Tunnel transport problem for open multilayer nitride nanostructures with an applied constant magnetic field and time-dependent potential: An exact solution

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(Received 6 March 2024; revised 8 July 2024; accepted 10 July 2024; published 24 July 2024)

A quantum mechanical theory for description of electronic quasistationary states in two-dimensional (2D) layered semiconductor nanostructure was developed. A constant external magnetic field B was directed along semiconductor layers. The presence of internal electric fields F originated from the potential barriers and wells in semiconductor layers was taken into account. The vector of F was directed perpendicularly to semiconductor layers. The interaction of tunneled electrons with a time-dependent electromagnetic fields was analyzed in detail. An approach to obtain an exact solution to the problem based on a combination of the Lewis-Riesenfeld method for complete Schrödinger equation and scattering theory was developed. Application of the scattering theory in combination with the S-matrix and transfer-matrix methods allowed us to calculate the electron quasistationary spectra. Effects of magnetic field on the resonant energy and width of electronic quasistationary states as well as on the electronic conductivity were also discussed.

DOI: 10.1103/PhysRevB.110.045438

I. INTRODUCTION

Many types of modern optoelectronic devices, in their functional features, are based on electronic transitions within the conduction band of multilayer nanostructures, which are formed by alternating semiconductor films of various thicknesses [1–3]. For example, for a coherent electron transport in quantum cascade lasers or detectors [4–7], it is necessary to take into account the interaction of a tunneled flow with an electromagnetic field that is of harmonic dependence on time [8–11]. Another important aspect is the presence of constant fields that can directly affect the electron tunneling process. In these systems, the applied constant external electric field is treated as a correction mechanism that ensures consistency between adjacent cascades of quantum cascade lasers based on arsenide semiconductors [12,13]. For nitride nanostructures, an important factor is the internal electric field caused by piezoelectric and spontaneous polarizations [14–16]. In planar nitride nanostructures, its consideration is necessary to calculate the spectral parameters of electrons [17,18]. In our previous paper, the solutions of the complete Schrödinger equation with a time-dependent potential and the presence of a constant magnetic field was discussed [19].

In general, three different approaches have been proposed for finding solutions of the complete Schrödinger equation with a time-dependent Hamiltonian. There are many papers discussing these approaches.

The first approach was developed in 1969 in the fundamental paper by Lewis and Riesenfeld [20]. This approach deals with searching for quantum mechanical invariants. Nowadays, it is actively applied in studies of coherent states in dissipative systems [21], time-dependent harmonic oscillators [22], and inverted harmonic oscillators [23] associated with time-dependent forms of the Hamiltonian.

The second approach deals with the studies of the Hamiltonian systems containing quadratic and linear terms, both in momentum and coordinates [24]. This approach is based on application of the Lie algebra method together with unitary transformations used in investigations of quantum mechanical evolution operators in the presence of a scalar linear potential [25] and for the time-dependent quantum harmonic oscillator [26]. In addition, methods of Kramers-Heisenberger transformation [27], and special constructions for evolution operators [28] are used.

The third approach is based on the application of Airy wave packets [29] and Feynman path integral methodology for solution of the classical equations of motion in periodic Hamiltonians with a time-dependent quantum oscillators [30]. Moreover, the above approaches in combination with a unitary transformation were developed [31].

The combined approaches also expanded the application of the Lewis-Riesenfeld method for the non-Hermitian

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Hamiltonians and the problem of parity-time (PT)-symmetrically coupled oscillators [32] or time-dependent optical potentials [33]. These approaches are treated as a bridge that makes further development of theoretical methods for different applications possible. It is especially important in the presence of external variables and constant fields that occur in optoelectronic devices [34].

For quantum cascade lasers and detectors, cases of using a magnetic field are quite rare, however, it should be noted that in these cases the voltage vectors of the electric and magnetic fields were mutually perpendicular or parallel [35–37]. It is also worth highlighting a group of problems [38–40] aimed at studying the influence of a magnetic field or quadratic potential on the electronic spectrum in closed nanosystems by finding solutions of the stationary problem for the Schrödinger equation. Regarding open nanosystems, there are several papers [41–44] where the influence of the magnetic field on the transparency coefficient of nanosystems and the electronic spectrum was studied. It should be noted that for open nanosystems the use of approximative numerical approaches and approximations of the local Hamiltonian is problematic due to the known problems with normalization of the electron wave function to the Dirac delta function. Thus, it should be concluded that the theoretical problem of describing the tunneling process of electrons interacting with an electromagnetic field in the presence of constant electric and magnetic fields applied to a nanosystem still remains open, which, in turn, hinders the possibilities of the practical use of such mechanisms.

This paper consists of four sections. The first section is devoted to the statement of the problem, and a two-dimensional (2D) layered GaN/AIN semiconductor nanosystem is considered. Next, we present the complete solution of the Schrödinger equation for the time-dependent Hamiltonian describing the interaction of electrons with electromagnetic fields obtained based on the application of the Lewis-Riesenfeld method. This part of the paper is sequential in nature, which is done to demonstrate how the Lewis-Riesenfeld method is applied in this subject area. The proposed theory makes it possible to describe the tunneling transport of electrons in multilayer open nanosystems with an applied constant magnetic and time-dependent electromagnetic field. Theoretical results make it possible to study electronic quasistationary states and the influence of external fields on them, providing a significant advantage over theoretical models built for closed nanosystems, which only allows calculations of the stationary electronic spectrum and oscillator strengths of quantum transitions. The second section of the paper is devoted to presenting results that testify to the necessity of using methods of quantum scattering theory, the *S*-matrix method in particular, to solve the spectral problem for quasistationary electronic states in the Lewis-Riesenfeld method. The third section of the paper focuses on analytical calculations of the electronic conductivity of the nanosystem under study. In the fourth section, we analyze the spectral parameters of electronic quasistationary states and electronic conductivity of the nanosystem depending on the constant magnetic field. Calculations were performed for the time intervals limited by the relaxation time, that is, under conditions close to the operation of nanodevices, such as quantum cascade lasers and detectors. Calculations based on theory show how a magnetic field can influence the spectral parameters of electron quasistationary states in open multilayer nanosystems. This, in turn, affects their electronic conductivity. This is important for such nanodevices as quantum cascade detectors that use multilayer nanosystems as their working elements. In addition, as will be shown in the present paper, a magnetic field will be used to adjust both the frequency of electronic transitions (operating frequency) and the absorption or emission band in nanodevices. The results of calculations carried out in compliance with the established conditions, when the coherent tunneling regime is maintained in real nanosystems, will have direct practical significance for the subject area of nanophysics, in the operation of quantum cascade lasers and detectors, in particular.

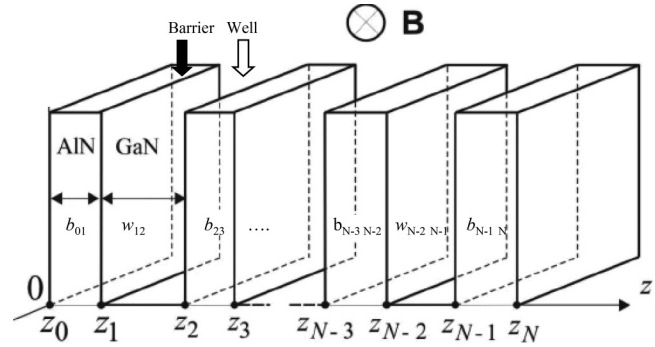

FIG. 1. Energetic schema diagram of studied 2D layered GaN/AIN semiconductor nanosystem with N layers stretched along the z axis. Symbols b and w denote the width of barriers and wells, respectively. In the general case, these widths may be different along the z axis. The coordinates z_1, z_2, .., z_N correspond to the boundaries between the AlN and GaN layers of the nanosystem.

II. ELECTRON TUNNELING IN LAYERED SEMICONDUCTOR NANOSYSTEMS. SOLUTION OF THE SCHRÖDINGER EQUATION WITH APPLICATION OF THE LEWIS-RIESENFELD METHOD

Figure 1 presents the schema of studied 2D layered GaN/AIN semiconductor nanosystem with N layers stretched along the z axis. Here, the potential wells and potential barriers are created on the basis of GaN and AlN semiconductors, respectively. This nanosystem is considered to be placed in the external environment of a GaN semiconductor, corresponding to the material of the potential wells. In this regard, the nanosystem is open—electrons can tunnel through it and their states are quasistationary.

The problem was analyzed in the presence of a constant external magnetic field (**B**) directed along semiconductor layers (perpendicularly to z and, accordingly, to the direction of propagation of tunneled electrons) and internal electric fields (**F**) originating from the potential barriers and wells in semiconductors. The internal electric fields (**F**) are directed along the z axis. It is caused by the appearance of spontaneous and piezoelectric polarization in the semiconductor layers and has different signs in the potential wells and barriers (see, e.g., Ref. [16]). Therefore, the directions of the external magnetic

field **B** and the internal electric field **F** are mutually perpendicular. That is, it corresponds to the case experimentally studied in Ref. [37]. Taking into account this statement of the problem, we choose the gauge fixing of the vector potential of the magnetic field in the form of the Landau gauge [45]: $\mathbf{A}(\mathbf{r}) = Bz\mathbf{e}_x = (Bz,\ 0,\ 0)$. In this case, the magnetic field will modify the momentum of the electron in the x direction.

The general space $[\mathbf{r} = (x, y, z)]$ and time (t)-dependent problem of electron tunneling through the system requires a solution of the complete Schrödinger equation,

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = H(\mathbf{r},t)\,\Psi(\mathbf{r},t), \tag{1}$$

where $H(\mathbf{r},t)$ is the time-dependent Hamiltonian, which, considering the gauge fixing of the vector potential of the magnetic field, looks as follows:

$$H(\mathbf{r},t) = \frac{1}{2m(z)}\left[\left(p_x - \frac{e}{c}Bz\right)^2 + p_y^2 + p_z^2\right] + U_{e-em}(z,t) + U(z) + U_e(z). \tag{2}$$

All components $U_{e-em}(z,t)$; $U_e(z)$; $U(z)$ of this Hamiltonian that affect the electron tunneling process are described in detail below.

The generation or absorption of an electromagnetic field in quantum electronic transitions is the basis for the operation of quantum cascade nanodevices [1–6]. The appropriate operating frequency of these nanodevices is their main operating characteristic, so taking into account the contribution of the electromagnetic field is a key point for this type of nanosystem. In the statement of the problem, it was assumed that a monoenergetic flow of noninteracting electrons with concentration n_0 entered the layered nanosystem from the left side (which corresponds to the moment $t = 0$, the start of the tunneling process). The energy of electrons in the flow was considered to be close to the energy of an arbitrary level of the nanosystem formed due to dimensional quantization. The energies of these levels can be determined by solving the Schrödinger equation with the stationary part of the Hamiltonian (2). This problem statement corresponds to the operating principle of nanodevices [3–5], where an electron flow is injected into an arbitrary electronic level of the nanosystem. As a result, both quantum electronic transitions with the emission and absorption of an electromagnetic field become possible. The case of electron injection into the first level corresponds to a situation where we are dealing only with a detector quantum transition. As a result of quantum transitions between the electronic levels of the nanosystem, a time-dependent electromagnetic field is generated, the frequency ω of which corresponds to the energy of the electronic transition and, accordingly, the energy value $\hbar\omega$ is commensurate with the energy of the electronic spectrum of the nanosystem. For the steady-state process of the ballistic electron transport, it is assumed that the electromagnetic field is present in the nanosystem at any moment of time, including at $t = 0$. As a result, we consider the one-electron problem, and it is necessary to take into account the presence of the electromagnetic field with a harmonic time dependence of its electrical component on the electron tunneling process in the nanosystem under study. The interaction of the single electron in flow with this time-dependent electromagnetic field was described in the dipole approximation [46]. Finally, the time t and position z dependence of energy of the electron interaction with a variable electromagnetic field U_{e-em} was described as

$$U_{e-em}(z,t) = -2e\{z[\theta(z) - \theta(z - z_N)] + z_N\theta(z - z_N)\}\xi\cos\omega t, \tag{3}$$

where $\theta(z)$ is the Heaviside unit function, ξ is the amplitude of the electromagnetic field electrical component, and ω is its frequency. In Eq. (3), it is taken into account that the electric component of the electromagnetic field is of a harmonic dependence on time [47]. In addition, we consider the electromagnetic field to be weak, contrary to the case considered in Ref. [9]. Consequently, the contribution of the electrical component to the electron momentum in the z direction is assumed to be negligibly small. In this problem, the dynamics of electrons in a nanosystem are determined by their interaction with the electromagnetic field. This case is fundamentally different from the case of a stationary Hamiltonian when we are limited to only calculating the electronic spectrum and cannot study the dynamics of electrons in a nanosystem. Thus, the Hamiltonian component (3) cannot be neglected in consideration of the time-dependent dynamics of electrons in the nanosystem. On the other hand, attempting to apply perturbation theory methods to electronic quasistationary states in an open nanosystem, while treating the Hamiltonian component (3) as a small perturbation, leads to problems with the use of the electronic function normalized to the Dirac delta function [48].

In the absence of internal (**F**) and external (**B**) fields, the energetic scheme is defined by rectangular potentials. In this case, the potential energy of the electron $U(z)$ is defined as

$$U(z) = U_0\sum_{p=0}^{N}[\theta(z - z_{2p}) - \theta(z - z_{2p+1})], \tag{4}$$

where U_0 is the height of the potential barrier calculated in the model with rectangular potential wells and barriers.

In the absence of an electric field in the external environment, the energy of electron interaction U_e with internal electric field (**F**) is described by the following equation [19]:

$$U_e(z) = e\sum_{p=0}^{N}(-1)^p F_{p+1}\left(z - \frac{F_p}{F_{p+1}}z_p\right) \times [\theta(z - z_p) - \theta(z - z_{p+1})]; \quad F_0 = F_{N+1} = 0; \quad U_e(z)|_{z<0\ (z>z_N)} = 0. \tag{5}$$

The position-dependent effective mass of an electron can be presented as

$$m(z) = m_w[\theta(-z) + \theta(z - z_N)] + \sum_{p=1}^{N} m_p[\theta(z - z_p) - \theta(z - z_{p+1})]; \quad m_p = \begin{cases} m_w, & p - \text{odd (wells)} \\ m_b, & p - \text{even (barriers)}, \end{cases} \tag{6}$$

where m_w and m_b are effective electron masses in the material of potential wells and barriers, respectively. The problems

related with the position-dependent effective mass of electrons in multilayer systems were thoroughly studied previously [49,50]. For this case, we have

$$-\frac{\hbar^2}{2}\frac{\partial}{\partial z}\frac{1}{m(z)}\frac{\partial}{\partial z}=\begin{cases}-\frac{\hbar^2}{2m_w}\frac{d^2}{dz^2}, & \text{wells}\\ -\frac{\hbar^2}{2m_b}\frac{d^2}{dz^2}, & \text{barriers}\end{cases} \tag{7}$$

The Hamiltonian given by Eq. (2) in each layer is Hermitian (see, e.g., Ref. [51]) and it is worth making a separation of the variables in Eq. (1):

$$\Psi(\mathbf{r},t)=\exp(ik_x x)\exp(ik_y y)\Phi(z,t). \tag{8}$$

Having substituted Eq. (8) into Eq. (1) and taking into account Eqs. (2) and (6), we have

$$\begin{aligned}i\hbar\frac{\partial\Psi(z,t)}{\partial t}&=\left(\frac{P_z^2}{2m_p}+\frac{\hbar^2}{2m_p}\left[\left(k_x-\frac{eBz}{\hbar}\right)^2+k_y^2\right]+U_0+(-1)^p\right.\\&\left.\times e(F_{p+1}z-F_pz_p)-2ez\xi\cos\omega t\right)\Psi(z,t),\end{aligned} \tag{9}$$

where

$$\frac{P_z^2}{2m_p}=-\frac{\hbar^2}{2m_p}\frac{d^2}{dz^2}. \tag{10}$$

The presence of the term U_0 in Eq. (9) and the following expressions means that this equation is written for the nanosystem layer corresponding to the potential barrier. In the case of a layer corresponding to a potential well, the term U_0 is absent.

