^{yy}_{\pm\mathbf{Q},\omega}$ susceptibilities in the local reference frame (denoted by bars here and below), which are not fixed by Ward identities, by (in general frequency-dependent) in-plane and out-of-plane components of temporal spin stiffnesses $\chi_{2,\omega}$ and $\chi_{1,\omega}$ according to

$$\bar{\chi}^{xx}_{\mathbf{0},\omega} = \frac{m^2}{hm + \chi_{2,\omega}\omega^2}, \quad \bar{\chi}^{yy}_{\pm\mathbf{Q},\omega} = \frac{m^2}{hm + \chi_{1,\omega}\omega^2}, \tag{28}$$

and assume $\bar{\chi}_{\mathbf{0},\omega} = \bar{\chi}^{zx}_{\mathbf{0},\omega} = 0$. From this we obtain the frequency dependence of the uniform susceptibilities in the local reference frame

$$\bar{\chi}^{ab}_{\mathbf{0},\omega} = \begin{pmatrix} md_{2,\omega} & \omega\tilde{d}_{2,\omega} & 0 \\ -\omega\tilde{d}_{2,\omega} & h\tilde{d}_{2,\omega} & 0 \\ 0 & 0 & \bar{\chi}^{zz}_{\mathbf{0},\omega} \end{pmatrix}_{x,y,z}, \tag{29}$$

where $d_{i,\omega} = m/(hm + \omega^2\chi_{i,\omega})$, $\tilde{d}_{i,\omega} = d_{i,\omega}\chi_{i,\omega}$, and the index x, y, z refers to the respective spin S^x, S^y, S^z

reference frame. The blocks of the local susceptibility at the momenta $\mathbf{q} = \pm\mathbf{Q}$ are conveniently written in the basis $S^+_\mathbf{q}, S^y_\mathbf{q}, S^-_\mathbf{q}$ ($S^\pm_\mathbf{q} = S^z_\mathbf{q} \pm iS^x_\mathbf{q}$ are the circular in-plane spin components) as

$$\bar{\chi}^{ab}_{-\mathbf{Q},\omega} = \begin{pmatrix} \frac{\mathrm{i}h}{\omega}\bar{\chi}^{+y}_{-\mathbf{Q},\omega} & \bar{\chi}^{+y}_{-\mathbf{Q},\omega} & \bar{\chi}^{+-}_{-\mathbf{Q},\omega} \\ -\mathrm{i}\omega\tilde{d}_{1,\omega} & md_{1,\omega} & \bar{\chi}^{y-}_{-\mathbf{Q},\omega} \\ h\tilde{d}_{1,\omega} & -\mathrm{i}\omega\tilde{d}_{1,\omega} & \frac{\mathrm{i}h}{\omega}\bar{\chi}^{y-}_{-\mathbf{Q},\omega} \end{pmatrix}_{+,y,-}, \tag{30}$$

the susceptibility $\bar{\chi}_{\mathbf{q}=\mathbf{Q},\omega} = (\bar{\chi}_{\mathbf{q}=-\mathbf{Q},\omega})^+$ can be obtained via Hermitian conjugate. In the basis $(S^{+,y,-}_{-\mathbf{Q}}, S^{+,y,-}_{\mathbf{0}}, S^{+,y,-}_{\mathbf{Q}})$ we find then in the global reference frame

$$\chi^{ab}_{\mathbf{q}\mathbf{q}',\omega} = \begin{pmatrix} 0 & 0 & \bar{\chi}^{zz}_{\mathbf{0},\omega}+md_{2,\omega} & 0 & i\omega\tilde{d}_{2,\omega} & 0 & \bar{\chi}^{zz}_{\mathbf{0},\omega}-md_{2,\omega} & 0 & 0 \\ 0 & d_{1,\omega}m & 0 & -i\omega\tilde{d}_{1,\omega} & 0 & 0 & 0 & 0 & 0 \\ \bar{\chi}^{+-}_{2\mathbf{Q},\omega} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & h\tilde{d}_{1,\omega} & 0 & i\tilde{d}_{1,\omega}\omega & 0 \\ 0 & 0 & i\omega\tilde{d}_{2,\omega} & 0 & h\tilde{d}_{2,\omega} & 0 & -i\omega\tilde{d}_{2,\omega} & 0 & 0 \\ 0 & -i\omega\tilde{d}_{1,\omega} & 0 & h\tilde{d}_{1,\omega} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\chi}^{+-}_{2\mathbf{Q},\omega} \\ 0 & 0 & 0 & 0 & 0 & i\omega\tilde{d}_{1,\omega} & 0 & d_{1,\omega}m & 0 \\ 0 & 0 & \bar{\chi}^{zz}_{\mathbf{0},\omega}-md_{2,\omega} & 0 & -i\omega\tilde{d}_{2,\omega} & 0 & \bar{\chi}^{zz}_{\mathbf{0},\omega}+md_{2,\omega} & 0 & 0 \end{pmatrix}_{+,y,-}. \tag{31}$$

As one can see from the limit $\omega \to 0$ followed by $h \to 0$, the uniform static transverse susceptibilities are given by $\chi^{+-}_{\mathbf{q}=0,\omega\to 0} = 2\chi^{xx}_{\mathbf{q}=0,\omega\to 0} = 2\chi^{zz}_{\mathbf{q}=0,\omega\to 0} = \chi_{1,0}$ and $\chi^{yy}_{\mathbf{q}=0,\omega\to 0} = \chi_{2,0}$, while similarly to the commensurate case the uniform susceptibilities vanish for $h \to 0$ at finite ω.

Performing the inversion of the susceptibilities in S^x, S^y, S^z basis, we obtain

$$\bar{\kappa}^{ab}_{\mathbf{0},\omega} = \frac{1}{m}\begin{pmatrix} h & -\omega & 0 \\ \omega & m\chi^{-1}_{2,\omega} & 0 \\ 0 & 0 & m\big(\bar{\chi}^{zz}_{\mathbf{0},\omega}\big)^{-1} \end{pmatrix}_{x,y,z},$$
$$\bar{\kappa}^{ab}_{-\mathbf{Q},\omega} = \frac{1}{m}\begin{pmatrix} -\mathrm{i}d^{+-}_\omega\bar{\chi}^{+y}_{-\mathbf{Q},\omega} & 0 & \omega d^{+-}_\omega\tilde{d}_{1,\omega} \\ 2\mathrm{i}\omega & h & 0 \\ -\omega\bar{\chi}^2_\omega d^{+-}_\omega & 2\mathrm{i}\omega & -\mathrm{i}d^{+-}_\omega\bar{\chi}^{y-}_{-\mathbf{Q},\omega} \end{pmatrix}_{+,y,-}, \tag{32}$$

where $\bar{\chi}^2_\omega = (\bar{\chi}^{+y}_{-\mathbf{Q},\omega}\bar{\chi}^{y-}_{-\mathbf{Q},\omega} - \bar{\chi}^{yy}_{-\mathbf{Q},\omega}\bar{\chi}^{+-}_{-\mathbf{Q},\omega})/m$, $d^{+-}_\omega = 4\omega/(\bar{\chi}^{+-}_{-\mathbf{Q},\omega} + h\bar{\chi}^2_\omega)$.

The entire discussion of Sec. III A concerning the frequency dependence of the off-diagonal components in the commensurate case is applicable to the incommensurate case, except that the $\bar{\kappa}^{zz}_{\mathbf{0},\omega}$ and $\bar{\kappa}^{\pm\pm}_{\mathbf{Q},\omega}$ components acquire their own dynamics in the latter case.

2. Momentum dependence of susceptibilities near $q = 0, \mathbf{Q}$

To obtain susceptibilities at the momenta close to $\mathbf{q} = \mathbf{0}$ we consider leading contributions to the momentum dependence of the inverse susceptibility matrix (including charge components), allowed by symmetry [28],

$$\bar{\bar{\kappa}}^{ab}_{\mathbf{q},\omega} = m^{-1}\begin{pmatrix} h + A_{nl}q_nq_l & \omega & C_nq_n & D_nq_n \\ -\omega & m\chi^{-1}_{2,\omega} & 0 & 0 \\ -C_nq_n & 0 & d^{0z}\bar{\chi}^{00}_0 & -d^{0z}\bar{\chi}^{0z}_0 \\ -D_nq_n & 0 & -d^{0z}\bar{\chi}^{0z}_0 & d^{0z}\bar{\chi}^{zz}_0 \end{pmatrix}_{x,y,z,0}, \tag{33}$$

where $d^{0z} = m/[\bar{\chi}^{00}_0\bar{\chi}^{zz}_0 - (\bar{\chi}^{0z}_0)^2]$, such that the inversion of Eq. (33) at $\mathbf{q} = 0$ yields in the spin sector the result (29), the double overline stands for the quantities in the local reference frame which include charge component. For momenta near $\pm\mathbf{Q}$ we neglect the coupling of charge and spin components, which appears to be small numerically, and obtain

$$\bar{\kappa}^{ab}_{-\mathbf{Q}+\mathbf{q},\omega} = m^{-1}\begin{pmatrix} -\mathrm{i}d^{+-}_\omega\bar{\chi}^{+y}_{-\mathbf{Q},\omega} & 0 & \omega d^{+-}_\omega\tilde{d}_{1,\omega} \\ 2\mathrm{i}\omega & h + B_{nl}q_nq_l & 0 \\ -\omega\chi^2_\omega d^{+-}_\omega & 2\mathrm{i}\omega & -\mathrm{i}d^{+-}_\omega\bar{\chi}^{y-}_{-\mathbf{Q},\omega} \end{pmatrix}_{+,y,-}. \tag{34}$$

We further apply Ward identities (10) and (19) to determine the coefficients A_{nl}, B_{nl}, C_n, D_n (see Appendix E). The resulting susceptibilities written in the global reference frame read as

$$\chi^{xx,zz}_{\mathbf{q}\pm\mathbf{Q},\mathbf{q}\pm\mathbf{Q},\omega} = \frac{1}{4}\big(\bar{\chi}^{xx}_{\mathbf{q},\omega} + \bar{\chi}^{zz}_{\mathbf{0},\omega} + \bar{\chi}^{+-}_{2\mathbf{Q},\omega}\big). \tag{35}$$

The respective susceptibilities in the local reference frame

$$\bar{\chi}^{xx}_{\mathbf{q},\omega} = \frac{m^2}{hm + \chi_{2,\omega}\omega^2 + \rho_{2,m}q_m^2}, \tag{36}$$

$$\chi^{yy}_{\pm\mathbf{Q},\pm\mathbf{Q},\omega} = \bar{\chi}^{yy}_{\pm\mathbf{Q},\pm\mathbf{Q},\omega} = \frac{m^2}{hm + \chi_{1,\omega}\omega^2 + \rho_{1,m}q_m^2}, \tag{37}$$

where $\rho_{1,n} = 2M^{xx,zz}_{q\to 0;nn}$ and $\rho_{2,n} = M^{yy}_{q\to 0;nn} + \Delta\rho_{2,n}$ are the out-of-plane and in-plane spatial spin stiffnesses (we consider the $q_{x,y}$ coordinates which diagonalize the current correlators $M^{0;aa}_{nl}$ over the spatial indices n, l), $\Delta\rho_{2,n}$ is the contribution to the in-plane spatial stiffness

$$\Delta\rho_{2,n} = \frac{m^2}{2}\lim_{q\to 0}\partial^2_{q_n}\left(\frac{1}{\bar{\chi}^{xx}_{\mathbf{q},0}} - \bar{\kappa}^{xx}_{\mathbf{q},0}\right) = \frac{1}{\bar{\chi}^{zz}_0}\big(D_n\bar{\chi}^{0z}_0 + C_n\bar{\chi}^{zz}_0\big)^2, \tag{38}$$

which explicit form is given by Eq. (44) below and which originates from the coupling of the x component of the susceptibility with $0, z$ components at finite momenta. The factor of $\frac{1}{4}$ in Eq. (35) occurs due to passing to global reference frame and expresses the fact that one Goldstone mode of $\bar{\chi}^{xx}_{\mathbf{q}=0,0}$ in the local reference frame is split into four modes of $\chi^{xx,zz}_{\pm\mathbf{Q},0}$ in the global reference frame.

Although the results (28), (36), and (37) formally coincide with those of Refs. [15,28,29], there are two important differences: the temporal stiffnesses $\chi_{1,2,\omega}$ are in general frequency dependent in our approach since their frequency dependence is not fixed by Ward identities, and the spatial stiffnesses $\rho_{1,2}$ are expressed via the respective gauge kernels, including the correction term in Eq. (17) and the contribution of the $0, z$ modes $\Delta\rho_{2,n}$.

On approaching the spin-symmetric phase of the chargon sector, the spin symmetry in this sector tends to be restored, and the susceptibility $\bar{\chi}^{zz}_{\mathbf{0},\omega} = md_{2,\omega}$ is also divergent. Apart from that, near the paramagnetic phase the Goldstone mode at $\mathbf{q} = 0$ in the local reference frame is mirrored to the wave vector $2\mathbf{Q}$, such that $\bar{\chi}^{+-}_{2\mathbf{Q}} = 2md_{2,\omega}$. Adding the respective momentum dependencies we find in this limit the in-plane susceptibilities in the global reference frame

$$\chi^{xx,zz}_{\mathbf{q}\pm\mathbf{Q},\mathbf{q}\pm\mathbf{Q},\omega} = \frac{m^2}{hm + \chi_{2,\omega}\omega^2 + \rho_{2,m}q_m^2}, \tag{39}$$

which have the same form as the out-of-plane susceptibilities (37); the number of Goldstone modes in local and global reference frames coincides in this case.

