Editors' Suggestion

Giant microwave absorption in the vortex lattice in *s*-wave superconductors

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In this article we study microwave absorption in superconductors in the presence of a vortex lattice. We show that in addition to the conventional absorption mechanism associated with the vortex core motion, there is another mechanism of microwave absorption, which is caused by the time dependence of the quasiparticle density of states outside the vortex cores. This mechanism exists even in the absence of vortex motion. It provides the dominant contribution to microwave absorption at sufficiently small magnetic fields. At low frequencies, the dissipative part of the microwave conductivity $\sigma(\omega)$ is proportional to the inelastic relaxation time $\tau_{\rm in}$, which is typically much larger than the elastic relaxation time $\tau_{\rm el}$. At high frequencies $\sigma(\omega)$ is proportional to the quasiparticle diffusion time across the intervortex distance $\tau_{\rm D}$, which is still larger than $\tau_{\rm el}$.

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I. INTRODUCTION

Type-II superconductors placed in a magnetic field normal to the plane, which is weaker than the upper critical field H_{c2}, host Abrikosov vortices whose density is set by the condition that the average flux of the magnetic field per vortex is equal to the flux quantum $\Phi_0 = \frac{\pi\hbar c}{e}$ [1]. In the absence of pinning, the Magnus force on the vortices induced by the transport current through the sample causes vortex motion and dissipation. The corresponding dissipative conductivity has been extensively studied since the work of Bardeen and Stephen [2–6] and is described by the formula

$$\sigma_{BS} \sim \sigma_n \frac{H_{c2}}{H}, \tag{1}$$

where σ_n is the normal-state conductivity and H_{c2} is the upper critical field. In the Bardeen-Stephen theory, dissipation arises from the friction force caused by vortex motion. Equation (1) may be obtained by expressing the friction force-per-unit line length of the vortex as $\boldsymbol{F} = -\eta_{BS}\boldsymbol{v}_v$, where $\eta_{BS} = \Phi_0 H_{c2}\sigma_n/c^2$ is the vortex viscosity, and $\boldsymbol{v}_v$ is the vortex velocity. In the flux flow regime the latter is given by

$$\boldsymbol{v}_v = c\,\frac{\boldsymbol{E}\times\boldsymbol{H}}{H^2}. \tag{2}$$

Equating the rate of viscous energy dissipation to Joule heat $\frac{H}{\Phi_0}\eta_{BS}\boldsymbol{v}_v^2 = \sigma_{BS}E^2$, one obtains Eq. (1).

In the presence of disorder the vortex lattice is pinned, and is capable of supporting a dissipationless current density which is smaller than the critical current J_c. Therefore, Eq. (1) is relevant only in the nonlinear (flux flow) regime where the transport current significantly exceeds J_c. On the other hand, ac electric fields $\boldsymbol{E} = \boldsymbol{E}_0\cos(\omega t)$ induce dissipation in superconductors even in the linear regime. In this case, the microwave absorption coefficient is controlled by the dissipative ac conductivity $\sigma(\omega)$. The latter can be evaluated using the relation

$$T\overline{\dot{S}} = \tfrac{1}{2}\sigma(\omega)\boldsymbol{E}_0^2, \tag{3}$$

where T is the temperature, $\dot{S}$ is the entropy production rate per unit volume, and the overline $\overline{(\ldots)}$ indicates averaging over time. In most articles on microwave absorption, the dissipative conductivity in the linear regime is evaluated phenomenologically using the Bardeen-Stephen expression for the vortex viscosity η_{BS} and the vortex velocity $\boldsymbol{v}'_v(t)$, which is modified by the pinning forces (see, for example, Refs. [7–9]). The corresponding result, denoted by $\sigma'_{BS}(\omega)$ below, is proportional to the elastic momentum relaxation time τ_{el} in the normal state.

In this article we describe a new mechanism of microwave absorption in superconductors in the mixed state. This mechanism is caused by the spectral flow of the quasiparticle energy levels in the presence of an ac electric field, and exists even in the absence of vortex motion. We show that in a broad range of physical parameters the dissipative part of the conductivity caused by this mechanism can be parametrically larger than $\sigma'_{BS}(\omega)$.

The origin of this mechanism can be traced to a fundamental difference between the quasiparticle kinetics in superconductors and electron kinetics in normal metals. In superconductors, the density of quasiparticle states $\nu(\epsilon, t)$ may depend on time, which means that energies of individual single-particle states change in time. In the adiabatic approximation, the quasiparticles occupying these levels also travel in energy space. As a result, a nonequilibrium quasiparticle distribution is created, whose relaxation generates entropy and provides a mechanism for microwave absorption.

The time dependence of $\nu(\epsilon, t)$ arises because the microwave electric field induces oscillations of the superfluid momentum

$$\boldsymbol{p}_s(\boldsymbol{r}, t) = \frac{\hbar}{2}\left[\nabla\chi(\boldsymbol{r}, t) - \frac{2e}{\hbar c}\boldsymbol{A}(\boldsymbol{r}, t)\right]. \tag{4}$$

Here $\chi(\boldsymbol{r}, t)$ is the phase of the order parameter $\boldsymbol{A}(\boldsymbol{r}, t)$—the vector potential, and e, $\hbar$, and c are, respectively, the electron charge, Planck's constant, and the speed of light. The microwave field induces acceleration of the condensate $\dot{\boldsymbol{p}}_s(\boldsymbol{r}, t) \propto e\boldsymbol{E}(t)$. Since the density of states $\nu(\epsilon, t)$ is a scalar,