Introducing the definitions of the magnetic length $LL=\sqrt{\hbar/eB}$ of the center of the cyclotron orbit $Z_0^{(p)}$,

$$\tilde{Z}_0^{(p)}=(-1)^p\frac{m_p}{m_{p+1}}Z_0^{(p+1)}-2k_xL^2, \tag{11}$$

and new definitions

$$\begin{aligned}&\upsilon=\frac{z-\tilde{Z}_0^{(p)}}{L};\quad \frac{d}{dz}=\frac{1}{L}\frac{d}{d\upsilon};\quad \frac{d^2}{dz^2}=\frac{1}{L^2}\frac{d^2}{d\upsilon^2};\\&\frac{P_z^2}{2m_p}=\frac{P_\upsilon^2}{2m_pL^2};\quad P_\upsilon=-i\hbar\frac{d}{d\upsilon},\end{aligned} \tag{12}$$

the Schrödinger equation (9) can be rewritten as follows:

$$\begin{aligned}i\hbar\frac{\partial\Psi(\upsilon,t)}{\partial t}=\left[\frac{P_\upsilon^2}{2m_pL^2}+\frac{\hbar^2k_x^2}{2m_p}+\frac{\hbar^2k_y^2}{2m_p}+U_0\right.\\+(-1)^{p+1}eF_pz_p-\frac{\hbar^2\tilde{Z}_0^{(p)}}{2m_pL^4}+\frac{\hbar^2}{2m_pL^2}\upsilon^2\\\left.-2e\left(\upsilon L+\tilde{Z}_0^{(p)}\right)\xi\cos\omega t\right]\Psi(\upsilon,t).\end{aligned} \tag{13}$$

The grouping of components that are clearly dependent on the coordinate υ and time t variables gives the following equation for the Hamiltonian:

$$H_p(t)=\frac{P_\upsilon^2}{2m_pL^2}+W(t)+\frac{\hbar^2}{2m_pL^2}\upsilon^2-2e\upsilon L\xi\cos\omega t, \tag{14}$$

where

$$W(t)=W_0-2e\tilde{Z}_0^{(p)}\xi\cos\omega t \tag{15}$$

and

$$W_0=\frac{\hbar^2k_x^2}{2m_p}+\frac{\hbar^2k_y^2}{2m_p}+U_0+(-1)^{p+1}eF_pz_p-\frac{\hbar^2\tilde{Z}_0^{(p)}}{2m_pL^4}. \tag{16}$$

The derived Hamiltonian [Eq. (14)], despite the presence of quadratic terms in coordinate and momentum, is not similar to that analyzed in papers published earlier, where time-dependent magnetic field [20], mass $m=m(t)$, or frequency $\omega=\omega(t)$ [20,52–54] were introduced. In our case, the time-dependent component of the Hamiltonian H_p is proportional to the value of $\xi\cos\omega t$ [Eq. (14)]. Using the relations

$$\begin{aligned}\dot{P}_\upsilon&=-\frac{\partial H_p(t)}{\partial\upsilon}=\frac{\hbar^2}{m_pL^2}\upsilon-2eL\xi\cos\omega t;\\\dot{\upsilon}&=\frac{\partial H_p(t)}{\partial P_\upsilon}=\frac{P_\upsilon}{m_pL^2};\\\ddot{\upsilon}&=\frac{\dot{P}_\upsilon}{m_pL^2}=\left(\frac{\hbar}{m_pL^2}\right)^2\upsilon-\frac{2e\xi}{m_pL}\cos\omega t,\end{aligned} \tag{17}$$

the solutions of classical equations for electron motion can be presented as follows:

$$\begin{aligned}\upsilon(t)=&C_1\exp\left(\frac{\hbar t}{m_pL^2}\right)+C_2\exp\left(-\frac{\hbar t}{m_pL^2}\right)\\&+\frac{2em_pL^3\xi}{\hbar^2+(m_p\omega L^2)^2},\end{aligned} \tag{18}$$

where C_1 and C_2 are the integration constants.

To find solutions to the complete Schrödinger equation (13), taking into account the form of Hamiltonian (14), we will use the Lewis-Riesenfeld method [20]. The method, conceived by the authors of the pioneering paper [20], aims to find solutions to the complete Schrödinger equation for a time-dependent oscillator. It is based on considering the time-dependent Hermitian Hamiltonian of the system $H(t)$. In this case, it is assumed that there also exists a time-dependent Hermitian operator $I(t)$, which is treated as an invariant. The action of this invariant on the wave function of the system gives another form of the solution of the complete Schrödinger equation. Provided that such an invariant does not involve time differentiation, the eigenfunctions of this invariant are determined as those of the corresponding states which satisfy the complete Schrödinger equation. Following the general principle, we will look for the invariant $I_p(t)$ for the Schrödinger equation written for an arbitrary layer of the nanosystem under study. The following equation is used for this purpose:

$$\frac{dI_p(t)}{dt}=\frac{\partial I_p(t)}{\partial t}+\frac{1}{i\hbar}[I_p(t),H_p(t)]=0. \tag{19}$$

Since the Hamiltonian in Eq. (14) contains terms linear and quadratic in coordinate υ and quadratic in momentum P_υ, the invariant should be found in its most general form [20], that is,

$$\begin{aligned}I_p(t)=&\alpha(t)P_\upsilon^2+\beta(t)\upsilon+\gamma(t)\upsilon^2\\&+\delta(t)(\upsilon P_\upsilon+P_\upsilon\upsilon)+\Delta(t),\end{aligned} \tag{20}$$

where $\alpha(t)$, $\beta(t)$ $\gamma(t)$, $\delta(t)$, υ, and $\Delta(t)$ are time-dependent coefficients.

Having calculated $\partial I(t)/\partial t$, all commutators, and substituting the results into Eq. (19), the following equation was obtained:

$$\left[\frac{\partial\alpha(t)}{\partial t}+\frac{2\delta(t)}{m_pL^2}\right]P_\upsilon^2+\left[\frac{\partial\beta(t)}{\partial t}+4eL\xi\delta(t)\cos\omega t\right]\upsilon + \left[\frac{\partial\gamma(t)}{\partial t}-\frac{2\hbar^2\delta(t)}{m_pL^2}\right]\upsilon^2+\left[4eL\xi\alpha(t)\cos\omega t+\frac{\beta(t)}{m_pL^2}\right]P_\upsilon + \left[\frac{\partial\delta(t)}{\partial t}+\frac{\hbar^2\alpha(t)}{m_pL^2}-\frac{\gamma(t)}{m_pL^2}\right](\upsilon P_\upsilon+P_\upsilon\upsilon)+\frac{\partial\Delta(t)}{\partial t}=0. \tag{21}$$

Equation (21) is satisfied if all expressions in brackets, which are coefficients at P_υ; P_υ^2; υ; υ^2; $\upsilon P_\upsilon + P_\upsilon\upsilon$, as well as the value $\partial\Delta(t)/\partial t$ are equal to zero.

It allows obtaining the following system of first order differential equations:

$$\frac{\partial\alpha(t)}{\partial t}+\frac{2\delta(t)}{m_pL^2}=0, \tag{22}$$

$$\frac{\partial\beta(t)}{\partial t}+4eL\xi\delta(t)\cos\omega t=0, \tag{23}$$

$$\frac{\partial\gamma(t)}{\partial t}-\frac{2\hbar^2\delta(t)}{m_pL^2}=0, \tag{24}$$

$$\frac{\partial\delta(t)}{\partial t}+\frac{\hbar^2\alpha(t)}{m_pL^2}-\frac{\gamma(t)}{m_pL^2}=0, \tag{25}$$

$$4eL\xi\alpha(t)\cos\omega t+\frac{\beta(t)}{m_pL^2}=0, \tag{26}$$

$$\frac{\partial\Delta(t)}{\partial t}=0. \tag{27}$$

From Eqs. (26) and (27), it follows that

$$\alpha(t)=-\frac{\beta(t)}{4em_p\xi L^3\cos\omega t};\quad \Delta(t)=\Delta_0. \tag{28}$$

Taking into account the invariant dimension in Eq. (20), we choose the value of Δ_0 as follows:

$$\Delta_0=\begin{cases}0, & \text{wells}\\ W_0, & \text{barriers.}\end{cases} \tag{29}$$

Differentiating the Eq. (25) and using Eqs. (22) and (24) to exclude $\partial\alpha(t)/\partial t$ and $\partial\gamma(t)/\partial t$ from it, we have obtained

$$\frac{\partial^2\delta(t)}{\partial t^2}+\frac{\hbar^2}{m_pL^2}\frac{\partial\alpha(t)}{\partial t}-\frac{1}{m_pL^2}\frac{\partial\gamma(t)}{\partial t}=0;$$

$$\frac{\partial\gamma(t)}{\partial t}=\frac{2\hbar^2\delta(t)}{m_pL^2};\quad \frac{\partial\alpha(t)}{\partial t}=-\frac{2\delta(t)}{m_pL^2}. \tag{30}$$

It results in the following equation:

$$\frac{\partial^2\delta(t)}{\partial t^2}-\left(\frac{2\hbar}{m_pL^2}\right)^2\delta(t)=0, \tag{31}$$

where the solutions can be represented as follows:

$$\delta(t)=A_p\exp\left(-\frac{2\hbar}{m_pL^2}t\right)+B_p\exp\left(\frac{2\hbar}{m_pL^2}t\right), \tag{32}$$

where A and B are the integration constants.

Returning to Eqs. (22) and (24) and taking into account Eq. (32), the following equations were obtained:

$$\frac{\partial\alpha(t)}{\partial t}=-\frac{2}{m_pL^2}\left[A_p\exp\left(-\frac{2\hbar}{m_pL^2}t\right)+B_p\exp\left(\frac{2\hbar}{m_pL^2}t\right)\right];$$

$$\frac{\partial\gamma(t)}{\partial t}=\frac{2\hbar^2}{m_pL^2}\left[A_p\exp\left(-\frac{2\hbar}{m_pL^2}t\right)+B_p\exp\left(\frac{2\hbar}{m_pL^2}t\right)\right]. \tag{33}$$

Taking into account Eq. (28), the following general solutions were obtained:

$$\alpha(t)=\frac{1}{\hbar}\left[A_p\exp\left(-\frac{2\hbar}{m_pL^2}t\right)-B_p\exp\left(\frac{2\hbar}{m_pL^2}t\right)\right]+\alpha_0;$$

$$\beta(t)=-4em_p\xi L^3\left\{\frac{1}{\hbar}\left[A_p\exp\left(-\frac{2\hbar}{m_pL^2}t\right)-B_p\exp\left(\frac{2\hbar}{m_pL^2}t\right)\right]+\alpha_0\right\}\cos\omega t;$$

$$\gamma(t)=\hbar\left[-A_p\exp\left(-\frac{2\hbar}{m_pL^2}t\right)+B_p\exp\left(\frac{2\hbar}{m_pL^2}t\right)\right]+\gamma_0. \tag{34}$$

where α_0 and γ_0 are arbitrary constants.

For $t\in(0,+\infty)$, the solutions of Eqs. (32) and (34) are limited within the range $[\alpha(t),\ \beta(t),\ \gamma(t),\ \delta(t),\ \Delta(t)]<+\infty$. Then, in these relations, it should be assumed that $B=0$.

Therefore, we have the following system of equations:

$$\alpha(t)=\frac{A_p}{\hbar}\exp\left(-\frac{2\hbar}{m_pL^2}t\right)+\alpha_0; \tag{35}$$

$$\beta(t)=-4em_p\xi L^3\left\{\frac{A_p}{\hbar}\exp\left(-\frac{2\hbar}{m_pL^2}t\right)+\alpha_0\right\}\cos\omega t; \tag{36}$$

$$\gamma(t)=-\hbar A_p\exp\left(-\frac{2\hbar}{m_pL^2}t\right)+\gamma_0; \tag{37}$$

$$\delta(t)=A_p\exp\left(-\frac{2\hbar}{m_pL^2}t\right); \tag{38}$$

$$\Delta(t)=\Delta_0. \tag{39}$$

Substituting the obtained coefficients $\alpha(t)$, $\beta(t)$ $\gamma(t)$, $\delta(t)$, υ, and $\Delta(t)$ into the expression for the invariant [Eq. (20)], the following equation for the invariant $I_p(t)$ was obtained:

$$I_p(t)=\left[\frac{A_p}{\hbar}\exp\left(-\frac{2\hbar}{m_pL^2}t\right)+\alpha_0\right]P_\upsilon^2 - 4em_p\xi L^3\cos\omega t\left[\frac{A_p}{\hbar}\exp\left(-\frac{2\hbar}{m_pL^2}t\right)+\alpha_0\right]\upsilon + \left[-\hbar A_p\exp\left(-\frac{2\hbar}{m_pL^2}t\right)+\gamma_0\right]\upsilon^2 + \left[A_p\exp\left(-\frac{2\hbar}{m_pL^2}t\right)\right](\upsilon P_\upsilon+P_\upsilon\upsilon)+\Delta_0. \tag{40}$$

Tacking into account that

$$\alpha_0=\frac{1}{2m_pL^2};\quad A_p=\frac{\hbar}{2m_pL^2}=\hbar\alpha_0;\quad \gamma_0=\frac{\hbar^2}{2m_pL^2}, \tag{41}$$

the equation for the invariant $I_p(t)$ may be rewritten as follows:

$$I_p(t) = \frac{1}{2m_pL^2}\left[\exp\left(-\frac{2\hbar}{m_pL^2}t\right)+1\right]P_\upsilon^2 - 2e\xi L\upsilon\cos\omega t\left[\exp\left(-\frac{2\hbar}{m_pL^2}t\right)+1\right] + \frac{\hbar^2}{2m_pL^2}\left[1-\exp\left(-\frac{2\hbar}{m_pL^2}t\right)\right]\upsilon^2 + \frac{\hbar}{2m_pL^2}\exp\left(-\frac{2\hbar}{m_pL^2}t\right)(\upsilon P_\upsilon + P_\upsilon\upsilon) + W_0. \quad (42)$$

The next step is to find the eigenvalues and eigenfunctions of the obtained invariant. The equation for determining the eigenvalues λ and eigenfunctions Φ_λ of the operator $I_p(t)$ is as follows:

$$I_p(t)\Phi_\lambda(\upsilon,t) = \lambda\Phi_\lambda(\upsilon,t); \quad \left\{\frac{1}{2m_pL^2}\left[1+\exp\left(-\frac{2\hbar}{m_pL^2}t\right)\right]P_\upsilon^2 - 2e\xi L\upsilon\cos\omega t\left[1+\exp\left(-\frac{2\hbar}{m_pL^2}t\right)\right] + \frac{\hbar^2}{2m_pL^2}\left[1-\exp\left(-\frac{2\hbar}{m_pL^2}t\right)\right]\upsilon^2 + \frac{\hbar}{2m_pL^2} \times \exp\left(-\frac{2\hbar}{m_pL^2}t\right)(\upsilon P_\upsilon + P_\upsilon\upsilon) + W_0\right\}\Phi_\lambda(\upsilon,t) = 0. \quad (43)$$

Thus, we obtain an equation for determining the function Φ_λ. Then, after simple but rather cumbersome transformations, the following equation was obtained:

$$\frac{\partial^2\Phi_\lambda}{\partial\upsilon^2} + \frac{2i}{1+\exp\left(\frac{2\hbar}{m_pL^2}t\right)}\upsilon\frac{\partial\Phi_\lambda}{\partial\upsilon} + \left(-\tanh\left(\frac{\hbar}{m_pL^2}t\right)\upsilon^2 + \frac{4em_p\xi L^3}{\hbar^2}\upsilon\cos\omega t - \left\{\frac{2m_pL^2(W_0-\lambda)}{\hbar^2\left[1+\exp\left(-\frac{2\hbar}{m_pL^2}t\right)\right]} - \frac{i}{1+\exp\left(\frac{2\hbar}{m_pL^2}t\right)}\right\}\right)\Phi_\lambda = 0. \quad (44)$$