C. Explicit expressions for spin stiffnesses

In the following we represent the modified gauge kernel as $M_{q;\mu\rho} = K_{q;\mu\rho} - K^C_{q;\mu\rho}$, where the two contributions correspond to the first and second terms in Eq. (17). Considering again the local reference frame, we represent the local susceptibility $\bar{\chi}_q$ and the current-spin kernels $\tilde{K}_{q;0\rho}$ and $\tilde{K}_{q;\mu 0}$ (the tilde corresponds to transforming spin, but not the current variables to the local reference frame) via their U-irreducible counterparts $\bar{\phi}_q$, $\tilde{\phi}_{q,\mu}$, $\tilde{\phi}^{\mathrm{t}}_{q,\mu}$ by the relations

$$\bar{\chi}_q = (1 - \bar{\phi}_q\hat{U})^{-1}\bar{\phi}_q = \bar{\phi}_q(1 - \hat{U}\bar{\phi}_q)^{-1},$$
$$\tilde{K}_{q;0\rho} = (1 - \bar{\phi}_q\hat{U})^{-1}\tilde{\phi}_{q,\rho}, \quad \tilde{K}_{q;\mu 0} = \tilde{\phi}^{\mathrm{t}}_{q,\mu}(1 - \hat{U}\bar{\phi}_q)^{-1}, \tag{40}$$

which are similar to those used previously for the dynamic spin susceptibility [30,33–36], $\hat{U} = 2U\,\mathrm{diag}(1, 1, 1, -1)$. Accordingly, the kernel K_q can be split into the U-irreducible paramagnetic part K^{irr}_q, the diamagnetic part K^d_q, and the U-reducible ladder K^L_q part,

$$K_{q;\mu\rho} = K^{\mathrm{irr}}_{q;\mu\rho} + K^d_{q;\mu\rho} + K^L_{q;\mu\rho}, \tag{41}$$

where

$$K^L_{q,\mu\rho} = \tilde{\phi}^{\mathrm{t}}_{q,\mu}(1 - \hat{U}\bar{\phi}_q)^{-1}\hat{U}\tilde{\phi}_{q,\rho}. \tag{42}$$

The explicit form of the U-irreducible parts K^{irr} and K^d is specified in Sec. IV. This splitting is itself exact and does not rely on some approximation. The correction term is represented as

$$K^C_{q;\mu\rho} = \tilde{\phi}^{\mathrm{t}}_{q,\mu}(1 - \hat{U}\bar{\phi}_q)^{-1}\bar{\kappa}_q(1 - \bar{\phi}_q\hat{U})^{-1}\tilde{\phi}_{q,\rho}, \tag{43}$$

$\bar{\kappa}_q$ is considered as 4×4 matrix with zero-charge components.

Using the explicit form of the inverse susceptibilities (33) and (34), evaluating the respective irreducible susceptibilities $\bar{\phi}_q$, one can find that the correction term (43) removes all singular terms, originating from Goldstone modes of the chargon susceptibility, contained in $K^L_{q,\mu\rho}$ [which occur from the inverse matrix in Eq. (42)]. These contributions were previously removed in Refs. [15,28,29] on the basis of their vanishing in the limit $\mathbf{q} \to 0$ at $\omega = 0$. This is, however, not fully correct since in the absence of external staggered field smallness of the kernels $\tilde{\phi}^{\mathrm{t}}_{q,\mu}$ and $\tilde{\phi}_{q,\rho}$, proportional to $\mathbf{q}$ in this limit, is in fact *compensated* by smallness of the denominator in Eq. (42), originating from the presence of Goldstone modes and yields the contribution, which is in general *finite*. Omitting these contributions implies either implicit introduction of infinitesimally small staggered field or fixing the Coulomb gauge $q_mA_m = 0$ in the nonlinear sigma model approach. Because of the non-Abelian form of the gauge field, the latter gauge fixing is expected, however, to produce nontrivial ghost field contributions.

In our approach this potentially singular contribution is compensated by the $K^C_{q,nn}$ term. Using the explicit form of the irreducible bubble matrices ϕ_q, determined from Eq. (33), and the explicit form of the correction

$$\Delta\rho_{2,n} = \frac{1}{\bar{\chi}^{zz}_0}\big[\tilde{\phi}^{yz}_{0,n0} + 2U\big(\tilde{\phi}^{yz}_{0,n0}\bar{\chi}^{zz}_0 - \tilde{\phi}^{y0}_{0,n0}\bar{\chi}^{0z}_0\big)\big]^2, \tag{44}$$

we find that the sum $K^L - K^C + \Delta\rho$ is finite and does not depend explicitly on momentum and staggered magnetic field h in the limit $\mathbf{q} \to 0$, due to cancellation of the terms $O(\mathbf{q}^2)$ and $O(h)$. Therefore, the numerical analysis below can be performed in zero external magnetic field. Keeping only ladder terms produces incorrect results in this case. Summing all contributions we finally obtain the respective spatial spin stiffnesses

$$\rho_{1,n} = 2\rho_n = 2\big(K^{\mathrm{irr},xx}_{0,nn} + K^{d,xx}_{0,nn}\big), \tag{45}$$

$$\begin{aligned}\rho_{2,n} = {}& K^{\mathrm{irr},yy}_{0,nn} + K^{d,yy}_{0,nn} + 2U\big[\tilde{\phi}^{\mathrm{t},y0}_{0,0n}(2U\bar{\chi}^{00}_0 - 1)\tilde{\phi}^{0y}_{0,n0} \\ & -4U\tilde{\phi}^{\mathrm{t},y0}_{0,0n}\bar{\chi}^{0z}_0\tilde{\phi}^{zy}_{0,n0} + \tilde{\phi}^{\mathrm{t},yz}_{0,0n}(2U\bar{\chi}^{zz}_0 + 1)\tilde{\phi}^{zy}_{0,n0}\big].\end{aligned} \tag{46}$$

This result can be obtained alternatively by differentiating the irreducible susceptibilities over momenta following Refs. [28,29,37] and using the respective Ward identities for spin-current vertices, which follow from Eq. (10) (see Appendix E). In case of static mean-field theory the obtained result coincides with that proposed in Refs. [15,28,29,37]. At the same time, it generalizes previous consideration to include dynamic effects beyond static mean-field theory. The cancellation of momentum q and external staggered field-dependent terms also shows independence of the spin stiffnesses on the gauge-fixing conditions.

IV. DYNAMICAL MEAN-FIELD THEORY APPROACH

To apply above derived identities for a particular system with long-range magnetic order, we consider a two-dimensional one-band Hubbard model on a square lattice (1) with hopping $t_{ij} = t$ between nearest neighbors (which is used as a unit of energy) and $t_{ij} = -t'$ for next-nearest neighbors. As it is discussed in Refs. [38–41], the incommensurate magnetically ordered states in the hole-doped Hubbard model are thermodynamically unstable within the mean-field approach, which yields a phase separation [40,41] of incommensurate magnetic order into domains with incommensurate and commensurate magnetic states. Therefore, for detail analysis of spin stiffnesses in a broad doping range, we consider below the DMFT approach [42], extended to describe incommensurate spin spiral order in the chargon sector.

The respective approach for the doped Hubbard model was developed previously in Refs. [29,30,43–45]. The main difference of dynamic from static mean-field theory is in the frequency dependence of the self-energy and interaction vertices (see the discussion in Refs. [30,46]). In the following, we use the approach of Ref. [30], which considers passing to local reference frame with the order parameter aligned along the z axis. As we showed in our previous work [30] for different doping levels $x = 1 - n$ (n is the concentration of electrons) both commensurate and incommensurate (spiral) long-range magnetic orders are present in DMFT solution of this model. This feature makes this model well suited for the study of low-energy magnetic excitations in strongly correlated quasi-two-dimensional electronic system. All calculations in the following study were performed for $t'/t = 0.15$, $U = 7.5t$, and the temperature $T = 0.1t$.

A. Gauge kernels in the local reference frame

The magnetic susceptibilities and gauge kernels are represented in DMFT as sum of the ladder diagrams with local particle-hole irreducible vertices [30,42,47]. For practical calculations we evaluate the components of the gauge kernel in the local reference frame. The correspondence between the paramagnetic parts of the gauge kernels in the global and local reference frame is given by (see Appendix D 3)

$$\begin{aligned}K^{yy}_{q,\mu\nu} &= \bar{K}^{yy}_{q,\mu\nu,++} + \bar{K}^{00}_{q,\mu\nu,--} + \bar{K}^{y0}_{q,\mu\nu,+-} + \bar{K}^{0y}_{q,\mu\nu,-+}, \\ K^{xx}_{q,\mu\nu} &= \sum_{s=\pm}\big[\bar{K}^{xx}_{q,\mu\nu,ss} - is\big(\bar{K}^{xz}_{q,\mu\nu,ss} - \bar{K}^{zx}_{q,\mu\nu,ss}\big) + \bar{K}^{zz}_{q,\mu\nu,ss}\big].\end{aligned} \tag{47}$$

We further split each term in Eqs. (47) according to the representation (41). The U-irreducible part of the kernels in the local coordinate frame can be then represented as the sum of the bare bubble and the U-irreducible vertex correction

$$\bar{K}^{\mathrm{irr}}_{q,\mu\rho,ss'} = \sum_\nu \bar{K}^0_{q;\mu\rho,ss'}(\nu) + \bar{K}^\phi_{q;\mu\rho,ss'}. \tag{48}$$

The bare current-current bubble $\bar{K}^0$ in the local reference frame reads as

$$\bar{K}^{0;\alpha\beta}_{q;\mu\rho,ss'}(\nu) = \sum_{\mathbf{k}} T^{\mu,s}_{\mathbf{k},\mathbf{q}}\mathrm{Tr}[\sigma^\alpha G_k \sigma^\beta G_{k+q}]T^{\rho,s'}_{\mathbf{k},\mathbf{q}}, \tag{49}$$

where the trace is taken with respect to spin variables, $G^{\sigma\sigma'}_k$ is the electron Green's function in the local reference frame,

$$\begin{aligned}T^{\mu\pm}_{\mathbf{k},\mathbf{q}} &= \big(T^\mu_{\mathbf{k}-\mathbf{Q}/2,\mathbf{q}} \pm T^\mu_{\mathbf{k}+\mathbf{Q}/2,\mathbf{q}}\big)/2 \ \ (\alpha,\beta = 0, y), \\ T^{\mu\pm}_{\mathbf{k},\mathbf{q}} &= T^\mu_{\mathbf{k}+s\mathbf{Q}/2,\mathbf{q}} \ \ (\alpha,\beta = x, z),\end{aligned} \tag{50}$$

$T^\mu_{\mathbf{k},\mathbf{q}} = (t^\mu_{\mathbf{k}} + t^\mu_{\mathbf{k}+\mathbf{q}})/2$ are the current vertices, $t^m_{\mathbf{k}} = \partial\epsilon_{\mathbf{k}}/\partial k^m$, $t^0_k \equiv \mathrm{i}$. The same Eqs. (47) with $T^0_{\mathbf{k},\mathbf{q}} = T^{0,+}_{\mathbf{k},\mathbf{q}} = \mathrm{i}$, $T^{0,-}_{\mathbf{k},\mathbf{q}} = 0$ are applicable for spin susceptibilities. The ladder and correction terms in the local reference frame (including transforming the current variables to this reference frame) are given by Eqs. (42) and (43) with the replacement $\tilde{\phi} \to \bar{\phi}$, where we use the particle-hole U-irreducible bubbles in the local reference frame

$$\bar{\phi}_{q,\rho,s} = \sum_\nu \gamma_q(\nu)\bar{K}^0_{q,0\rho,0s}(\nu), \tag{51}$$

$$\bar{\phi}^{\mathrm{t}}_{q,\mu,s} = \sum_\nu \bar{K}^0_{q,\mu 0,s0}(\nu)\gamma^{\mathrm{t}}_q(\nu), \tag{52}$$

$T^{0,0}_{\mathbf{k},\mathbf{q}} \equiv \mathrm{i}$, and matrix multiplications are assumed here and in the following expressions and triangular vertices are defined by

$$\gamma_q(\nu) = \sum_{\nu'}\big[\hat{I} - \bar{K}^0_{q;00,++}\tilde{\Phi}_\omega\big]^{-1}_{\nu'\nu}, \tag{53}$$

$$\gamma^{\mathrm{t}}_q(\nu) = \sum_{\nu'}\big[\hat{I} - \tilde{\Phi}_\omega\bar{K}^0_{q;00,++}\big]^{-1}_{\nu\nu'}. \tag{54}$$

In Eqs. (53) and (54) the inversion is performed with respect to the frequency and spin indices (considered as multi-index), $\bar{K}^0_{q;00}$ without frequency argument is considered as the diagonal matrix with respect to frequencies, $\tilde{\Phi}_\omega = \hat{\Phi}_\omega - \hat{U}$, $\hat{\Phi}_\omega$ is the particle-hole irreducible local vertex, expressed in the spin basis.

The U-irreducible vertex correction to the kernel can be represented as

$$\bar{K}^\phi_{q;\mu\rho,ss'} = \sum_\nu \bar{K}^0_{q,\mu 0,s0}(\nu)\widetilde{\gamma}_{q;\rho,s'}(\nu), \tag{55}$$

where

$$\widetilde{\gamma}_{q;\rho,s'}(\nu) = \sum_{\nu',\nu''} \big[\hat{I} - \tilde{\Phi}_\omega \bar{K}^0_{q;00}\big]^{-1}_{\nu\nu''} \big[\tilde{\Phi}_\omega \bar{K}^0_{q,0\rho,0s'}\big]_{\nu''\nu'}. \quad (56)$$

The diamagnetic part is expressed through its counterpart in the local reference frame as

$$K^d_{\mu\nu} = \bar{K}^{d,00}_{\mu\nu,+} + \bar{K}^{d,yy}_{\mu\nu,-}, \quad (57)$$

where

$$\bar{K}^{d,ab}_{\mu\nu,s} = -\delta_{ab}(1-\delta_{\mu 0})(1-\delta_{\nu 0}) \sum_k T^{\mu\nu,s}_{\mathbf{k}} \mathrm{Tr}[\sigma^a G_k], \quad (58)$$

$T^{\mu\nu\pm}_{\mathbf{k}} = (t^{\mu\nu}_{\mathbf{k}-\mathbf{Q}/2} \pm t^{\mu\nu}_{\mathbf{k}+\mathbf{Q}/2})/2$, $t^{mn}_k = \partial^2 \epsilon_{\mathbf{k}}/(\partial k^m \partial k^n)$.

Below we consider the results of the numerical calculations, performed within CT-QMC method, realized in iQIST package [48], keeping 160 (positive and negative) fermionic Matsubara frequencies. We also use corrections on the finiteness of frequency box (see Refs. [35,36]). We have verified that for static self-energies and vertices the mean-field results for the susceptibilities are reproduced (see also Appendix C). In the DMFT calculations below for simplicity we consider zero external staggered magnetic field $h = 0$. The wave vector $\mathbf{Q}$ of magnetic instability in the chargon sector is obtained from the minimum of smallest eigenvalue of $1 - U\phi_{\mathbf{Q},0}$ (see Appendix F).

