# Erratum: Local Ward identities for collective excitations in fermionic systems with spontaneously broken symmetries [Phys. Rev. B 106, 155105 (2022)]

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We found a mistake in the original paper, namely, the statement that the second derivative of the $\Gamma$ functional with respect to the gauge field returns the gauge kernel [Eq. (10) of the original paper]. The Ward identities (WIs) presented in the original paper are nonetheless correct, provided that one does not interpret the object labeled as $K_{\mu\nu}$ as the gauge kernel but as the second derivative with respect to the gauge field of the generating functional $\Gamma$ computed at zero fields. The physical meaning, the usefulness, and the computation of such object are left for future research. In the following, we present a revised derivation of the WIs for the *actual* gauge kernel $K_{\mu\nu}$, both for the case of a U(1) and of an SU(2) symmetry.

## I. U(1) SYMMETRY

We consider a fermionic theory, described by the Grassmann fields $\psi$ and $\overline{\psi}$ and coupled to a U(1) gauge field $A_\mu$, and governed by the action $\mathcal{S}[\psi, \overline{\psi}, A_\mu]$. We define the U(1) gauge kernel as

$$K_{\mu\nu}(x,x') \equiv \langle j_\mu(x) j_\nu(x')\rangle_c + \langle P_{\mu\nu}(x,x')\rangle, \tag{1}$$

with $j_\mu(x) \equiv \frac{\delta\mathcal{S}}{\delta A_\mu(x)}|_{A_\mu=0}$ the paramagnetic current operator, $P_{\mu\nu}(x,x') \equiv -\frac{\delta^2\mathcal{S}}{\delta A_\mu(x)\delta A_\nu(x')}|_{A_\mu=0}$ the diamagnetic current operator, and where the (connected) average is taken by means of the path integral defined by $\mathcal{S}$. The gauge kernel can also be defined as the second derivative at zero source fields of the $\mathcal{G}$-generating functional of the original paper [Eq. (1)]

$$K_{\mu\nu}(x,x') \equiv -\left.\frac{\delta^2\mathcal{G}[J,J^*,A_\mu]}{\delta A_\mu(x)\,\delta A_\nu(x')}\right|_{J=J^*=A_\mu=0}. \tag{2}$$

The U(1) gauge kernel obeys the relation

$$\partial_\mu K_{\mu\nu}(x,x') = 0, \tag{3}$$

even in presence of spontaneous symmetry breaking. This can be readily derived from Eq. (7) of the original paper. The above relation is somewhat obvious as it implies charge conservation, which must be fulfilled even in the superconducting state [1]. Fourier transforming Eq. (3), one obtains

$$i\omega K_{0\nu}(\mathbf{q},\omega) - iq_\alpha K_{\alpha\nu}(\mathbf{q},\omega) = 0, \tag{4}$$

where $\alpha$ is only a spatial index and a sum over it is implied. Here and henceforth we use Einstein's convention for repeated indices, unless otherwise specified. Note that for the derivative in Eq. (3) to make sense, $A_\mu$ needs to be defined on a *continuous* space-time, which implies, in case of a lattice system, that Eq. (4) is valid only for those values of $|\mathbf{q}|$ that are much smaller than the inverse lattice spacing. Setting $\nu = 0$, $\mathbf{q} = \mathbf{0}$, and $\nu = \beta$ (with $\beta$ also a spatial index), $\omega = 0$, respectively, one gets

$$K_{00}(\mathbf{q}=\mathbf{0},\omega) = 0, \tag{5a}$$

$$K_{\alpha\beta}(\mathbf{q},\omega=0) \simeq J_{\alpha\beta}(\mathbf{q}) - \frac{[J_{\alpha\gamma}(\mathbf{q})q_\gamma][J_{\beta\delta}(\mathbf{q})q_\delta]}{J_{\gamma\delta}(\mathbf{q})q_\gamma q_\delta}, \tag{5b}$$

where $J_{\alpha\beta}(\mathbf{q})$ is a function approaching the superfluid stiffness for $\mathbf{q}\to\mathbf{0}$ in the superconducting state. Equation (5b) is the most general form that $K_{\alpha\beta}(\mathbf{q},0)$ can take to obey $q_\alpha K_{\alpha\beta}(\mathbf{q},0) = 0$. The above relations clearly show that Eqs. (19) of the original paper cannot be correct if the correlator $K$ appearing there is the gauge kernel. While $\delta\mathcal{G}/\delta A_\mu = \delta\Gamma/\delta A_\mu$ descends from the properties of the Legendre transform, a similar relation does not apply to the second derivative of the $\mathcal{G}$ and $\Gamma$ functionals with respect to the gauge field. The correct form of Eq. (10) is