This equation may be rewritten and looks more convenient,

$$\Phi_\lambda'' + a\upsilon\Phi_\lambda' + (\tilde\alpha\upsilon^2 + \tilde\beta\upsilon + \tilde\gamma)\Phi_\lambda = 0, \quad (45)$$

by introducing the following notation:

$$a(t) = \frac{2i}{\left[1+\exp\left(\frac{2\hbar}{m_pL^2}t\right)\right]}, \quad (46)$$

$$\tilde\alpha(t) = -\tanh\left(\frac{\hbar}{m_pL^2}t\right), \quad (47)$$

$$\tilde\beta(t) = \frac{4em_p\xi L^3}{\hbar^2}\cos\omega t, \quad (48)$$

$$\tilde\gamma(t) = -\frac{2m_pL^2(W_0-\lambda)}{\hbar^2\left[1+\exp\left(-\frac{2\hbar}{m_pL^2}t\right)\right]} + \frac{i}{1+\exp\left(\frac{2\hbar}{m_pL^2}t\right)}. \quad (49)$$

Now we can introduce the new variable

$$s(t) = \frac{1}{2}\left(-i \pm \sqrt{\tanh\left(\frac{\hbar}{m_pL^2}t\right)-1}\right), \quad (50)$$

which is the solution of such an auxiliary equation:

$$4s^2 + 4is - \tanh\left(\frac{\hbar}{m_pL^2}t\right) = 0. \quad (51)$$

Let us now take the following expression as a solution of the equation:

$$\Phi_\lambda(\upsilon) = u_\lambda(\upsilon)\exp(s\upsilon^2), \quad (52)$$

and substituting it into Eq. (45) we can obtain

$$u_\lambda''(\upsilon) + (a+4s)\upsilon u_\lambda'(\upsilon) + [\tilde\beta\upsilon + \tilde\gamma + 2s]u_\lambda(\upsilon) = 0. \quad (53)$$

By substitution of expressions:

$$u_\lambda(\upsilon) = \eta_\lambda(\psi)\exp\left(-\frac{\tilde\beta}{a+4s}\upsilon\right); \quad \psi = \sqrt{|a+4s|}\left(\upsilon - \frac{2\tilde\beta}{(a+4s)^2}\right) \quad (54)$$

into Eq. (53) a differential equation in canonical form is obtained [55]:

$$\frac{d^2\eta_\lambda}{d\upsilon^2} + \psi\frac{d\eta_\lambda}{d\upsilon} + \frac{\tilde\beta^2 + (a+4s)^2(\tilde\gamma+2s)}{(a+4s)^3}\eta_\lambda = 0. \quad (55)$$

The solutions of this equation can be represented as a linear combination of the Hermite polynomial $H_n(x) = H(n,x)$ and the confluent Kummer hypergeometric function ${}_1F_1(a,b,z) = M(a,b,z)$:

$$\eta_\lambda(\psi) = \exp\left(-\frac{\psi^2}{2}\right)\left[C_1H\left(\frac{\tilde\beta^2 + (a+4s)^2(\tilde\gamma+2s)}{(a+4s)^3} - 1, \frac{\psi}{\sqrt{2}}\right) + C_2M\left(\frac{1}{2} - \frac{\tilde\beta^2 + (a+4s)^2(\tilde\gamma+2s)}{2(a+4s)^3}, \frac{1}{2}, \frac{\psi^2}{2}\right)\right]. \quad (56)$$

The expression for the function $\Phi_\lambda(\upsilon)$ is obtained from Eqs. (52) and (54):

$$\Phi_\lambda(\upsilon) = \exp\left[-\left(\frac{a}{2}+s\right)\upsilon^2 + \frac{\tilde\beta}{a+4s}\upsilon - \frac{2\tilde\beta^2}{(a+4s)^3}\right] \times \left\{C_1H\left[\frac{\tilde\beta^2 + (a+4s)^2(\tilde\gamma+2s)}{(a+4s)^3} - 1, \times\sqrt{\frac{|a+4s|}{2}}\left(\upsilon - \frac{2\tilde\beta}{(a+4s)^2}\right)\right] + C_2M\left[\frac{1}{2} - \frac{\tilde\beta^2 + (a+4s)^2(\tilde\gamma+2s)}{2(a+4s)^3}, \frac{1}{2}, \times\frac{(a+4s)}{2}\left(\upsilon - \frac{2\tilde\beta}{(a+4s)^2}\right)^2\right]\right\}, \quad (57)$$

where the coefficients C_1 and C_2 in Eq. (57) should be determined.

According to the Lewis-Riesenfeld theory, the electron wave function in an arbitrary pth layer of a nanosystem can be represented as follows:

$$\Psi_\lambda^{(p)}(\upsilon,t)=\exp\left[i\delta_\lambda^{(p)}(t)\right]\Phi_\lambda^{(p)}(\upsilon,t), \tag{58}$$

where $\delta_\lambda^{(p)}(t)$. This is the phase factor. Having substituted Eq. (58) into the complete Schrödinger equation (1), the following equation is obtained:

$$\hbar\frac{d\delta_\lambda^{(p)}(t)}{dt}=\left\langle\Phi_\lambda^{(p)}(\upsilon,t)\middle|i\hbar\frac{\partial}{\partial t}-H_p(t)\middle|\Phi_\lambda^{(p)}(\upsilon,t)\right\rangle. \tag{59}$$

Generally, it is assumed that the system of functions $\{\Phi_\lambda(\upsilon,t)\}$ is orthogonal, and the spectrum of values λ is stationary and discrete. These assumptions were applied for the problems related with studies of quantum oscillators (see, e.g., Refs. [56–61]) and for the problems with homologous Hamiltonians. That is, it made a fundamental difference between obtaining the spectrum λ itself (usually defined as $\lambda=n+1/2,\ n\in\mathrm{Z}$) and performing the calculation of the phase factor $\delta_\lambda(t)$.

In our case, the nanosystem is open, which mostly means that the spectrum is quasistationary, and the numbers λ themselves are complex. In addition, in the general case, the system of functions $\{\Phi_\lambda(\upsilon,t)\}$ is not orthogonal. The total electron wave function in an open nanosystem is normalized to the Dirac delta function, i.e.,

$$\begin{aligned}\int_{-\infty}^{+\infty}\Psi_\lambda(\upsilon,t)\Psi_{\lambda'}^*(\upsilon,t)d\upsilon&=\int_{-\infty}^{+\infty}\exp\left[i\delta_\lambda(t)\right]\Phi_\lambda(\upsilon,t)\\&\quad\times\exp\left[-i\delta_{\lambda'}(t)\right]\Phi_{\lambda'}^*(\upsilon,t)d\upsilon\\&=\delta(\lambda'-\lambda).\end{aligned} \tag{60}$$

The integration Eq. (60) is done sequentially within all separate layers of the nanosystem $[z_{p-1},\ z_p],\ p\in\overline{1..N}$ with thickness $\Delta_p=z_p-z_{p-1}$.

In the external regions attached at the left and right sides of the nanosystem, the electronic Hamiltonian simplifies and looks like:

$$H(t)=-\frac{\hbar^2}{2m_w}\frac{\partial^2}{\partial z^2}-2ez\xi\cos\omega t. \tag{61}$$

A similar problem was solved previously in our paper [19]. However, we obtained the necessary solution in the external regions without resorting to finding a quantum mechanical invariant.

In the present paper, the calculation of an invariant were accounted and these intermediate calculations are presented in Appendix A.

In the region located at the left border of the nanosystem, the integration over the z coordinate occurs in the region $(-\infty,0)$. Substituting Eq. (61) into Eq. (59), the following relation was obtained:

$$\begin{aligned}\hbar\frac{d\delta_\lambda^{(0)}(t)}{dt}&=\left\langle\Phi_\lambda^{(0)}(\upsilon,t)\middle|i\hbar\frac{\partial}{\partial t}-H(t)\middle|\Phi_\lambda^{(0)}(\upsilon,t)\right\rangle;\\&\quad\times\left\langle\Phi_\lambda^{(0)}(\upsilon,t)\middle|i\hbar\frac{\partial}{\partial t}-H(t)\middle|\Phi_\lambda^{(0)}(\upsilon,t)\right\rangle\\&=\int_{-\infty}^{0}\exp\left\{\frac{i}{\hbar}[(\varphi_1^*-\varphi_1)z^2+[\varphi_2^*(t)-\varphi_2(t)]z]\right\}\\&\quad\times\left|i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m_w}\frac{\partial^2}{\partial z^2}+2ez\xi\cos\omega t\right|dz\\&=I_1^{(0)}(t)\frac{\partial\varphi_2(t)}{\partial t}+\frac{1}{2m_w}I_2^{(0)}(t)+2e\xi I_3^{(0)}(t)\cos\omega t.\end{aligned} \tag{62}$$

The function $\delta_\lambda^{(0)}$ can be evaluated as follows:

$$\begin{aligned}\delta_\lambda^{(0)}(t)=\int\hbar^{-1}\Bigg(&I_1^{(0)}(t)\frac{\partial\varphi_2(t)}{\partial t}+\frac{1}{2m_w}I_2^{(0)}(t)\\&+2e\xi I_3^{(0)}(t)\cos\omega t\Bigg)dt.\end{aligned} \tag{63}$$

In the region located at the right border of the nanosystem, the integration over the z coordinate occurs in the region $(-\infty,0)$ $(d,+\infty)$, where d is the total thickness of the system. Having substituted Eq. (61) into Eq. (59), the following relation was obtained:

$$\begin{aligned}\hbar\frac{d\delta_\lambda^{(N+1)}(t)}{dt}&=\left\langle\Phi_\lambda^{(N+1)}(\upsilon,t)\middle|i\hbar\frac{\partial}{\partial t}-H(t)\middle|\Phi_\lambda^{(N+1)}(\upsilon,t)\right\rangle;\\&\quad\times\left\langle\Phi_\lambda^{(N+1)}(\upsilon,t)\middle|i\hbar\frac{\partial}{\partial t}-H(t)\middle|\Phi_\lambda^{(N+1)}(\upsilon,t)\right\rangle\\&=\int_{d}^{+\infty}e^{\frac{i}{\hbar}\{(\varphi_1^*-\varphi_1)z^2+[\varphi_2^*(t)-\varphi_2(t)]z\}}\\&\quad\times\left|i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m_w}\frac{\partial^2}{\partial z^2}+2ez\xi\cos\omega t\right|dz\\&=I_1^{(N+1)}(t)\frac{\partial\varphi_2(t)}{\partial t}+\frac{1}{2m_w}I_2^{(N+1)}(t)\\&\quad+2e\xi I_3^{(N+1)}(t)\cos\omega t.\end{aligned} \tag{64}$$

The function $\delta_\lambda^{(N+1)}$ can be obtained as follows:

$$\begin{aligned}\delta_\lambda^{(N+1)}(t)=\int\hbar^{-1}\Bigg(&I_1^{(N+1)}(t)\frac{\partial\varphi_2(t)}{\partial t}+\frac{1}{2m_w}I_2^{(N+1)}(t)\\&+2e\xi I_3^{(N+1)}(t)\cos\omega t\Bigg)dt.\end{aligned} \tag{65}$$

The more detailed information on calculation of the invariants $I_1^{(0)}(t)$, $I_2^{(0)}(t)$, $I_3^{(0)}(t)$, and $I_1^{(N+1)}(t)$, $I_2^{(N+1)}(t)$, $I_3^{(N+1)}(t)$ is presented in Appendix A.

III. AN ELECTRON SPECTRAL PROBLEM: S-MATRIX METHOD

The boundary conditions for functions $\Phi_\lambda^{(p)}(z,t)$ [Eq. (57)] and their flows of probability can be stated as follows:

$$\begin{aligned}&\Phi_\lambda^{(p)}(z,t)|_{z\to z_p-0}=\Phi_\lambda^{(p)}(z,t)|_{z\to z_p+0};\\&\frac{1}{m_p}\frac{\partial\Phi_\lambda^{(p)}(z,t)}{\partial z}\bigg|_{z\to z_p-0}=\frac{1}{m_{p+1}}\frac{\partial\Phi_\lambda^{(p+1)}(z,t)}{\partial z}\bigg|_{z\to z_p+0}.\end{aligned} \tag{66}$$

Having presented the functions $\Phi_\lambda^{(0)}(z,t)$ and $\Phi_\lambda^{(N+1)}(z,t)$ as

$$\Phi_\lambda^{(0)}(z,t) = a_\lambda^{(0)}(t)\exp\left(-\frac{i}{\hbar}\left[\varphi_1 z^2 + \varphi_2(t)z\right]\right); \quad \Phi_\lambda^{(N+1)}(z,t) = a_\lambda^{(N+1)}(t)\exp\left(-\frac{i}{\hbar}\left[\varphi_1 z^2 + \varphi_2(t)z\right]\right), \tag{67}$$

the expressions for φ_1 and $\varphi_2(t)$ are defined in Eq. (A10), shown in Appendix A. Further applying sequentially boundary conditions [Eq. (66)] according to the matrix transfer method, we can obtain

$$\begin{bmatrix} a_\lambda^{(0)}(t) \\ 0 \end{bmatrix} = \begin{bmatrix} T_{11}^{(0,N+1)}(t) & T_{12}^{(0,N+1)}(t) \\ T_{21}^{(0,N+1)}(t) & T_{22}^{(0,N+1)}(t) \end{bmatrix} \begin{bmatrix} a_\lambda^{(N+1)}(t) \\ 0 \end{bmatrix} \exp\left(-\frac{i}{\hbar}[\varphi_1 d^2 + \varphi_2(t)d]\right);$$

$$\begin{bmatrix} T_{11}^{(0,N+1)}(t) & T_{12}^{(0,N+1)}(t) \\ T_{21}^{(0,N+1)}(t) & T_{22}^{(0,N+1)}(t) \end{bmatrix} = \begin{bmatrix} \tau_{11}^{(0,1)}(t) & \tau_{12}^{(0,1)}(t) \\ \tau_{21}^{(0,1)}(t) & \tau_{22}^{(0,1)}(t) \end{bmatrix} \prod_{i=2}^{N-1} \begin{bmatrix} \tau_{11}^{(i-1,i)}(t) & \tau_{12}^{(i-1,i)}(t) \\ \tau_{21}^{(i-1,i)}(t) & \tau_{22}^{(i-1,i)}(t) \end{bmatrix}^{-1}$$
$$\times \begin{bmatrix} \tau_{11}^{(i,i+1)}(t) & \tau_{12}^{(i,i+1)}(t) \\ \tau_{21}^{(i,i+1)}(t) & \tau_{22}^{(i,i+1)}(t) \end{bmatrix} \begin{bmatrix} \tau_{11}^{(N,N+1)}(t) & \tau_{12}^{(N,N+1)}(t) \\ \tau_{21}^{(N,N+1)}(t) & \tau_{22}^{(N,N+1)}(t) \end{bmatrix}. \tag{68}$$

All elements of the transfer matrix (68) depend on the eigenvalues λ, as indicated by Eqs. (43), (44), (49) (the dependence on λ is directly visible), (54)–(59), (66). The further details on elements of the transfer matrix can be found in Appendix B.

