Broken time reversal symmetry vestigial state for a two-component superconductor in two spatial dimensions

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We consider the vestigial phase with broken time reversal symmetry above the superconducting transition temperature of a two-component superconductor in two spatial dimensions. We show that, in contrast to three dimensions, a vestigial phase is in general allowed within Ginzburg-Landau theory. The vestigial phase occupies an increasing temperature region if the parameters in the Ginzburg-Landau theory give a larger energy difference between the broken time reversal symmetry phase and the other ordered phase.

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I. INTRODUCTION

Consider a superconductor with the order parameter (Φ_1, Φ_2) belonging to a two-dimensional (2D) representation (see, e.g., Ref. [1]). Here, $\Phi_{1,2}$ are complex fields. Under gauge transformation by χ, the order parameter transforms as $(\Phi_1', \Phi_2') = e^{i\chi}(\Phi_1, \Phi_2)$, while spatial symmetry operations result in a transformation of (Φ_1, Φ_2) among themselves. For example, $(\Phi_1', \Phi_2') = [\Phi_1 \cos(\phi) + \Phi_2 \sin(\phi), \Phi_2 \cos(\phi) - \Phi_1 \sin(\phi)]$ under rotations by ϕ, with $\phi = \frac{2\pi}{n}$ where n is an integer. We shall be mainly interested in $n = 3$ and $n = 6$ for trigonal and hexagonal systems, respectively. Such multidimensional order parameters have been considered extensively since superfluid ^{3}He [2], heavy fermion superconductors [3,4], and also more recently in many other superconducting systems [5,6], as well as Bose-Einstein condensates with internal degrees of freedom [7].

When the order parameter acquires a nonzero expectation value, the system is in the superconducting phase and spontaneously breaks the U(1) gauge symmetry. If the order parameter belongs to a multidimensional representation, additional symmetry must also be broken. In the above-mentioned two-dimensional representation example, depending on the microscopic details, the energy minimum can be achieved by having $(\Phi_1, \Phi_2) \propto (1, 0)$, $(0, 1)$ (or their rotated counterparts), or $(\Phi_1, \Phi_2) \propto (1, \pm i)$. In the former case, the order parameter breaks gauge invariance and rotational invariance, whereas in the latter, it breaks gauge invariance as well as time reversal invariance [under which $\Phi_{1,2} \to \Phi_{1,2}^*$, so $(1, i) \to (1, -i)$]. In each case, as opposed to the case of an order parameter Φ belonging to a one-dimensional representation, there is additional symmetry breaking other than the gauge symmetry.

The high-temperature phase of the system has all symmetries intact, and will be called the symmetric phase. Within mean-field theory, a single-phase transition (usually second order) at the superconducting transition temperature separates the symmetric and superconducting phases: Both the gauge and rotational (or time reversal) symmetries are broken at this phase transition. In principle at least, in a more complex scenario such as when fluctuations are included, these broken symmetries do not have to occur at the same time. In particular, one can have a phase where, say, the rotational (or time reversal) symmetry is broken, whereas the gauge symmetry is still intact.

This intermediate state is characterized by the vanishing of the expectation values $\langle \Phi_{1,2} \rangle = 0$, whereas some higher-order combinations of $\Phi_{1,2}$ acquire finite expectation values. For instance, $\langle \Phi_1^* \Phi_1 - \Phi_2^* \Phi_2 \rangle \neq 0$, which breaks rotational symmetry, or $i\langle (\Phi_1^* \Phi_2 - \Phi_2^* \Phi_1) \rangle \neq 0$, which breaks time reversal. In additional to the above scenarios, one can have the possibilities that some other symmetry-violating combinations of $\Phi_{1,2}$ acquiring nonzero expectation values. For example, we can have $\langle \Phi_1^2 + \Phi_2^2 \rangle \neq 0$ even though $\langle \Phi_{1,2} \rangle = 0$. In this latter case, while the order parameter is not preserved under general gauge transformations, it is preserved under a special transformation $\chi \to \chi + \pi$, and thus describes $4e$ pairing [8]. Such phases, often called "vestigial" phases or a phase with "composite" or "higher-order" parameters, are gaining attention in the recent literature [9–16], though they have been investigated already in the past in similar [17–20] and related (e.g., Refs. [21–25]) contexts. Besides superconductivity, these exotic phases are also relevant to other, e.g., magnetic, systems [26–28].

In a previous paper [29], considering three spatial dimensions, we show that, within a Ginzburg-Landau theory with thermal fluctuations, such a vestigial phase is in general not stable, except for the case of extreme gradient energy terms in the free energy. This is because, when the temperature is

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lowered so that the completely symmetric phase is no longer the free-energy stable minimum, either (i) no saddle point corresponding to the vestigial phase exists, so that one has a direct second-order phase transition to the superfluid phase with broken rotational (time reversal) symmetry as well as gauge symmetry, or (ii) the vestigial phase with a composite order parameter is only a saddle point but fails to be a free-energy minimum. Instead, the free-energy minimum occurs in the region where the expectation value of $\Phi_{1,2}$ is/are finite. The system thus makes a joint first-order phase transition into the superconducting case. A similar situation can be shown to occur for a multicomponent Bose gas [30,31].

In this paper, we consider instead two spatial dimensions. We show that the situation becomes quite different. Case (i) above remains a possibility, but for case (ii), the saddle point does become stable in general for a finite range in temperature, due to a very different free-energy landscape. A similar strong dependence of the stability of vestigial phases on the spatial dimensionality has also been found in, e.g., Refs. [24,25].