$$\frac{\delta^2\Gamma}{\delta A_\mu(x)\delta A_\nu(x')}\bigg|_{A_\mu=\phi=0} = -K_{\mu\nu}(x,x') + \int_{x'',x'''} L_{\mu,a}(x,x'')\chi^{-1}_{ab}(x'',x''')L_{\nu,b}(x',x'''), \tag{6}$$

with

$$L_{\mu,a}(x,x') = -\frac{\delta^2\mathcal{G}}{\delta A_\mu(x)\delta J_a(x')}\bigg|_{A_\mu=J=0}. \tag{7}$$

A proof of this relation is given in Appendix. This proves that the $K$ correlator in Eq. (10) of the original paper is not the gauge kernel. Thus, in Eqs. (15)–(20) of the original paper one should replace $K_{\mu\nu}(\mathbf{q},\omega)$ with (minus) the right-hand side of Eq. (6).

Equation (20) of the original paper, despite being correct if one properly replaces $K_{\mu\nu}(\mathbf{q},\omega)$, is not particularly useful as it is not clear how to compute the second derivative of the $\Gamma$ functional with respect to the gauge field within a diagrammatic approach. In the following, we present a revised and more useful derivation of the Ward identities making use of the sole $\mathcal{G}$ functional.

We consider Eq. (7) of the original paper using the parametrization in Eqs. (8). We write $J_1(x) = h + \delta J_1(x)$, where $h$ is a uniform symmetry-breaking field. Taking the derivative of Eq. (7) of the original paper once with respect to $A_\mu$ and once with respect to $J_2$ and setting $A_\mu = J_2 = \delta J_1 = 0$, we obtain

$$\partial_\mu K^h_{\mu\nu}(x,x') = -2hL^h_{\nu,2}(x',x), \tag{8a}$$

$$\partial_\mu L^h_{\mu,2}(x,x') = -2h\chi^h_{22}(x,x') + 2\varphi^h_0\delta(x-x'), \tag{8b}$$

where $K^h_{\mu\nu}(x,x')$, $\chi^h_{ab}(x,x') \equiv -\frac{\delta^2\mathcal{G}}{\delta J_a(x)\delta J_b(x')}|_{\delta J_1=J_2=A_\mu=0}$, $\varphi^h_0 \equiv -\frac{\delta\mathcal{G}}{\delta J_1(x)}|_{\delta J_1=J_2=A_\mu=0}$, and $L^h_{\mu,a}(x,x')$ are the gauge kernel, susceptibility, condensate fraction, and $L$ correlator, respectively [see Eq. (7)], computed in presence of a uniform explicit symmetry-breaking field $h$. Combining the two equations above, we obtain

$$\partial_\mu\partial'_\nu K^h_{\mu\nu}(x,x') = 4h^2\left(\chi^h_{22}(x,x') - \frac{\varphi^h_0}{h}\delta(x-x')\right), \tag{9}$$

where $\partial'_\nu$ is a derivative over $x'$.

Considering the global U(1) symmetry of the problem, one can obtain a relation similar to Eq. (7) of the original paper:

$$\int_x \left[\frac{\delta\mathcal{G}}{\delta J_2(x)}J_1(x) - \frac{\delta\mathcal{G}}{\delta J_1(x)}J_2(x)\right] = 0, \tag{10}$$

where $\int_x$ is an integral over the spatiotemporal coordinates. From the above relation, one readily obtains

$$\chi^h_{22}(\mathbf{q}=\mathbf{0},\omega=0) = \frac{\varphi^h_0}{h}. \tag{11}$$