It allows us to calculate the phase factor $\delta_\lambda^{(p)}(t)$ in the solution $\Psi_\lambda^{(p)}(\upsilon,t)$ for the corresponding arbitrary layer inside the nanosystem:

$$\begin{aligned}
\delta_\lambda^{(p)}(t) &= \hbar^{-1}\int \left\langle \Phi_\lambda^{(p)}\left(\frac{z_i - \tilde{Z}_0^{(p)}}{L}, t\right)\middle| i\hbar\frac{\partial}{\partial t} - H(t)\middle|\Phi_\lambda^{(p)}\left(\frac{z_i - \tilde{Z}_0^{(p)}}{L}, t\right)\right\rangle dt \\
&\times \left\langle \Phi_\lambda^{(p)}\left(\frac{z_i - \tilde{Z}_0^{(p)}}{L}, t\right)\middle| i\hbar\frac{\partial}{\partial t} - H(t)\middle|\Phi_\lambda^{(p)}\left(\frac{z_i - \tilde{Z}_0^{(p)}}{L}, t\right)\right\rangle \\
&= \int_{z_p}^{z_{p+1}} \left(\Phi_\lambda^{(p)}\left(\frac{z - \tilde{Z}_0^{(p)}}{L}, t\right)\right)^* \left| i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m_p}\frac{\partial^2}{\partial z^2} - W_0 + 2e\tilde{Z}_0^{(p)}\xi\cos\omega t - \frac{\hbar^2}{2m_p L^2}\left(\frac{z - \tilde{Z}_0^{(p)}}{L}\right)^2\right. \\
&\left. + 2e\left(\frac{z - \tilde{Z}_0^{(p)}}{L}\right)L\xi\cos\omega t\middle|\Phi_\lambda^{(p)}\left(\frac{z - \tilde{Z}_0^{(p)}}{L}, t\right)dz.\right.
\end{aligned} \tag{69}$$

The integration in this equation can be performed exactly, but we do not present the resulting expressions because of their cumbersome form.

For determining the electronic spectrum, it is necessary to calculate the S matrix. The solutions to the full Schrödinger equation in regions external to the nanosystem can be presented as follows:

$$\Psi(z,t) = \exp\left\{\pm ikz + \frac{2ie\xi z}{\hbar}\frac{\sin\omega t}{\omega} + \frac{2ie\xi k}{m_w}\frac{\cos\omega t}{\omega^2} + f(t)\right\}, \tag{70}$$

where $f(t)$ is an unknown function, and the sign "±" refers to the solution corresponding to the incident (reflected) wave, relatively the nanosystem.

Substituting Eq. (70) into the complete Schrödinger equation with the Hamiltonian presented by Eq. (61), we have obtained

$$i\hbar\frac{df(t)}{dt} = \frac{\hbar^2}{2m_w}\left(k + \frac{2e\xi}{\hbar}\frac{\sin\omega t}{\omega}\right)^2 - \frac{2e\hbar k\xi}{m_w}\frac{\sin\omega t}{\omega};$$
$$f(t) = -\frac{i\hbar k^2}{2m_w}t - \frac{ie^2}{2\hbar m_w}\frac{\xi^2(2\omega t - \sin 2\omega t)}{\omega^3}. \tag{71}$$

The general solution for the wave function in the region at the left border of the nanosystem can be written as follows:

$$\begin{aligned}
&\Psi^{(0)}(z,t) \\
&= A^{(0)}(\lambda,t)\left(e^{i\left(k + \frac{2e\xi}{\hbar}\frac{\sin\omega t}{\omega}\right)z} e^{i\left[\frac{\hbar k^2}{2m_w}t + \frac{2e\xi k}{m_w}\frac{\cos\omega t}{\omega^2} - \frac{e^2}{2\hbar m_w}\frac{\xi^2(2\omega t - \sin 2\omega t)}{\omega^3}\right]}\right. \\
&\left. - S(\lambda,t)e^{-i\left(k - \frac{2e\xi}{\hbar}\frac{\sin\omega t}{\omega}\right)z} e^{i\left[\frac{\hbar k^2}{2m_w}t - \frac{2e\xi k}{m_w}\frac{\cos\omega t}{\omega^2} - \frac{e^2}{2\hbar m_w}\frac{\xi^2(2\omega t - \sin 2\omega t)}{\omega^3}\right]}\right).
\end{aligned} \tag{72}$$

Similarly, the wave function in the region at the right border of the nanosystem can be written as follows:

$$\begin{aligned}
\Psi_\lambda^{(N+1)}(z,t) = A^{(N+1)}(\lambda,t)&\left(e^{-i\left(k - \frac{2e\xi}{\hbar}\frac{\sin\omega t}{\omega}\right)z}\right. \\
&\times e^{i\left[\frac{\hbar k^2}{2m_w}t - \frac{2e\xi k}{m_w}\frac{\cos\omega t}{\omega^2} - \frac{e^2}{2\hbar m_w}\frac{\xi^2(2\omega t - \sin 2\omega t)}{\omega^3}\right]} \\
&- S(\lambda,t)e^{i\left(k + \frac{2e\xi}{\hbar}\frac{\sin\omega t}{\omega}\right)z} \\
&\left.\times e^{i\left[\frac{\hbar k^2}{2m_w}t + \frac{2e\xi k}{m_w}\frac{\cos\omega t}{\omega^2} - \frac{e^2}{2\hbar m_0}\frac{\xi^2(2\omega t - \sin 2\omega t)}{\omega^3}\right]}\right).
\end{aligned} \tag{73}$$

Having applied to the wave function $\Psi_\lambda^{(0)}(z,t)$, $\Psi_\lambda^{(p)}(z,t)$, $\Psi_\lambda^{(N+1)}(z,t)$ boundary conditions similar to those presented in Eq. (66) and applying the

transfer matrix method, the following relation were obtained:

$$\begin{bmatrix} A^{(0)}(\lambda,t)e^{\frac{2ie\xi k}{m_w}\frac{\cos\omega t}{\omega^2}} \\ -A^{(0)}(\lambda,t)S(\lambda,t)e^{-\frac{2ie\xi k}{m_w}\frac{\cos\omega t}{\omega^2}} \end{bmatrix} = e^{i[\delta_\lambda^{(N)}(t)-\delta_\lambda^{(1)}(t)]}\begin{bmatrix} T_{11}^{(1,N+1)}(t) & T_{12}^{(1,N+1)}(t) \\ T_{21}^{(1,N+1)}(t) & T_{22}^{(1,N+1)}(t) \end{bmatrix} \times \begin{bmatrix} -A^{(N+1)}(\lambda,t)S(\lambda,t)e^{i\left(k+\frac{2e\xi}{\hbar}\frac{\sin\omega t}{\omega}\right)z_N}e^{\frac{2ie\xi k}{m_w}\frac{\cos\omega t}{\omega^2}} \\ A^{(N+1)}(\lambda,t)e^{-i\left(k-\frac{2e\xi}{\hbar}\frac{\sin\omega t}{\omega}\right)z_N}e^{-\frac{2ie\xi k}{m_w}\frac{\cos\omega t}{\omega^2}} \end{bmatrix}. \tag{74}$$

Here, the factors $\exp\{i[\frac{\hbar k^2}{2m_w}t - \frac{e^2}{2\hbar m_w}\frac{\xi^2(2\omega t-\sin 2\omega t)}{\omega^3}]\}$ have been canceled on both sides of the equality.

In addition, the presence of a multiplier $\exp\{i[\delta_\lambda^{(1)}(t) - \delta_\lambda^{(N)}(t)]\}$ confirms that the transfer matrix depends only on the phase multiplier in the first and last layers of the nanosystem. This is more evident from the transformation of the product of matrices that form the general transfer matrix:

$$\begin{aligned} &\prod_{i=2}^{N-1} e^{-i\delta_\lambda^{(i-1)}(t)}\begin{bmatrix} \tau_{11}^{(i-1,i)}(t) & \tau_{12}^{(i-1,i)}(t) \\ \tau_{21}^{(i-1,i)}(t) & \tau_{22}^{(i-1,i)}(t) \end{bmatrix}^{-1} \\ &\quad\times e^{i\delta_\lambda^{(i)}(t)}\begin{bmatrix} \tau_{11}^{(i,i+1)}(t) & \tau_{12}^{(i,i+1)}(t) \\ \tau_{21}^{(i,i+1)}(t) & \tau_{22}^{(i,i+1)}(t) \end{bmatrix} \\ &= e^{i[\delta_\lambda^{(N)}(t)-\delta_\lambda^{(1)}(t)]}\begin{bmatrix} T_{11}^{(1,N+1)}(t) & T_{12}^{(1,N+1)}(t) \\ T_{21}^{(1,N+1)}(t) & T_{22}^{(1,N+1)}(t) \end{bmatrix}, \end{aligned} \tag{75}$$

where the elements $\tau_{11}^{(i,i+1)}(t)$; $\tau_{12}^{(i,i+1)}(t)$; $\tau_{21}^{(i,i+1)}(t)$; $\tau_{22}^{(i,i+1)}(t)$ were calculated as those presented in Appendix B [Eqs. (B2)].

From the matrix in Eq. (74), we can obtain a quadratic equation for determining the S matrix:

$$\begin{aligned} &S^2(\lambda,t) + \frac{1}{T_{11}^{(1,N+1)}(t)}\Big[T_{21}^{(1,N+1)}(t)e^{\frac{4ie\xi k}{m_w}\frac{\cos\omega t}{\omega^2}} \\ &\quad - T_{12}^{(1,N+1)}(t)e^{-2ikz_N}e^{-\frac{4ie\xi k}{m_w}\frac{\cos\omega t}{\omega^2}}\Big] \\ &\quad\times S(\lambda,t) - \frac{T_{22}^{(1,N+1)}(t)}{T_{11}^{(1,N+1)}(t)}e^{-2ikz_N} = 0; \end{aligned}$$

$$\begin{aligned} S(\lambda,t) = &-\frac{1}{2T_{11}^{(1,N+1)}(t)}\Big[T_{21}^{(1,N+1)}(t)e^{\frac{4ie\xi k}{m_w}\frac{\cos\omega t}{\omega^2}} \\ &- T_{12}^{(1,N+1)}(t)e^{-2ikz_N}e^{-\frac{4ie\xi k}{m_w}\frac{\cos\omega t}{\omega^2}}\Big] \\ &\pm\Bigg\{\frac{1}{4\big[T_{11}^{(1,N+1)}(t)\big]^2}\Big[T_{21}^{(1,N+1)}(t)e^{\frac{4ie\xi k}{m_w}\frac{\cos\omega t}{\omega^2}} \\ &- T_{12}^{(1,N+1)}(t)e^{-2ikz_N}e^{-\frac{4ie\xi k}{m_w}\frac{\cos\omega t}{\omega^2}}\Big]^2 \\ &+ \frac{T_{22}^{(1,N+1)}(t)}{T_{11}^{(1,N+1)}(t)}e^{-2ikz_N}\Bigg\}^{1/2}. \end{aligned} \tag{76}$$

Since the elements of the transfer matrix (68), (74) depend on the eigenvalue λ, the resulting S matrix also depends on it.

According to the scattering theory [48], determination of the electronic quasistationary spectrum $\lambda_n = E_n - i\Gamma_n/2$ deals with finding solutions to the dispersion equation in the complex plane at an arbitrary moment in time t:

$$S^{-1}(E_n - i\Gamma_n/2, t) = 0. \tag{77}$$

This makes it possible to determine the resonance energies E_n and resonance Γ_n widths of electronic quasistationary states. The detailed methodology for finding spectral parameters using Eq. (77) is the following. The Schrödinger equation was solved with the stationary part of the Hamiltonian (2) to find the stationary electronic spectrum. The desired quantum transition was selected and the frequency ω and value ξ from the stationary spectrum (the selection of these quantities will be discussed in direct calculations below) were calculated. Then, fixing a value of time t, the S matrix and dispersion equation (77) was evaluated. The real $[\mathrm{Re}(S^{-1}(E_n - i\Gamma_n/2, t)) = 0]$ and imaginary $[\mathrm{Im}(S^{-1}(E_n - i\Gamma_n/2, t)) = 0]$ parts of this dispersion equation were solved numerically to find the resonance energies E_n and resonant widths Γ_n, respectively. Examples of calculations of these dispersion equations are given in the Supplemental Material [62]. The obtained values $E_n, \omega_{nm}, \xi_{nm}$ were used to calculate the electronic conductivity.

IV. CALCULATION OF ELECTRONIC CONDUCTIVITY

The calculation of the electronic conductivity requires evaluation of the electron flow density in the media to the left and right border of the nanostructure based on the relation

$$J(E,z,t) = \frac{e\hbar n_0}{2im_w}\left[\Psi^*(z,t)\frac{\partial\Psi(z,t)}{\partial z} - \Psi(z,t)\frac{\partial\Psi^*(z,t)}{\partial z}\right]. \tag{78}$$

Tacking into account Eqs. (72) and (73), the following relation were obtained for the left border of the nanostructure,

$$\begin{aligned} &J(E,z,t)|_{z=0} \\ &= \frac{e\hbar n_0}{m_w}|A^{(0)}(\lambda,t)|^2[k(1-|S(\lambda,t)|^2) \\ &\quad + \frac{2e\xi}{\hbar}\frac{\sin\omega t}{\omega}\left[S^*(\lambda,t) - \exp\left(-\frac{4e\xi k}{m_w}\frac{\cos\omega t}{\omega^2}\right)\right] \\ &\quad\times\left[\exp\left(\frac{4e\xi k}{m_w}\frac{\cos\omega t}{\omega^2}\right) - S(\lambda,t)\right], \end{aligned}$$

and the right border of the nanostructure:

$$\begin{aligned} &J(E,z,t)|_{z=z_N} \\ &= \frac{e\hbar n_0}{2m_w}|A^{(N+1)}(\lambda,t)|^2\Bigg\{2k - \frac{4e\xi}{\hbar}\frac{\sin\omega t}{\omega} - \frac{3e\xi}{\hbar}\frac{\sin\omega t}{\omega} \\ &\quad\times\exp\left[-i\left(\frac{4e\xi k}{m_w}\frac{\cos\omega t}{\omega^2} + \left(k + \frac{2e\xi}{\hbar}\frac{\sin\omega t}{\omega}\right)z_N\right)\right] \\ &\quad\times\left[\exp\left(\frac{2ie\xi}{\hbar}\frac{\sin\omega t}{\omega}z_N\right)S^*(\lambda,t)\right. \\ &\quad\left.+\exp\left[2i\left(\frac{4e\xi k}{m_w}\frac{\cos\omega t}{\omega^2} + \left(k + \frac{2e\xi}{\hbar}\frac{\sin\omega t}{\omega}\right)z_N\right)\right]\right. \\ &\quad\left.- 2|S(\lambda,t)|^2\left(k + \frac{e\xi}{\hbar}\frac{\sin\omega t}{\omega}\right)\right\}. \end{aligned} \tag{79}$$

The energy received or transferred to an electromagnetic field over a time t, which for real quantum cascade devices [2–4] is close to period $T = 2\pi/\omega$, through the interaction of an electron with this field, can be calculated as

$$E_{e-em} = \int_0^t dt \int_0^{z_N} J(E,z,t)\,\xi(z,t)dz = z_N \int_0^t \xi(t)\,j(E,t)dt. \tag{80}$$

Here, $j(E,t)$ is the reduced current density:

$$j(E,t) = \frac{1}{z_N}\int_0^{z_N} J(E,z,t)dz. \tag{81}$$

Since the phase of the reduced current density is generally shifted, relatively the phase of the electric component of the electromagnetic field $\xi(t)$ by the amount $\Delta\phi$ of the complex electronic conductivity should be considered:

$$\sigma = \mathrm{Re}\sigma + i\mathrm{Im}\sigma. \tag{82}$$

The value σ is directly caused by the induced current:

$$i(E,\,t) = \mathrm{Re}(\sigma)\xi(t) = 2\xi\mathrm{Re}(\sigma)\cos(\omega t). \tag{83}$$

From Eqs. (80) and (83), the following relation can be obtained:

$$E_{e-em} = 4z_N\xi^2\mathrm{Re}(\sigma)\int_0^t \cos^2\omega t\,dt = z_N\xi^2\left(2t + \frac{\sin 2t\omega}{\omega}\right)\mathrm{Re}(\sigma). \tag{84}$$