We shall mainly be studying the broken time reversal symmetry state. Our approach, based on Ginzburg-Landau analysis, differs significant from the existing treatments of the vestigial problem in 2D. References [14,24,32–34] generally focus on topological excitations, and no amplitude fluctuations were included. References [14,32,33] depict all transitions (superfluid and vestigial) as Berezinskii-Kosterlitz-Thouless (BKT) type. Reference [35] does consider amplitude fluctuations of the order parameter, but still with the sum of the amplitudes (corresponding to our $|\Phi_\uparrow|^2 + |\Phi_\downarrow|^2$) fixed to a constant. These works are mostly numerical. In contrast, we obtain an effective ϕ^4 theory analytically for this Ising transition in terms of the parameters entering the Ginzburg-Landau theory.

Our calculations will be presented in Sec. II. In Sec. III, we shall include also a short discussion on the nematic case and the 4e state as well as conclusions.

II. THEORY FOR VESTIGIAL ORDER

For the reminder of this paper, it would be convenient to employ the "spin-$\frac{1}{2}$" notation $\Phi_{\uparrow,\downarrow} = \frac{1}{\sqrt{2}}[\Phi_1 \pm i\Phi_2]$. Under $\frac{2\pi}{n}$ rotation, they transform as $(\Phi_\uparrow, \Phi_\downarrow) \to (e^{-\frac{2\pi i}{n}}\Phi_\uparrow, e^{\frac{2\pi i}{n}}\Phi_\downarrow)$. Specializing to trigonal and hexagonal systems, we write down an effective Hamiltonian density $\mathcal{H}$ that is consistent with rotational, gauge, and time reversal symmetries,

$$\mathcal{H} = \mathcal{H}_K + \mathcal{H}_{\text{int}}, \tag{1}$$

with the "kinetic" part

$$\mathcal{H}_K = \sum_{s=\uparrow,\downarrow}\left[\alpha\Phi_s^*\Phi_s + K\left(\frac{\partial\Phi_s^*}{\partial x_i}\frac{\partial\Phi_s}{\partial x_i}\right)\right] \tag{2}$$

and the interacting part

$$\mathcal{H}_{\text{int}} = \frac{g_1}{2}(|\Phi_\uparrow|^4 + |\Phi_\downarrow|^4) + g_2(|\Phi_\uparrow|^2|\Phi_\downarrow|^2). \tag{3}$$

Here, x_i, with $i = 1, 2$, are the spatial coordinates. The interaction term in Eq. (3) is the most general quartic term allowed by symmetry. Note however that the "kinetic part" adopted in (2) is fully isotropic (invariant under separate rotations of space x_i and order parameter Φ_s). In general, more complicated gradient terms are allowed (see, e.g., Ref. [1]), which we opt to ignore. We remind the readers that Ref. [29] established the absence of the vestigial phase in three spatial dimensions when the gradient energy is of this form. Here, $\alpha = \alpha(T)$ is positive (negative) above (below) a mean-field transition temperature which we shall label as T_0, thus $\alpha(T) \approx \alpha'(T - T_0)$ with $\alpha' > 0$.

So far we choose to view this model as representing a two-component order parameter. One may switch back to the $\Phi_{1,2}$ basis and see that the quartic term now contains a contribution $\Phi_1^{*2}\Phi_2^2 + \Phi_2^{*1}\Phi_1^2$ [29]. With $\mathcal{H}_K$ fully isotropic, the model can thus alternatively be viewed as a special case of two superconductors $\Phi_{1,2}$ coupled together via a "four-electron" (two Cooper pairs) tunneling term $\Phi_1^{*2}\Phi_2^2 + \Phi_2^{*1}\Phi_1^2$ [32,33] (among others).

Mean-field theory amounts to simply assuming uniform Φ_s and minimizing $\mathcal{H}$. The system is in the completely symmetric (normal) phase with $\Phi_{\uparrow,\downarrow} = 0$ if $\alpha > 0$. For $\alpha < 0$, we have the one of the following: (i) $\Phi_\uparrow \neq 0$, $|\Phi_\uparrow|^2 = \frac{|\alpha|}{g_1}$ (or $\uparrow\leftrightarrow\downarrow$), with free-energy density $-\frac{\alpha^2}{2g_1}$, with time reversal symmetry broken, or (ii) $|\Phi_\uparrow| = |\Phi_\downarrow| = \frac{|\alpha|}{2(g_1+g_2)}$, with free energy $-\frac{\alpha^2}{2(g_1+g_2)}$, with rotational symmetry broken (nematic). The stability of these mean-field states requires $g_1 > 0$, $g_2 > -g_1$. The broken time reversal symmetry state has lower energy when $g_2 > g_1 > 0$. We shall focus on this region unless otherwise stated.

At finite temperatures, we need to consider the partition function [36,37] $Z \equiv \int_{\Phi_s} e^{-\int d^2x\mathcal{H}/T}$, where $\int_{\Phi_s}$ means sum over all configurations of $\Phi_s(\vec{r})$. We employ the Hartree-Fock (HF) approximation. The effective Hamiltonian density becomes

$$\mathcal{H}_{\text{eff}} = \mathcal{H}_K - h_\uparrow\Phi_\uparrow^*\Phi_\uparrow - h_\downarrow\Phi_\downarrow^*\Phi_\downarrow, \tag{4}$$

where $h_{\uparrow,\downarrow}$ are the self-energies (not to be confused with external magnetic fields), which are to be obtained self-consistently. In the calculations below, we take the equivalent procedure regarding $h_{\uparrow,\downarrow}$ as variational parameters, treat the free energy as a functional of these parameters, and minimize. $h_\uparrow \neq h_\downarrow$ signals that $\langle\Phi_\uparrow^*\Phi_\uparrow\rangle \neq \langle\Phi_\downarrow^*\Phi_\downarrow\rangle$ [hence $i\langle(\Phi_1^*\Phi_2 - \Phi_2^*\Phi_1)\rangle \neq 0$]. $h_\uparrow - h_\downarrow$ thus serves as an order parameter for the broken Z_2 symmetry.