This, combined with the properties $\chi^h_{22}(-\mathbf{q},\omega) = \chi^h_{22}(\mathbf{q},-\omega) = \chi^h_{22}(\mathbf{q},\omega)$, implies that the Fourier transform of the right-hand side of Eq. (9) is at least quadratic in frequency and/or momentum. We can thus assume the following small-$\mathbf{q}$ and small-$\omega$ expansions of $\chi^h_{22}(\mathbf{q},\omega)$:

$$\chi^h_{22}(\mathbf{q},\omega) \simeq \frac{4(\varphi^h_0)^2}{-\chi^h_n\omega^2 + J^h_{\alpha\beta}q_\alpha q_\beta + 4h\varphi^h_0}. \tag{12}$$

Taking the Fourier transform of Eq. (9), one gets

$$\lim_{\omega\to0} K^h_{00}(\mathbf{0},\omega) = -4h^2\,\frac{1}{2}\partial^2_\omega\chi^h_{22}(\mathbf{0},\omega)\big|_{\omega\to0} = -\chi^h_n = -4(\varphi^h_0)^2\,\frac{1}{2}\partial^2_\omega\left(\frac{1}{\chi^h_{22}(\mathbf{0},\omega)}\right)\bigg|_{\omega\to0}, \tag{13a}$$

$$\lim_{\mathbf{q}\to\mathbf{0}} K^h_{\alpha\beta}(\mathbf{q},0) = -4h^2\,\frac{1}{2}\partial^2_{q_\alpha q_\beta}\chi^h_{22}(\mathbf{q},0)\big|_{\mathbf{q}\to\mathbf{0}} = J^h_{\alpha\beta} = 4(\varphi^h_0)^2\,\frac{1}{2}\partial^2_{q_\alpha q_\beta}\left(\frac{1}{\chi^h_{22}(\mathbf{q},0)}\right)\bigg|_{\mathbf{q}\to\mathbf{0}}. \tag{13b}$$

Finally, taking the $h\to0$ limit, we obtain

$$\chi_n = \lim_{h\to0}\lim_{\omega\to0} K^h_{00}(\mathbf{0},\omega) = -4(\varphi_0)^2\,\frac{1}{2}\partial^2_\omega\left(\frac{1}{\chi_{22}(\mathbf{0},\omega)}\right)\bigg|_{\omega\to0}, \tag{14a}$$

$$J_{\alpha\beta} = \lim_{h\to0}\lim_{\mathbf{q}\to\mathbf{0}} K^h_{\alpha\beta}(\mathbf{q},0) = 4(\varphi_0)^2\,\frac{1}{2}\partial^2_{q_\alpha q_\beta}\left(\frac{1}{\chi_{22}(\mathbf{q},0)}\right)\bigg|_{\mathbf{q}\to\mathbf{0}}, \tag{14b}$$

where $\chi_n$ and $J_{\alpha\beta}$ are the coefficients of the expansion (12) obtained in absence of a symmetry-breaking field. In particular, $J_{\alpha\beta}$ is the superfluid stiffness. Also, $\varphi_0$ and $\chi_{22}(\mathbf{q},\omega)$ are the condensate fraction and susceptibility obtained for zero $h$.

The equations above are a more useful version of the Ward identities presented in the original paper. The calculation of $\lim_{h\to 0} K^h_{\mu\nu}$ can be performed in the same way as one computes the gauge kernel, introducing a symmetry-breaking field, and sending it to zero at the end of the calculation. The equations above also clearly prove that the dynamical [Eq. (14a)] or static [Eq. (14b)] limits do not commute with the $h \to 0$ limit in the gauge kernel. Inverting their order, one would obtain Eqs. (5) instead of those above.

## II. SU(2) SYMMETRY

Most of the considerations above also apply to the case of a spontaneously broken SU(2) symmetry, with minor modifications. The SU(2) gauge kernel has a similar definition to its U(1) counterpart. Also in this case, the SU(2) gauge kernel can only be obtained by taking the derivative of the $\mathcal{G}$ functional (and not from the $\Gamma$ functional).

Relations similar to Eqs. (6) and (7) can be obtained for a system with SU(2) symmetry, making, also in this case, Eq. (40) of the original paper not useful for practical calculations. In the following, we derive WIs for the actual SU(2) gauge kernel.