The energy E_{e-em} of electron interaction with an electromagnetic field may be calculated as the sum of the energies of flows of noninteracting electrons that emanate from both sides of the nanosystem:

$$E_{e-em} = \frac{\hbar\omega t}{e}\Big\{\big[J(E+\hbar\omega,z,t)|_{z=z_N} - J(E-\hbar\omega,z,t)|_{z=z_N}\big] - [J(E+\hbar\omega,z,t)|_{z=0} - J(E-\hbar\omega,z,t)|_{z=0}]\Big\}. \tag{85}$$

Equating Eqs. (84) and (85), the electronic conductivity can be defined as follows:

$$\sigma^{(e)} = \mathrm{Re}(\sigma) = \frac{\hbar^2\omega n_0 t}{2\xi^2 z_N m_w(2t\omega + \sin 2t\omega)} \times \Big\{\big[J(E+\hbar\omega,z,t)|_{z=z_N} - J(E-\hbar\omega,z,t)|_{z=z_N}\big] - [J(E+\hbar\omega,z,t)|_{z=0} - J(E-\hbar\omega,z,t)|_{z=0}]\Big\}. \tag{86}$$

Here, the quantities $E \pm \hbar\omega$ specify the following replacement: $k \to k_\pm = \sqrt{2m_w(E \pm \hbar\omega)}/\hbar$ in Eq. (79). The frequencies included in the calculated relations, in particular, in the expressions for electronic conductivity (86), are calculated as follows: $\omega_{nm} = (E_m - E_n)\hbar^{-1}$, where n and m are the numbers of the corresponding electronic levels of the nanosystem. The amplitude values of the electrical component of the electromagnetic field are calculated as follows: $\xi = \xi_{nm} = 10^{-2}(E_m - E_n)/ed$. It corresponds to the case of a weak electromagnetic field. This case is also satisfied by the following condition: $e\xi d \ll E_{nm}$ (the maximum energy of interaction of electrons with the electromagnetic field should be significantly smaller than the energy of electronic transitions). This condition can be controlled, for example, by changing the geometric design of the nanosystem. It should be noted that the values of ω_{nm} and ξ_{nm} are also different for different values of magnetic field induction. These values are also different when calculating the values of electronic conductivity that arise in various quantum transitions with different energy values. Also, we emphasize that the frequencies ω_{nm} and values ξ_{nm} used in the calculation of electronic conductivity are derived from the E_n values obtained from the S matrix, not from the stationary problem. Thus, when calculating the frequency ω and the amplitude ξ, which are substituted into expressions (76) and (77), we initially use the spectrum obtained from the stationary part of the Hamiltonian (2). Subsequently, when calculating the resonance energies E_n and widths Γ_n for quasistationary S-matrix poles, the entirety of the Hamiltonian (2) was considered. This is supported by the elements of the transfer matrix included in expression (76), which, according to relations (66) and the expressions provided in Appendix B, correspond to the entire Hamiltonian (2).

V. EXAMPLE OF CALCULATION OF SPECTRAL CHARACTERISTICS AND ELECTRONIC CONDUCTIVITY

Generally, the optoelectronic devices are based on the ballistic transport of a monoenergetic beam of electrons. For maintaining the coherence of the tunneled flow, it is important to account for the finite lifetime of electronic quasistationary states.

The relaxation time τ_{rel} is of importance in the operation of real nanodevices. Since such nanodevices are based on the ballistic transport of electrons, the relaxation time sets the limit at which the electron system has not yet been removed from the coherent state due to dissipation processes, due to electron-phonon interaction, in particular. The developed theory is applicable to calculations of the spectrum and resonance widths of electronic quasistationary states within any time interval. That is, such calculations can be performed within the relevant interval of values commensurate with the relaxation time τ_{rel}. It should be noted that such calculations cannot be made within the framework of theory based on stationary solutions of the Schrödinger equation [40]. In our case, we calculate the relaxation time τ_{rel} directly for a practical purpose: establishing the time values for which the study of tunnel transport is relevant, that is, while maintaining its coherence and the ballistic tunneling regime. The immediate purpose of considering tunnel transport with reference to relaxation time values is due to the practical purpose of applying the theory within the same timeframe that occurs in experimentally implemented devices. However, since the mechanisms that determine the relaxation time are not directly included in the Hamiltonian of the system under study, this allows us to consider the tunneling process in an arbitrary time interval, taking into account the dependence of the Hamiltonian term on frequency ω.

The relaxation time was calculated using the Matthiessen equation:

$$\tau_{\mathrm{rel}} = (\tau_{\mathrm{ph}-e}^{-1} + \tau_B^{-1} + \tau_U^{-1} + \tau_M^{-1})^{-1}. \tag{87}$$

Here, τ_{ph-e} is the scattering time due to electron-phonon interaction, τ_B is the contribution of boundary scattering relaxation time at the effective thickness z_N of the system, τ_U is the relaxation time due Umklapp scattering, and τ_M is the contribution related with the mass-difference scattering (due to difference of the effective masses of electrons in well and barrier layers). The more detailed analysis of these components can be found in Ref. [63]. Direct calculations of relaxation times for multilayer nanosystems with a detailed analysis of the components included in the Matthiessen equation are also presented in Ref. [64]. The estimation testifies that for the studied nanostructure relaxation time, τ_{rel} is of order 8.2 ps. This value of the relaxation time is close in order of magnitude to the lifetime of electronic quasistationary states. It should be noted that the relaxation time has a formally similar designation to the elements of the transfer matrix presented in relation (75), however, these quantities are in no way related with each other.

The theory was applied to calculate the electronic spectral parameters for different time values including $t < \tau_{rel}$ (coherent tunneling mode) and $t > \tau_{rel}$ (the case when quasistationary states can be destroyed due to dissipation mechanisms). We draw the reader's attention to the fact that all calculated values of the spectral parameters of the electron and electronic conductivity depend on time. They are determined by the time dependence of the Hamiltonian, which describes the interaction of electrons with an electromagnetic field.

As an example, a three-barrier layered nanosystem with GaN potential wells with 4 nm and 10 nm thickness, and AlN potential barriers with 3 nm thickness were analyzed. The effective electron masses in the potential well and barriers were equal to $m_w = 0.2m_e$ and $m_b = 0.4m_e$, respectively. Here m_e is the mass of a free electron. Since constant internal electric and external magnetic fields have a significant impact on the tunneling process and the electronic potential scheme of the nanosystem, it should be analyzed by introducing the effective potential into consideration. The potential scheme of the nanosystem without magnetic field was calculated according to general principles [14–16]. Next, the effective potential of a nanosystem is obtained from Eqs (4), (5), (9) and characterizes the total potential energy of an electron in electric and magnetic fields (Fig. 2):

$$U_{\text{eff}}(z) = U(z) + U_e(z) + \frac{\hbar^2}{2m(z)}\left(k_x - \frac{eBz}{\hbar}\right)^2. \qquad (88)$$

Examples of calculated potential schemes for an electron $U_{\text{eff}}(z)$ accounting for the contribution of the internal electric field for various values of magnetic fields B are shown in Fig. 2. It should be noted that an increase of the magnetic field B resulted in the increase of the potentials $U_{\text{eff}}(z)$.

Moreover, there are a number of features that could not be established within the framework of a stationary problem [38,40] as the mainstream method used in such types of problems. Such theoretical models are limited mainly by the possibility of calculating the electronic spectrum. As calculations of the resonant energies and widths of electronic quasistationary states demonstrate the dependence on the magnitude of the magnetic field induction B for different moments of time t, expressed in units of relaxation time τ_{rel}, using a magnetic field, it is possible to adjust the operating frequency

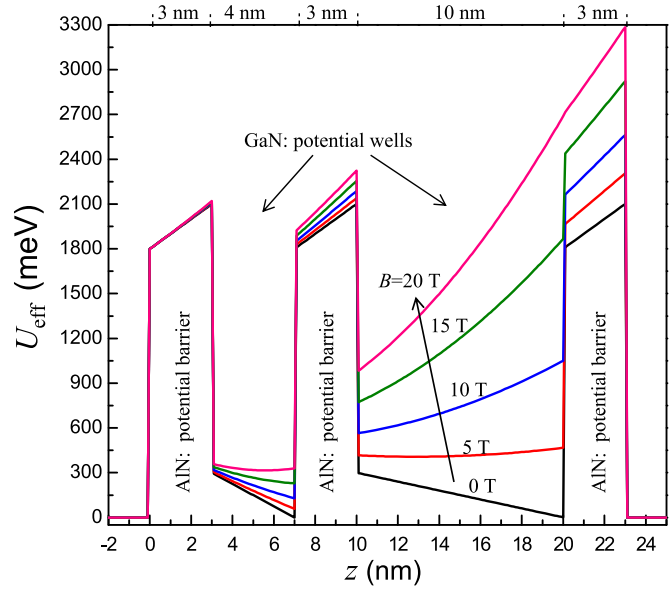

FIG. 2. The electronic potential profile of the nanosystem $U_{\text{eff}}(z)$, calculated using relation (88) (for $k_x = 0$) for different values of magnetic field induction B.

and other parameters of nanodevices operating on quantum transitions between electronic states of nanosystems. This confirms the importance of the proposed theoretical results for the nanoscience subject area, since they make it possible to study electronic quasistationary states under conditions closer to the conditions of actually existing nanodevices.

As can be seen from Figs. 3(a)–3(d), the resonance energies of the electronic levels are of quadratic dependence on the magnetic field induction. A characteristic feature of these dependencies is the formation of anticrossings (bottleneck effects) between neighboring energy levels, designated as (I)–(III). The presence of anticrossings testifies a change in the localization of electrons in the nanosystem in their quasistationary states under the influence of a magnetic field. As can be seen from Fig. 3, the energy dependencies on the magnetic field also differ for different values of time t. Thus, anticrossings that occur at $t = 0.2\tau_{rel}$ for values of time $t = 0.5\tau_{rel}$ are shifted to the left in the scale of magnetic field induction values; when $t = 1.0\tau_{rel}$, anticrossing between the second and third electronic levels disappears, and anticrossings (I) and (III) shift to the left; when $t = 1.5\tau_{rel}$, only anticrossing (I) remains between the first and second electronic levels. In addition, the localization of an electron in a nanosystem depends not only on the induction of a magnetic field but is different for different moments of time t, which results from the influence of a time-dependent electromagnetic field. In addition, it should be noted that when the energy values, which are the same at $B = 0$, are reached, as the value of B increases, the greater the calculated value of the moment t (the permissible values of which are less or comparable with the relaxation time τ_{rel}). Thus, a magnetic field makes it possible to control the operating frequency of nanodevices operating on quantum electronic transitions. Because of it, a very important aspect is to be taken into account: the relaxation time and tunneling time of the electron flow in comparison with this value. It should be noted that within the framework of the developed

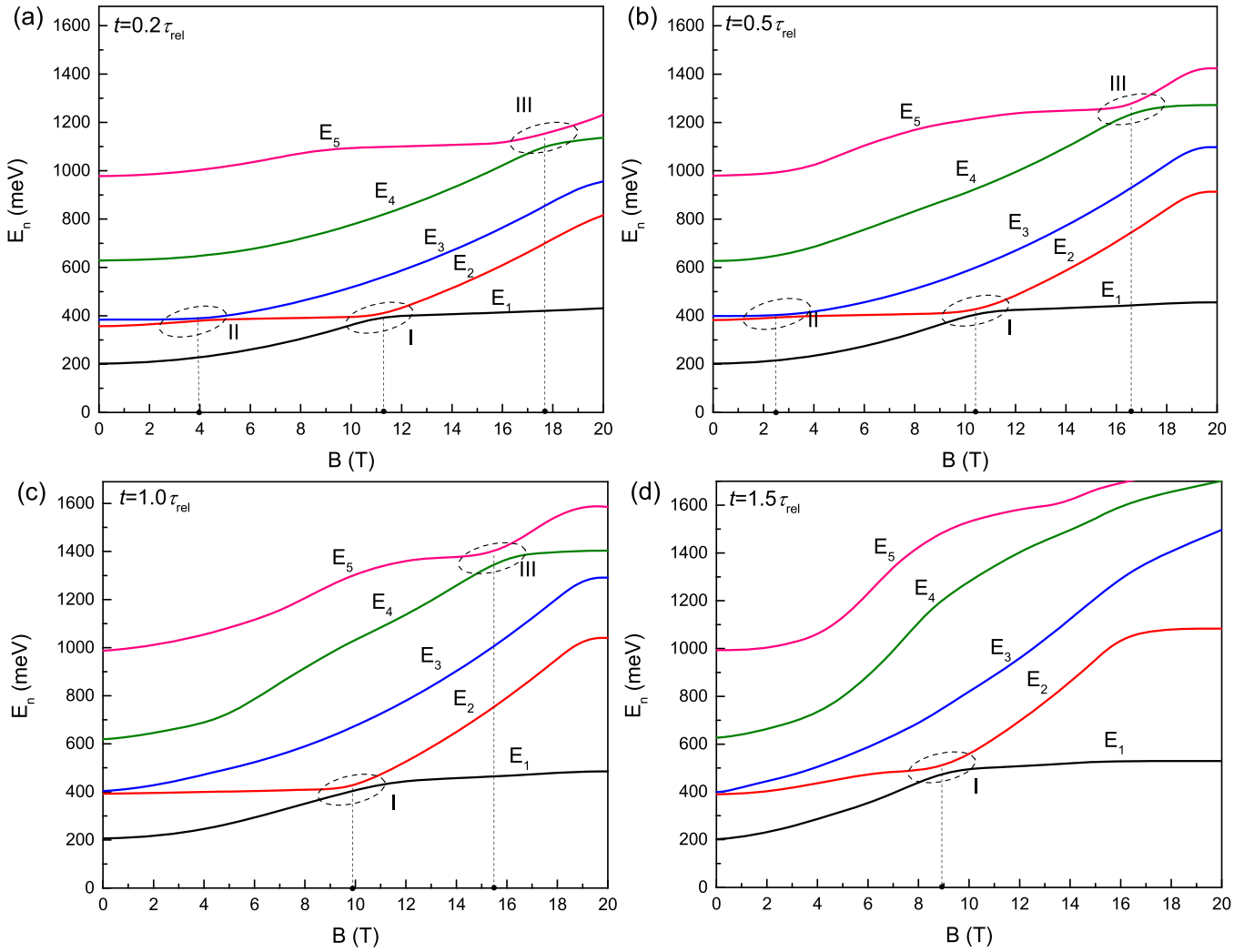

FIG. 3. The energies of the first five electronic levels E_n localized in a nanostructure as a function of the magnetic field induction B for different times t taken in terms of relaxation time τ_{rel}: $0.2\tau_{\text{rel}}$ (a); $0.5\tau_{\text{rel}}$ (b); $1.0\tau_{\text{rel}}$ (c); $1.5\tau_{\text{rel}}$ (d).

theory, it is not possible to carry out the limiting transition $t \to 0$ to a stationary problem when electrons do not interact with the electromagnetic field. However, as calculations of the electronic spectrum at $t = 0.2\tau_{\text{rel}}$ testify, they correlate quite well with the results obtained in the model of a closed nanosystem, which allows us to compare formally the results of both methods.