After Fourier transform,

$$\mathcal{H}_{\text{eff}} = \sum_{s,\vec{k}}\Phi_{\vec{k},s}^*(\alpha + Kk^2 - h_s)\Phi_{\vec{k},s}, \tag{5}$$

where $\vec{k}$ represents the wave vector. We thus have the expectation values

$$\langle\Phi_{\vec{k},s}\Phi_{\vec{k},s'}^*\rangle = TG_s(\vec{k})\delta_{s,s'}, \tag{6}$$

with the "Green's function"

$$G_s(\vec{k}) = \frac{1}{\alpha + Kk^2 - h_s}. \tag{7}$$

For the vestigial phase, we must have $\alpha - h_s > 0$.

The free-energy density is, within the HF approximation,

$$\mathcal{F} = \frac{T}{L^2}\sum_{\vec{k},s}[\ln(\alpha + Kk^2 - h_s) + h_s G_s(\vec{k})] + g_1\left[\left(\frac{T}{L^2}\sum_{\vec{k}} G_\uparrow(\vec{k})\right)^2 + \left(\frac{T}{L^2}\sum_{\vec{k}} G_\downarrow(\vec{k})\right)^2\right] + g_2\left[\left(\frac{T}{L^2}\sum_{\vec{k}} G_\uparrow(\vec{k})\right)\times\left(\frac{T}{L^2}\sum_{\vec{k}} G_\downarrow(\vec{k})\right)\right]. \quad (8)$$

This expression is ultraviolet divergent, both because of the $\ln(\alpha + Kk^2 - h_s)$ and interaction terms. These divergences are also present even for $F \equiv F_0$ where we set h_s to be zero [and thus replace $G_s(\vec{k})$ by $G_0(\vec{k}) \equiv \frac{1}{\alpha + Kk^2}$]. However, we note that insertion of the Hartree-Fock self-energies $(2g_1 + g_2)\frac{T}{L^2}\sum_{\vec{k}} G_0(\vec{k})$ to the propagators G_s or G_0 would amount to replacing α by $\alpha + (2g_1 + g_2)\frac{T}{L^2}\sum_{\vec{k}} G_0(\vec{k})$, which can be regarded as a redefinition of α. Using this renormalized $\alpha(T)$, the difference of the free-energy density between the phase under consideration and F_0 can then be written as [29]

$$\Delta\mathcal{F} = \frac{T}{L^2}\sum_{\vec{k},s}[\ln(\alpha + Kk^2 - h_s) - \ln(\alpha + Kk^2) + h_s G_s(\vec{k})] + g_1\left[\left(\frac{T}{L^2}\sum_{\vec{k}}[G_\uparrow(\vec{k}) - G_0(\vec{k})]\right)^2 + \left(\frac{T}{L^2}\sum_{\vec{k}}[G_\downarrow(\vec{k}) - G_0(\vec{k})]\right)^2\right] + g_2\left[\left(\frac{T}{L^2}\sum_{\vec{k}}[G_\uparrow(\vec{k}) - G_0(\vec{k})]\right)\times\left(\frac{T}{L^2}\sum_{\vec{k}}[G_\downarrow(\vec{k}) - G_0(\vec{k})]\right)\right]. \quad (9)$$

This expression is ultraviolet convergent, and the contributions giving rise to finite $\Delta\mathcal{F}$ arise only for small wave vectors when α and h_s are small, as it should be.

The momentum sums can be easily evaluated. For 3D, we reproduce the result in Ref. [29]. In the present case, we get

$$\Delta\mathcal{F} = -\frac{T}{4\pi K}\left\{\left[\alpha\ln\left(1 - \frac{h_\uparrow}{\alpha}\right) + h_\uparrow\right] + \left[\alpha\ln\left(1 - \frac{h_\downarrow}{\alpha}\right) + h_\downarrow\right]\right\} + \frac{T^2 g_1}{(4\pi K)^2}\left\{\left[\ln\left(1 - \frac{h_\uparrow}{\alpha}\right)\right]^2 + \left[\ln\left(1 - \frac{h_\downarrow}{\alpha}\right)\right]^2\right\} + \frac{T^2 g_2}{(4\pi K)^2}\left[\ln\left(1 - \frac{h_\uparrow}{\alpha}\right)\ln\left(1 - \frac{h_\downarrow}{\alpha}\right)\right]. \quad (10)$$

An important point to note is that, in contrast to the three-dimensional case [29], this free energy diverges to $+\infty$ due to the g_1 term (since $g_1 > 0$) when $h_s \to \alpha_-$. Hence there is no "falling off" to the unphysical $(\alpha - h_s < 0)$ region, in contrast to Ref. [29], and stable nontrivial minima can exist within the physical $h_s < \alpha$ region. See Fig. 1.