The analog of Eq. (7) is, in the SU(2)-symmetric case,

$$\partial_\mu\left(\frac{\delta\mathcal{G}}{\delta A^a_\mu(x)}\right) - \varepsilon^{a\ell m}\left[\frac{\delta\mathcal{G}}{\delta J^\ell(x)}J^m(x) + \frac{\delta\mathcal{G}}{\delta A^\ell_\mu(x)}A^m_\mu(x)\right] = 0. \tag{15}$$

We now set $J_a(x) = h_a(\mathbf{x}) + \delta J_a(x)$, where $h_a(\mathbf{x})$ is a symmetry-breaking field. Its spatial dependence must take the same form as the spin condensate $S^h_a(\mathbf{x}) = -\delta\mathcal{G}/\delta J_a(x)|_{A_\mu=\delta J=0}$. In essence, for every point $\mathbf{x}$, we have $S^h_a(\mathbf{x}) = (\varphi^h_0/h)h_a(\mathbf{x})$, where $h = \sqrt{h_a(\mathbf{x})h_a(\mathbf{x})}$ and $\varphi^h_0$ are constants.

With some algebra, we can derive from Eq. (15) the following set of equations:

$$\partial_\mu K^{h;ab}_{\mu\nu}(x,x') = \varepsilon^{a\ell m}L^{h;b}_{\nu,\ell}(x',x)h_m(\mathbf{x}) + \varepsilon^{ab\ell}B^{h;\ell}_\nu(x)\delta(x-x'), \tag{16a}$$

$$\partial_\mu L^{h;a}_{\mu,b}(x,x') = \varepsilon^{a\ell m}\chi^h_{\ell b}(x,x')h_m(\mathbf{x}) - \varepsilon^{ab\ell}S^h_\ell(\mathbf{x})\delta(x-x'), \tag{16b}$$

$$\partial_\mu B^{h;a}_\mu(x) = \varepsilon^{a\ell m}S^h_\ell(\mathbf{x})h_m(\mathbf{x}) = 0, \tag{16c}$$

where the first (second) equation has been obtained deriving (15) with respect to $A_\nu(x')$ [$J_b(x')$] and setting $A_\mu = \delta J = 0$. The third equation can be derived from (15) setting $A_\mu = \delta J = 0$. Its right-hand side is zero because we have assumed $S^h_a(\mathbf{x})$ and $h_a(\mathbf{x})$ to be parallel. We have also defined

$$L^{h;a}_{\mu,b}(x,x') = -\left.\frac{\delta^2\mathcal{G}}{\delta A^a_\mu(x)\delta J^b(x')}\right|_{A_\mu=\delta J=0}, \tag{17a}$$

$$B^{h,a}_\mu(x) = -\left.\frac{\delta\mathcal{G}}{\delta A^a_\mu(x)}\right|_{A_\mu=\delta J=0}. \tag{17b}$$

$K^{h;ab}_{\mu\nu}$ and $\chi^h_{ab}$ are the gauge kernel and spin susceptibility in presence of a symmetry-breaking field.

Combining all Eqs. (16), we get

$$\begin{aligned}\partial_\mu\partial'_\nu K^{h;ab}_{\mu\nu}(x,x') &= \varepsilon^{a\ell m}\varepsilon^{bpr}\chi^h_{p\ell}(x,x')h_m(\mathbf{x})h_r(\mathbf{x}') + (2\delta_{a\ell}\delta_{bm} - \delta_{am}\delta_{b\ell} - \delta_{ab}\delta_{\ell m})S^h_\ell(\mathbf{x})h_m(\mathbf{x})\delta(x-x')\\ &= \varepsilon^{a\ell m}\varepsilon^{bpr}\chi^h_{p\ell}(x,x')h_m(\mathbf{x})h_r(\mathbf{x}') + h\,\varphi^h_0\left(\frac{h_a(\mathbf{x})h_b(\mathbf{x})}{h^2} - \delta_{ab}\right).\end{aligned} \tag{18}$$

Similarly to the previous section, one can derive a functional identity for $\mathcal{G}$ descending from the *global* SU(2) symmetry of the system, reading as

$$\int_x \varepsilon^{a\ell m}\frac{\delta\mathcal{G}}{\delta J_\ell(x)}J_m(x) = 0. \tag{19}$$

Taking a functional derivative with respect to $J_n(x')$, multiplying by $\varepsilon^{bnp}h_p(\mathbf{x}')$, summing over $n$ and $p$, setting $\delta J_a = A^a_\mu = 0$, and integrating over $x'$, we obtain

$$\int_x\int_{x'}\varepsilon^{a\ell m}\varepsilon^{bnp}\chi^h_{\ell n}(x,x')h_m(\mathbf{x})h_p(\mathbf{x}') = \frac{\varphi^h_0}{h}\int_x(\delta_{ab}\delta_{\ell m} - \delta_{a\ell}\delta_{bm})h_\ell(\mathbf{x})h_m(\mathbf{x}). \tag{20}$$