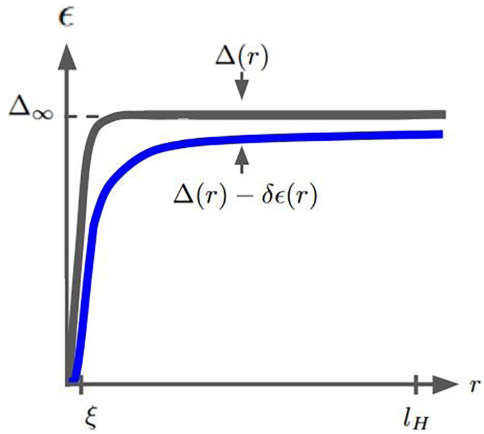

FIG. 1. The black line denotes the dependence of the modulus of the order parameter $\Delta(r)$ on the distance from the vortex core r. The blue line shows the r dependence of the edge of the quasiparticle spectrum.

its time derivative can have a linear dependence on $\dot{\boldsymbol{p}}_s(\boldsymbol{r})$ only in the presence of a dc superfluid momentum $\bar{\boldsymbol{p}}_s(\boldsymbol{r})$; $\dot{\nu}(\boldsymbol{r}) \propto \dot{\boldsymbol{p}}_s(\boldsymbol{r}) \cdot \bar{\boldsymbol{p}}_s(\boldsymbol{r})$. In the case of a flux lattice, the dc superfluid momentum is the equilibrium superfluid momentum about the vortex cores.

The nonequilibrium quasiparticle distribution created by the spectral flow can relax via two channels: inelastic scattering and quasiparticle diffusion. The relative importance of these two channels depends on frequency of the microwave field ω. In typical superconductors the inelastic relaxation time τ_{in} significantly exceeds the diffusion time across the vortex plaquette τ_{D}. Therefore, inelastic relaxation is caused by those quasiparticles for which diffusion cannot lead to full equilibration. These quasiparticles have energies below the threshold of percolation between different vortex plaquettes. The percolation threshold ϵ^* arises because in the presence of a superfluid momentum $\bar{\boldsymbol{p}}_s(\boldsymbol{r})$ the energy gap of the quasiparticle states is shifted down from the value of the order parameter Δ by $\delta\epsilon(\bar{p}_s(\boldsymbol{r}))$, as illustrated in Fig. 1. The nonequilibrium quasiparticles with energies below ϵ^* are trapped inside the vortex plaquettes. Because of pinning the vortex lattice is distorted, and the spatial distribution of the superfluid momentum $\bar{\boldsymbol{p}}_s(\boldsymbol{r})$ is not symmetric about the vortex core. Therefore, the nonequilibrium part of the distribution function cannot relax completely by diffusion across the plaquettes, and ultimately its relaxation is achieved by inelastic scattering processes. Thus, the low-frequency conductivity $\sigma \sim K\tau_{\text{in}}$ is proportional to the inelastic relaxation time and a parameter K characterizing the degree of lattice distortion. Since τ_{in} may exceed the elastic relaxation time τ_{el} by many orders of magnitude, this contribution typically is much larger than σ'_{BS}. [1]

At higher frequencies the value of the conductivity is controlled by the quasiparticle diffusion time across a vortex lattice plaquette τ_D, which is assumed to be much smaller that the inelastic relaxation time but larger than the elastic relaxation time $\tau_{\text{el}} \ll \tau_{\text{D}} \ll \tau_{\text{in}}$. In this case the value of the conductivity σ is still larger than σ'_{BS}.

For simplicity, we focus on the case where the thickness d of the superconducting film is smaller than the skin length, and the microwave electric field $\boldsymbol{E}(t)$ is spatially uniform in the film. We also assume that the distance between vortices, which is of order of the magnetic length $l_H = \hbar c/|e|H$, is smaller than the Pearl length [13], so that the magnetic field is also uniform in the film.

The consideration below is organized as follows. In Sec. II we present a general kinetic theory of quasiparticle dynamics in superconductors with time-dependent density of states. In Sec. III we discuss the spatial distribution of the condensate acceleration $\dot{\boldsymbol{p}}_s(\boldsymbol{r})$ and show that it has a significant component outside the vortex cores, which exists even in the absence of vortex motion. In Sec. IV we use our formalism to evaluate microwave conductivity arising from this component, and show that in Sec. IV we use our formalism to evaluate the microwave conductivity arising from this component, and show that it becomes dominant at sufficiently small magnetic fields. We discuss the results in Sec. V.

II. QUASIPARTICLE KINETICS IN THE PRESENCE OF SPECTRAL FLOW

In this section we present a general description of diffusive quasiparticle kinetics in superconductors in the presence of spectral flow and inelastic relaxation.

In the diffusive regime, where the characteristic spatial scales exceed the elastic mean-free path $v_F\tau_{\text{el}}$, the quasiparticle distribution function $n(\epsilon, \boldsymbol{r}, t)$ depends only on energy ϵ, coordinate $\boldsymbol{r}$, and time t. In the presence of spectral flow, its evolution equation has the form

$$\partial_t n(\epsilon, \boldsymbol{r}, t) + v_\nu(\epsilon, \boldsymbol{r}, t)\partial_\epsilon n(\epsilon, \boldsymbol{r}, t) + \nabla_i[D_{ij}(\epsilon, \boldsymbol{r}, t)\nabla_j n(\epsilon, \boldsymbol{r}, t)] = I_{\text{in}}\{n\}. \tag{5}$$

Here, $D_{ij}(\epsilon, \boldsymbol{r}, t)$ is the diffusion tensor, [2] and $I_{\text{in}}\{n\}$ is the collision integral, for which we will use the relaxation-time approximation

$$I_{\text{in}}\{n\} = -\frac{n(\epsilon) - n_F(\epsilon)}{\tau_{\text{in}}}, \tag{6}$$

with $n_F(\epsilon)$ being the Fermi distribution function.