B. DMFT approach in the antiferromagnetic phase

We consider first the antiferromagnetic order in the chargon sector, which occurs at half-filling or small doping. We have verified that the dynamic transverse uniform susceptibility of chargons vanishes at zero staggered field, as it is required by Ward identities. To determine the (frequency-dependent) temporal stiffness, we use the relation (23), $\chi_\omega = m^2/(\chi^{yy}_{\mathbf{Q}}\omega^2)$. Since the susceptibility $\chi^{yy}_{\mathbf{Q}}$ at finite frequency changes continuously with the staggered field, the stiffness χ_ω, obtained in the zero-field calculation, can be also considered as a limit $h \to 0$ of the respective uniform susceptibility.

If no dynamical effects were present, one would obtain almost frequency-independent temporal stiffness χ in the low-frequency limit. In DMFT approach the temporal stiffness χ_ω becomes essentially dynamic and does not necessarily reduce to a constant. The calculated frequency dependencies of temporal stiffness χ_ω for various hole dopings x are presented in Fig. 1(a). Near half-filling $x \to 0$ the frequency dependence of temporal stiffness becomes less significant and its static limit is almost equal to the mean-field results for half-filling. With increase of the doping a peak develops at $\omega = 0$. The change of static temporal stiffness limit χ_0 with hole doping is opposite to its high-frequency limit at $\omega_n \gg t$: the latter susceptibility decreases with doping, while the former increases. As a result, the frequency dependence becomes more pronounced with doping. In Fig. 1(b) we present various susceptibility components of chargon sector, computed in the AFM state at finite hole doping $x = 0.05$. One can see that the considered components fulfill the Ward identities. It is important to emphasize the computed nondiagonal components are parameter free and present solely due to the presence of the long-range magnetic order in the chargon sector.

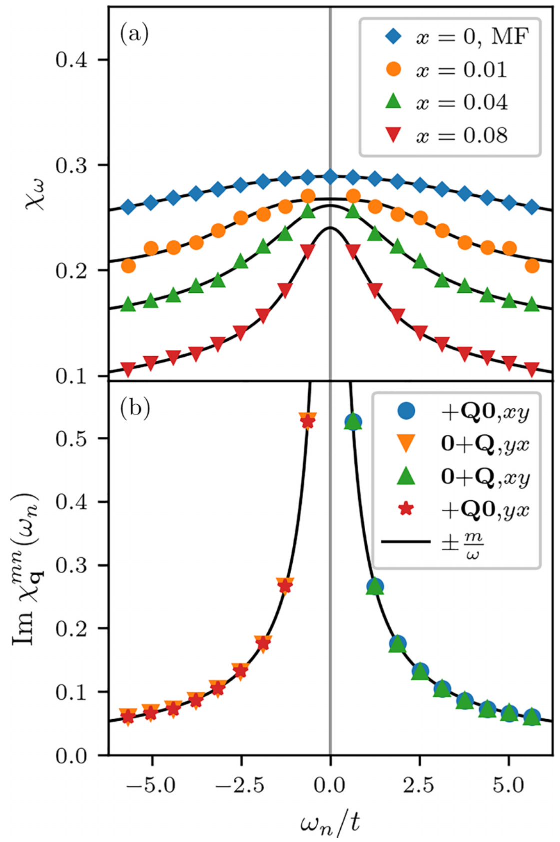

FIG. 1. (a) Frequency dependence of the temporal stiffness χ_ω for the AFM case for various fillings in DMFT approach, the result of mean-field (MF) approach at half-filling is presented for comparison. (b) Susceptibility components for AFM case at $x = 0.05$ compared to the Ward identities result (24).

The momentum dependence of inverse susceptibility, obtained from the current-current correlation function, is compared to the full momentum dependence obtained in DMFT in Fig. 2 for two doping levels. We see that the results match each other up to quadratic terms $O(q^2)$ as it follows from the Ward identities. Note that although the obtained relations are not meant to be correct apart from the quadratic order $\mathbf{q}^2$, the $\mathbf{q}$-dependent current-current correlation function gives better agreement with the inverse susceptibility than constant value of spin stiffness ρ away from half-filling, where inverse susceptibility is well described by constant stiffness in a wide range of wave vectors.

In the commensurate case the contributions, containing vertex corrections, namely, the U-irreducible term K^ϕ, the ladder contribution K^L, the correction term K^C [Eq. (43)], and the contribution $\Delta\rho_{2,n}$ vanish at $q = 0$ since the bubbles $\bar{K}^0_{0;0\rho}(\nu)$ and $\bar{K}^0_{0;\mu 0}(\nu)$, together with the respective bubbles $\phi_{0,\rho}(\nu)$ and $\phi^{\mathrm{t}}_{0,\mu}(\nu)$, vanish because of the oddness of the function, which is summed in Eq. (49) with respect to $\mathbf{k} \to \mathbf{k} + (\pi, \pi)$ (cf. Refs. [28,29]). Therefore, these terms do not contribute to the spin stiffnesses in the commensurate case. The situation in this respect is the same as for the optical conductivity in DMFT, where it was argued that the vertex corrections vanish [42,49].

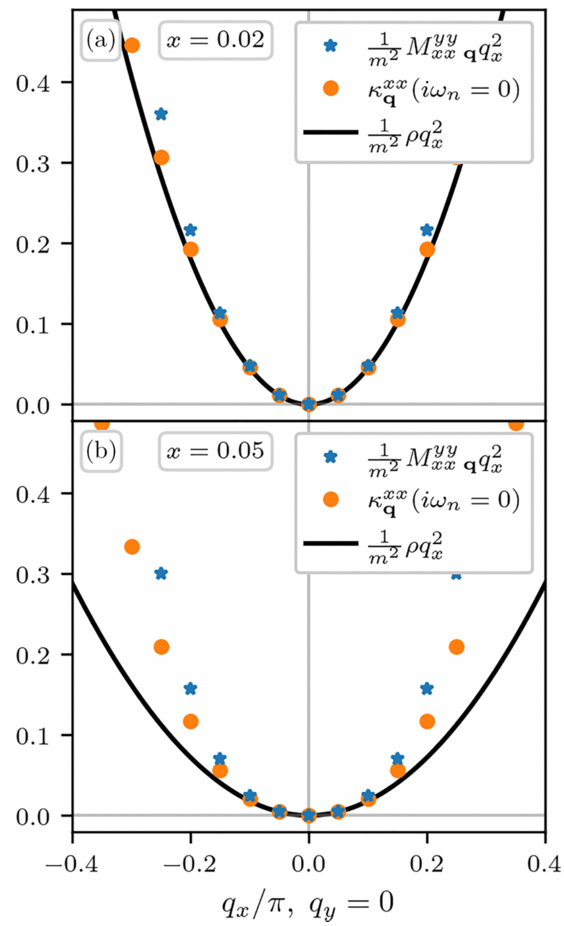

FIG. 2. Wave-vector dependence of the inverse susceptibility κ in the AFM case for hole-doping levels $x = 0.02$ (a) and $x = 0.05$ (b). Orange circles correspond to the result of the DMFT calculation, solid lines correspond to the long-wavelength limit $\mathbf{q} \to 0$, obtained from Ward identities, stars are the respective dependence, obtained with account of the momentum dependence of the gauge kernels.

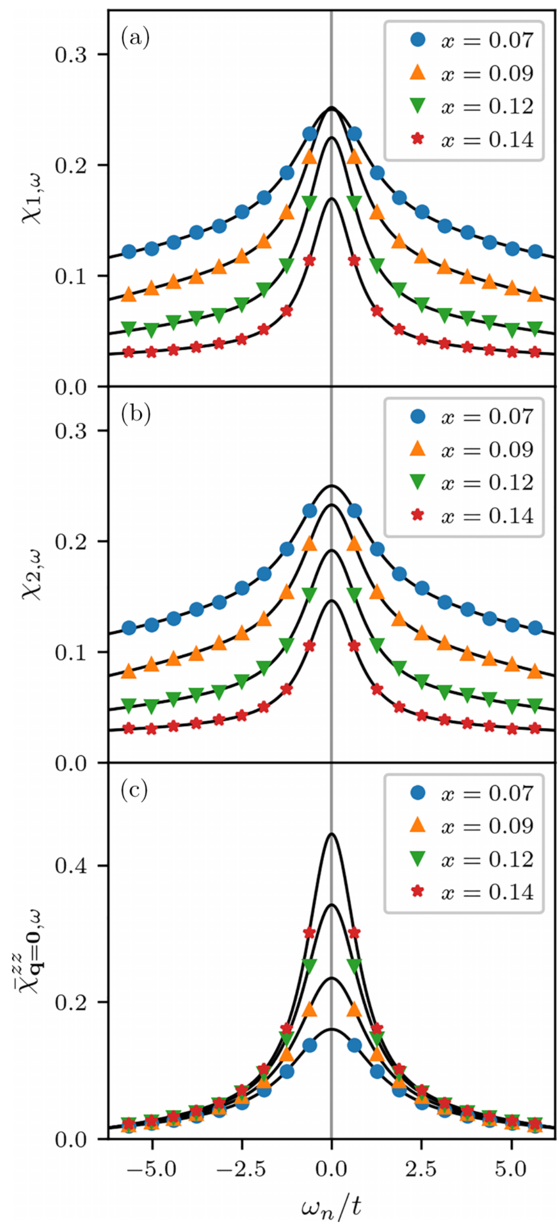

FIG. 3. The frequency dependencies of the out-of-plane $\chi_{1,\omega}$ (a) and in-plane $\chi_{2,\omega}$ (b) temporal stiffnesses in the incommensurate case, as well as the component of the susceptibility $\bar{\chi}^{zz}_{\mathbf{0},\omega}$ (c) for various fillings.

C. DMFT approach in the incommensurate phase

According to the Eq. (31) in the incommensurate case the frequency dependence of the susceptibilities is characterized by two temporal stiffnesses $\chi_{1,2,\omega}$ and the uniform dynamic susceptibility $\bar{\chi}^{zz}_{\mathbf{0},\omega}$ in the local coordinate frame. Using our definition of Eq. (28) we extract temporal spin stiffnesses as $\chi_{1,\omega} = m^2/(\chi^{yy}_{\mathbf{Q},\omega_n}\omega_n^2)$ and $\chi_{2,\omega_n} = 2m^2/(\chi^{xx}_{\mathbf{0},\omega_n}\omega_n^2)$. The results of the numerical calculations are shown in Fig. 3. We find that in the incommensurate phase away from half-filling the frequency dependence of temporal stiffnesses becomes even more essential than in the AFM phase. With the increase of doping the temporal stiffnesses $\chi_{1,2}(\omega_n)$ develop a sharp peak at $\omega_n = 0$, with different height for longitudinal and transverse channels. The susceptibility $\bar{\chi}^{zz}_{\mathbf{0},\omega}$ (together with $\bar{\chi}^{+-}_{2\mathbf{Q},\omega}$ component, not shown) continuously increases with doping, and its zero-frequency component diverges at the incommensurate magnetic to paramagnetic transition, which provides a possibility of passing from Eq. (35) to (39) and formation of a single soft mode on the paramagnetic side. We have verified that the components of the susceptibility $\bar{\chi}_{\mathbf{0},\omega}$ and $\bar{\chi}_{\pm\mathbf{Q},\omega}$ in the local reference frame, computed within DMFT, coincide with the corresponding analytical results (30), including the off-diagonal contributions to the local susceptibility $\bar{\chi}^{\pm y}$, which have parameter-free analytic form $\pm m/\omega$.

The momentum dependencies of inverse susceptibilities are shown and compared to the results of Ward identities in Fig. 4. It can be seen that in the incommensurate case two Goldstone modes result in four different spatial stiffnesses, corresponding to the in-plane and out-of-plane fluctuations along q_x and q_y directions. The obtained spin stiffnesses appear to be equal to the long-wavelength limit of the current correlation functions $M_{\mathbf{q}\to 0}$. At finite doping the susceptibility is not well described by quadratic dependence, apart from the small-$|\mathbf{q}|$ values. Also in the incommensurate case using the momentum-dependent current-current correlation functions

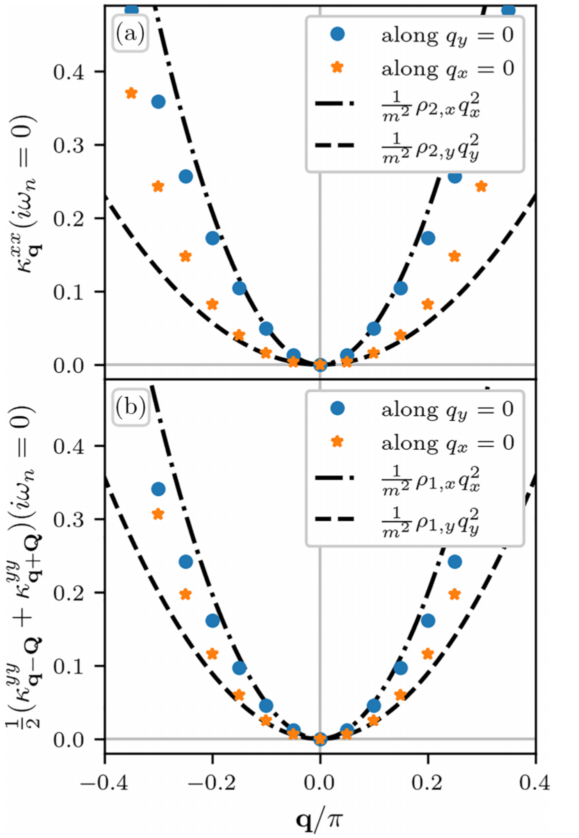

FIG. 4. Wave-vector dependence of the in-plane $\bar{\kappa}^{xx}$ (a) and out-of-plane $\bar{\kappa}^{yy}$ (b) components of the static inverse susceptibility in the local reference frame in the incommensurate case for hole-doping levels $x = 0.13$. Blue circles and orange stars correspond to the DMFT result in different directions, dashed-dotted and dotted lines show the long-wavelength limit $\mathbf{q} \to 0$, following from the Ward identities.