### A. Spiral order

In the case of spiral order, we have $h(\mathbf{x}) = h(\cos(\mathbf{Q}\cdot\mathbf{x}), \sin(\mathbf{Q}\cdot\mathbf{x}), 0)$ and $S^h(\mathbf{x}) = \varphi_0^h(\cos(\mathbf{Q}\cdot\mathbf{x}), \sin(\mathbf{Q}\cdot\mathbf{x}), 0)$. Equation (20) then gives

$$\widetilde{\chi}^h_{22}(\mathbf{0}, 0) = \frac{\varphi_0^h}{h}, \tag{21a}$$

$$\widetilde{\chi}^h_{33}(\mathbf{Q}, 0) = \widetilde{\chi}^h_{33}(-\mathbf{Q}, 0) = \frac{\varphi_0^h}{h}, \tag{21b}$$

where $\widetilde{\chi}^h_{ab}$ is the susceptibility in a *rotated* spin basis, and it is connected to $\chi^h_{ab}$ with relations similar to Eqs. (55) of the original paper. Setting $a = b = 1, 2, 3$ in Eq. (16), Fourier transforming and using the above relations, one obtains

$$\lim_{\omega\to 0} K^{h;11}_{00}(\mathbf{0}, \omega) = \lim_{\omega\to 0} K^{h;22}_{00}(\mathbf{0}, \omega) = -\frac{h^2}{4}\partial^2_\omega \widetilde{\chi}^h_{33}(\mathbf{Q}, \omega)|_{\omega\to 0}, \tag{22a}$$

$$\lim_{\mathbf{q}\to \mathbf{0}} K^{h;11}_{\alpha\beta}(\mathbf{q}, 0) = \lim_{\mathbf{q}\to \mathbf{0}} K^{h;22}_{\alpha\beta}(\mathbf{q}, 0) = -\frac{h^2}{4}\partial^2_{q_\alpha q_\beta} \widetilde{\chi}^h_{33}(\mathbf{q}, 0)|_{\mathbf{q}\to \mathbf{Q}}, \tag{22b}$$

$$\lim_{\omega\to 0} K^{h;33}_{00}(\mathbf{0}, \omega) = -\frac{h^2}{2}\partial^2_\omega \widetilde{\chi}^h_{22}(\mathbf{0}, \omega)|_{\omega\to 0}, \tag{22c}$$

$$\lim_{\mathbf{q}\to \mathbf{0}} K^{h;33}_{\alpha\beta}(\mathbf{q}, 0) = -\frac{h^2}{2}\partial^2_{q_\alpha q_\beta} \widetilde{\chi}^h_{22}(\mathbf{q}, 0)|_{\mathbf{q}\to \mathbf{0}}, \tag{22d}$$

where, as in the original paper, $K^{ab}_{\mu\nu}(\mathbf{q}, \omega)$ is the translational invariant component of the SU(2) gauge kernel.

Assuming the following forms for the susceptibilities,

$$\widetilde{\chi}^h_{22}(\mathbf{q}, \omega) \simeq \frac{(\varphi_0^h)^2}{-\chi^{h;\square}_{\rm dyn}\omega^2 + J^{h;\square}_{\alpha\beta} q_\alpha q_\beta + h\varphi_0^h}, \tag{23a}$$

$$\widetilde{\chi}^h_{33}(\mathbf{q}, \omega) \simeq \sum_{\eta=\pm} \frac{(\varphi_0^h)^2/2}{-\chi^{h;\perp}_{\rm dyn}\omega^2 + J^{h;\perp}_{\alpha\beta}(q-\eta Q)_\alpha (q-\eta Q)_\beta + h\varphi_0^h/2}, \tag{23b}$$

one can prove, following the steps performed in the U(1)-symmetric case, the final form of the Ward identities for a spiral magnet