The second term in Eq. (5) describes the motion of quasiparticles in energy space caused by the spectral flow. For sufficiently slow spatial variations of the superfluid momentum $\boldsymbol{p}_s$ and other system parameters, the "level velocity" $v_\nu(\epsilon, \boldsymbol{r}, t)$ can be expressed in terms of the time derivative of the local density of states $\nu(\epsilon, \boldsymbol{r}, t)$. This relation has the same

[1] The mentioned above mechanism of absorption has been discussed in the context of microwave absorption in superconductors in the presence of spatially uniform supercurrent [10,11] and the resistance of superconductors in flux-flow regime [12].

[2] Apart from the crystalline anisotropy, it acquires additional anisotropy in the presence of a superfluid momentum. In isotropic systems, it may be expressed in terms of the longitudinal and transverse diffusion coefficients in the form $D_{ij}(\epsilon, \boldsymbol{r}, t) = n_i n_j D_\parallel(\epsilon, \boldsymbol{r}, t) + (\delta_{ij} - n_i n_j)D_\perp(\epsilon, \boldsymbol{r}, t)$, where $\boldsymbol{n}$ is a unit vector along $\boldsymbol{p}_s$. It can be shown that this anisotropy can be substantial only in clean superconductors, whereas in the dirty regime it is small.

form as that in uniform systems [10], and may be obtained using conservation of the number of energy levels. From the continuity equation for the spectral current in energy space, $\partial_t \nu(\epsilon, \boldsymbol{r}, t) + \partial_\epsilon[v_\nu(\epsilon, \boldsymbol{r}, t)\nu(\epsilon, \boldsymbol{r}, t)] = 0$, one gets

$$v_\nu(\epsilon, \boldsymbol{r}, t) = -\frac{1}{\nu(\epsilon, \boldsymbol{r}, t)} \int_0^\epsilon d\tilde{\epsilon}\, \partial_t \nu(\tilde{\epsilon}, \boldsymbol{r}, t). \tag{7}$$

Similarly, the diffusion coefficient $D(\epsilon, \boldsymbol{r}, t)$ in this approximation is assumed to have the same dependence on $\boldsymbol{p}_s$ and ϵ as in a uniform superconductor $D(\epsilon, \bar{\boldsymbol{p}}_s(\boldsymbol{r}))$.

Entropy production in the system is caused by diffusion of quasiparticles as well as their inelastic relaxation. The dissipative part of the macroscopic microwave conductivity $\sigma(\omega)$ can be obtained by equating the Joule heating losses to the energy dissipation rate Eq. (3). Linearizing with respect to small deviations from the equilibrium distribution function $n = n_F(\epsilon) + \delta n$, the entropy production rate is given by

$$T\dot{S} = T \int d\epsilon d\boldsymbol{r} \nu(\epsilon, \boldsymbol{r}, t)\bigg[\nabla_i \delta n D_{ij}(\epsilon, \boldsymbol{r}, t)\nabla_j \delta n(\epsilon, \boldsymbol{r}, t) + \frac{\delta n^2(\epsilon, \boldsymbol{r}, t)}{\tau_{\rm in} n_F(\epsilon)(1 - n_F(\epsilon))}\bigg]. \tag{8}$$

Here $n_F(\epsilon)$ is the Fermi distribution function.

The description of quasiparticle kinetics in the presence of spectral flow given by Eqs. (5)–(8) has a broad range of applicability. It describes both clean and dirty superconductors, and does not assume a particular pairing symmetry. On the other hand, the criteria of their applicability, and specific values of $\nu(\epsilon, \boldsymbol{r}, t)$, $v_\nu(\epsilon, \boldsymbol{r}, t)$, and $D_{ij}(\epsilon, \boldsymbol{r}, t)$ depend on the pairing symmetry and other parameters of the system. In the clean regime $\Delta\tau_{\rm el} \gg 1$, Eqs. (5) and (7) have been derived in Ref. [10] from the conventional Boltzmann kinetic equation for quasiparticle distribution function [14]. We note that Eq. (5) has the same form as Eq. (A32) in Ref. [5], which describes the time evolution of the quasiparticle distribution function f in the "dirty limit" $\Delta\tau_{\rm el} \ll 1$, [3] which was derived by Larkin and Ovchinnikov [15]. However, the local relation Eq. (7) between the level velocity $v_\nu(\epsilon, \boldsymbol{r}, t)$ and the time derivative of the density of states can be derived from the Larkin-Ovchinnikov equations only when the characteristic length scale of spatial inhomogeneity exceeds the superconducting coherence length $\xi = \sqrt{D_n/\Delta}$, where $D_n = v_F^2 \tau_{\rm el}/3$, with v_F being the Fermi velocity, is the normal metal diffusion coefficient.

In situations where the spectral flow is caused by the pair-breaking effect of the condensate momentum $\boldsymbol{p}_s$, the level velocity in Eq. (7) may be expressed in the form

$$v_\nu(\epsilon, \boldsymbol{p}_{\rm s}) = \dot{\boldsymbol{p}}_s \cdot \boldsymbol{V}(\epsilon, \boldsymbol{p}_{\rm s}), \tag{9}$$

where $\boldsymbol{V}(\epsilon, \boldsymbol{p}_s)$ denotes the level sensitivity to changes in $\boldsymbol{p}_s$, and is given by

$$\boldsymbol{V}(\epsilon, \boldsymbol{p}_{\rm s}) = -\frac{1}{\nu(\epsilon, p_{\rm s})} \int_0^\epsilon d\epsilon \frac{\partial \nu(\epsilon, \boldsymbol{p}_{\rm s})}{\partial \boldsymbol{p}_{\rm s}}. \tag{10}$$

For s-wave superconductors, the $\boldsymbol{p}_s$ dependence of the density of states and level sensitivity was determined in Refs. [10,11].