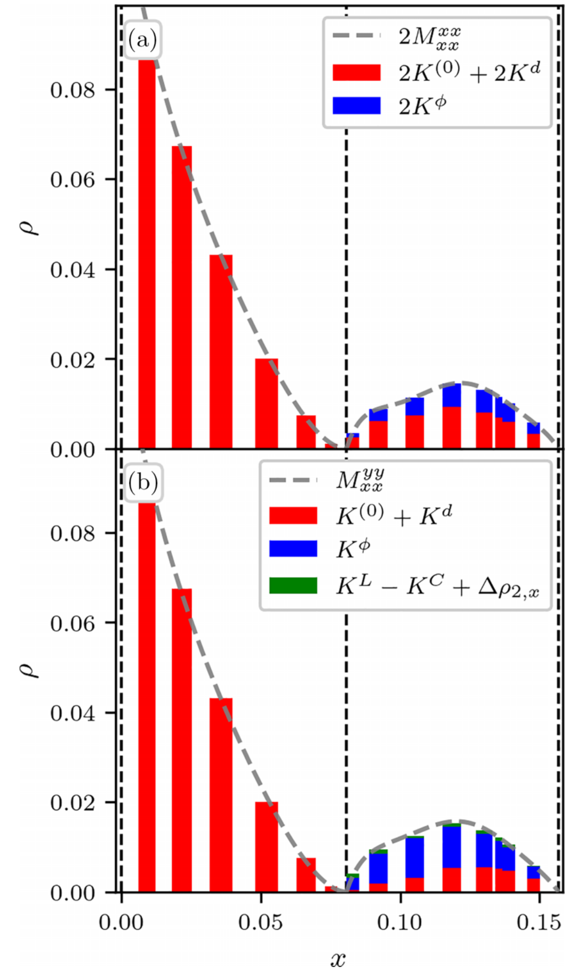

FIG. 5. The bar graph, showing various contributions to the spatial out-of-plane spin stiffness $\rho_{1,x}$ (a) and the in-plane spin stiffness $\rho_{2,x}$ (b) Red part is the sum of bare paramagnetic and diamagnetic contributions, blue part is the U-irreducible vertex correction K^{ϕ}, the remaining green part, which originates from the U-reducible vertex correction $K^L - K^C$, together with the contribution $\Delta\rho_{2,n}$, is negligibly small for the in-plane mode and vanishes for the out-of-plane mode. Vertical dashed lines mark the commensurate-incommensurate and incommensurate-paramagnetic transitions.

$M_{\mathbf{q}}$ instead of the respective spin stiffnesses does not significantly improve the description of the momentum dependence, in contrast to the commensurate case. This feature might be a consequence of the suppression of bubble contributions and nonvanishing vertex corrections in the incommensurate case (see below).

As well as in the commensurate case, the vertex corrections $K^{\phi,L,C}_{\mathbf{0},0,yy}$ and $\Delta\rho_{2,y}$ to the respective spin stiffnesses $\rho_{n,y}$, determined by the components of the modified kernel M_{yy}, vanish by symmetry since they are related to the components of the current along the direction of $Q_y = \pi$ component. We also find vanishing of vertex corrections $K^{L,xx}_{\mathbf{0},0,xx} - K^{C,xx}_{\mathbf{0},0,xx}$ to the respective spin stiffnesses $\rho_{1,x}$. The respective contributions to the M_{xx} components of the modified kernel at $i\omega_n = 0$, which determine $\rho_{1,x}$ and $\rho_{2,x}$ stiffnesses, are plotted as a functions of x in Fig. 5. We find the U-reducible contribution $K^L - K^C + \Delta\rho_{2,n}$ (containing also the contribution of the z component $\Delta\rho_{2,n}$) is negligibly small. Despite partial cancellation of the bubble part of paramagnetic response K^0_q and its diamagnetic part K^d, for M^{xx}_{xx} component these bare contributions constitute substantial part of the spatial spin stiffness. Yet, the U-irreducible vertex correction contribution K^{ϕ}, which occurs due to dynamic effects, is also nonzero in this case, and provides approximately $\frac{1}{3}$ to $\frac{1}{4}$ amount of the respective spin stiffness. On the other hand, for M^{yy}_{xx} component in the incommensurate case the U-irreducible vertex part provides the major part of the spin stiffness. Therefore, neglecting dynamic vertex corrections in that case would yield a dramatic underestimate of spin stiffness.

Finally, we provide the doping dependence of the respective spatial spin stiffnesses in Fig. 6. One can see that in the DMFT approach with the deviation from half-filling the spin stiffnesses first decrease and vanish at the commensurate-incommensurate transition. This vanishing of spin stiffness can be easily understood considering the incommensurate case in the vicinity of such a transition, where Goldstone modes are present at both wave vectors $\mathbf{q} = \pm\mathbf{Q}$ in the global reference frame [see Eq. (35)]. Therefore, approaching the commensurate phase necessarily yields q^4 behavior of the inverse susceptibility, which implies vanishing of the spin

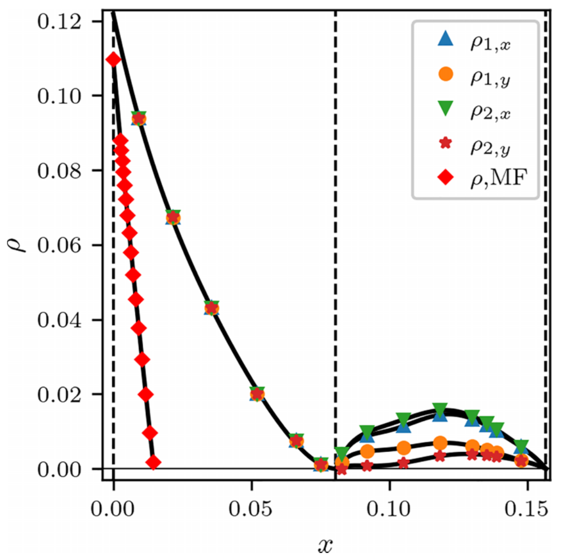

FIG. 6. Spatial stiffness as a function of doping level x in DMFT (triangles, circles, stars) and mean-field (rhombus) approaches. Vertical dashed lines mark the commensurate-incommensurate and incommensurate-paramagnetic transitions.

stiffness (see Appendix F). With further increase of doping the spin stiffnesses first increase, and then decrease again to zero at the incommensurate magnetic-paramagnetic transition, where their smallness compensates the smallness of the numerator in Eq. (39).

For comparison, we show the results of the mean-field approach in the antiferromagnetic phase, where stable magnetic excitations exist in this approach. One can see that the mean-field spin stiffness quickly decreases with doping, and vanishes at a few percent of doping. At larger dopings, we were not able to find stable incommensurate magnetic solution, in agreement with earlier results [38–41]. Quick disappearance of static long-range order in mean-field theory is reminiscent of the experimental data on high-T_c compounds. At the same time, the persistence of stable magnetic excitations in dynamical mean-field theory shows that dynamic nature of electronic correlations allows for preserving incommensurate magnetic order in the chargon sector in a broad doping range. With account of spinon fluctuations (see, e.g., Ref. [15]), this corresponds to the *short-range* magnetic order of the entire system. We note that strong suppression of correlation length near the commensurate-incommensurate transition was observed for $La_{2-x}Sr_xCuO_4$ in Ref. [50], which agrees with the obtained suppression of the spatial spin stiffnesses near this transition. Therefore, the dynamical mean-field theory appears more appropriate for description of short-range magnetic order.

V. CONCLUSIONS

In summary, we have derived corrected Ward identities for the frequency and momentum dependence of the susceptibilities in the commensurate and incommensurate magnetic phases. It was shown that Ward identities relate the temporal spin stiffnesses, which are in general dynamic, and spatial stiffnesses to the respective quantities, which can be extracted from the microscopic analysis. The obtained results for the spatial spin stiffnesses contain contributions of the bare paramagnetic and diamagnetic terms, as well as the two types (U-irreducible K^ϕ and U-reducible K^L) of vertex corrections, together with two additional correction terms K^C and $\Delta\rho$.

We have verified obtained identities numerically in the framework of dynamic mean-field theory for the two-dimensional Hubbard model with nearest- and next-nearest-neighbor hopping in the antiferromagnetic and incommensurate cases. In particular, we have obtained temporal and spatial spin stiffnesses for various hole dopings. Several important results were obtained. First, with increasing of doping, temporal stiffnesses acquire strong frequency dependence, their static limit differs significantly from the high-frequency asymptotics. Second, spatial stiffness was found to vanish at the transition from antiferromagnetic to the state with spiral magnetic order. Third, our calculations show that although there are no vertex corrections to the spatial spin stiffnesses in the antiferromagnetic case, these corrections are finite in the incommensurate state. Their effect is different for the different excitation modes and different spatial directions. Vertex corrections are absent along commensurate direction, while they are finite along incommensurate direction. U-irreducible dynamic vertex corrections contribute significantly to the spatial spin stiffness, while the effect of the U-reducible vertex corrections was found to be numerically small. The obtained vertex corrections to the spatial spin stiffness, which are absent in the static mean-field theory, seem to be the reason of stabilization of incommensurate long-range magnetic order in the chargon sector, obtained previously in Ref. [30]. Therefore, strong dynamic effects are essential for stabilization of magnetic order, especially incommensurate, at significant doping levels.

The incommensurate long-range magnetic order in the chargon sector, obtained in this study, is observed as the *short-range* magnetic order in cuprate high-T_c compounds. With obtained spin stiffnesses this short-range order can be straightforwardly described, considering fluctuations in the spinon sector according to the $1/N$ expansion for CP^{N-1} model [15,16,51–53] or $O(N)/[O(N-2)\times O(2)]$ nonlinear sigma model [54,55] for magnetic degrees of freedom. This represents the topic for the forthcoming study. The performed study shows that the *dynamic* effects in the chargon sector, as well as frequency dependence of the resulting temporal spin stiffness, are crucial for this study. The generalization and application of the obtained results to the frustrated magnetic systems is also of certain interest. The derivation of corrected Ward identities for systems with broken symmetry can also be applied for the superconductivity case in the future studies.

Note added in Proof. Recently, we were informed about the Erratum [55] to Ref. [28], which also suggests introducing small external, generally nonuniform, magnetic field for derivation of the corrected form of Ward identities for both, commensurate and incommensurate types of magnetic order.

ACKNOWLEDGMENT

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APPENDIX A: GENERAL DEFINITIONS AND DERIVATION OF THE CURRENT, SPIN SUSCEPTIBILITIES, AND GAUGE KERNEL

We use the definition of spin operators

$$S_i^\alpha = \tfrac{1}{2} c_i^+ \sigma^\alpha c_i. \tag{A1}$$

After the Fourier transformations

$$c_k = \sum_i e^{-\mathrm{i}kr_i} c_i, \;\; c_k^+ = \sum_i e^{\mathrm{i}kr_i} c_i^+ \tag{A2}$$

this results in the following expression for Fourier components:

$$S_q^\alpha = \sum_i e^{-\mathrm{i}qr_i} S_i^\alpha = \sum_k c_k^+ \frac{\sigma^\alpha}{2} c_{k+q}, \tag{A3}$$

and $S_q^{\alpha+} = S_{-q}^\alpha$. The spin susceptibility

$$\chi_{q,q'}^{\alpha,\beta} = -\langle\langle S_q^\alpha | S_{-q'}^\beta \rangle\rangle = \langle S_q^\alpha S_{-q'}^\beta \rangle - \delta_{q,0}\delta_{q',0}\langle S_{q=0}^\alpha \rangle \langle S_{q'=0}^\beta \rangle \tag{A4}$$

reads as

$$\begin{aligned}\chi_{q,q'}^{\alpha,\beta} &= -\frac{1}{4}\sum_{k,k'} \langle\langle c_k^+ \sigma^\alpha c_{k+q} | c_{k+q'}^+ \sigma^\beta c_k \rangle\rangle \\ &= \frac{1}{4}\sum_{k,k'} \big[\langle c_{k,\sigma}^+ \sigma_{\sigma,\sigma'}^\alpha c_{k+q,\sigma'} c_{k'+q',\sigma'''}^+ \sigma_{\sigma''',\sigma''}^\beta c_{k',\sigma''}\rangle \\ &\quad - \delta_{q,0}\delta_{q',0}\delta_{\sigma,\sigma'}\delta_{\sigma''',\sigma''}\langle c_{k,\sigma}^+ \sigma_{\sigma,\sigma'}^\alpha c_{k,\sigma'}\rangle \\ &\quad \times \langle c_{k',\sigma'''}^+ \sigma_{\sigma''',\sigma''}^\beta c_{k',\sigma''}\rangle\big].\end{aligned} \tag{A5}$$

The current operators are defined similarly to the U(1) case (see, e.g., Ref. [31]). We start with the respective part of the action

$$\mathcal{S}_0 = -\sum_{ij} c_i^+ t_{ij} \mathcal{R}_i^+ \mathcal{R}_j c_j \tag{A6}$$

and rewrite it in the form where Hermiticity is explicit and $j = i + \Delta_m$, where m indicated direction $m \in x, y$, assuming $t_{\Delta_m} = t_{-\Delta_m}$

$$\begin{aligned}\mathcal{S}_0 = &-\frac{1}{2}\sum_i \sum_{\Delta_m} t_{\Delta_m}(c_{i+\Delta_m}^+ \mathcal{R}_{i+\Delta_m}^+ \mathcal{R}_i c_i \\ &+ c_i^+ \mathcal{R}_i^+ \mathcal{R}_{i+\Delta_m} c_{i+\Delta_m}).\end{aligned} \tag{A7}$$

This part of the action can be rewritten in the form

$$\mathcal{S}_0 = -\frac{1}{2}\sum_i \sum_\Delta t_\Delta c_i^+ [e^{\Delta_m(\overleftarrow{\partial}_m + \mathrm{i}A_{im})} + e^{\Delta_m(\overrightarrow{\partial}_m - \mathrm{i}A_{im})}] c_i, \tag{A8}$$

where Δ enumerates all neighbors connected by finite hopping t_Δ, $\overleftarrow{\partial}_m$ and $\overrightarrow{\partial}_m$ are generators of left and right lattice translations in the directions $m = x, y$, respectively. A_{im} is defined according to (3).