$$\chi^\perp_{\rm dyn} = \lim_{h\to 0}\lim_{\omega\to 0} K^{h;11}_{00}(\mathbf{0}, \omega) = \lim_{h\to 0}\lim_{\omega\to 0} K^{h;22}_{00}(\mathbf{0}, \omega) = -\frac{(\varphi_0)^2}{2}\partial^2_\omega\left(\frac{1}{\widetilde{\chi}^{33}(\mathbf{Q}, \omega)}\right)\bigg|_{\omega\to 0}, \tag{24a}$$

$$J^\perp_{\alpha\beta} = \lim_{h\to 0}\lim_{\mathbf{q}\to \mathbf{0}} K^{h;11}_{\alpha\beta}(\mathbf{q}, 0) = \lim_{h\to 0}\lim_{\omega\to 0} K^{h;22}_{\alpha\beta}(\mathbf{q}, 0) = \frac{(\varphi_0)^2}{2}\partial^2_{q_\alpha q_\beta}\left(\frac{1}{\widetilde{\chi}^{33}(\mathbf{q}, 0)}\right)\bigg|_{\mathbf{q}\to \mathbf{Q}}, \tag{24b}$$

$$\chi^\perp_{\rm dyn} = \lim_{h\to 0}\lim_{\omega\to 0} K^{h;33}_{00}(\mathbf{0}, \omega) = -(\varphi_0)^2\partial^2_\omega\left(\frac{1}{\widetilde{\chi}^{22}(\mathbf{0}, \omega)}\right)\bigg|_{\omega\to 0}, \tag{24c}$$

$$J^\square_{\alpha\beta} = \lim_{h\to 0}\lim_{\mathbf{q}\to \mathbf{0}} K^{h;33}_{\alpha\beta}(\mathbf{q}, 0) = (\varphi_0)^2\partial^2_{q_\alpha q_\beta}\left(\frac{1}{\widetilde{\chi}^{22}(\mathbf{q}, 0)}\right)\bigg|_{\mathbf{q}\to \mathbf{0}}. \tag{24d}$$

Also in this case, the static or dynamic limit does not commute with the $h \to 0$ limit.

### B. Néel order

The Ward identities in the case of Néel order can be straightforwardly derived following the steps performed in the previous subsection. Assuming $S^h(\mathbf{x}) = \varphi_0^h(-1)^{\mathbf{x}}(1, 0, 0)$ and $h(\mathbf{x}) = h(-1)^{\mathbf{x}}(1, 0, 0)$, they read as

$$\chi_{\rm dyn} = \lim_{h\to 0}\lim_{\omega\to 0} K^{h;22}_{00}(\mathbf{0}, \omega) = -(\varphi_0)^2\partial^2_\omega\left(\frac{1}{\chi^{33}(\mathbf{Q}, \omega)}\right)\bigg|_{\omega\to 0}, \tag{25a}$$

$$J_{\alpha\beta} = \lim_{h\to 0}\lim_{\mathbf{q}\to \mathbf{0}} K^{h;22}_{\alpha\beta}(\mathbf{q}, 0) = (\varphi_0)^2\partial^2_{q_\alpha q_\beta}\left(\frac{1}{\chi^{33}(\mathbf{q}, 0)}\right)\bigg|_{\mathbf{q}\to \mathbf{Q}}, \tag{25b}$$

where now $\mathbf{Q}$ takes the form $(\pi, \pi)$. The above equations remain valid upon exchanging the index "2" with "3."

### III. EXPLICIT CALCULATION FOR A SPIRAL MAGNET

Section III of the original paper is fully correct, as the microscopic expressions for the spin stiffnesses and dynamical susceptibilities correspond to those derived in presence of a small symmetry-breaking field that is sent to zero at the end of the calculation.

A misprint is present in Eq. (A4) of the original paper. Its correct form is

$$\kappa_\alpha^{31}(\mathbf{0}) = -\frac{\Delta}{4}\int_{\mathbf{k}}\left\{\left[\frac{h_{\mathbf{k}}}{e_{\mathbf{k}}}\left(\partial_{k_\alpha}g_{\mathbf{k}}\right)+\left(\partial_{k_\alpha}h_{\mathbf{k}}\right)\right]\frac{f'(E^+_{\mathbf{k}})}{e_{\mathbf{k}}}+\left[\frac{h_{\mathbf{k}}}{e_{\mathbf{k}}}\left(\partial_{k_\alpha}g_{\mathbf{k}}\right)-\left(\partial_{k_\alpha}h_{\mathbf{k}}\right)\right]\frac{f'(E^-_{\mathbf{k}})}{e_{\mathbf{k}}}+\frac{h_{\mathbf{k}}}{e^2_{\mathbf{k}}}\left(\partial_{k_\alpha}g_{\mathbf{k}}\right)\frac{f(E^-_{\mathbf{k}})-f(E^+_{\mathbf{k}})}{e_{\mathbf{k}}}\right\}, \tag{26}$$

that is, compared to the original paper, the prefactor is $\Delta$ and not $\Delta^2$.