Below, we use Eqs. (5)–(10) to study microwave absorption in films of type-II s-wave superconductors in the presence of a pinned vortex lattice. We show that in the London regime, where the intervortex distance $\sim l_H$ exceeds the core radius $\sim\xi$, with the exception of small temperatures $T \ll \Delta$, the main contribution to the microwave absorption comes from quasiparticles which reside at distances $\sim l_H$ from the vortex cores.

III. SPATIAL DISTRIBUTION OF THE CONDENSATE ACCELERATION

At low frequencies in the London regime, where the order parameter outside the vortex cores $\Delta(T)$, which we denote as Δ in formulas below, is approximately uniform, the quasiparticle density of states depends on the local instantaneous condensate momentum $\boldsymbol{p}_s(\boldsymbol{r}, t)$, and its rate of change is proportional to the local condensate acceleration $\dot{\boldsymbol{p}}_s(\boldsymbol{r}, t)$. In thin films, the Pearl length [13], which characterizes the screening of the magnetic field, practically always exceeds the intervortex distance $\sim l_H$. Therefore, in this regime we can neglect the small inhomogeneity of the magnetic field $\boldsymbol{H}(\boldsymbol{r})$ caused by the diamagnetic currents.

In the presence of a microwave field $\boldsymbol{E}(t)$ the vortex positions $\boldsymbol{r}_a(t)$ become time dependent, and the condensate acceleration is given by

$$\dot{\boldsymbol{p}}_s(\boldsymbol{r}, t) = \hbar\hat{z} \times \sum_a \bigg[\frac{2(\boldsymbol{r} - \boldsymbol{r}_a(t))(\boldsymbol{r} - \boldsymbol{r}_a(t)) \cdot \dot{\boldsymbol{r}}_a(t)}{|\boldsymbol{r} - \boldsymbol{r}_a(t)|^4} - \frac{\dot{\boldsymbol{r}}_a(t)}{|\boldsymbol{r} - \boldsymbol{r}_a(t)|^2}\bigg] + e\boldsymbol{E}(t). \tag{11}$$

Thus, only in the absence of vortex displacement the condensate acceleration is given by the second term, $e\boldsymbol{E}(t)$; the modification caused by the motion of the vortices is described by the first term.

Let us consider the spatial distribution of $\dot{\boldsymbol{p}}_s(\boldsymbol{r}, t)$ inside the plaquette of a given vortex a, that is at $|\boldsymbol{r} - \boldsymbol{r}_a| \lesssim l_H$. The term a in the sum in Eq. (11), which corresponds to the motion of the native core $\dot{\boldsymbol{r}}_a$, produces a near-field contribution which decays rapidly with the distance from the core

$$\dot{\boldsymbol{p}}_s^{(n)}(\boldsymbol{\rho}_a) = \hbar\hat{z}\bigg[\frac{2\boldsymbol{\rho}_a\,(\boldsymbol{\rho}_a \cdot \dot{\boldsymbol{r}}_a) - |\boldsymbol{\rho}_a|^2 \dot{\boldsymbol{r}}_a}{|\boldsymbol{\rho}_a|^4}\bigg]. \tag{12}$$

Here we introduce the notation $\boldsymbol{\rho}_a = \boldsymbol{r} - \boldsymbol{r}_a$. The remaining contribution, which is produced by the motion of the other vortices together with the second term, does not fall off with distance and may be approximated by a constant for $|\boldsymbol{r} - \boldsymbol{r}_a(t)| \lesssim l_H$. We refer to it as the far-field contribution and denote it by $\dot{\boldsymbol{p}}_s^{(f)}$. By order of magnitude, $\dot{\boldsymbol{p}}_s^{(f)}$ coincides with the spatial average of the condensate acceleration in the system, $\langle \dot{\boldsymbol{p}}_s(t)\rangle$. In the linear approximation in the microwave field and at sufficiently low frequencies, where the viscous

[3] A more general formulation of the kinetic equation for quasiparticles in dirty limit involves two distribution functions f and f_1, where the distribution function f_1 is responsible for the charge imbalance. We neglected the latter because electromagnetic field absorption in uniform superconductors does not create a charge imbalance. In the weakly inhomegeneous regime considered here the generation of charge imbalance is small in the spatial gradients. At the same time, in the presence of pair breaking, it relaxes relatively quickly by local elastic scattering.

forces are negligible in comparison to pinning, $\langle \dot{\boldsymbol{p}}_s(t)\rangle$ is related to the electric field by the Campbell formula

$$\dot{\boldsymbol{p}}_s^{(f)} \sim \langle \dot{\boldsymbol{p}}_s(t)\rangle = \frac{\lambda_L^2}{\lambda_C^2(H)}\, e\boldsymbol{E}(t). \tag{13}$$

Here, λ_L is the London length, and $\lambda_C(H)$ is the Campbell [16] length. The latter depends on the pinning strength and characterizes the macroscopic superfluid density of the system.[4] Similarly, the typical vortex velocity $\dot{\boldsymbol{r}}_a$, which determines the magnitude of $\dot{\boldsymbol{p}}_s^{(n)}(\boldsymbol{u}_a)$, may be estimated as

$$\dot{\boldsymbol{r}}_a \sim \boldsymbol{v}_v'(t) = c\,\frac{\boldsymbol{E}(t)\times\boldsymbol{H}}{H^2}\left(1-\frac{\lambda_L^2}{\lambda_C^2(H)}\right). \tag{14}$$

The second term in the brackets on the right-hand side describes the pinning-induced reduction of the average vortex velocity from the value in Eq. (2).