Expansion of the free energy in terms of $h_z \equiv (h_\uparrow - h_\downarrow)/2$ and $h_0 \equiv (h_\uparrow + h_\downarrow)/2$ gives

$$\Delta\mathcal{F} = a h_z^2 + b h_z^4 + \gamma h_0 h_z^2 + c h_0^2, \quad (11)$$

where

$$a = T I_2[1 + T(2g_1 - g_2)I_2], \quad (12)$$

$$b = \tfrac{3}{2}T I_4 + 2T^2 g_1\left(I_3^2 + 2I_2 I_4\right) + T^2 g_2\left(I_3^2 - 2I_2 I_4\right), \quad (13)$$

$$\gamma = 4T I_3 + 2T^2(6g_1 - g_2)I_2 I_3, \quad (14)$$

$$c = T I_2[1 + T(2g_1 + g_2)I_2]. \quad (15)$$

Here, $I_2 = \frac{1}{4\pi K\alpha}$ and generally $I_n = \frac{1}{(n-1)4\pi K\alpha^{n-1}}$ for $n \geqslant 2$.

The coefficient a changes sign at T at T_2 where

$$0 = 1 + (2g_1 - g_2)\frac{T_2}{4\pi K\alpha(T_2)}, \quad (16)$$

signaling a phase transition (at T_2 if second order). This transition thus exists only when $g_2 - 2g_1 > 0$. Equation (11) implies $h_0 = -\frac{\gamma}{2c}h_z^2$. Eliminating h_0, the effective coefficient for h_z^4 becomes $b - \frac{\gamma^2}{4c}$. The value of this coefficient at T_2 is given by $b_{\rm eff} = \frac{T_2}{4\pi K\alpha^3(T_2)}\frac{6g_1 - g_2}{24 g_2}$ hence positive only when $g_2 < 6g_1$. Hence the transition is second order only when $g_2 < 6g_1$ [38]. See the Appendix for further analysis on this point. Below we shall confine ourselves only to this parameter regime. Since α is rapidly varying with temperature near T_0, Eq. (16) implies

$$T_2 \approx T_0\left[1 + \frac{(g_2 - 2g_1)}{4\pi K\alpha'}\right], \quad (17)$$

hence a transition temperature increasing from T_0 linearly with $g_2 - 2g_1$ when the latter is positive. Below T_2, $h_z^2 \approx -\frac{a'}{2b_{\rm eff}}(T - T_2)$, with $a' = -\frac{T_2}{4\pi K\alpha^3(T_2)}\alpha'$.

The above has assumed that the transition is to a state with uniform $h_{0,z}$. One can also consider the free energy F for the

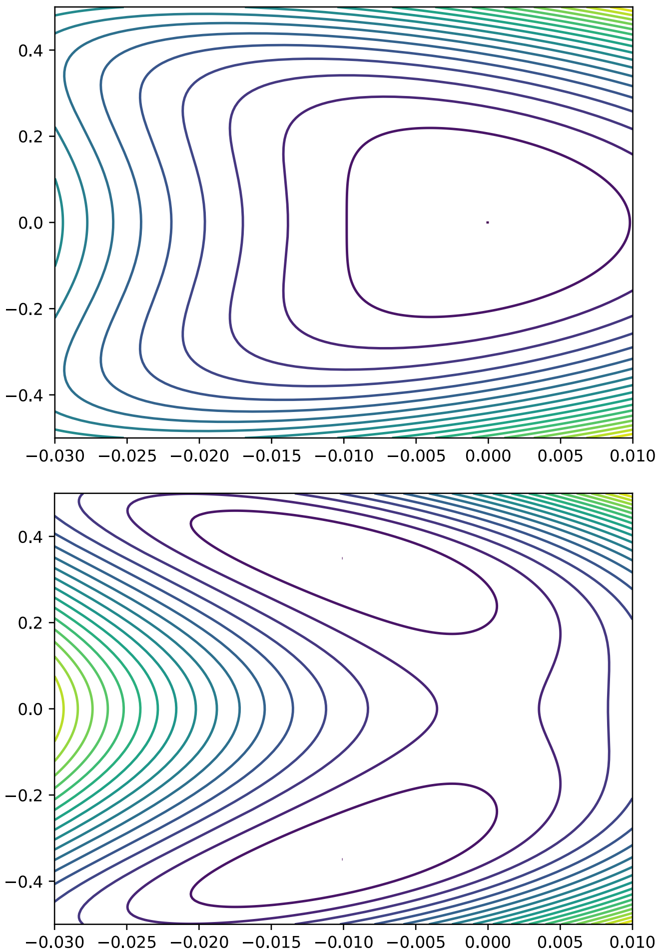

FIG. 1. Example contour plots of the free energy in Eq. (10). Abscissa: $[\ln(1-\frac{h_\uparrow}{\alpha})+\ln(1-\frac{h_\downarrow}{\alpha})]/2 \equiv x$; ordinate: $[\ln(1-\frac{h_\uparrow}{\alpha})-\ln(1-\frac{h_\downarrow}{\alpha})]/2 \equiv y$. Upper diagram: Symmetric phase, with free-energy minimum located at $h_\uparrow = h_\downarrow$, hence $y=0$. Lower diagram: Broken symmetry phase, with two degenerate minima, at $h_\uparrow \neq h_\downarrow$, $y \neq 0$.