If we consider only slowly varying fields A_{im} we can neglect the contributions coming from derivatives of A_{im} fields. In this case action simplifies to the usual form of Peierls substitution,

$$\mathcal{S}_0 \approx -\frac{1}{2}\sum_i \sum_\Delta t_\Delta (c_{i+\Delta}^+ e^{+\mathrm{i}\Delta_m A_{im}} c_i + c_i^+ e^{-\mathrm{i}\Delta_m A_{im}} c_{i+\Delta}). \tag{A9}$$

It can be further expanded in series with respect to the powers of A_{im} field

$$\begin{aligned}\mathcal{S}_0 = &-\frac{1}{2}\sum_{i,\Delta} t_\Delta (c_{i+\Delta}^+ c_i + c_i^+ c_{i+\Delta}) \\ &-\frac{\mathrm{i}}{2}\sum_i\sum_\Delta t_\Delta \Delta_m (c_{i+\Delta}^+ A_{im} c_i - c_i^+ A_{im} c_{i+\Delta}) \qquad \text{(A10)} \\ &+\frac{1}{4}\sum_i\sum_\Delta t_\Delta \Delta_m \Delta_n (c_{i+\Delta}^+ A_{im} A_{i,n} c_i \\ &+ c_i^+ A_{im} A_{i,n} c_{i+\Delta}) + \cdots .\end{aligned} \tag{A11}$$

Then we get current operator which is explicitly Hermitian,

$$J_{im}^\alpha = \frac{\mathrm{i}}{2}\sum_\Delta t_\Delta \Delta_m \left(c_{i+\Delta}^+ \frac{\sigma^\alpha}{2} c_i - c_i^+ \frac{\sigma^\alpha}{2} c_{i+\Delta} \right), \tag{A12}$$

with the respective Fourier transform

$$J_{q,m}^\alpha = \frac{1}{2}\sum_k \big(t_{\mathbf{k}}^m - t_{-\mathbf{k}-\mathbf{q}}^m\big) c_k^+ \frac{\sigma^\alpha}{2} c_{k+q}, \tag{A13}$$

where $t_{\mathbf{k}}^m = -\mathrm{i}\sum_{r_{ij}} t_{ij} r_{ij}^m e^{-\mathrm{i}kr_{ij}} = \partial\epsilon_{\mathbf{k}}/\partial k^m$, $\epsilon_{\mathbf{k}} = \sum_{r_{ij}} e^{-\mathrm{i}kr_{ij}} t_{ij}$. In the presence of inversion symmetry $\epsilon_{\mathbf{k}} = \epsilon_{-\mathbf{k}}$, $t_{-\mathbf{k}}^m = -t_{\mathbf{k}}^m$, and

$$J_{q,m}^\alpha = \sum_k T_{\mathbf{k},\mathbf{q}}^m c_k^+ \frac{\sigma^\alpha}{2} c_{k+q}, \tag{A14}$$

where $T_{\mathbf{k},\mathbf{q}}^m = (t_{\mathbf{k}}^m + t_{\mathbf{k}+\mathbf{q}}^m)/2$. It can be easily verified that $J_{q,m}^{\alpha+} = J_{-q,m}^\alpha$, as it should be for a Fourier transform of a Hermitian operator.

Defining $T_{\mathbf{k},\mathbf{q}}^0 = \mathrm{i}$, the gauge kernel (including its temporal, i.e., spin components) then takes the form

$$\begin{aligned}K_{q,q';\mu,\nu}^{\alpha,\beta} &= \sum_j \int d\tau \frac{\delta W}{\delta A_{i,\mu}^\alpha \delta A_{j,\nu}^\beta} e^{iq(x_j - x_i)} \\ &= \langle\langle J_{q,\mu}^\alpha | J_{-q',\nu}^\beta \rangle\rangle + K_{\mu\nu}^d,\end{aligned}$$

where

$$\begin{aligned}&\langle\langle J_{q,\mu}^\alpha | J_{-q',\nu}^\beta \rangle\rangle \\ &\quad = -\frac{1}{4}\sum_{k,k'} T_{\mathbf{k},\mathbf{q}}^\mu \langle c_{k,\sigma}^+ \sigma_{\sigma,\sigma'}^\alpha c_{k+q,\sigma'} c_{k'+q',\sigma'''}^+ \sigma_{\sigma''',\sigma''}^\beta c_{k',\sigma''}\rangle T_{\mathbf{k}',\mathbf{q}'}^\nu,\end{aligned} \tag{A15}$$

$$K_{\mu\nu}^d = -\frac{1}{4}\sum_k t_{\mathbf{k}}^{\mu\nu} \langle c_k^+ \sigma^0 c_k \rangle (1 - \delta_{\mu 0})(1 - \delta_{\nu 0}) \tag{A16}$$

are the paramagnetic and diamagnetic contributions to the current correlation function $t_{\mathbf{k}}^{mn} = \partial^2 \epsilon_{\mathbf{k}}/(\partial k_m \partial k_n)$.

APPENDIX B: GAUGE TRANSFORMATION AND WARD IDENTITIES

The dependence of the action on the matrix $\mathcal{R}$ has the form

$$S[c, c^+, \mathcal{R}] = \sum_{ij} \int_0^\beta d\tau\, c_i^+ \left[\left(\frac{\partial}{\partial \tau} - \mu + \mathcal{R}_i^+ \frac{\partial}{\partial \tau} \mathcal{R}_i \right) \delta_{ij} - t_{ij} \mathcal{R}_i^+ \mathcal{R}_j \right] c_j + U \sum_i \int_0^\beta d\tau\, n_{i\uparrow} n_{i\downarrow}. \quad \text{(B1)}$$

The gauge transformation parametrized by SU(2) field $\mathcal{V}_i(\tau)$ applied to the fields A_μ reads as

$$A_{\mu i} \to \mathcal{V}_i A_{\mu i} \mathcal{V}_i^+ + \mathrm{i} \mathcal{V}_i \partial_\mu \mathcal{V}_i^+. \quad \text{(B2)}$$

For infinitesimal transformations we obtain Eq. (8) of the main text.

1. Ward identities for the functional *W*

The Ward identities are derived from the condition $\delta W[\mathcal{R}[\mathcal{V}]] = 0$. The main Ward identity for the functional W was derived in Ref. [28]. The variation of the functional W takes the form (integration over imaginary time and summation over lattice indices are implicitly assumed)

$$\delta W = \frac{\delta W}{\delta A_\mu^a} \frac{\delta A_\mu^a}{\delta \mathcal{V}^b} \delta \mathcal{V}^b + \frac{\delta W}{\delta A_\mu^a} \frac{\delta A_\mu^a}{\delta (\partial_\nu \mathcal{V}^b)} \delta(\partial_\nu \mathcal{V}^b). \quad \text{(B3)}$$

The variational derivatives of the fields $A_{\mu i}(\tau)$ over gauge field $\mathcal{V}^a$ take the form

$$\frac{\delta A_\mu^a}{\delta(\partial_\nu \mathcal{V}^b)} = -\delta_{a,b} \delta_{\mu,\nu}, \quad \frac{\delta A_\mu^a}{\delta \mathcal{V}^b} = \varepsilon_{abc} A_\mu^c. \quad \text{(B4)}$$

Then the condition $\delta W = 0$ yields

$$\partial_\mu \left(\frac{\delta W}{\delta A_\mu^a} \right) - \varepsilon_{abc} \frac{\delta W}{\delta A_\mu^b} A_\mu^c = 0. \quad \text{(B5)}$$

Equation (B5) coincides with that derived in Ref. [28]. It describes motion of spins in the external magnetic and current fields. We note that this equation does not contain explicitly the internal (mean) fields. By differentiating over A_ν^d we obtain the equation

$$\partial_{\mu,x} K_{\mu,x;\nu,x'}^{ad} + \epsilon_{acb} A_{\mu,x}^c K_{\mu,x;\nu,x'}^{bd} + \varepsilon_{abd} j_{\nu,x}^b \delta_{x,x'} = 0, \quad \text{(B6)}$$

where $j_{\nu,x}^b = -\delta W / \delta A_{\nu,x}^b$ is the spin current (including the $\nu = 0$ spin density component). For $\nu = n > 0$ the last term in the left-hand side vanishes in the equilibrium and we find Eq. (9) of the main text. Restricting the nonzero component of the gauge field in the equilibrium to $A_{0,x}$ only (which is proportional to the external nonuniform magnetic field), setting $\nu = 0$ in Eq. (B6), we find Eq. (10) of the main text.

In the commensurate antiferromagnetic case the transverse susceptibilities χ^{ab} with $a, b = x, y$ are decoupled from χ^{zz}. However, the diagonal susceptibilities $\chi^{xx,yy}$ are coupled to the off-diagonal ones $\chi^{xy,yx}$ by the dynamic terms. The general form of the transverse susceptibility at $\mathbf{q}, \mathbf{q}' = \mathbf{0}, \mathbf{Q}$, allowed by the Ward identity (10), together with the conjugated one, obtained by the interchange $x \leftrightarrow x'$ in the basis $a, b = x, y$ takes the form

$$\chi_{\mathbf{q}\mathbf{q}',\omega}^{ab} = \frac{1}{\omega^2} \left(\begin{array}{cc:cc} h r_{\mathbf{Q},\omega}^y & h^2 \chi_{\mathbf{Q},\omega}^{yx} & -\omega h \chi_{\mathbf{Q},\omega}^{yx} & \omega r_{\mathbf{Q},\omega}^y \\ h^2 \chi_{\mathbf{Q},\omega}^{xy} & h r_{\mathbf{Q},\omega}^x & -\omega r_{\mathbf{Q},\omega}^x & \omega h \chi_{\mathbf{Q},\omega}^{xy} \\ \hdashline \omega h \chi_{\mathbf{Q},\omega}^{xy} & \omega r_{\mathbf{Q},\omega}^x & \omega^2 \chi_{\mathbf{Q},\omega}^{xx} & \omega^2 \chi_{\mathbf{Q},\omega}^{xy} \\ -\omega r_{\mathbf{Q},\omega}^y & -\omega h \chi_{\mathbf{Q},\omega}^{yx} & \omega^2 \chi_{\mathbf{Q},\omega}^{yx} & \omega^2 \chi_{\mathbf{Q},\omega}^{yy} \end{array} \right), \quad \text{(B7)}$$

where $r_{\mathbf{q},\omega}^a = m - h \chi_{\mathbf{q},\omega}^{aa}$, $\chi_{\mathbf{Q},\omega}^{ab} \equiv \chi_{\mathbf{QQ},\omega}^{ab}$ and the 2×2 blocks correspond to the values $\mathbf{q}, \mathbf{q}' = \mathbf{0}, \mathbf{Q}$. The staggered components $\chi_{\mathbf{Q},\omega}$ are not fixed by Ward identities, and have to be determined from the microscopic theory. In the main text we express them in Eq. (23) through the frequency dependence of χ_ω, which appears to be equal to the uniform transverse susceptibility according to Eq. (24).

2. Ward identities for the Legendre transformed functional Γ

Using Eq. (13), we obtain the Ward identity

$$\partial_\mu \left(\frac{\delta \Gamma}{\delta A_{\mu,x}^a} \right) - \varepsilon_{abc} \left[\frac{\delta \Gamma}{\delta A_{m,x}^b} A_{m,x}^c - \phi_x^b \left(i A_0^c + \frac{\delta \Gamma}{\delta \phi_x^c} \right) \right] = 0, \quad \text{(B8)}$$

which is equivalent to that used in Refs. [15,28,29]:

$$\partial_\mu \left(\frac{\delta \Gamma}{\delta A_{\mu,x}^a} \right) - \varepsilon_{abc} \left(\frac{\delta \Gamma}{\delta A_{\mu,x}^b} A_{\mu,x}^c + \frac{\delta \Gamma}{\delta \phi_x^b} \phi_x^c \right) = 0. \quad \text{(B9)}$$

Differentiating over $\phi_{x'}^d$ and $A_{n,x'}^d$ we obtain

$$\partial_m \frac{\delta^2 \Gamma}{\delta A_{m,x}^a \delta \phi_{x'}^d} + \varepsilon_{abc} \phi_x^b \frac{\delta^2 \Gamma}{\delta \phi_x^c \delta \phi_{x'}^d} = i \left(\partial_\tau \delta_{ad} - i \varepsilon_{adc} A_{0,x}^c \right) \delta_{xx'}, \quad \text{(B10)}$$

$$\partial_m \frac{\delta^2 \Gamma}{\delta A_{m,x}^a \delta A_{n,x'}^d} + \varepsilon_{abc} \phi_x^b \frac{\delta^2 \Gamma}{\delta \phi_x^c \delta A_{n,x'}^d} = 0. \quad \text{(B11)}$$

From Eqs. (B10) and (B11) we obtain the Ward identity (18) for the second derivatives.