Using the expression for the typical vortex velocity $\boldsymbol{v}_v'(t)$ in Eq. (14), the Bardeen-Stephen contribution to the conductivity for a pinned vortex lattice may be estimated as

$$\frac{\sigma_{BS}'}{\sigma_{BS}} \sim \left(1-\frac{\lambda_L^2}{\lambda_C^2(H)}\right)^2. \tag{15}$$

Assuming that the pinning strength is determined by the vortex cores and is independent of the magnetic field, the ratio of the Campbell and London length may be expressed in the form [16,17]

$$\frac{\lambda_C^2(H)}{\lambda_L^2} = 1 + \frac{\Phi_0 H d}{8\pi\lambda_L^2 k}. \tag{16}$$

Here k is the average "spring constant," which relates the average pinning force on the vortex $\boldsymbol{F}_{\rm pin} = -k\delta\boldsymbol{r}$ to the average vortex displacement $\delta\boldsymbol{r}$.

The difference $\lambda_C^2(H)/\lambda_L^2 - 1$ characterizes the effectiveness of pinning. At perfect pinning, $k\to\infty$, we have $\lambda_C^2(H)/\lambda_L^2 - 1 \to 0$ and $\langle\dot{\boldsymbol{p}}_s\rangle \to e\boldsymbol{E}$. At finite k, pinning becomes more effective as the magnetic field is reduced; $\lambda_C^2(H)/\lambda_L^2 - 1$ decreases. In particular at strong pinning, where the condensation energy of the vortex core changes by a factor of the order unity when the vortex is displaced by a distance ξ from the equilibrium position, we have $k \sim d\Delta^2\nu_n$. In this case the pinning effectiveness from Eq. (16) may be estimated as

$$\frac{\lambda_C^2(H)}{\lambda_L^2} - 1 \sim \frac{H}{H_{c2}}. \tag{17}$$

Here we used the fact that for dirty superconductors $\lambda_L^{-2} \sim \Delta\sigma_n/\hbar c^2 \sim e^2\nu_n D_n\Delta/\hbar c^2$.

In the next section, we evaluate the dissipation arising from the two contributions to the condensate acceleration. The dissipation caused by the near-field part of the condensate acceleration $\dot{\boldsymbol{p}}_s^{(n)}(\boldsymbol{\rho}_a)$ is localized to the vortex cores and corresponds to the Bardeen-Stephen contribution to the conductivity, Eq. (15). The dissipation due to the far-field part $\dot{\boldsymbol{p}}_s^{(f)}$ arises from quasiparticles outside the vortex cores. We will show that these quasiparticles produce the dominant contribution to the conductivity in a wide interval of physical parameters.

[4]Equation (13) reflects the fact that the time derivative of the superfluid transport current can be expressed in two equivalent forms, $\langle\dot{\boldsymbol{j}}\rangle = \frac{c^2}{4\pi e\lambda_L^2}\langle\dot{\boldsymbol{p}}_s(t)\rangle = \frac{c^2}{4\pi e\lambda_C^2(H)}e\boldsymbol{E}(t)$.

IV. ESTIMATES OF MICROWAVE CONDUCTIVITY

We now apply Eqs. (5), (6), and (8) to evaluate the microwave conductivity $\sigma(\omega)$ in the London regime, where the intervortex distance $\sim l_H$ significantly exceeds the vortex core size $\sim\xi$. We focus on the contribution to $\sigma(\omega)$, which arises from quasiparticles residing outside the vortex cores, where the level velocity is described by Eqs. (9) and (10). For simplicity we assume $T \gtrsim \Delta$ and focus on the dirty limit $\Delta\tau_{\rm el} \ll 1$. In this case the dependence of the quasiparticle density of states on the local superfluid momentum $\bar{p}_s(\boldsymbol{r})$ is characterized by the dimensionless parameter

$$\eta(p_s) = D_n p_s^2/\Delta. \tag{18}$$

The gap in the quasiparticle spectrum is lowered from Δ by the amount

$$\delta\epsilon(p_s) \sim \Delta\eta^{2/3}(p_s). \tag{19}$$

For $\epsilon > \Delta + \delta\epsilon(p_s)$, the sensitivity $\boldsymbol{V}(\epsilon,\boldsymbol{p}_s)$ and $\nu(\epsilon,p_s)$ are rapidly decreasing functions of energy ϵ [10]. Therefore, the dominant contribution to the microwave conductivity arises from quasiparticles with energies $|\epsilon-\Delta| \lesssim \delta\epsilon(p_s)$. In this energy interval the values of the level sensitivity $\boldsymbol{V}(\epsilon,\boldsymbol{p}_s)$ and the density of states $\nu(\epsilon,p_s)$ may be estimated as [10]

$$\boldsymbol{V}(\epsilon,\boldsymbol{p}_s) \sim D_n\boldsymbol{p}_s\eta^{-1/3}(p_s), \tag{20a}$$

$$\nu(\epsilon,p_s) \sim \nu_n\eta^{-1/3}(p_s), \tag{20b}$$

where ν_n is the normal-state density of states at the Fermi level.