case where the self-energies h_s vary with position. If these fields have wave vector $\vec{Q}$, then the free energy has the form

$$\Delta\mathcal{F} = a(Q)h_z(\vec{Q})h_z(-\vec{Q}) + \cdots, \tag{18}$$

with

$$a(Q) = TI_2(Q)[1 + T(2g_1 - g_2)I_2(Q)], \tag{19}$$

where

$$I_2(Q) \equiv \frac{1}{L^2}\sum_{\vec{k}} \frac{1}{(\alpha + Kk_+^2)(\alpha + Kk_-^2)}, \tag{20}$$

with $\vec{k}_\pm = \vec{k} \pm \frac{\vec{Q}}{2}$. $I_2(Q) = I_2$ if $Q = 0$, decreases with increasing Q or α, and is positive definite if $\alpha > 0$. Hence if $2g_1 - g_2 > 0$, $a(Q)$ is positive for any Q and positive α. If $2g_1 - g_2 < 0$, $a(Q) > 0$ for all Q's at high temperatures, and at T_2, $a(Q)$ changes sign at $Q = 0$ with $a(Q) > 0$ at $Q \neq 0$, verifying that the transition is to the uniform state.

The above considerations show that, for long-wavelength fluctuations of h_z, the free-energy density has the form

$$\Delta\mathcal{F} = ah_z^2 + \tilde{K}(\vec{\nabla}h_z)^2 + b_{\text{eff}}h_z^4. \tag{21}$$

The coefficient $\tilde{K}$ can be obtained from an expansion of $a(Q)$ at small Q. Using $I_2(Q) = I_2 - \alpha\frac{KQ^2}{2}I_4(0)$, we obtain thus

$$\tilde{K} = -\frac{\alpha K}{2}I_4[1 + 2T(2g_1 - g_2)I_2]. \tag{22}$$

Near $a = 0$ [T_2 given in Eq. (16)], $\tilde{K} \approx \frac{T}{12\pi\alpha} > 0$ [39]. Equation (21) represents an effective Hamiltonian for a second-order Ising transition, with $\tilde{K} > 0$ and $b_{\text{eff}} > 0$ (the latter holds if $g_2 < 6g_1$, as already mentioned).

The above considerations find the minimum of the free energy in h_z. More precisely, h_z is itself a fluctuating quantity and the free-energy density in Eq. (21) should be regarded as the effective Hamiltonian density for $h_z(\vec{r})$. We thus obtained an effective ϕ^4 theory for the Ising transition where h_z plays the role of the order parameter for the Z_2 transition. The considerations so far thus give, upon lowering of temperature, an Ising transition from a completely symmetric phase to a Z_2 broken symmetry phase ($h_z \neq 0$ yet with $\langle\Phi_s\rangle = 0$) at $T_2 > T_0$ (thus $\alpha > 0$) given by Eq. (17) if $g_2 > 2g_1$. This is our vestigial phase. In this region, $h_\uparrow \neq h_\downarrow$, but both $\Phi_\uparrow$ and $\Phi_\downarrow$ have vanishing expectation values. Correlations between Φ_s at different positions decay exponentially in space: $\langle\Phi_s^*(\vec{r})\Phi_s(\vec{r}')\rangle \propto e^{-\frac{|\vec{r}-\vec{r}'|}{\lambda_s}}$. Moreover, due to the finite h_z, $\lambda_\uparrow \neq \lambda_\downarrow$. Upon lowering of the temperature, $h_{z,0}$ both grows in magnitude, whereas α decreases. Within the above considerations, at temperature where $\alpha = h_\uparrow$, the system makes a transition to the state with $\langle\Phi_\uparrow\rangle \neq 0$ but $\langle\Phi_\downarrow\rangle = 0$ or vice versa, a state just as $T = 0$. At this temperature, $\alpha - h_\downarrow > 0$ so that $\langle\Phi_\downarrow^*(\vec{r})\Phi_\downarrow(\vec{r}')\rangle$ still decays exponentially. Furthermore, for $g_2 < 2g_1$, h_s vanishes, the system goes from the symmetric phase to the state $\langle\Phi_\uparrow\rangle \neq 0$ but $\langle\Phi_\downarrow\rangle = 0$ or vice versa at T_0, where α vanishes [40].

At finite T, the phase with long-range order just described is due to the artifact that phase fluctuation of Φ_s was not considered. Mermin-Wagner theorem states that this long-range order is destroyed in 2D. However, quasi-long-range order [43] is allowed. For the phase diagram, the simplest possibility is that the above-mentioned phase with long-range order is instead characterized by power-law correlations, thus instead of finite expectation value for $\Phi_\uparrow$, we have simply $\langle\Phi_\uparrow^*(\vec{r})\Phi_\uparrow(\vec{r}')\rangle \propto \frac{1}{|\vec{r}-\vec{r}'|^\eta}$. The resulting phase diagram is as given in Fig. 2(a).