### IV. NOTE

In Ref. [2] the expressions for the spin stiffnesses and dynamical susceptibilities derived in the original paper have been used. Even though it was not explicitly stated, they have been derived applying a small symmetry-breaking field to the system and sending it to zero *after* performing the dynamical or static limit. The expressions in Ref. [2] are therefore correct within the (renormalization group improved) random phase approximation employed in the paper.

During the review process of this Erratum, Ref. [3] appeared, which presents a similar derivation of the Ward identities above, and, as also discussed here, identifies the presence of an infinitesimal symmetry-breaking field as crucial to obtain the correct formulas for the spin stiffnesses.

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### APPENDIX: DERIVATION OF EQ. (6)

In this Appendix, we present a derivation of Eq. (6). The identity

$$\frac{\delta\Gamma}{\delta A_\mu(x)} = \frac{\delta\mathcal{G}}{\delta A_\mu(x)} \tag{A1}$$

descends directly from the properties of the Legendre transform that connects $\mathcal{G}$ and $\Gamma$. Taking a further derivative with respect to the gauge field in the above equation, we get

$$\frac{\delta^2\Gamma}{\delta A_\mu(x)\delta A_\nu(x')} = \frac{\delta^2\mathcal{G}}{\delta A_\mu(x)\delta A_\nu(x')} + \int_{x''}\frac{\delta^2\mathcal{G}}{\delta A_\mu(x)\delta J_a(x'')}\frac{\delta J_a(x'')}{\delta A_\nu(x')}. \tag{A2}$$

Setting the fields to zero, we obtain

$$\left.\frac{\delta^2\Gamma}{\delta A_\mu(x)\delta A_\nu(x')}\right|_{A_\mu=\phi=0} = -K_{\mu\nu}(x,x') - \int_{x''}L_{\mu,a}(x,x'')\left.\frac{\delta J_a(x'')}{\delta A_\nu(x')}\right|_{A_\mu=\phi=0}, \tag{A3}$$

with $L_{\mu,a}(x,x'')$ defined as in Eq. (7). At this point, one needs to express the source field $J_a(x)$ in terms of the "classical" field $\phi_a(x)$, which $\Gamma$ depends on. Since we only need its derivative with respect to the gauge field at zero sources, we are allowed to expand the relation connecting $\phi_a(x)$ and $J_a(x)$ to linear order in the fields:

$$\phi_a(x) \equiv -\frac{\delta\mathcal{G}}{\delta J_a(x)} \approx \varphi_{0,a} + \int_{x'}[\chi_{ab}(x,x')J_b(x') + L_{\mu,a}(x',x)A_\mu(x')] + \cdots, \tag{A4}$$

with $\varphi_{0,a} \equiv -\frac{\delta\mathcal{G}}{\delta J_a(x)}|_{J=A_\mu=0}$ the condensate fraction. Solving for $J_a(x)$, we obtain

$$\left.\frac{\delta J_a(x)}{\delta A_\mu(x')}\right|_{A_\mu=\phi=0} = -\int_{x''}\chi^{-1}_{ab}(x,x'')L_{\mu,b}(x',x''), \tag{A5}$$

with the inverse susceptibility $\chi_{ab}^{-1}(x, x')$ obeying

$$\int_{x''} \chi_{ac}(x, x'')\chi_{cb}^{-1}(x'', x') = \delta(x - x')\delta_{ab}. \tag{A6}$$

Inserting Eq. (A5) into (A3), we obtain Eq. (6).

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[3] I. A. Goremykin and A. A. Katanin, companion paper, Antiferromagnetic and spin spiral correlations in the doped two-dimensional Hubbard model: Gauge symmetry, Ward identities, and dynamical mean-field theory analysis, Phys. Rev. B **110**, 085153 (2024).