Additional information can be obtained by analyzing the calculated resonance widths of electronic quasistationary states, the dependencies on the magnetic field induction of which are presented in Figs. 4(a)–4(d). As can be seen from Fig. 4 for values B, which depend on the electronic energies $E_n(B)$, presented in Fig. 3, anticrossings take place; in the dependencies $\Gamma_n(B)$ crossings of curves for neighboring levels are observed. In general, the effect of resonant widths on the magnitude of the magnetic field induction is more complex than the corresponding resonant energies and is not likely to increasingly effect their values throughout the entire range of changes in B. Thus, for all calculated values t, the resonant width of the first quasistationary state Γ_1 before the formation of crossing with the resonant width of the second quasistationary state Γ_2 insufficiently depends on the magnetic field; after crossing, it increases sharply, reaching its maximum value. The resonant width of the second quasistationary state Γ_2, on the contrary, at $t \leqslant \tau_{\text{rel}}$ reaches its maximum values before the formation of crossing with the resonant width of the first quasistationary state, subsequently, insufficiently depending on B. The resonant width of the third quasistationary state Γ_3 at $t \leqslant \tau_{\text{rel}}$ has the greatest values for low values of the magnetic field induction B. At $t \leqslant \tau_{\text{rel}}$, Γ_2 and Γ_3 both reach their maximum values only at $B = 20\,\text{T}$. The resonant width of the fourth quasistationary state Γ_4 is similar to that mentioned above, deepening insufficiently on the magnetic field induction for all values t, increasing sharply at the end of the interval of values B. And vice versa, the resonant width of the fifth quasistationary state Γ_5 increases almost over the entire interval of values of B, sharply decreasing by its end.

To sum up, the obtained results testify that the influence of a magnetic field can be used to correct two more important characteristics of nanodevices: the lifetime of electrons in a working quasistationary state, being associated with the resonant width $\tau_n = \hbar^{-1}\Gamma_n$, and also changes the emission or absorption band width $\Gamma_{nm} = \Gamma_n + \Gamma_m$ in the $n\rangle \to m\rangle$ transition.

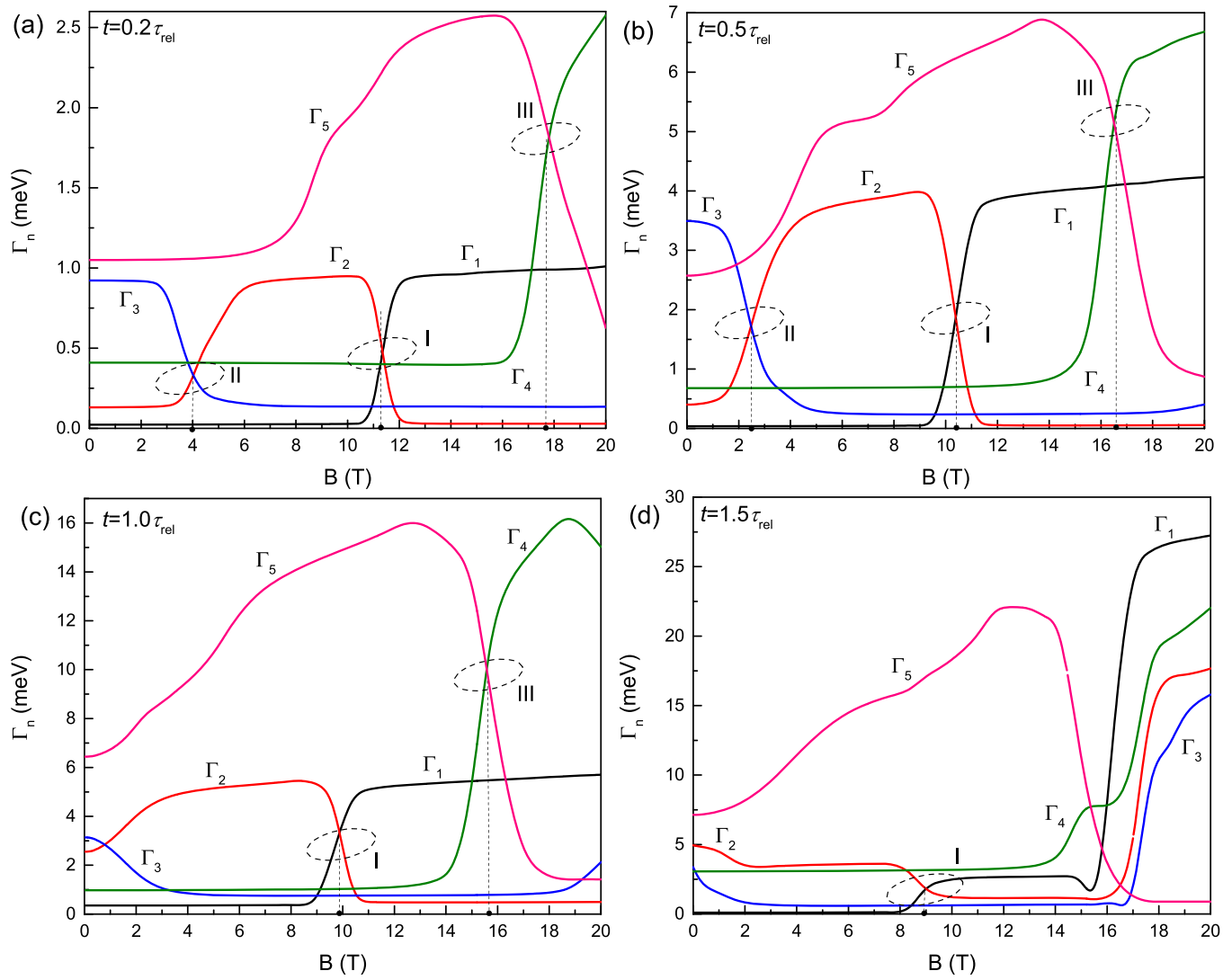

FIG. 4. The resonance widths of the first five electronic levels Γ_n localized in a nanostructure as a function of the magnetic field induction B. Calculations were performed for different values of time t taken in terms of relaxation time τ_{rel}: $0.2\tau_{\text{rel}}$ (a); $0.5\tau_{\text{rel}}$ (b); $1.0\tau_{\text{rel}}$ (c); $1.5\tau_{\text{rel}}$ (d).

In the nanodevices such as quantum cascade detectors and lasers, their operation consists of injecting an electron beam into a nanosystem. The energy of the electrons must coincide with the energy of one of the electronic quasi-stationary states of this nanosystem. There are no restrictions on the consideration of quantum transitions that can arise in the nanosystem under study. In subsequent electronic transitions, a tunneling current, being associated with electronic conductivity, is formed. Electronic conductivity serves as a measure of the intensity of such quantum transitions. For calculations of electronic conductivity, the quantum transitions from the first quasistationary electronic state to the remaining quasistationary states were considered. Calculations of electronic conductivity, formed in detector electronic transitions, were performed for the same time values as in calculations of resonance energies (Fig. 3) and widths (Fig. 4). The results of these calculations are presented in Figs. 5(a)–5(d). For convenience, conductivity values are presented in relative units, and for ease of scaling, natural logarithms of these values are taken. As seen in Fig. 5, a magnetic field B can be applied to increase the electronic conductivity of the nanosystem for the calculated moments of time $t \leqslant \tau_{\text{rel}}$. However, it should be noted that different ranges of induction B values are acceptable to achieve this goal. For example, for conductivity σ_{12} it is important to use the vicinity of the induction values (from 8 T to 13 T), in which anticrossings of the energies of the first and second quasistationary states (Fig. 3) and crossings from the resonant widths (Fig. 4) take place [marked in the figures as (I)]. Conductivity σ_{13} increases with the value of induction B in the range from 4 T to 11 T at $t = 0.2\tau_{\text{rel}}$; in the range from 4 T to 9 T at $t = 0.5\tau_{\text{rel}}$; and in the range from 2 T to 7 T at $t = 1.0\tau_{\text{rel}}$. As for the use of a magnetic field for conductivities σ_{14} and σ_{15}, despite the fact that an increase of induction B over almost the entire range of calculations leads to an increase of the values of these conductivities, they are, however, much less than conductivities σ_{12} and σ_{13}, which negates the possibility of their use. For this reason, it is not worth using a magnetic field in the case when $t = 1.5\tau_{\text{rel}}$, since in this case all the conductivities are small or they are all proportional to each other (this is also true for large

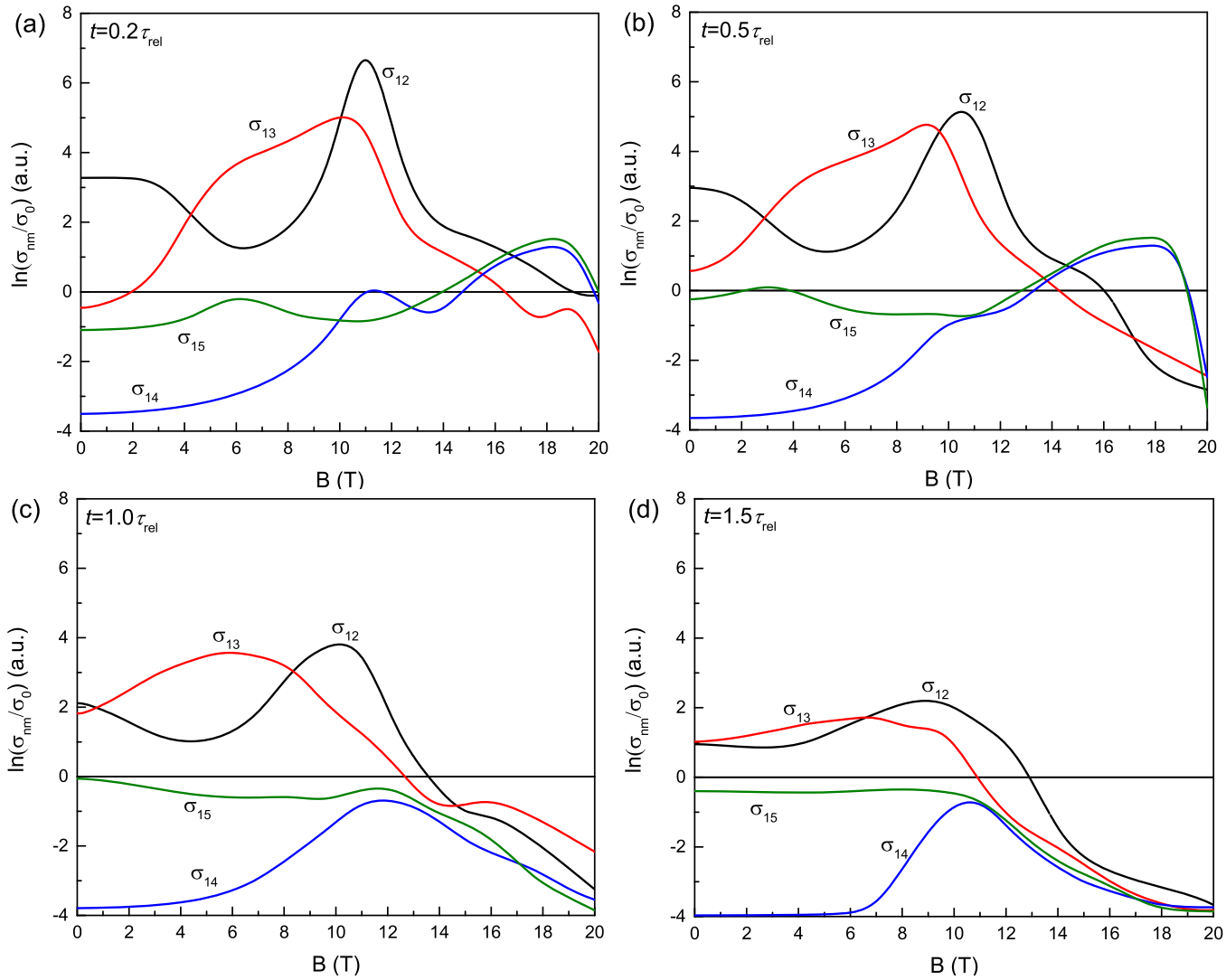

FIG. 5. The logarithms of dimensionless electronic conductivity quantities $\ln(\sigma_{1m}/\sigma_0)$ arising in detector quantum transitions $1\rangle \to m\rangle$, $m = 2;\ 3;\ 4;\ 5$ as a function of the magnetic field induction B. Calculations were performed for different values of time t taken in terms of relaxation time $\tau_{\rm rel}$: $0.2\tau_{\rm rel}$ (a); $0.5\tau_{\rm rel}$ (b); $1.0\tau_{\rm rel}$ (c); $1.5\tau_{\rm rel}$ (d). Here $\sigma_0 = 1$ S/cm.

induction values $B > 14$ T at $t \leqslant \tau_{\rm rel}$). Thus, to summarize, we should highlight the fact that the use of a magnetic field to increase electronic conductivity is reasonable only for quantum transitions $1\rangle \to 2\rangle$ and $1\rangle \to 3\rangle$, and only under the condition that $t \leqslant \tau_{\rm rel}$. Thus, the results presented in Figs. 3–5, in addition to demonstrating the dependence on the magnetic field induction, also illustrate an important aspect of the effect of a time-dependent electromagnetic field on the electronic quasistationary spectrum and electronic conductivity of a nanosystem. This dependence clearly shows differences between these quantities when calculated for different values of frequency ω and amplitude ξ [see also the text after Eq. (86)].

The strong time dependence of resonance energies E_n and widths Γ_n for larger magnetic field inductions B is associated with changes in the localization of electronic quasistationary states in the nanosystem. This is indicated by the presence of anticrossings in the dependence of resonance energies on magnetic field induction and crossings in similar dependencies of resonance widths. In our opinion, as the value of time t increases, the electrons manage to tunnel more into the output quantum well, the bottom of which rises significantly with an increase in the magnitude of the magnetic induction (as seen in Fig. 2), leading to an increase in the energy values of the electron levels. Similarly, this leads to a decrease in the lifetime of electronic quasistationary states. Since the lifetimes of electronic quasistationary states are related to the resonance width of the level as $\tau_n = \hbar^{-1}\Gamma_n$, this results in an increase in the values of resonance widths and, consequently, in absorption or emission bands. This is directly reflected in a decrease in conductivity values, as confirmed by the dependencies shown in Fig. 5.

VI. SUMMARY

A theory of the tunneling transport of electrons through an open multilayer nanosystem has been developed in the presence of the influence of a constant internal electric field, a constant magnetic field directed perpendicular to the direction of electron motion and taking into account the interaction

of tunneled electrons with the time-dependent electromagnetic field. For this purpose, the complete Schrödinger equation with a time-dependent potential is considered. The solution of this equation was obtained by applying the Lewis-Riesenfeld method together with a representation of the wave function in the way is specific for scattering theory. This result demonstrates the possibility of applying this approach to problems studying tunnel transport in open nanosystems. The use of boundary conditions for the wave function and flows of its probability at the boundaries of the layers of the nanosystem made it possible to unambiguously determine the electronic wave function and the S matrix from poles of which the electronic quasistationary spectrum-resonant energies and widths are determined. The use of the obtained wave functions made it possible to perform analytical calculations of the electronic conductivity of the nanosystem.

Calculations of the spectral characteristics of quasistationary states and electronic conductivity, performed using the proposed theory for the parameters of a three-barrier nitride nanostructure, made it possible to obtain the main results:

(1) With increasing magnetic field induction, the energies of electronic levels also increase, forming anticrossings between neighboring energy levels. The positions of these anticrossings, as the value of the moment t taken in the calculations approaches the value of the corresponding relaxation time τ_{tel}, shift to the left in the scale of magnetic field induction values; some of these anticrossings disappear altogether at $t \geqslant \tau_{\text{tel}}$;

(2) With an increase in the magnitude of the magnetic field induction, the dependencies of the resonant widths form crossings for the same induction values as the formation of anticrossings in the dependencies of the resonant energies took place. These crossings shift in scale and disappear in the same way as anticrossings of resonant energies—this happens for the same values of magnetic field induction and time t. The presence of resonance-dependent interval widths for which their values increase makes it possible to influence the lifetime of electronic quasistationary states by changing the magnetic field induction, as well as change the widths of absorption and emission bands in functioning nanodevices;

(3) Calculations of the electronic conductivity of a nanosystem depending on the magnetic field induction testified that, by changing the induction value, it is possible to increase the value of the electronic conductivity formed in the electronic transitions $1\rangle \to 2\rangle$ and $1\rangle \to 3\rangle$, however, this will be effective only under the condition that the tunneling mode occurs in the time interval not exceeding the relaxation time.