Another possibility is that, due to thermal fluctuations of the phase, there is always a vestigial Z_2 broken symmetry phase that lies between the completely symmetric phase and the quasi-long-range order phase, even for the region $g_2 < 2g_1$. This possibility has been raised in a few theoretical calculations based on models which are related to though not the same as the one we have in this paper [24,32,33,35] (though there are also related studies where such a phase is absent [34]). The resulting vestigial phase again only has short-range order, but since Z_2 is broken, the decaying lengths $\lambda_{\uparrow,\downarrow}$ are thus unequal. This phase is indistinguishable from our vestigial phase described by $h_z \neq 0$, though the physical picture giving rise to this broken Z_2 symmetry seems quite

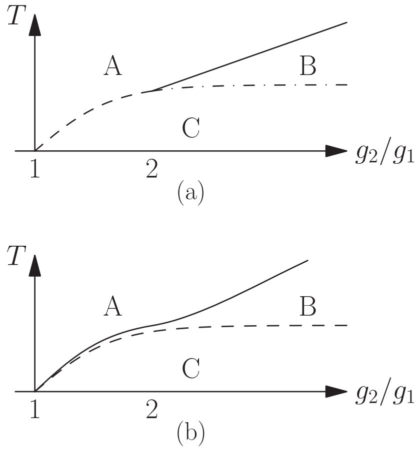

FIG. 2. Possible phase diagrams. Region A is the symmetric phase. Region B is the vestigial phase with broken Z_2 symmetry, but with only short-range order for both $\Phi_\uparrow$ and $\Phi_\downarrow$. $\langle \Phi_s^*(\vec{r})\Phi_s(\vec{r}\,')\rangle \propto e^{-\frac{|\vec{r}-\vec{r}\,'|}{\lambda_s}}$, but $\lambda_\uparrow \neq \lambda_\downarrow$. In region C, one of the Φ_s has quasi-long-range order while the other has only a short-range correlation. Solid lines: Ising transitions; dashed or dotted-dashed lines: BKT transitions.

different. The resulting phase diagram is sketched qualitatively in Fig. 2(b). [44] For both Figs. 2(a) and 2(b), the phase transition temperatures all vanish at $g_2 = g_1$. At this point there the symmetry is enhanced to SO(3), which forbids any order at finite temperature in two spatial dimensions, a fact also pointed out in Ref. [35].

III. CONCLUSION

Starting from a Ginzburg-Landau theory for a two-component superconductor, we show that the vestigial broken time reversal symmetry state with no superconducting order parameter is possible in two spatial dimensions, provided that the parameters lie in the suitable region. This is in strong contrast to the case in three spatial dimensions [29], where such a phase is in general not possible except for some extreme situations.

Similar calculations can be extended also to vestigial nematic order, governed by an order parameter $\vec{h} = (h_x, h_y)$. We have already shown in Ref. [29] that the vestigial nematic state is generally unstable in three spatial dimensions. Back to the present case of two spatial dimensions, calculations similar to Sec. II can also be carried out. For example, we still have Eq. (11), etc., if we exchange h_z there by $|\vec{h}|$, provided we also replace $g_{1,2}$ there by $\frac{g_1+g_2}{2}$ and g_1, respectively (cf. also Ref. [29]), hence $2g_1 - g_2$ in Eq. (12) by g_2. A vestigial nematic state thus requires $g_2 < 0$. b_{eff} is now proportional to $\frac{2g_1+3g_2}{g_1}$. The effective gradient energy has the form $\tilde{K}(\partial_i h_j)(\partial_i h_j)$ [note our Eq. (2) has no "spin-orbit" coupling] with coefficient $\tilde{K}$ given by the same as the expression below Eq. (22). Instead of an Ising transition, we expect a Kosterlitz-Thouless transition for $\vec{h}$ itself when $g_2 > -\frac{2}{3}g_1$, but a more complicated scenario is feasible if this inequality is not satisfied.

If $g_2 < 0$, we can also have $4e$ superconductivity with "pairing" between fields $\Phi_\uparrow$ and $\Phi_\downarrow$. The vestigial $4e$ state now corresponds to quasi-long-range order of the product $\Phi_\uparrow\Phi_\downarrow$ ($\propto \Phi_1^2 + \Phi_2^2$) but without quasi-long-range order of either Φ_s. When the gradient term is simply taken as in (2), the calculations for the effective free energy are entirely parallel to that of the nematic phase, as has already been pointed out in Refs. [13–15]. Discussions above also apply to this case with appropriate substitutions.

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APPENDIX: ORDER OF PHASE TRANSITION

We analyze this phase transition without expansion in $h_{0,z}$. We define $x_s = -\ln(1 - \frac{h_s}{\alpha})$, where we have chosen the sign so that x_s is an increasing function of h_s. All $h_s < \alpha$, hence $-\infty < x_s < \infty$ are acceptable. Employing $x = (x_\uparrow + x_\downarrow)/2$, and $y = (x_\uparrow - x_\downarrow)/2$, Eq. (10) can be written as

$$\frac{\Delta\mathcal{F}}{\alpha T/(4\pi K)} = 2[x + e^{-x}\cosh(y) - 1] + (2\tilde{g}_1 + \tilde{g}_2)x^2 + (2\tilde{g}_1 - \tilde{g}_2)y^2, \tag{A1}$$

where $\tilde{g}_{1,2} = \frac{Tg_{1,2}}{4\pi\alpha K}$.

The stationary point conditions are

$$0 = (1 - e^{-x}\cosh y) + (2\tilde{g}_1 + \tilde{g}_2)x \tag{A2}$$

and

$$0 = e^{-x}\sinh y - (\tilde{g}_2 - 2\tilde{g}_1)y. \tag{A3}$$

The second equation is trivially satisfied by $y = 0$. For $y \neq 0$, we solve for x using this second equation and substitute back to the first to yield a single equation for y,

$$G = \frac{\frac{y}{\tanh y} - \frac{\alpha(T)}{\alpha(T_2)}}{\ln\left[\frac{\sinh y}{y}\frac{\alpha(T)}{\alpha(T_2)}\right]}, \tag{A4}$$