APPENDIX C: MEAN-FIELD THEORY AT HALF-FILLING

As an example demonstrating the importance of the external staggered field for obtaining correct uniform susceptibility we consider the mean-field approximation for considering the chargon sector at half-filling. The mean-field equations were derived in Refs. [56,57]; in Ref. [30] it was shown how they can be obtained in the local reference frame; they read as

$$m = \frac{1}{2} \sum_{\mathbf{k}} \frac{\Delta}{E_{\mathbf{k}}} \left[f\left(E_{\mathbf{k}}^v\right) - f\left(E_{\mathbf{k}}^c\right) \right],$$
$$n = \sum_{\mathbf{k}} \left[f\left(E_{\mathbf{k}}^v\right) + f\left(E_{\mathbf{k}}^c\right) \right] = 1, \quad \text{(C1)}$$

where $\Delta = Um + h$, $E_{\mathbf{k}} = \sqrt{(\varepsilon_{\mathbf{k}}^-)^2 + \Delta^2}$, $\varepsilon_{\mathbf{k}}^\pm = (\varepsilon_{\mathbf{k}} \pm \varepsilon_{\mathbf{k}+\mathbf{Q}})/2$, $E_k^{c,v} = \varepsilon_{\mathbf{k}}^+ \pm E_{\mathbf{k}} - \mu$, h is the external staggered

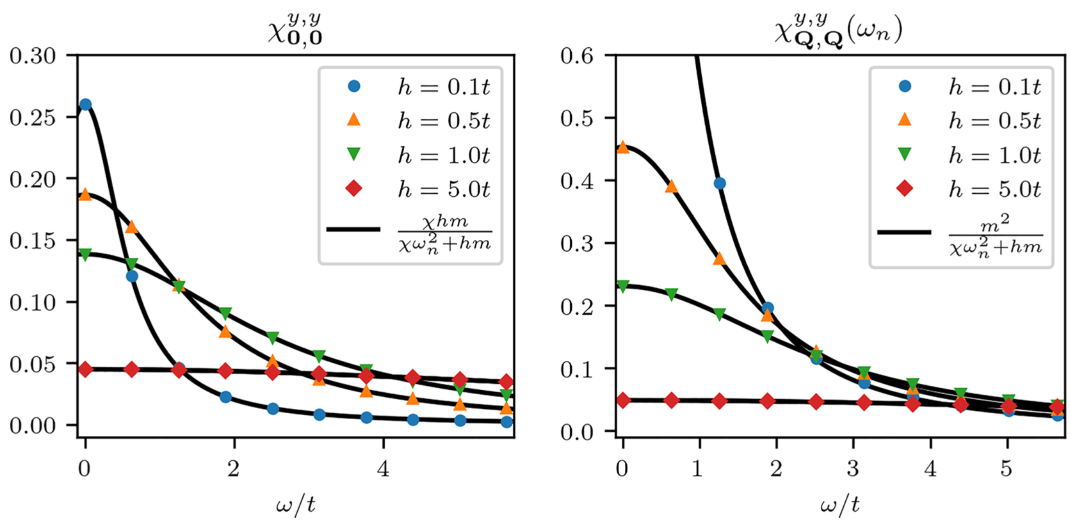

FIG. 7. The frequency dependence of the uniform χ^{yy}_{00} and $\chi^{yy}_{\mathbf{QQ}}$ transverse susceptibilities for the half-filled Hubbard model in the mean-field approximation.

magnetic field. The transverse susceptibility is given by [56,57]

$$\chi^{+-}_{\mathbf{qq},\omega} = \frac{\chi^{+-,0}_{\mathbf{qq},\omega}\left(1-U\chi^{+-,0}_{\mathbf{q+Q},\mathbf{q+Q},\omega}\right)+U\left(\chi^{+-,0}_{\mathbf{q},\mathbf{q+Q},\omega}\right)^2}{\left(1-U\chi^{+-,0}_{\mathbf{qq},\omega}\right)\left(1-U\chi^{+-,0}_{\mathbf{q+Q},\mathbf{q+Q},\omega}\right)-U^2\left(\chi^{+-,0}_{\mathbf{q},\mathbf{q+Q},\omega}\right)^2}, \tag{C2}$$

$$\chi^{+-}_{\mathbf{q},\mathbf{q+Q},\omega} = \frac{\chi^{+-,0}_{\mathbf{q},\mathbf{q+Q},\omega}}{\left(1-U\chi^{+-,0}_{\mathbf{qq},\omega}\right)\left(1-U\chi^{+-,0}_{\mathbf{q+Q},\mathbf{q+Q},\omega}\right)-U^2\left(\chi^{+-,0}_{\mathbf{q},\mathbf{q+Q},\omega}\right)^2}, \tag{C3}$$

where

$$\chi^{+-,0}_{\mathbf{qq},\omega} = \frac{1}{4}\sum_{\mathbf{k}}\left(1-\frac{\varepsilon^-_{\mathbf{k}}\varepsilon^-_{\mathbf{k+q}}-\Delta^2}{E^-_{\mathbf{k}}E^-_{\mathbf{k+q}}}\right) \times\left[\frac{f(E^v_{\mathbf{k}})-f(E^c_{\mathbf{k+q}})}{i\omega-E^c_{\mathbf{k+q}}+E^v_{\mathbf{k}}}+\frac{f(E^c_{\mathbf{k}})-f(E^v_{\mathbf{k+q}})}{i\omega-E^v_{\mathbf{k+q}}+E^c_{\mathbf{k}}}\right], \tag{C4}$$

$$\chi^{+-,0}_{\mathbf{q},\mathbf{q+Q},\omega} = \frac{1}{4}\sum_{\mathbf{k}}\frac{\Delta(E^-_{\mathbf{k}}+E^-_{\mathbf{k+q}})}{E^-_{\mathbf{k}}E^-_{\mathbf{k+q}}} \times\left[\frac{f(E^c_{\mathbf{k}})-f(E^v_{\mathbf{k+q}})}{i\omega-E^v_{\mathbf{k+q}}+E^c_{\mathbf{k}}}-\frac{f(E^v_{\mathbf{k}})-f(E^c_{\mathbf{k+q}})}{i\omega-E^c_{\mathbf{k+q}}+E^v_{\mathbf{k}}}\right], \tag{C5}$$

and $+-$ refer (only in this subsection) to $S^x \pm iS^y$ basis. For $h=0$ we find vanishing susceptibility $\chi^{+-}_{\mathbf{00},\omega}$, which occurs due to cancellation of the first and second terms in the numerator of Eq. (C2). At the same time, for finite staggered field the cancellation does not occur, and in the limit $\omega \to 0$ the first term in the numerator remains finite (being proportional to the staggered field), while the second term, related to the susceptibility $\chi^{+-,0}_{\mathbf{q},\mathbf{q+Q},\omega}$, vanishes. The resulting static uniform susceptibility is given by the random phase approximation (RPA) equation

$$\chi^{+-}_{\mathbf{00},\omega\to 0} = \frac{\chi^{+-,0}_{\mathbf{00},0}}{1-U\chi^{+-,0}_{\mathbf{00},0}}, \tag{C6}$$

in agreement with the derivation of the nonlinear sigma model in Ref. [25] and previous Ward identity approach [28,29]. The latter approach, however, derived Eq. (C6) for *zero* external magnetic field, which, as we explain in the discussion above, is not correct. The reason for this discrepancy is in our opinion missing the off-diagonal terms [like the second terms in the numerator and denominator of Eq. (C2)] in the approach of Refs. [28,29].

In Fig. 7 we show field dependence of several components of dynamical susceptibility for the two-dimensional Hubbard model with $U=7.5t$, $t'/t=0.15$, and their compliance to the form, dictated by Ward identities (24). Note that apart from explicit field dependence in Eq. (24), the static susceptibility χ also implicitly depends on magnetic field, saturating, however, at finite value in the limit $h\to 0$.

APPENDIX D: TRANSFORMATION TO LOCAL COORDINATE FRAME

1. Transformation of the fermion operators and electron Green's functions

The transformation to the local reference frame is obtained by [30]

$$d=\begin{pmatrix} d_\uparrow \\ d_\downarrow\end{pmatrix} = R^\theta\begin{pmatrix} c_\uparrow \\ c_\downarrow\end{pmatrix} = R^\theta c, \tag{D1}$$

where

$$R^\theta = \exp\left(i\frac{\theta}{2}\sigma^y\right) = \begin{pmatrix}\cos(\theta/2) & \sin(\theta/2)\\ -\sin(\theta/2) & \cos(\theta/2)\end{pmatrix}. \tag{D2}$$

The corresponding transformation in momentum space reads as [30]

$$c_{k\uparrow} = (D_{k-\mathbf{Q}/2,+} + D_{k+\mathbf{Q}/2,-})/\sqrt{2}, \tag{D3}$$

$$c_{k\downarrow} = (D_{k-\mathbf{Q}/2,+} - D_{k+\mathbf{Q}/2,-})/(\sqrt{2}i), \tag{D4}$$

where we introduce the operators (Grassmann variables)

$$D_{k,\pm} = (d_{k,\uparrow} \pm i d_{k\downarrow})/\sqrt{2}. \tag{D5}$$

We furthermore define

$$c_{k,\pm} = (c_{k\uparrow} \pm i c_{k\downarrow})/\sqrt{2} = D_{k\mp\mathbf{Q}/2,\pm}.$$

The D operators reduce the single-electron Green's functions to the 2×2 form:

$$\langle\langle c_{k,+}|c^+_{k,+}\rangle\rangle = \langle\langle D_{k-\mathbf{Q}/2,+}|D^+_{k-\mathbf{Q}/2,+}\rangle\rangle = \mathcal{G}_{k-\mathbf{Q/2},++}, \tag{D6}$$

$$\langle\langle c_{k-\mathbf{Q},-}|c^+_{k-\mathbf{Q},-}\rangle\rangle = \langle\langle D_{k-\mathbf{Q}/2,-}|D^+_{k-\mathbf{Q}/2,-}\rangle\rangle = \mathcal{G}_{k-\mathbf{Q}/2,--}, \tag{D7}$$

$$\langle\langle c_{k,+}|c^+_{k-\mathbf{Q},-}\rangle\rangle = \langle\langle D_{k-\mathbf{Q}/2,+}|D^+_{k-\mathbf{Q}/2,-}\rangle\rangle = \mathcal{G}_{k-\mathbf{Q}/2,+-}, \tag{D8}$$

$$\langle\langle c_{k-\mathbf{Q},-}|c^+_{k,+}\rangle\rangle = \langle\langle D_{k-\mathbf{Q}/2,-}|D^+_{k-\mathbf{Q}/2,+}\rangle\rangle = \mathcal{G}_{k-\mathbf{Q}/2,-+}. \tag{D9}$$

The inverse of the lattice Green's function $\mathcal{G}_{k,\alpha\alpha'} = -\langle T D_{k\alpha}(\tau) D^+_{k\alpha'}(0)\rangle$ reads as [30]

$$\mathcal{G}^{-1}_{k,\alpha\alpha'} = \begin{pmatrix} \phi_\nu - \epsilon_{\mathbf{k}+\mathbf{Q}/2} & -(\Sigma_{\nu,\uparrow} - \Sigma_{\nu,\downarrow})/2 \\ -(\Sigma_{\nu,\uparrow} - \Sigma_{\nu,\downarrow})/2 & \phi_\nu - \epsilon_{\mathbf{k}-\mathbf{Q}/2} \end{pmatrix}, \tag{D10}$$

$\phi_\nu = i\nu + \mu - (\Sigma_{\nu,\uparrow} + \Sigma_{\nu,\downarrow})/2$. The above-mentioned relations allow for obtaining transformation rules from global to the local coordinate frame:

$$\begin{aligned} c^+_{k\sigma}\sigma^\pm_{\sigma\sigma'}c_{k+q,\sigma'} &= 2D_{k\pm\mathbf{Q}/2,\mp}D_{k\mp\mathbf{Q}/2+q,\pm} \\ &= d^+_{k\pm\mathbf{Q}/2}\sigma^\pm d_{k\mp\mathbf{Q}/2+q}, \end{aligned} \tag{D11}$$

$$\begin{aligned} c^+_{k\sigma}\sigma^y_{\sigma\sigma'}c_{k+q,\sigma'} &= \sum_{\alpha=\pm}\alpha D_{k+\alpha\mathbf{Q}/2,\alpha}D_{k+\alpha\mathbf{Q}/2+q,\alpha} \\ &= [d^+_{k+\mathbf{Q}/2}(\sigma^y + \sigma^0)d_{k+\mathbf{Q}/2+q} \\ &\quad + d^+_{k-\mathbf{Q}/2}(\sigma^y - \sigma^0)d_{k-\mathbf{Q}/2+q}]/2, \end{aligned} \tag{D12}$$

$$\begin{aligned} c^+_{k\sigma}\sigma^0_{\sigma\sigma'}c_{k+q,\sigma'} &= \sum_{\alpha=\pm} D_{k+\alpha\mathbf{Q}/2,\alpha}D_{k+\alpha\mathbf{Q}/2+q,\alpha} \\ &= [d^+_{k+\mathbf{Q}/2}(\sigma^0 + \sigma^y)d_{k+\mathbf{Q}/2+q} \\ &\quad + d^+_{k-\mathbf{Q}/2}(\sigma^0 - \sigma^y)d_{k-\mathbf{Q}/2+q}]/2, \end{aligned} \tag{D13}$$

where $\sigma^\pm = \sigma^z \pm i\sigma^x$.

2. Transformation of the current and spin operators

Let us consider the transformation of current operators (A14). Let us introduce the notation

$$T^{\mu\pm}_{\mathbf{k},\mathbf{q}} = \frac{T^\mu_{\mathbf{k}-\mathbf{Q}/2,\mathbf{q}} \pm T^\mu_{\mathbf{k}+\mathbf{Q}/2,\mathbf{q}}}{2}. \tag{D14}$$

We obtain the following transformation rule using Eqs. (D11)–(D13):

$$\hat{j}^{\mu 0}_q = \sum_k d^+_k\left(T^{\mu+}_{\mathbf{k},\mathbf{q}}\frac{\sigma^0}{2} + T^{\mu-}_{\mathbf{k},\mathbf{q}}\frac{\sigma^y}{2}\right)d_{k+q}, \tag{D15}$$

$$\hat{j}^{\mu x}_q = \frac{1}{2}\sum_{k,\alpha=\pm} T^\mu_{\mathbf{k}+\alpha\mathbf{Q}/2,\mathbf{q}}d^+_k\left(\frac{\sigma^x}{2} + \alpha i\frac{\sigma^z}{2}\right)d_{k+q+\alpha\mathbf{Q}}, \tag{D16}$$

$$\hat{j}^{\mu y}_q = \sum_k d^+_k\left(T^{\mu-}_{\mathbf{k},\mathbf{q}}\frac{\sigma^0}{2} + T^{\mu+}_{\mathbf{k},\mathbf{q}}\frac{\sigma^y}{2}\right)d_{k+q}, \tag{D17}$$

$$\hat{j}^{\mu z}_q = \frac{1}{2}\sum_{k,\alpha=\pm} T^\mu_{\mathbf{k}+\alpha\mathbf{Q}/2,\mathbf{q}}d^+_k\left(\frac{\sigma^z}{2} - \alpha i\frac{\sigma^x}{2}\right)d_{k+q+\alpha\mathbf{Q}}. \tag{D18}$$

For spin and charge operators we have $T^{\mu=0}_{\mathbf{k},\mathbf{q}} = T^{\mu=0,+}_{\mathbf{k},\mathbf{q}} = \mathrm{i}$, $T^{\mu=0,-}_{\mathbf{k},\mathbf{q}} = 0$.