In future studies, it would be desirable to investigate tunneling processes in systems of electrons interacting with acoustic and optical phonons. This would allow us to develop a consistent theory of electron-phonon interactions in open multilayer nanosystems for arbitrary temperature values. As a result, we would be able to investigate decoherence effects under more realistic conditions.

We express our hope that the theoretical and computational results being obtained will be useful both to researchers engaged in theoretical studies of problems for time-dependent potentials in the complete Schrödinger equation and to researchers whose direct sphere of scientific interests is focused on the study of tunnel transport in multilayer nanosystems in the presence of external and internal fields impact, as well as dissipative factors.

ACKNOWLEDGMENTS

We acknowledge funding from the Ministry of Education and Science of Ukraine (Grant DI 247-22 No. 0122U001859) and the National Academy of Sciences of Ukraine (KPKVK 398FQ).

APPENDIX A: WAVE FUNCTION OF AN ELECTRON IN AREAS EXTERNAL TO THE NANOSYSTEM

In the absence of magnetic field outside the system, the electronic Hamiltonian is significantly simplified and looks like

$$i\hbar\frac{\partial\Psi(z,t)}{\partial t} = \left[-\frac{\hbar^2}{2m_w}\frac{\partial^2}{\partial z^2} - 2ez\xi\cos\omega t\right]\Psi(z,t). \quad \text{(A1)}$$

The solution of this equation could be established without searching for a quantum mechanical invariant.

The more general case of this equation is under generation of an electromagnetic field in quantum transitions of all potential wells, which was also previously analyzed [19].

We are looking for an invariant in the following form:

$$I(t) = a(t)P + b(t)z + c(t). \quad \text{(A2)}$$

The electronic Hamiltonian looks as follows:

$$H(t) = \frac{p^2}{2m_w} + f(t)z;$$
$$p = -i\hbar\frac{\partial}{\partial z}; \quad f(t) = -2e\xi\cos\omega t. \quad \text{(A3)}$$

Having substituted expressions Eqs. (A2) and (A3) into Eq. (19) and calculating all commutators, the following equation can be obtained:

$$\left(\frac{\partial a}{\partial t} + \frac{b}{m_w}\right)p + \frac{\partial b}{\partial t}z + \frac{\partial c}{\partial t} - af(t) = 0. \quad \text{(A4)}$$

Then the following system of equations can be obtained:

$$\begin{bmatrix} \frac{\partial a}{\partial t} + \frac{b}{m_w} = 0; \\ \frac{\partial b}{\partial t} = 0; \\ \frac{\partial c}{\partial t} - af(t) = 0. \end{bmatrix} \Rightarrow \begin{bmatrix} a = -\frac{b_0}{m_w}t + a_0; \\ b = b_0; \\ c = \int\left(-\frac{b_0}{m_w}t + a_0\right)f(t)dt + c_0. \end{bmatrix}$$

$$c = \int\left(-\frac{b_0}{m_w}t + a_0\right)f(t) + c_0$$
$$= -2e\xi\int\left(-\frac{b_0}{m_w}t + a_0\right)\cos\omega t\,dt + c_0$$
$$= \frac{2e\xi}{m_w\omega^2}[b_0\cos\omega t + (b_0t - a_0m_w)\omega \times \sin\omega t] + c_0. \quad \text{(A5)}$$

Without loss of generality, the following quantities as integration constants can be used:

$$a_0 = \frac{1}{2m_w}; \quad b_0 = \Omega; \quad c_0 = 0, \quad \text{(A6)}$$

where $\Omega = (E_n - i\Gamma_n/2)/\hbar$ is the frequency corresponding to the energy level formed by size quantization in the system, which is affected by the electrons of the monoenergetic beam.

Therefore, we have

$$a = \frac{1}{2m_w}(1 - 2\Omega t);$$
$$b = \Omega/\hbar;$$
$$c = \frac{e\xi}{m_w\omega^2}[2\Omega\cos\omega t + (2\Omega t - 1)\omega\sin\omega t]. \quad \text{(A7)}$$

Using Eq. (A2), we find

$$I(t) = \frac{p}{2m_w} + \Omega z + \frac{e\xi}{m_w\omega^2}[2\Omega\cos\omega t + (2\Omega t - 1)\omega\sin\omega t]. \quad \text{(A8)}$$

Now the geometric phase is obtained by solving the eigenvalue equation of the invariant in Eq. (A8) and taking into account (A2):

$$\left\{ -\frac{i\hbar}{2m_w}\frac{\partial}{\partial z} + \Omega z + \frac{e\xi}{m_w\omega^2}[2\Omega\cos\omega t + (2\Omega t - 1)\omega\sin\omega t] - \lambda \right\}\Phi_\lambda(z,t) = 0. \quad \text{(A9)}$$

From here, we have

$$\Phi_\lambda(z,t) = \exp\left(-\frac{i}{\hbar}[\varphi_1 z^2 + \varphi_2(t)z]\right); \quad \varphi_1 = m_w\Omega;$$
$$\varphi_2(t) = -\frac{2e\xi}{\omega^2}[2\Omega\cos\omega t + (2\Omega t - 1)\omega\sin\omega t] + 2m_w\lambda. \quad \text{(A10)}$$

The calculated expressions for $I_1^{(0)}(t)$, $I_2^{(0)}(t)$, $I_3^{(0)}(t)$ in Eq. (62) are as follows:

$$I_1^{(0)}(t) = \int_{-\infty}^{0} e^{\frac{i}{\hbar}[(\varphi_1^* - \varphi_1)z^2 + [\varphi_2^*(t) - \varphi_2(t)]z]}dz = \int_{-\infty}^{0} e^{\frac{2i}{\hbar}\{\mathrm{Im}(\varphi_1)z^2 + \mathrm{Im}[\varphi_2(t)]z\}}dz = \frac{1}{2i}\sqrt{\frac{\pi\hbar}{2\mathrm{Im}(\varphi_1)}}\exp\left(-\frac{\mathrm{Im}[\varphi_2(t)]^2}{2\hbar\mathrm{Im}(\varphi_1)}\right)\mathrm{erfc}\left(-i\frac{\mathrm{Im}[\varphi_2(t)]}{\sqrt{2\hbar\mathrm{Im}(\varphi_1)}}\right), \quad \text{(A11)}$$

$$I_2^{(0)}(t) = -\int_{-\infty}^{0}\frac{2i\hbar\varphi_1 + [2z\varphi_1 + \varphi_2(t)]^2}{\hbar^2}e^{\frac{2i}{\hbar}[\mathrm{Im}(\varphi_1)z^2 + \mathrm{Im}[\varphi_2(t)]z]}dz$$
$$= \frac{1}{8}\sqrt{\frac{\pi\hbar}{2[\mathrm{Im}(\varphi_1)]^5}}\exp\left(-\frac{\mathrm{Im}[\varphi_2(t)]^2}{2\hbar\mathrm{Im}(\varphi_1)}\right)\left\{1 + \mathrm{erf}\left(i\sqrt{\frac{\mathrm{Im}(\varphi_1)}{2\hbar}}\frac{\mathrm{Im}[\varphi_2(t)]}{\mathrm{Im}(\varphi_1)}\right)\right\}\{[\varphi_2(t)\varphi_1^* - \varphi_1\varphi_2^*(t)]^2 + 4\hbar|\varphi_1|^2\mathrm{Im}(\varphi_1)\}$$
$$+ \frac{\hbar\varphi_1\{\varphi_1\mathrm{Im}[\varphi_2(t)] - 2\varphi_2(t)\mathrm{Im}(\varphi_1)\}}{2[\mathrm{Im}(\varphi_1)]^2}, \quad \text{(A12)}$$

$$I_3^{(0)}(t) = \int_{-\infty}^{0} z e^{\frac{2i}{\hbar}\{\mathrm{Im}(\varphi_1)z^2 + \mathrm{Im}[\varphi_2(t)]z\}}dz$$
$$= \frac{\hbar}{4\mathrm{Im}(\varphi_1)} - \frac{\mathrm{Im}[\varphi_2(t)]}{4i\mathrm{Im}(\varphi_1)}\sqrt{\frac{\pi\hbar}{2\mathrm{Im}(\varphi_1)}}\exp\left(-\frac{\mathrm{Im}[\varphi_2(t)]^2}{2\hbar\mathrm{Im}(\varphi_1)}\right)\left\{1 + \mathrm{erf}\left(i\sqrt{\frac{\mathrm{Im}(\varphi_1)}{2\hbar}}\frac{\mathrm{Im}[\varphi_2(t)]}{\mathrm{Im}(\varphi_1)}\right)\right\}. \quad \text{(A13)}$$

Here, $\mathrm{erf}(x)$ and $\mathrm{erfc}(x)$ are the error function and the complementary error function, respectively.

The calculated expressions for $I_1^{(N+1)}(t)$, $I_2^{(N+1)}(t)$, $I_3^{(N+1)}(t)$ in relation (64) are as follows:

$$I_1^{(N+1)}(t) = \int_{d}^{+\infty} e^{\frac{i}{\hbar}\{(\varphi_1^* - \varphi_1)z^2 + [\varphi_2^*(t) - \varphi_2(t)]z\}}dz$$
$$= -\frac{1}{2}\sqrt{\frac{\pi\hbar}{2\mathrm{Im}(\varphi_1)}}\exp\left(-\frac{\mathrm{Im}[\varphi_2(t)]^2}{2\hbar\mathrm{Im}(\varphi_1)}\right)\left(i + \mathrm{erfi}\left\{\frac{2d\mathrm{Im}(\varphi_1) + \mathrm{Im}[\varphi_2(t)]}{\sqrt{2\hbar\mathrm{Im}(\varphi_1)}}\right\}\right), \quad \text{(A14)}$$

$$I_2^{(N+1)}(t) = -\int_{d}^{0}\frac{2i\hbar\varphi_1 + [2z\varphi_1 + \varphi_2(t)]^2}{\hbar^2}e^{\frac{2i}{\hbar}\{\mathrm{Im}(\varphi_1)z^2 + \mathrm{Im}[\varphi_2(t)]z\}}dz$$
$$= \frac{1}{4\mathrm{Im}^{5/2}(\varphi_1)}\exp\left(-\frac{\mathrm{Im}[\varphi_2(t)]^2}{2\hbar\mathrm{Im}(\varphi_1)}\right)\left\{2\hbar\exp\left(\frac{\{2d\mathrm{Im}(\varphi_1) + \mathrm{Im}[\varphi_2(t)]\}^2}{2\hbar\mathrm{Im}(\varphi_1)}\right)\varphi_1\sqrt{\mathrm{Im}(\varphi_1)}\right.$$
$$\times\{2[d\varphi_1 + \varphi_2(t)]\mathrm{Im}(\varphi_1) - \varphi_1\mathrm{Im}[\varphi_2(t)] + i\sqrt{2\pi}\left(\sqrt{\hbar}\mathrm{Im}(\varphi_1)\left[\hbar\varphi_1^2 + \left[2i\hbar\varphi_1 + \varphi_2^2(t)\right]\mathrm{Im}(\varphi_1)\right]\right.$$
$$\left.\left. - 2\varphi_1\varphi_2(t)\mathrm{Im}(\varphi_1)\mathrm{Im}[\varphi_2(t)] + \sqrt{\hbar}\varphi_1^2\mathrm{Im}^2[\varphi_2(t)]\right) + \sqrt{2\pi\hbar}\,\mathrm{erfi}\left\{\frac{2d\mathrm{Im}(\varphi_1) + \mathrm{Im}[\varphi_2(t)]}{\sqrt{2\hbar\mathrm{Im}(\varphi_1)}}\right\}\right.$$
$$\left.\times\left\{\left[2i\hbar\varphi_1 + \varphi_2^2(t)\right]\mathrm{Im}^2(\varphi_1) + \varphi_1^2\mathrm{Im}^2[\varphi_2(t)] - \varphi_1\mathrm{Im}(\varphi_1)[\hbar\varphi_1 + 2\varphi_2(t)\mathrm{Im}[\varphi_2(t)]]\right\}\right\}, \quad \text{(A15)}$$

$$I_3^{(N+1)}(t) = \int_d^{+\infty} z e^{\frac{2i}{\hbar}\{\mathrm{Im}(\varphi_1)z^2 + \mathrm{Im}[\varphi_2(t)]z\}} dz \tag{A16}$$

$$= \frac{1}{8\mathrm{Im}^{3/2}(\varphi_1)} \exp\left(-\frac{\mathrm{Im}[\varphi_2(t)]^2}{2\hbar\mathrm{Im}(\varphi_1)}\right) \times \left(-2\hbar\sqrt{\mathrm{Im}(\varphi_1)} \exp\left(\frac{\{2d\mathrm{Im}(\varphi_1) + \mathrm{Im}[\varphi_2(t)]\}^2}{2\hbar\mathrm{Im}(\varphi_1)}\right) + \sqrt{2\pi\hbar}\mathrm{Im}[\varphi_2(t)]\mathrm{erfi}\left\{\frac{2d\mathrm{Im}(\varphi_1) + \mathrm{Im}[\varphi_2(t)]}{\sqrt{2\hbar\mathrm{Im}(\varphi_1)}}\right\}\right). \tag{A17}$$

Here $\mathrm{erfi}(x)$ is a complex part of the error function.

APPENDIX B: THE ELEMENTS OF THE TRANSFER MATRIX

The elements of the transfer matrix are as follows:

$$\tau_{11}^{(0,1)}(t) = H\left[\frac{\tilde{\beta}^2 + (a+4s)^2(\tilde{\gamma}+2s)}{(a+4s)^3} - 1,\ -\sqrt{\frac{|a+4s|}{2}}\left(\frac{\tilde{Z}_0^{(0)}}{L} + \frac{2\tilde{\beta}}{(a+4s)^2}\right)\right];$$

$$\tau_{12}^{(0,1)}(t) = M\left[\frac{1}{2} - \frac{\tilde{\beta}^2 + (a+4s)^2(\tilde{\gamma}+2s)}{2(a+4s)^3}, \frac{1}{2},\ \frac{(a+4s)}{2}\left(\frac{\tilde{Z}_0^{(0)}}{L} + \frac{2\tilde{\beta}}{(a+4s)^2}\right)^2\right];$$

$$\tau_{21}^{(0,1)}(t) = \frac{1}{m_w}\left\{\left[-\left(\frac{a+2s}{L}\right) + \frac{\tilde{\beta}}{a+4s}\right]H\left[\frac{\tilde{\beta}^2 + (a+4s)^2(\tilde{\gamma}+2s)}{(a+4s)^3} - 1,\ -\sqrt{\frac{|a+4s|}{2}}\left(\frac{\tilde{Z}_0^{(p)}}{L} + \frac{2\tilde{\beta}}{(a+4s)^2}\right)\right]\right.$$
$$\left. + \frac{1}{L}\sqrt{\frac{|a+4s|}{2}}\left[\frac{\tilde{\beta}^2 + (a+4s)^2(\tilde{\gamma}+2s)}{(a+4s)^3} - 1\right]H\left[\frac{\tilde{\beta}^2 + (a+4s)^2(\tilde{\gamma}+2s)}{(a+4s)^3} - 2\ -\sqrt{\frac{|a+4s|}{2}}\left(\frac{\tilde{Z}_0^{(p)}}{L} + \frac{2\tilde{\beta}}{(a+4s)^2}\right)\right]\right\}$$
$$\times \exp\left[-\left(\frac{a}{2}+s\right)\left(\frac{\tilde{Z}_0^{(p)}}{L}\right)^2 - \frac{\tilde{\beta}}{a+4s}\frac{\tilde{Z}_0^{(p)}}{L} - \frac{2\tilde{\beta}^2}{(a+4s)^3}\right];$$

$$\tau_{22}^{(0,1)}(t) = \frac{1}{m_w}\left\{\left[-\left(\frac{a+2s}{L}\right) + \frac{\tilde{\beta}}{a+4s}\right]M\left[\frac{1}{2} - \frac{\tilde{\beta}^2 + (a+4s)^2(\tilde{\gamma}+2s)}{2(a+4s)^3}, \frac{1}{2}, \frac{(a+4s)}{2}\left(\frac{\tilde{Z}_0^{(0)}}{L} + \frac{2\tilde{\beta}}{(a+4s)^2}\right)^2\right]\right.$$
$$+ \frac{(a+4s)}{L}\left(\frac{\tilde{Z}_0^{(p)}}{L} + \frac{2\tilde{\beta}}{(a+4s)^2}\right)\left[\frac{1}{2} - \frac{\tilde{\beta}^2 + (a+4s)^2(\tilde{\gamma}+2s)}{2(a+4s)^3}\right]M\left[\frac{3}{2} - \frac{\tilde{\beta}^2 + (a+4s)^2(\tilde{\gamma}+2s)}{2(a+4s)^3}, \frac{3}{2},\right.$$
$$\left.\left. \times\ \frac{(a+4s)}{2}\left(\frac{\tilde{Z}_0^{(0)}}{L} + \frac{2\tilde{\beta}}{(a+4s)^2}\right)^2\right]\right\} e^{-\left(\frac{a}{2}+s\right)\left(\frac{\tilde{Z}_0^{(0)}}{L}\right)^2 - \frac{\tilde{\beta}}{a+4s}\frac{\tilde{Z}_0^{(0)}}{L} - \frac{2\tilde{\beta}^2}{(a+4s)^3}}. \tag{B1}$$