with $G \equiv \frac{2\tilde{g}_1+\tilde{g}_2}{\tilde{g}_2-2\tilde{g}_1}$. Since we have $0 < 2\tilde{g} < \tilde{g}_2$, G decreases with increasing $\tilde{g}_2/\tilde{g}_1$. For $2\tilde{g}_1 < \tilde{g}_2 < 6\tilde{g}$, G lies between 2 and $+\infty$. For $6\tilde{g}_1 < \tilde{g}_2$, G lies between 1 and 2. The graphical solution shows that for $2 < G < \infty$, y vanishes for $\alpha > \alpha(T_2)$. A nontrivial solution for y starts from zero and grows with decreasing $\alpha < \alpha(T_2)$, thus a typical second-order phase transition at $T = T_2$. For $1 < G < 2$, finite y solutions already exist at some $\alpha(T) > \alpha(T_2)$, and with decreasing $\alpha(T)$, one obtains two solutions, one with y decreasing and the other increasing with decreasing α. $\alpha(T_2)$ is the point at which the decreasing solution approaches $y = 0$, which shows a typical first-order transition behavior. Note however in contrast to 3D, we have a local free-energy minimum, not just a saddle point.

[1] M. Sigrist and K. Ueda, Phenomenological theory of unconventional superconductivity, Rev. Mod. Phys. **63**, 239 (1991).

[2] A. J. Leggett, A theoretical description of the new phases of liquid ^{3}He, Rev. Mod. Phys. **47**, 331 (1975).

[3] G. E. Volovik and L. P. Gor'kov, Superconducting classes in heavy-fermion systems, Zh. Eksp. Teor. Fiz. **88**, 1412 (1985) [Sov. Phys. JETP **61**, 843 (1985)].

[4] J. A. Sauls, The order parameter for the superconducting phases of UPt_3, Adv. Phys. **43**, 113 (1994).

[5] S. Yonezawa, Nematic superconductivity in doped Bi_2Se_3 topological superconductors, Condens. Matter **4**, 2 (2018).

[6] S. K. Ghosh, M. Smidman, T. Shang, J. F. Annett, A. D. Hillier, J. Quintanilla, and H. Yuan, Recent progress on superconductors with time-reversal symmetry breaking, J. Phys.: Condens. Matter **33**, 033001 (2021).

[7] D. M. Stamper-Kurn and M. Ueda, Spinor Bose gases: Symmetries, magnetism, and quantum dynamics, Rev. Mod. Phys. **85**, 1191 (2013).

[8] More complex situations such as 6e, etc., pairing are possible, but we shall not go into those in this paper.

[9] R. M. Fernandes, P. P. Orth, and J. Schmalian, Intertwined vestigial order in quantum materials: Nematicity and beyond, Annu. Rev. Condens. Matter Phys. **10**, 133 (2019).

[10] M. Hecker and J. Schmalian, Vestigial nematic order and superconductivity in the doped topological insulator $Cu_xBi_2Se_3$, npj Quantum Inf. **3**, 26 (2017).

[11] C.-W. Cho, J. Shen, J. Lyu, O. Atanov, Q. Chen, S. H. Lee, Y. S. Hor, D. J. Gawryluk, E. Pomjakushina, M. Bartkowiak, M. Hecker, J. Schmalian, and R. Lortz, Z_3-vestigial nematic order due to superconducting fluctuations in the doped topological insulators $Nb_xBi_2Se_3$ and $Cu_xBi_2Se_3$, Nat. Commun. **11**, 3056 (2020).

[12] V. Grinenko, D. Weston, R. Caglieris, C. Wuttke, C. Hess, T. Gottschall, I. Maccari, D. Gorbunov, S. Zherlitsyn, J. Wosnitza, A. Rydh, K. Kihou, C.-H. Lee, R. Sarkar, S. Dengre, J. Garaud, A. Charnukha, R. Hühne, K. Nielsch, B. Büchner, H.-H. Klauss, and E. Babaev, State with spontaneously broken time-reversal symmetry above the superconducting phase transition, Nat. Phys. **17**, 1254 (2021).

[13] R. M. Fernandes and L. Fu, Charge-4e superconductivity from multicomponent nematic pairing: Application to twisted bilayer graphene, Phys. Rev. Lett. **127**, 047001 (2021).

[14] S.-K. Jian, Y. Huang, and H. Yao, Charge-4e superconductivity from nematic superconductors in two and three dimension, Phys. Rev. Lett. **127**, 227001 (2021).

[15] M. Hecker, R. Willa, J. Schmalian, and R. M. Fernandes, Cascade of vestigial orders in two-component superconductors: Nematic, ferromagnetic, s-wave charge-4e, and d-wave charge-4e states, Phys. Rev. B **107**, 224503 (2023).

[16] P. P. Poduval and M. S. Scheurer, Vestigial singlet pairing in a fluctuating magnetic triplet superconductor and its implications for graphene superlattices, Nat. Commun. **15**, 1713 (2024).

[17] S. Ashhab, Superfluid vs ferromagnetic behavior in a Bose gas of spin-1/2 atoms, J. Low Temp. Phys. **140**, 51 (2005).

[18] S. S. Natu and E. J. Mueller, Pairing, ferromagnetism, and condensation of a normal spin-1 Bose gas, Phys. Rev. A **84**, 053625 (2011).

[19] J. Radić, S. S. Natu, and V. Galitski, Stoner ferromagnetism in a thermal pseudospin-1/2 Bose gas, Phys. Rev. Lett. **113**, 185302 (2014).

[20] M. H. Fischer and E. Berg, Fluctuation and strain effects in a chiral p-wave superconductor, Phys. Rev. B **93**, 054501 (2016).