Although in the presence of superfluid momentum the diffusion tensor is anisotropic, for $\Delta\tau_{\rm el} \ll 1$ the anisotropy is negligible, and we set $D_{ij} = \delta_{ij}D(\epsilon,\bar{p}_s)$. The value of the diffusion coefficient in the relevant energy interval may be estimated as

$$D(\bar{p}_s) \sim D_n\eta^{1/3}. \tag{21}$$

This estimate can be obtained[5] using the Larkin-Ovchinnikov equations [15].

A. Low-frequency regime

At the lowest frequencies, the microwave conductivity is dominated by inelastic relaxation processes. The reason is that diffusion cannot lead to full relaxation of the nonequilibrium quasiparticle density for quasiparticles with energies below the percolation threshold ϵ^*. Such quasiparticles are trapped inside a plaquette of a particular vortex. The size of the trapping region $r(\epsilon)$ for energy ϵ may be estimated using Eqs. (18)

[5]In general the diffusion coefficient is expressed in terms of the quasiclassical Green's functions g^R and g^A (see the Appendix in Ref. [5]), which must be calculated using Usadel's equation [18]. Since the Usadel's equation has already been solved in the relevant regime [11], we will omit the details of this calculation.

and (19) and the dependence of the superfluid momentum $\bar{p}_s(r)$ on the distance r to the vortex core. Since for $r \lesssim l_H$

$$\bar{p}_s(r) \sim \frac{\hbar}{r}, \tag{22}$$

we get

$$r(\epsilon) \sim \xi \left(\frac{\Delta}{\Delta - \epsilon} \right)^{3/4}. \tag{23}$$

The size of the trapping region increases with the quasiparticle energy ϵ and becomes of the order of the inter-vortex distance $\sim l_H$ as ϵ approaches the percolation threshold ϵ^* whose value may be estimated as

$$\Delta - \epsilon^* \sim \Delta \left(\frac{\xi}{l_H} \right)^{4/3}. \tag{24}$$

At frequencies smaller than the inverse diffusion time across the intervortex distance τ_{D}^{-1}, the nonequilibrium distribution function of the trapped quasiparticles becomes spatially uniform and depends only on the energy. This part of the distribution (zero mode of diffusion) can relax only via inelastic processes.

To describe this slow time relaxation of the zero mode we linearize Eq. (5) with respect to δn and average the result over the area of spatial confinement at energy ϵ in the ath vortex. This yields

$$\left(\partial_t + \frac{1}{\tau_{\mathrm{in}}} \right) \langle \delta n(\epsilon, t) \rangle_a = -\langle \dot{\boldsymbol{p}}_s(\boldsymbol{r}) \cdot \boldsymbol{V}(\epsilon, \bar{\boldsymbol{p}}_s(\boldsymbol{r})) \rangle_a \partial_\epsilon n_F(\epsilon), \tag{25}$$

where $\langle \ldots \rangle_a$ denotes averaging over the ϵ-dependent confinement region of the ath vortex.

It is important to note that in the case of perfectly symmetric lattice the right-hand side of Eq. (25) vanishes. However in the presence of disorder, the superfluid momentum around the vortices is asymmetric, and this term is nonzero. Thus, $\delta \langle n(\epsilon) \rangle_a$ and $\langle \dot{\boldsymbol{p}}_s(\boldsymbol{r}) \cdot \boldsymbol{V}(\epsilon, \bar{\boldsymbol{p}}_s(\boldsymbol{r})) \rangle_a$ are random quantities which fluctuate from plaquette to plaquette. Equations (20a) and (22) show that these quantities are dominated by distances from the core, which are of the order of the radius of the trapping region in Eq. (23). In this region $\dot{\boldsymbol{p}}_s(\boldsymbol{r})$ may be approximated by $\dot{\boldsymbol{p}}_s^{(f)}(\boldsymbol{r})$ in Eq. (13). Making this approximation, substituting $\delta \langle n(\epsilon) \rangle_a$ from Eq. (25) into Eq. (8), averaging over plaquettes, and using Eq. (3), we obtain the following result for the microwave conductivity $\sigma(\omega)$ at $T \gtrsim \Delta$:

$$\frac{\sigma(\omega)}{\sigma'_{BS}} \sim \frac{\tau_{\mathrm{in}}}{\tau_{\mathrm{el}}} \frac{1}{\left[1 + (\omega \tau_{\mathrm{in}})^2\right]} \frac{H}{H_{c2}} \frac{\lambda_L^4(H)}{\left(\lambda_C^2(H) - \lambda_L^2\right)^2} \times \int_0^{\epsilon^*} d\epsilon \frac{\left\langle \langle \nu(\epsilon, \bar{p}_s(\boldsymbol{r})) \rangle_a \langle \boldsymbol{n} \cdot \boldsymbol{V}(\epsilon, \boldsymbol{r}) \rangle_a^2 \right\rangle}{T \nu_n v_{\mathrm{F}}^2}, \tag{26}$$

where the outer brackets indicate averaging over plaquettes, and we introduced the unit vector $\boldsymbol{n}$ in the direction of $\boldsymbol{E}$. To arrive at this expression we used Eqs. (1), (15), the Einstein relation for the normal-state conductivity, $\sigma_n = e^2 D_n \nu_n$, and the approximation $\partial_\epsilon n_F(\epsilon) \approx -1/4T$ for $\epsilon \lesssim \epsilon^*$. We note that for the zero-mode limit only the second term in the brackets in Eq. (8) contributes to the entropy production rate.