3. Transformation of the gauge kernel

In the local coordinate frame the momentum is conserved and all correlation functions become diagonal with respect to the wave vectors $\mathbf{q}$, $\mathbf{q}'$, so they depend on the only wave vector $\mathbf{q}$, which greatly simplifies the calculations. Introducing the paramagnetic space-time part of the kernel in the local reference frame

$$\bar{K}^{ab}_{q;\mu\nu,ss'} = \sum_{k,k'} T^{\mu,s}_{\mathbf{k}}\langle\langle \hat{d}^+_k\sigma^a\hat{d}_{k+q}|\hat{d}^+_{k'+q}\sigma^b\hat{d}_{k'}\rangle\rangle T^{\nu,s'}_{\mathbf{k}'},$$
$$a, b = 0, y \tag{D19}$$

$$\bar{K}^{ab}_{q;\mu\nu,ss} = \sum_{k,k'} T^\mu_{\mathbf{k}+s\mathbf{Q}/2} \times \langle\langle \hat{d}^+_k\sigma^a\hat{d}_{k+q+\alpha\mathbf{Q}}|\hat{d}^+_{k'+q+\alpha\mathbf{Q}}\sigma^b\hat{d}_{k'}\rangle\rangle T^\nu_{\mathbf{k}'+s\mathbf{Q}/2},$$
$$a, b = x, z \tag{D20}$$

($s, s' = \pm$), the components of the kernel in the global reference frame can be expressed as given by Eqs. (47) of the main text. On the other hand, the diamagnetic part is expressed as given by Eq. (57) of the main text with

$$\bar{K}^{d,ab}_{\mu\nu,s} = -\delta_{ab}\sum_k T^{\mu\nu,s}_k\langle\hat{d}^+_k\sigma^a\hat{d}_k\rangle, \tag{D21}$$

which is equivalent to Eq. (58).

APPENDIX E: APPLICATION OF WARD IDENTITIES FOR OBTAINING SUSCEPTIBILITIES IN THE INCOMMENSURATE PHASE

In the case of spiral incommensurate order, an explicit form of Ward identities is conveniently written in the local coordinate frame (see Appendix D), where the spins are aligned along the local z axis (see also Refs. [28–30]). The susceptibilities in the local coordinate frame are diagonal with respect to momenta. The corresponding $\mathbf{q} = 0$ block of the susceptibilities in the local reference frame, determined by Ward identity (10) in the basis $S^x_{\mathbf{q}}$, $S^y_{\mathbf{q}}$, $S^z_{\mathbf{q}}$, takes the form

$$\bar{\chi}^{ab}_{\mathbf{q}=\mathbf{0},\omega} = \frac{1}{\omega^2}\begin{pmatrix} \omega^2\bar{\chi}^{xx}_{\mathbf{0},\omega} & \omega\bar{r}^x_{\mathbf{0},\omega} & \omega^2\bar{\chi}^{xz}_{\mathbf{0},\omega} \\ -\omega\bar{r}^x_{\mathbf{0},\omega} & h\bar{r}^x_{\mathbf{0},\omega} & h\omega\bar{\chi}^{xz}_{\mathbf{0},\omega} \\ \omega^2\bar{\chi}^{zx}_{\mathbf{0},\omega} & -h\omega\bar{\chi}^{zx}_{\mathbf{0},\omega} & \omega^2\bar{\chi}^{zz}_{\mathbf{0},\omega} \end{pmatrix}_{x,y,z}, \tag{E1}$$

where $\bar{r}^a_{\mathbf{q},\omega} = m - h\bar{\chi}^{aa}_{\mathbf{q},\omega}$ and the bars stand for the local coordinate frame and the index x, y, z refers to the respective spin reference frame. The blocks at the momenta $\mathbf{q} = \pm\mathbf{Q}$ are written in the basis $S^+_{\mathbf{q}}, S^y_{\mathbf{q}}, S^-_{\mathbf{q}}$ ($S^\pm_{\mathbf{q}} = S^z_{\mathbf{q}} \pm iS^x_{\mathbf{q}}$) as

$$\bar{\chi}^{ab}_{\mathbf{q}=-\mathbf{Q},\omega} = \frac{1}{\omega^2}\begin{pmatrix} ih\omega\bar{\chi}^{+y}_{-\mathbf{Q},\omega} & \omega^2\bar{\chi}^{+y}_{-\mathbf{Q},\omega} & \omega^2\bar{\chi}^{+-}_{-\mathbf{Q},\omega} \\ -i\omega\bar{r}^y_{-\mathbf{Q},\omega} & \omega^2\bar{\chi}^{yy}_{-\mathbf{Q},\omega} & \omega^2\bar{\chi}^{y-}_{-\mathbf{Q},\omega} \\ h\bar{r}^y_{-\mathbf{Q},\omega} & -i\omega\bar{r}^y_{-\mathbf{Q},\omega} & ih\omega\bar{\chi}^{y-}_{-\mathbf{Q},\omega}\end{pmatrix}_{+,y,-}. \tag{E2}$$

The susceptibility $\bar{\chi}_{\mathbf{q}=\mathbf{Q},\omega}$ can be obtained via Hermitian conjugate as $\bar{\chi}_{\mathbf{q}=\mathbf{Q},\omega} = \bar{\chi}^+_{\mathbf{q}=-\mathbf{Q},\omega}$. The components of the susceptibility $\bar{\chi}^{ab}_{\mathbf{0},\omega}$ with $a, b = x, z$ and $\bar{\chi}^{yy}_{\alpha\mathbf{Q},\omega}$ are not fixed by Ward identities, and have to be determined from the microscopic theory. The form of the $\bar{\chi}^{xx}_{\mathbf{0},\omega}$ and $\bar{\chi}^{yy}_{\alpha\mathbf{Q},\omega}$ components is parametrized by the frequency dependence of the temporal stiffnesses in Eq. (28) of the main text. The frequency dependence of the component $\bar{\chi}^{zz}_{\mathbf{0},\omega}$ is determined directly from the microscopic approach since it is nonsingular in the ordered phase, but has a finite discontinuity at $\omega = 0$. Finally, the components $\bar{\chi}^{xz,zx}_{\mathbf{0},\omega}$ vanish by symmetry.

To determine momentum dependencies of susceptibilities, we consider the general form of the susceptibilities [Eqs. (33) and (34) of the main text], which includes the zeroth (charge) components allowed by the symmetry [28,29]. Relating the corresponding coefficients to those in the momentum dependent $\bar{\kappa}_{\mathbf{q},\omega}$ and using again Ward identities (19) for the functional Γ, written in the local coordinate frame, we find equations for the coefficients A_{nl} and B_{nl}:

$$\bar{\kappa}^{yy}_{\pm\mathbf{Q}+\mathbf{q},\omega} = \frac{h}{m} + B_{nl}\frac{q_nq_l}{m} = \frac{h}{m} + \frac{2q_nq_l}{m^2}M^{xx,zz}_{q;nl}, \tag{E3}$$

$$\bar{\kappa}^{xx}_{\mathbf{q},\omega} = \frac{h}{m} + \left(A_{nl} + \frac{1}{d^{0z}\chi^{zz}_0}D_nD_l\right)\frac{q_nq_l}{m} = \frac{h}{m} + \frac{q_nq_l}{m^2}M^{yy}_{q;nl}. \tag{E4}$$

Couplings C_n and D_n can also be computed from the knowledge of current-spin correlation functions. The nonzero off-diagonal components of the susceptibilities are fixed by the identity (10), which takes the form

$$m\delta_{ax} = \tilde{K}^{ay}_{q;0\nu}q_\nu + h\bar{\chi}^{ax}_q, \tag{E5}$$

$$m\delta_{ax} = -\tilde{K}^{ya}_{q;\nu0}q_\nu + h\bar{\chi}^{xa}_q, \tag{E6}$$

where in the kernels $\tilde{K}^{ya}_{q,\nu0}$ and $\tilde{K}^{ay}_{q,0\nu}$ we pass to the local coordinate frame with respect to the spin index a only. Using Eq. (40) of the main text, the identities (E5) and (E6) can be written as

$$q_\nu\tilde{\phi}^{t,ya}_{q;\nu0} = (2Um + h)\bar{\phi}^{xa}_q - m\delta_{a,x},$$
$$\tilde{\phi}^{ay}_{q;0\nu}q_\nu = m\delta_{a,x} - (2Um + h)\bar{\phi}^{ax}_q. \tag{E7}$$

Calculating the derivatives $(\partial_{q_n}\bar{\phi}^{xz}_q)_{q=0}$ and $(\partial_{q_n}\bar{\phi}^{x0}_q)_{q=0}$ from Eq. (33) of the main text, we find

$$(h + 2mU)\left(\partial_{q_n}\bar{\phi}^{xz}_q\right)_{q=0} = \frac{C_n\left(d^{0z}\bar{\chi}^{zz}_0 - 2mU\right) + D_nd^{0z}\bar{\chi}^{z0}_0}{d^{0z} - 4mU^2 - 2d^{0z}U\left(\bar{\chi}^{00}_0 - \bar{\chi}^{zz}_0\right)} = \tilde{\phi}^{t,yz}_{0;n0}, \tag{E8}$$

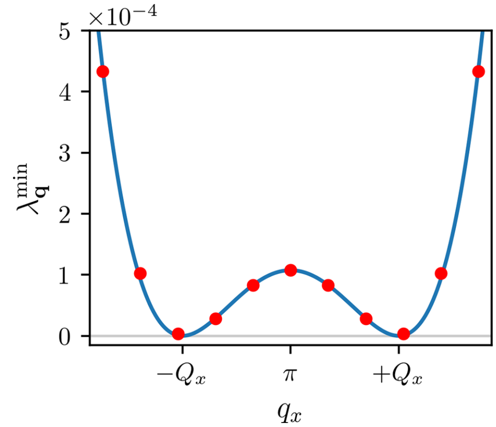

FIG. 8. The form of the dependence $\lambda^{\min}_{\mathbf{q}}$ along the border of the Brillouin zone $\mathbf{q} = (q_x, \pi)$ computed for a state with hole-doping level $x = 0.09$ (points); the line shows the fit by Eq. (F1).

$$(h + 2mU)\left(\partial_{q_n}\bar{\phi}^{x0}_q\right)_{q=0} = \frac{C_nd^{0z}\bar{\chi}^{z0}_0 + D_n\left(d^{0z}\bar{\chi}^{00}_0 + 2mU\right)}{d^{0z} - 4mU^2 - 2d^{0z}U\left(\bar{\chi}^{00}_0 - \bar{\chi}^{zz}_0\right)} = \tilde{\phi}^{t,y0}_{0;n0}. \tag{E9}$$

From this we obtain

$$C_n = \frac{\tilde{\phi}^{t,y0}_{0,n0}d^{0z}\bar{\chi}^{0z}_0 - \tilde{\phi}^{t,yz}_{0,n0}\left(d^{0z}\bar{\chi}^{00}_0 + 2mU\right)}{m},$$
$$D_n = \frac{\tilde{\phi}^{t,yz}_{0,n0}d^{0z}\bar{\chi}^{0z}_0 - \tilde{\phi}^{t,y0}_{0,n0}\left(d^{0z}\bar{\chi}^{zz}_0 - 2mU\right)}{m}. \tag{E10}$$

Using these results and assuming that the kernels $\phi^{yz,y0}_{n0}$ are nonvanishing only along one of the directions $q_{x,y}$ corresponding to the incommensurate direction $Q_{x,y} \neq \pi$, we obtain Eqs. (44) and (46) of the main text.

APPENDIX F: DETERMINATION OF STABLE MAGNETIC CONFIGURATION IN THE CHARGON SECTOR

To determine thermodynamically stable magnetic configuration we require positivity of the spectrum of magnetic excitations, determined by inverse susceptibility in the local coordinate frame $\bar{\chi}^{-1}_{\mathbf{q},i\omega_n}$, given by Eq. (40) of the main text. In particular, the minimal eigenvalue $\lambda^{\min}_{\mathbf{q}}$ of the matrix $1 - U\phi_{\mathbf{q},0}$ should be non-negative everywhere in the Brillouin zone. The points $\mathbf{q} = \pm\mathbf{Q}$ where this eigenvalue is equal to zero determine the ordering wave vector $\mathbf{Q}$ in chargon sector and correspond to Goldstone modes. To find these points we performed the iterative procedure where at every step we have executed a DMFT calculation for a trial magnetic wave vector $\mathbf{Q}^{(n)}$ and the wave vector of next iteration $\mathbf{Q}^{(n+1)}$ was determined by the $\mathbf{q}$ point, which provides $\min_{\mathbf{q}}\lambda^{\min}_{\mathbf{q}}$. This procedure rapidly converged to a stable magnetic wave vector $\mathbf{Q}$.

In Fig. 8 we show the typical form of the obtained momentum dependence of eigenvalues $\lambda^{\min}_{\mathbf{q}}$ along the direction $\mathbf{q} = (q_x, \pi)$. The obtained dependencies can be well approximated by

$$\lambda^{\min}_{\mathbf{q}} = C[(Q_x - \pi)^2 - (q_x - \pi)^2]^2, \tag{F1}$$

where C is a numerical constant. The out-of-plane spatial spin stiffness in the x direction $\rho_{1,x}$ is proportional to the second derivative of minimal eigenvalue

$$\rho_{1,x} \propto \left.\frac{\partial^2 \lambda^{\min}_{\mathbf{q}}}{\partial q_x^2}\right|_{q_x=Q_x} = 8C(\pi - Q_x)^2. \qquad \text{(F2)}$$

For $Q_x \to \pi$ (i.e., when the order in the chargon sector approaches commensurate one) the spin stiffness approaches zero.

The argumentation presented above is directly applicable also to the in-plane spatial spin stiffness in the x direction $\rho_{2,x}$, for which $\lambda^{\min}_{\mathbf{q}}$ should be considered near $\mathbf{q} = \pm(\pi - Q_x, 0)$ points, where the in-plane mode is located in the local coordinate frame. At the incommensurate-commensurate transition full symmetry of the susceptibilities should be restored. Thus, all four spin stiffnesses should become equal. This is only possible if all four spatial spin stiffness components vanish simultaneously. By continuity of the dependence of spatial stiffnesses on the doping level x, at the commensurate-incommensurate transition point at finite doping the spin stiffnesses therefore vanish approaching the transition also from the antiferromagnetic side.

[1] N. Plakida, *High-Temperature Cuprate Superconductors* (Springer, Heidelberg, 2010).