[21] E. Babaev, A. Sudbø, and N. W. Ashcroft, A superconductor to superfluid phase transition in liquid metallic hydrogen, Nature (London) **431**, 666 (2004).

[22] A. Kuklov, N. Prokof'ev, and B. Svistunov, Commensurate two-component bosons in an optical lattice: Ground state phase diagram, Phys. Rev. Lett. **92**, 050402 (2004).

[23] E. V. Herland, E. Babaev, and A. Sudbø, Phase transitions in a three dimensional $U(1) \times U(1)$ lattice London superconductor: Metallic superfluid and charge-4e superconducting states, Phys. Rev. B **82**, 134511 (2010).

[24] T. A. Bojesen, E. Babaev, and A. Sudbø, Time reversal symmetry breakdown in normal and superconducting states in frustrated three-band systems, Phys. Rev. B **88**, 220511 (2013).

[25] T. A. Bojesen, E. Babaev, and A. Sudbø, Phase transitions and anomalous normal state in superconductors with broken time reversal symmetry, Phys. Rev. B **89**, 104509 (2014).

[26] O. Gauthé and F. Mila, Thermal Ising transition in the spin-$1/2 J_1$-J_2 Heisenberg model, Phys. Rev. Lett. **128**, 227202 (2022).

[27] F.- F. Song and G.-M. Zhang, Tensor network approach to the fully frustrated XY model on a kagome lattice with a fractional vortex-antivortex pairing transition, Phys. Rev. B **108**, 014424 (2023).

[28] N. Francini and L. Janssen, Spin vestigial orders in extended Heisenberg-Kitaev models near hidden SU(2) points: Application to $Na_2Co_2TeO_6$, Phys. Rev. B **109**, 075104 (2024).

[29] P. T. How and S. K. Yip, Absence of Ginzburg-Landau mechanism for vestigial order in the normal phase above a two-component superconductor, Phys. Rev. B **107**, 104514 (2023).

[30] L. He, P. Gao, and Z.-Q. Yu, Normal-superfluid phase separation in spin-half bosons at finite temperature, Phys. Rev. Lett. **125**, 055301 (2020).

[31] P. T. How and S. K. Yip, Superfluid transition of a ferromagnetic Bose gas, Phys. Rev. Res. **6**, L022030 (2024)

[32] M. Zeng, L.-H. Hu, H.-Y. Hu, Y.-Z. You, and C. Wu, High-order time-reversal symmetry breaking normal state, Sci. China: Phys. Mech. Astron. **67**, 237411 (2024).

[33] Y.-H. Liu, J. Zhou, C. Wu, and F. Yang, Charge-4e superconductivity and chiral metal in 45°-twisted bilayer cuprates and related bilayers, Nat. Commun. **14**, 7926 (2023).

[34] F.-F. Song and G.-M. Zhang, Phase coherence of pairs of Cooper pairs as quasi-long-range order of half-vortex pairs in a two-dimensional bilayer system, Phys. Rev. Lett. **128**, 195301 (2022).

[35] I. Maccari, J. Carlström, and E. Babaev, Prediction of time-reversal-symmetry breaking fermionic quadrupling condensate in twisted bilayer graphene, Phys. Rev. B **107**, 064501 (2023).

[36] A. Z. Patashinskii and V. L. Pokrovskii, *Fluctuation Theory of Phase Transitions* (Pergamon Press, Oxford, UK, 1979).

[37] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon Press, Oxford, UK, 1989).

[38] The same calculation for 3D, with generally $I_n \equiv \frac{1}{L^d}\sum_{\vec{k}} \frac{1}{(\alpha+Kk^2)^n}$ and $d = 3$ instead of 2, shows that $b_{\rm eff}$ is always negative [given by $-\frac{T}{8\pi\alpha^{5/2}K^{3/2}}\frac{(g_2-2g_1)}{32g_2}$]. This result corroborates that in Ref. [29], showing that we can only have a joint first-order phase transition.

[39] It can be seen from Eq. (22) that $\tilde{K}$ is negative if $2g_1 > g_2$, and also at sufficiently high temperatures if $g_2 > 2g_1$. However, as explained already below Eq. (20), this does not lead to transitions to nonuniform states.

[40] This phase diagram cannot be quite right, even within mean-field theory for the Φ_s transitions. It predicts three second-order phase transition lines meeting at a point with finite angles with respect to each other, and the finite $\Phi_\uparrow$ phase occupies a region with an angle larger than π, violating the phase rules in Refs. [41,42] respectively. We however would not pursue this question further since the 2D nature requires us to consider phase fluctuations of Φ_s, to be discussed below.

[41] S. K. Yip, T. Li, and P. Kumar, Thermodynamic considerations and the phase diagram of superconducting UPt_3, Phys. Rev. B **43**, 2742 (1991).

[42] A. J. Leggett, Implications of the ^{3}He phase diagram below 3 mK, Prog. Theor. Phys. **51**, 1275 (1974).

[43] J. M. Kosterlitz and D. J. Thouless, Ordering, metastability and phase transitions in two-dimensional systems, J. Phys. C: Solid State Phys. **6**, 1181 (1973).

[44] In the scenarios of Refs. [32,33], the transition from the symmetric phase to the Z_2 broken phase is via a BKT transition, instead of an Ising transition we deduced for $g_2 > 2g_1$. In that case, for Fig. 2(b), there must also be an additional multicritical point separating the two regions.