Substituting the estimates from Eqs. (20), (22), and (23) into Eq. (26) we find that the dominant contribution to microwave conductivity arises from trapped quasiparticles with energies near the percolation threshold ϵ^* given by Eq. (24). Using Eq. (16), and the relation $\xi^2/l_H^2 \sim H/H_{c2}$, we can express the result in the form

$$\frac{\sigma(\omega)}{\sigma'_{BS}} \sim K \frac{\Delta}{T} \frac{\tau_{\mathrm{in}} \Delta}{[1 + (\omega \tau_{\mathrm{in}})^2]} \frac{\lambda_L^4(H)}{\left[\lambda_C^2(H) - \lambda_L^2\right]^2} \left(\frac{H}{H_{c2}} \right)^{5/3}, \tag{27}$$

where the parameter

$$K = \frac{\left\langle \langle \boldsymbol{n} \cdot \boldsymbol{V}(\epsilon^*, \boldsymbol{r}) \rangle_a^2 \right\rangle}{\langle \langle V(\epsilon^*, \bar{p}_s(\boldsymbol{r}) \rangle_a^2 \rangle} \tag{28}$$

characterizes the degree of spatial asymmetry of the condensate momentum $\boldsymbol{p}_s(\boldsymbol{r})$ inside the trapping areas in the pinned vortex lattice. The value of K is nonuniversal and depends on the details of the pinning potential. If the relative amplitude of fluctuations of $\boldsymbol{V}(\epsilon, \boldsymbol{r})$ is of order unity, and their correlation radius is on the order of the plaquette size, then $K \approx 1$. Using the estimate Eq. (17), we obtain for strong pinning

$$\frac{\sigma(\omega)}{\sigma'_{BS}} \sim \frac{\Delta}{T} \frac{\tau_{\mathrm{in}} \Delta}{[1 + (\omega \tau_{\mathrm{in}})^2]} \left(\frac{H_{c2}}{H} \right)^{1/3}. \tag{29}$$

Since in typical situations $\tau_{\mathrm{in}} \Delta \gg 1$, Eqs. (27) and (29) show that the inelastic relaxation gives the main contribution to the conductivity in the low-frequency regime.

B. High-frequency regime

At $\omega \tau_{\mathrm{in}} > 1$ the contribution of inelastic relaxation to $\sigma(\omega)$ in Eq. (27) decreases as $1/\omega^2$, and at sufficiently high frequencies the microwave conductivity is dominated by spatial diffusion of nonequilibrium quasiparticles inside the trapping regions.

The diffusive contribution to dissipation is present even in a perfect vortex lattice. Therefore, to estimate it we neglect the distortion of the vortex lattice. We assume the axially symmetric distribution of the superfluid momentum $\bar{\boldsymbol{p}}_s$ inside a given plaquette, Eq. (22), and write the diffusion equation Eq. (5) in polar coordinates r and θ. The dependence of the level velocity $v_\nu(r, \theta)$ in Eq. (9) and nonequilibrium quasiparticle density δn on the azimuth angle corresponds to the first angular harmonic

$$\delta n(\epsilon, r, \theta; t) = n_1(\epsilon, r; t) \cos \theta.$$

Substituting this form into the linearized Eq. (5) we get

$$\partial_t n_1(\epsilon, r; t) + \frac{1}{r} \partial_r (r D(\epsilon, \bar{p}_s(r)) \partial_r n_1(\epsilon, r; t)) - \frac{D(\epsilon, \bar{p}_s(r)) n_1(\epsilon, r; t)}{r^2} = -\langle \dot{p}_s^{(f)} \rangle V(\epsilon, \bar{p}_s(r)) \partial_\epsilon n_F. \tag{30}$$

Here we neglected the inelastic collision integral and focused on the long-distance part of the condensate acceleration given by Eq. (13).

For the relevant quasiparticle energies, the diffusion coefficient $D(\epsilon, \bar{p}_s(r))$ and the level sensitivity $V(\epsilon, \bar{p}_s(r))$ have a power-law dependence of r, which is given by Eqs. (18), (20a), (21), and (22). For a microwave field of frequency ω, the solution of this equation is characterized by the length scale L_ω corresponding to the diffusion distance of the relevant quasiparticles during the oscillation period, $D(\bar{p}_s(L_\omega))/L_\omega^2 = \omega$, and can be estimated using Eqs. (18), (21), and (22), giving

$$L_\omega \sim \xi \left(\frac{\Delta}{\omega}\right)^{3/8}. \tag{31}$$

In the regions $\xi \ll r \ll L_\omega$ and $L_\omega \ll r$, the solution to Eq. (30), $n_1(r)$, has a power-law dependence on r. This can be seen as follows. The diffusion coefficient decreases with distance from the core as $D(\bar{p}_s(r)) \propto 1/r^{2/3}$, as follows from Eqs. (18), (21), and (22). Therefore, in the spatial region $r \ll L(\omega)$ one may neglect $\partial_t n_1 = -i\omega n_1$. Because of the power-law dependence of the diffusion coefficient and the sensitivity on r, the resulting solution is a power of r. In the complementary region $r \gg L(\omega)$, one can neglect the diffusive terms in comparison to $\partial_t n_1 = -i\omega n_1$. In this region the solution n_1 still has a power-law dependence on r because of the power-law dependence of the sensitivity.