[2] J. J. Wagman, G. Van Gastel, K. A. Ross, Z. Yamani, Y. Zhao, Y. Qiu, J. R. D. Copley, A. B. Kallin, E. Mazurek, J. P. Carlo, H. A. Dabkowska, and B. D. Gaulin, Two-dimensional incommensurate and three-dimensional commensurate magnetic order and fluctuations in La_2BaCuO_4, Phys. Rev. B **88**, 014412 (2013).

[3] S.-W. Cheong, G. Aeppli, T. E. Mason, H. Mook, S. M. Hayden, P. C. Canfield, Z. Fisk, K. N. Clausen, and J. L. Martinez, Incommensurate magnetic fluctuations in La_2SrCuO_4, Phys. Rev. Lett. **67**, 1791 (1991).

[4] T. E. Mason, G. Aeppli, S. M. Hayden, A. P. Ramirez, and H. A. Mook, Low energy excitations in superconducting $La_{1.86}Sr_{0.14}CuO_4$, Phys. Rev. Lett. **71**, 919 (1993).

[5] M. Matsuda, K. Yamada, Y. Endoh, T. R. Thurston, G. Shirane, R. J. Birgeneau, M. A. Kastner, I. Tanaka, and H. Kojima, Spin fluctuations in superconducting $La_{1.85}Sr_{0.15}CuO_4$, Phys. Rev. B **49**, 6958 (1994).

[6] J. Schmalian, D. Pines, and B. Stojkovic, Weak pseudogap behavior in the underdoped cuprate superconductors, Phys. Rev. Lett. **80**, 3839 (1998); Microscopic theory of weak pseudogap behavior in the underdoped cuprate superconductors: General theory and quasiparticle properties, Phys. Rev. B **60**, 667 (1999).

[7] E. Z. Kuchinskii and M. V. Sadovskii, Models of the pseudogap state of two-dimensional systems, J. Exp. Theor. Phys. **88**, 968 (1999).

[8] F. Onufrieva, P. Pfeuty, and M. Kiselev, New scenario for high-T_c cuprates: Electronic topological transition as a motor for anomalies in the underdoped regime, Phys. Rev. Lett. **82**, 2370 (1999); F. Onufrieva and P. Pfeuty, Normal state pseudogap and $(\pi, 0)$ feature in the underdoped high-T_c cuprates: A microscopic theory, *ibid.* **82**, 3136 (1999).

[9] O. Gunnarsson, T. Schäfer, J. P. F. LeBlanc, E. Gull, J. Merino, G. Sangiovanni, G. Rohringer, and A. Toschi, Fluctuation Diagnostics of the Electron Self-Energy: Origin of the Pseudogap Physics, Phys. Rev. Lett. **114**, 236402 (2015).

[10] E. A. Stepanov, L. Peters, I. S. Krivenko, A. I. Lichtenstein, M. I. Katsnelson, and A. N. Rubtsov, Fluctuation diagnostics of the electron self-energy: Origin of the pseudogap physics, npj Quantum Mater. **3**, 54 (2018).

[11] W. Wu, M. S. Scheurer, S. Chatterjee, S. Sachdev, A. Georges, and M. Ferrero, Pseudogap and Fermi-surface topology in the two-dimensional Hubbard model, Phys. Rev. X **8**, 021048 (2018).

[12] M. S. Scheurer, S. Chatterjee, W. Wu, M. Ferrero, A. Georges, and S. Sachdev, Topological order in the pseudogap metal, Proc. Natl. Acad. Sci. USA **115**, E3665 (2018).

[13] F. Krien, P. Worm, P. Chalupa-Gantner, A. Toschi, and K. Held, Explaining the pseudogap through damping and antidamping on the Fermi surface by imaginary spin scattering, Commun. Phys. **5**, 336 (2022).

[14] D. Vilardi, P. M. Bonetti, and W. Metzner, Dynamical functional renormalization group computation of order parameters and critical temperatures in the two-dimensional Hubbard model, Phys. Rev. B **102**, 245128 (2020).

[15] P. M. Bonetti and W. Metzner, SU(2) gauge theory of the pseudogap phase in the two-dimensional Hubbard model, Phys. Rev. B **106**, 205152 (2022).

[16] A. Auerbach, *Interacting Electrons and Quantum Magnetism* (Springer, New York, 1994).

[17] E. Brezin and J. Zinn-Justin, Renormalization of the nonlinear σ model in $2 + \epsilon$ dimensions—application to the Heisenberg ferromagnets, Phys. Rev. Lett. **36**, 691 (1976); Spontaneous breakdown of continuous symmetries near two dimensions, Phys. Rev. B **14**, 3110 (1976).

[18] D. R. Nelson and R. A. Pelcovits, Momentum-shell recursion relations, anisotropic spins, and liquid crystals in $2 + \epsilon$ dimensions, Phys. Rev. B **16**, 2191 (1977).

[19] F. D. M. Haldane, Nonlinear field theory of large-spin Heisenberg antiferromagnets: Semiclassically quantized solitons of the one-dimensional easy-axis Néel state, Phys. Rev. Lett. **50**, 1153 (1983).

[20] S. Chakravarty, B. I. Halperin, and D. R. Nelson, Two-dimensional quantum Heisenberg antiferromagnet at low temperatures, Phys. Rev. B **39**, 2344 (1989).

[21] T. Dombre and N. Read, Nonlinear σ models for triangular quantum antiferromagnets, Phys. Rev. B **39**, 6797 (1989).

[22] P. Azaria, B. Delamotte, and T. Jolicoeur, Nonuniversality in helical and canted-spin systems, Phys. Rev. Lett. **64**, 3175 (1990); P. Azaria, B. Delamotte, and D. Mouhanna, Low-temperature properties of two-dimensional frustrated quantum antiferromagnets, *ibid.* **68**, 1762 (1992); SO(3) nonlinear σ model for a doped quantum helimagnet, Nucl. Phys. B **408**, 485 (1993).

[23] H. J. Schulz, Effective action for strongly correlated fermions from functional integrals, Phys. Rev. Lett. **65**, 2462 (1990);

Functional integrals for correlated electrons, in *Proceedings of NATO Advanced Research Workshop on the Physics and Mathematical Physics of the Hubbard Model*, NATO ASI Series B Vol. 343 (Springer Science + Business Media, New York, 1995), pp. 89–102.

[24] Z. Y. Weng, C. S. Ting, and T. K. Lee, Path-integral approach to the Hubbard model, Phys. Rev. B **43**, 3790 (1991).

[25] K. Sengupta and N. Dupuis, Effective action and collective modes in quasi-one-dimensional spin-density-wave systems, Phys. Rev. B **61**, 13493 (2000); Y. Tomio, N. Dupuis, and Y. Suzumura, Effect of nearest- and next-nearest neighbor interactions on the spin-wave velocity of one-dimensional quarter-filled spin-density-wave conductors, *ibid.* **64**, 125123 (2001).

[26] N. Dupuis, Spin fluctuations and pseudogap in the two-dimensional half-filled Hubbard model at weak coupling, Phys. Rev. B **65**, 245118 (2002); K. Borejsza and N. Dupuis, Antiferromagnetism and single-particle properties in the two-dimensional half-filled Hubbard model: A nonlinear sigma model approach, *ibid.* **69**, 085119 (2004).

[27] S. Sachdev, M. A. Metlitski, Y. Qi, and C. Xu, Fluctuating spin density waves in metals, Phys. Rev. B **80**, 155129 (2009).

[28] P. M. Bonetti, Local Ward identities for collective excitations in fermionic systems with spontaneously broken symmetries, Phys. Rev. B **106**, 155105 (2022).

[29] P. M. Bonetti, Long-range order, bosonic fluctuations, and pseudogap in strongly correlated electron systems, Ph.D. thesis, University of Stuttgart, 2022.

[30] I. A. Goremykin and A. A. Katanin, Commensurate and spiral magnetic order in the doped two-dimensional Hubbard model: Dynamical mean-field theory analysis, Phys. Rev. B **107**, 245104 (2023).

[31] H. Hafermann, E. G. C. P. van Loon, M. I. Katsnelson, A. I. Lichtenstein, and O. Parcollet, Collective charge excitations of strongly correlated electrons, vertex corrections, and gauge invariance, Phys. Rev. B **90**, 235105 (2014).

[32] S. Sachdev, A. V. Chubukov, and A. Sokol, Crossover and scaling in a nearly antiferromagnetic Fermi liquid in two dimensions, Phys. Rev. B **51**, 14874 (1995).

[33] J. A. Hertz and D. M. Edwards, Electron-magnon interactions in itinerant ferromagnetism. I. Formal theory, J. Phys. F: Met. Phys. **3**, 2174 (1973).

[34] A. A. Katanin, A. Toschi, and K. Held, Comparing pertinent effects of antiferromagnetic fluctuations in the two- and three-dimensional Hubbard model, Phys. Rev. B **80**, 075104 (2009).

[35] A. A. Katanin, Improved treatment of fermion-boson vertices and Bethe-Salpeter equations in nonlocal extensions of dynamical mean field theory, Phys. Rev. B **101**, 035110 (2020).

[36] A. A. Katanin, Generalized dynamical mean-field theory of two-sublattice systems with long-range interactions and its application to study charge and spin correlations in graphene, Phys. Rev. B **104**, 245142 (2021).

[37] P. M. Bonetti and W. Metzner, Spin stiffness, spectral weight, and Landau damping of magnons in metallic spiral magnets, Phys. Rev. B **105**, 134426 (2022).

[38] A. V. Chubukov and K. A. Musaelian, Magnetic phases of the two-dimensional Hubbard model at low doping, Phys. Rev. B **51**, 12605 (1995).

[39] A. P. Kampf, Collective excitations in itinerant spiral magnets, Phys. Rev. B **53**, 747 (1996).

[40] P. A. Igoshev, M. A. Timirgazin, A. A. Katanin, A. K. Arzhnikov, and V. Yu. Irkhin, Incommensurate magnetic order and phase separation in the two-dimensional Hubbard model with nearest- and next-nearest-neighbor hopping, Phys. Rev. B **81**, 094407 (2010).

[41] P. A. Igoshev, M. A. Timirgazin, V. F. Gilmutdinov, A. K. Arzhnikov, and V. Yu. Irkhin, Correlation effects and non-collinear magnetism in the doped Hubbard model, J. Magn. Magn. Mater. **383**, 2 (2015); Spiral magnetism in the single-band Hubbard model: The Hartree–Fock and slave-boson approaches, J. Phys.: Condens. Matter **27**, 446002 (2015).

[42] A. Georges, G. Kotliar, W. Krauth, and M. J. Rozenberg, Dynamical mean-field theory of strongly correlated fermion systems and the limit of infinite dimensions, Rev. Mod. Phys. **68**, 13 (1996).

[43] M. Fleck, A. I. Liechtenstein, A. M. Oleś, L. Hedin, and V. I. Anisimov, Dynamical mean-field theory for doped antiferromagnets, Phys. Rev. Lett. **80**, 2393 (1998).

[44] S. Goto, S. Kurihara, and D. Yamamoto, Incommensurate spiral magnetic order on anisotropic triangular lattice: Dynamical mean-field study in a spin-rotating frame, Phys. Rev. B **94**, 245145 (2016).

[45] P. M. Bonetti, J. Mitscherling, D. Vilardi, and W. Metzner, Charge carrier drop at the onset of pseudogap behavior in the two-dimensional Hubbard model, Phys. Rev. B **101**, 165142 (2020).

[46] G. Sangiovanni, A. Toschi, E. Koch, K. Held, M. Capone, C. Castellani, O. Gunnarsson, S.-K. Mo, J. W. Allen, H.-D. Kim, A. Sekiyama, A. Yamasaki, S. Suga, and P. Metcalf, Static versus dynamical mean-field theory of Mott antiferromagnets, Phys. Rev. B **73**, 205121 (2006).

[47] G. Rohringer, H. Hafermann, A. Toschi, A. A. Katanin, A. E. Antipov, M. I. Katsnelson, A. I. Lichtenstein, A. N. Rubtsov, and K. Held, Diagrammatic routes to nonlocal correlations beyond dynamical mean field theory, Rev. Mod. Phys. **90**, 025003 (2018).

[48] L. Huang, Y. Wang, Z. Y. Meng, L. Du, P. Werner, and X. Dai, iQIST: An open source continuous-time quantum Monte Carlo impurity solver toolkit, Comput. Phys. Commun. **195**, 140 (2015); L. Huang, iQIST v0.7: An open source continuous-time quantum Monte Carlo impurity solver toolkit, *ibid.* **221**, 423 (2017).

[49] A. Khurana, Electrical conductivity in the infinite-dimensional Hubbard model, Phys. Rev. Lett. **64**, 1990 (1990).

[50] S. Wakimoto, G. Shirane, Y. Endoh, K. Hirota, S. Ueki, K. Yamada, R. J. Birgeneau, M. A. Kastner, Y. S. Lee, P. M. Gehring, and S. H. Lee, Observation of incommensurate magnetic correlations at the lower critical concentration for superconductivity in $La_{2-x}Sr_xCuO_4 (x = 0.05)$, Phys. Rev. B **60**, R769(R) (1999).

[51] A. V. Chubukov, S. Sachdev, and T. Senthil, Quantum phase transitions in frustrated quantum antiferromagnets, Nucl. Phys. B **426**, 601 (1994).

[52] P. Azaria, P. Lecheminant, and D. Mouhanna, The massive CP^{N-1} model for frustrated spin systems, Nucl. Phys. B **455**, 648 (1995).

[53] M. Campostrini and P. Rossi, The 1/N expansion of two-dimensional spin models, Riv. Nuovo Cimento **16**, 1 (1993).

[54] A. N. Ignatenko, V. Yu. Irkhin, and A. A. Katanin, Universal properties of frustrated spin systems: $1/N$-expansion and renormalization group approaches, Nucl. Phys. B **814**, 439 (2009).

[55] P. M. Bonetti, companion paper, Erratum: Local Ward identities for collective excitations in fermionic systems with spontaneously broken symmetries [Phys. Rev. B 106, 155105 (2022)], Phys. Rev. B **110**, 079902 (2024).

[56] J. R. Schrieffer, X. G. Wen, and S. C. Zhang, Dynamic spin fluctuations and the bag mechanism of high-T_c superconductivity, Phys. Rev. B **39**, 11663 (1989).

[57] A. V. Chubukov and D. M. Frenkel, Renormalized perturbation theory of magnetic instabilities in the two-dimensional Hubbard model at small doping, Phys. Rev. B **46**, 11884 (1992).