Expressing $\langle \dot{p}_s^{(f)} \rangle$ in terms of the microwave field using Eq. (13), approximating $\partial_r n_1 \sim n_1/L_\omega$, as well as using Eqs. (18), (21), and (22), one finds from Eq. (30)

$$n_1 \sim eE\frac{\lambda_L^2}{\lambda_C^2(H)}\partial_\epsilon n_F \begin{cases} \frac{r^{7/3}}{\xi^{4/3}}, & r \ll L_\omega, \\ -i\frac{L_\omega^{8/3}}{r^{1/3}\xi^{4/3}}, & r \gg L_\omega. \end{cases} \tag{32}$$

Substituting Eq. (32) into the first term of the right-hand side of Eq. (8) we get

$$\frac{\sigma(\omega)}{\sigma'_{BS}} \sim \frac{\Delta}{T}\frac{\lambda_L^4}{\left(\lambda_C^2 - \lambda_L^2\right)^2}\left(\frac{H}{H_{c2}}\right)^{1/3} \begin{cases} 1, & \omega \ll \tau_D^{-1}, \\ (\omega\tau_D)^{-5/4}, & \omega \gg \tau_D^{-1}. \end{cases} \tag{33}$$

Here, the diffusion time of the relevant quasiparticles across the plaquette, $\tau_D = l_H^2/D(\bar{p}_s(l_H))$. It may be estimated as

$$\tau_D \sim \frac{\hbar}{\Delta}\left(\frac{l_H}{\xi}\right)^{8/3}. \tag{34}$$

Comparing Eq. (27) and Eq. (33) we conclude that, for $\omega < \tau_{\mathrm{D}}^{-1}$, diffusive relaxation gives the dominant contribution to the conductivity for $\omega \gtrsim \omega^*$, where the crossover frequency ω^* may be estimated as

$$\omega^* \sim \sqrt{\frac{K\Delta}{\tau_{\mathrm{in}}}}\left(\frac{H}{H_{c2}}\right)^{2/3} \sim (\tau_{\mathrm{in}}\tau_{\mathrm{D}})^{-1/2}. \tag{35}$$

At strong pinning, see Eq. (17), the expression for the conductivity in Eq. (33) simplifies to

$$\frac{\sigma(\omega)}{\sigma'_{BS}} \sim \frac{\Delta}{T}\left(\frac{H_{c2}}{H}\right)^{5/3} \begin{cases} 1, & \omega \ll \tau_D^{-1}, \\ (\omega\tau_D)^{-5/4}, & \omega \gg \tau_D^{-1}. \end{cases} \tag{36}$$

In this case, in a broad frequency interval $\omega^* < \omega < \tau_D^{-1}$, the ac conductivity is dominated by diffusion of quasiparticles outside the vortex cores for sufficiently weak magnetic fields $H \ll H_{c2}$.

Finally, we note that replacing $\langle \dot{p}_s^{(f)}(r) \rangle$ in Eq. (30) by $\dot{p}_s^{(n)}(r)$ from Eq. (12) (which has the same angular dependence) gives a contribution to dissipation, which is dominated by short distances from the core, and reproduces the Bardeen-Stephen result σ'_{BS}.

V. DISCUSSION OF THE RESULTS

We developed a theory of microwave absorption of type-II superconductors in the presence of a strongly pinned vortex lattice. In this case, in addition to the Bardeen-Stephen contribution to the dissipative conductivity σ'_{BS}, which is caused by the vortex motion and is described by Eqs. (1), (15), and (16), there is another contribution. This new contribution is caused by the quasiparticle spectral flow induced by the condensate acceleration in the presence of a microwave field. This contribution exists even in the absence of vortex displacements and exceeds σ'_{BS} in a wide interval of physical parameters. For $T \gtrsim \Delta$, this contribution is dominated by the quasiparticles residing outside the vortex cores.

At low frequencies the relaxation of the nonequilibrium distribution quasiparticles induced by the spectral flow is mediated by inelastic processes. As a result, the dissipative conductivity $\sigma(\omega)$ in this regime is controlled by the inelastic relaxation time τ_{in}, see Eq. (27). At $\omega \ll \tau_{\mathrm{in}}$ the lowest frequencies $\sigma(\omega)$ becomes proportional to τ_{in}. Therefore it can be parametrically larger than σ'_{BS}, which is proportional to the elastic mean-free time τ_{el}. The mechanism of absorption in this case is similar to the Debye mechanism of electromagnetic wave absorption is molecules [19]. The difference, however, is that the Debye absorption coefficient in molecules is quadratic in frequency, while the dissipative part of conductivity in superconductors $\sigma(\omega)$ is nonzero at $\omega \to 0$. The low-frequency conductivity is proportional to the degree of distortion of the spatial distribution of the superfluid momentum in the lattice, which is caused by the pinning and characterized by the parameter K in Eq. (28).

At frequencies above ω^* given by Eq. (35) the conductivity is dominated by diffusion of quasiparticles across the vortex. The characteristic time scale τ_D for this process is given by Eq. (34), and is still much larger than τ_{el}. In this regime the conductivity exhibits a nontrivial frequency dependence given by Eq. (33). The distortion of the vortex lattice induced by the pining becomes inessential in this regime.

At small temperatures $T \ll \Delta$, the quasiparticles reside only in the vortex cores. In this case the low-frequency contribution to conductivity, which is proportional to τ_{in}, still exists, but the coefficient K in Eq. (28) is determined by the deformation of the core.

We note that in the case of weak pinning there is another mechanism of low-frequency microwave absorption, which is proportional to τ_{in}, Ref. [17]. It is related to vortex motion and exists also in the flux-flow regime [12]. The main contribution to σ in this case comes from quasiparticles in the vortex cores.

Although the estimates for the microwave conductivity were obtained in the dirty regime $\Delta\tau_{\mathrm{el}} \ll 1$, qualitatively, our conclusions remain valid for arbitrary value of $\Delta\tau_{\mathrm{el}}$, provided $l_H \gg v_F\tau_{\mathrm{el}}$. In the clean case $\Delta\tau_{\mathrm{el}} \gg 1$, one would need to use different estimates for the sensitivities and relevant intervals, which can be found in Ref. [10].

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