Similarly, we have

$$\tau_{11}^{(i,i+1)}(t) = e^{-\left(\frac{a}{2}+s\right)\left(\frac{z_i - \tilde{Z}_0^{(i)}}{L}\right)^2 + \frac{\tilde{\beta}}{a+4s}\frac{z_i - \tilde{Z}_0^{(i)}}{L} - \frac{2\tilde{\beta}^2}{(a+4s)^3}} H\left[\frac{\tilde{\beta}^2 + (a+4s)^2(\tilde{\gamma}+2s)}{(a+4s)^3} - 1\sqrt{\frac{|a+4s|}{2}}\left(\frac{z_i - \tilde{Z}_0^{(i)}}{L} - \frac{2\tilde{\beta}}{(a+4s)^2}\right)\right];$$

$$\tau_{12}^{(i,i+1)}(t) = e^{-\left(\frac{a}{2}+s\right)\left(\frac{z_i - \tilde{Z}_0^{(i)}}{L}\right)^2 + \frac{\tilde{\beta}}{a+4s}\frac{z_i - \tilde{Z}_0^{(i)}}{L} - \frac{2\tilde{\beta}^2}{(a+4s)^3}} M\left[\frac{1}{2} - \frac{\tilde{\beta}^2 + (a+4s)^2(\tilde{\gamma}+2s)}{2(a+4s)^3}, \frac{1}{2}, \frac{(a+4s)}{2}\left(\frac{z_i - \tilde{Z}_0^{(i)}}{L} - \frac{2\tilde{\beta}}{(a+4s)^2}\right)^2\right];$$

$$\tau_{21}^{(i,i+1)}(t) = \frac{1}{m_w}\left\{\left[-(a+2s)\frac{z_i - \tilde{Z}_0^{(i)}}{L} + \frac{\tilde{\beta}}{a+4s}\right]H\left[\frac{\tilde{\beta}^2 + (a+4s)^2(\tilde{\gamma}+2s)}{(a+4s)^3} - 1,\ \sqrt{\frac{|a+4s|}{2}}\left(\frac{z_i - \tilde{Z}_0^{(i)}}{L} - \frac{2\tilde{\beta}}{(a+4s)^2}\right)\right]\right.$$
$$- \frac{\tilde{Z}_0^{(i)}}{L}\sqrt{\frac{|a+4s|}{2}}\left[\frac{\tilde{\beta}^2 + (a+4s)^2(\tilde{\gamma}+2s)}{(a+4s)^3} - 1\right]H\left[\frac{\tilde{\beta}^2 + (a+4s)^2(\tilde{\gamma}+2s)}{(a+4s)^3} - 2,\right.$$
$$\left.\left. \times\ \sqrt{\frac{|a+4s|}{2}}\left(\frac{z_i - \tilde{Z}_0^{(i)}}{L} - \frac{2\tilde{\beta}}{(a+4s)^2}\right)\right]\right\} e^{-\left(\frac{a}{2}+s\right)\left(\frac{z_i - \tilde{Z}_0^{(i)}}{L}\right)^2 + \frac{\tilde{\beta}}{a+4s}\frac{z_i - \tilde{Z}_0^{(i)}}{L} - \frac{2\tilde{\beta}^2}{(a+4s)^3}};$$

$$\tau_{22}^{(i,i+1)}(t) = \frac{1}{m_w}\left\{\left[-(a+2s)\frac{z_i-\tilde{Z}_0^{(i)}}{L}+\frac{\tilde{\beta}}{a+4s}\right]M\left[\frac{1}{2}-\frac{\tilde{\beta}^2+(a+4s)^2(\tilde{\gamma}+2s)}{2(a+4s)^3},\frac{1}{2},\frac{1}{2},\frac{(a+4s)}{2}\left(\frac{z_i-\tilde{Z}_0^{(i)}}{L}-\frac{2\tilde{\beta}}{(a+4s)^2}\right)^2\right]\right.$$
$$+\frac{(a+4s)}{L}\left(\frac{z_i-\tilde{Z}_0^{(i)}}{L}-\frac{2\tilde{\beta}}{(a+4s)^2}\right)\left[\frac{1}{2}-\frac{\tilde{\beta}^2+(a+4s)^2(\tilde{\gamma}+2s)}{2(a+4s)^3}\right]M\left[\frac{3}{2}-\frac{\tilde{\beta}^2+(a+4s)^2(\tilde{\gamma}+2s)}{2(a+4s)^3},\frac{3}{2},\right.$$
$$\left.\left.\times\frac{(a+4s)}{2}\left(\frac{z_i-\tilde{Z}_0^{(i)}}{L}-\frac{2\tilde{\beta}}{(a+4s)^2}\right)^2\right]\right\}e^{-\left(\frac{a}{2}+s\right)\left(\frac{z_i-\tilde{Z}_0^{(i)}}{L}\right)^2+\frac{\tilde{\beta}}{a+4s}\frac{z_i-\tilde{Z}_0^{(i)}}{L}-\frac{2\tilde{\beta}^2}{(a+4s)^3}}. \tag{B2}$$

[1] P. Ourednik, G. Picco, D. T. Nguyen, C. Spudat, and M. Feiginov, Large-signal dynamics of resonant-tunneling diodes, J. Appl. Phys. **133**, 014501 (2023).

[2] S. Alomari, Q. Al-Taai, M. Elksne, A. Al-Khalidi, E. Wasige, and J. Figueiredo, Speed limitations of resonant tunneling diode-based photodetectors, Opt. Express **31**, 18300 (2023).

[3] S. Markmann, M. Franckie, S. Pal, D. Stark, M. Beck, M. Fiebig, G. Scalari, and J. Faist, Two-dimensional spectroscopy on a THz quantum cascade structure, Nanophotonics **10**, 171 (2020).

[4] F. Kapsalidis, B. Schneider, M. Singleton, M. Bertrand, E. Gini, M. Beck, and J. Faist, Mid-infrared quantum cascade laser frequency combs with a microstrip-like line waveguide geometry, Appl. Phys. Lett **118**, 071101 (2021).

[5] M. Jeannin, E. Cosentino, S. Pirotta, M. Malerba, G. Biasiol, J.-M. Manceau, and R. Colombelli, Low intensity saturation of an ISB transition by a mid-IR quantum cascade laser, Appl. Phys. Lett **122**, 241107 (2023).

[6] P. Micheletti, J. Micheletti, T. Olariu, U. Senica, M. Beck, and G. Scalari, Regenerative terahertz quantum detectors, APL Photonics **6**, 106102 (2021).

[7] Y. Salamin, I.-C. Benea-Chelmus, Y. Fedoryshyn, W. Heni, D. L. Elder, L. R. Dalton, J. Faist, and J. Leuthold, Compact and ultra-efficient broadband plasmonic terahertz field detector, Nat. Commun. **10**, 5550 (2019).

[8] M. Tkach, J. O. Seti, and O. M. Voitsekhivska, Non-perturbation theory of electronic dynamic conductivity for two-barrier resonance tunnel nano-structure, Condens. Matter Phys. **14**, 43702 (2011).

[9] J. Z. Kaminski, S. E, and F. C. Felez, Phase control of resonant tunneling in nanostructures, Ann. Phys. (Berlin) **525**, 118 (2013).

[10] A. Semenov and A. Nitzan, Electron transfer in confined electromagnetic fields, J. Chem. Phys. **150**, 174122 (2019).

[11] D. Lozano-Gomez and D. G. Kelkar, Electron interaction with the environment in tunnel ionization, Ann. Phys. **415**, 168112 (2020).

[12] M. Franckie, D. O. Winge, J. Wolf, V. Liverini, E. Dupont, V. Trinite, J. Faist, and A. Wacker, Impact of interface roughness distributions on the operation of quantum cascade lasers, Opt. Express **23**, 5201 (2015).

[13] C. Ndebeka-Bandou, M. Rosch, K. Ohtani, M. Beck, and J. Faist, Negative free carrier absorption in terahertz quantum cascade lasers, Appl. Phys. Lett. **108**, 091102 (2016).

[14] F. Bernardini, V. Fiorentini, and D. Vanderbilt, Spontaneous polarization and piezoelectric constants of III-V nitrides, Phys. Rev. B **56**, R10024(R) (1997).

[15] F. Bernardini and V. Fiorentini, Macroscopic polarization and band offsets at nitride heterojunctions, Phys. Rev. B **57**, R9427(R) (1998).

[16] F. Bernardini and V. Fiorentini, Spontaneous versus piezoelectric polarization in III-V nitrides: Conceptual aspects and practical consequences, Phys. Status Solidi B **216**, 391 (1999).

[17] B. K. Ridley, W. J. Schaff, and L. F. Eastman, Theoretical model for polarization superlattices: Energy levels and intersubband transitions, J. Appl. Phys. **94**, 3972 (2003).

[18] S. V. Gryshchenko, M. V. Klymenko, O. V. Shulika, I. A. Sukhoivanovm, and V. V. Lysak, Temperature dependence of electron transport in GaN/AlGaN quantum cascade detectors, Superlattices Microstruct. **52**, 894 (2012).

[19] I. Boyko, M. Petryk, and N. Lebovka, Application of the Lewis-Riesenfeld quantum mechanical invariant method for description of electron tunneling transport in nitride multilayer quantum well nanostructures, Phys. Lett. A **489**, 129152 (2023).

[20] H. R. Lewis and W. B. Riesenfeld, An exact quantum theory of the time-dependent harmonic oscillator and of a charged particle in a time-dependent electromagnetic field, J. Math. Phys. **10**, 1458 (1969).

[21] A. Pereira, A. Lemos, and F. Brito, Squeezed coherent states for a free particle with time-varying mass, Eur. Phys. J. Plus **138**, 363 (2023).

[22] P. Patra, On the two-dimensional time-dependent anisotropic harmonic oscillator in a magnetic field, J. Math. Phys. **64**, 042105 (2023).

[23] S. Koh, Wigner function of a time-dependent inverted harmonic oscillator, New Physics: Sae Mulli **73**, 74 (2023).

[24] D. Laroze, G. Gutierrez, R. Rivera, and J. M. Yanez, Dynamics of a rotating particle under a time-dependent potential: exact quantum solution from the classical action, Phys. Scr. **78**, 015009 (2008).

[25] F. Fratini and L. Safari, Quantum mechanical evolution operator in the presence of a scalar linear potential: Discussion on the evolved state, phase shift generator and tunneling, Phys. Scr. **89**, 085004 (2014).

[26] F. Onah, E. García Herrera, J. Ruelas-Galván, G. Juárez Rangel, E. Real Norzagaray, and B. Rodríguez-Lara, A quadratic time-dependent quantum harmonic oscillator, Sci. Rep. **13**, 8312 (2023).

[27] H. Breuer, K. Dietz, and M. Holthaus, A remark on the Kramers-Henneberger transformation, Phys. Lett. A **165**, 341 (1992).

[28] A. P. Rau and K. Unnikrishnan, Evolution operators and wave functions in a time-dependent electric field, Phys. Lett. A **222**, 304 (1996).

[29] I. M. Besieris and P. Saari, Exact and paraxial broadband airy wave packets in free space and a temporally dispersive medium, Photonics (MDPI) **11**, 94 (2024).
[30] H. Cruz, D. Schuch, O. Castanos, and O. Rosas-Ortiz, Time-evolution of quantum systems via a complex nonlinear Riccati equation. I. Conservative systems with time-independent Hamiltonian, Ann. Phys. **360**, 44 (2015).
[31] D.-Y. Song, Unitary transformation for the system of a particle in a linear potential, Europhys. Lett. **62**, 622 (2003).
[32] A. Fring, T. Taira, and R. Tenney, Time-dependent C-operators as Lewis-Riesenfeld invariants in non-Hermitian theories, Phys. Lett. A **452**, 128458 (2022).
[33] A. Fring and R. Tenney, Time-independent approximations for time-dependent optical potentials, Eur. Phys. J. Plus **135**, 163 (2020).
[34] V. Evangelakos, E. Paspalakis, and D. Stefanatos, Efficient light transfer in coupled nonlinear triple waveguides using shortcuts to adiabaticity, Sci. Rep. **13**, 1368 (2023).
[35] A. Leuliet, A. Vasanelli, A. Wade, G. Fedorov, D. Smirnov, G. Bastard, and C. Sirtori, Electron scattering spectroscopy by a high magnetic field in quantum cascade lasers, Phys. Rev. B **73**, 085311 (2006).
[36] A. Wade, G. Fedorov, D. Smirnov, S. Kumar, B. S. Williams, Q. Hu, and J. L. Reno, Magnetic-field-assisted terahertz quantum cascade laser operating up to 225 K, Nat. Photonics **3**, 41 (2009).
[37] F. Valmorra, G. Scalari, K. Ohtani, M. Beck, and J. Faist, InGaAs/AlInGaAs Thz quantum cascade lasers operating up to 195 K in strong magnetic field, New J. Phys. **17**, 023050 (2015).
[38] C. S. Kim and O. Olendski, The effect on currents of anticrossings in the energy spectrum in quantum wells under crossed electric and magnetic fields, Semicond. Sci. Technol. **12**, 788 (1997).
[39] D.-Y. Song, Localization or tunneling in asymmetric double-well potentials, Ann. Phys. **362**, 609 (2015).
[40] B. F. Ramos, I. A. Pedrosa, and K. Bakke, Effects of a non-Hermitian potential on the Landau quantization, Int. J. Mod. Phys. A **34**, 1950072 (2019).
[41] Y. Guo, H. Wang, B.-L. Gu, and Y. Kawazoe, Electric-field effects on electronic tunneling transport in magnetic barrier structures, Phys. Rev. B **61**, 1728 (2000).
[42] Y. Guo, H. Wang, B.-L. Gu, and K. Y, Electron coherent tunneling in low-dimensional magnetic quantum structures, Physica E **8**, 146 (2000).
[43] H. Wang, Y. Zhang, and H. Xu, The effect of transverse wave vector and magnetic fields on resonant tunneling times in double-barrier structures, J. Appl. Phys. **101**, 023712 (2007).
[44] S. K. Sahu, S. Ipsita, P. K. Mahapatra, and D. K. Mishra, 2-D tunneling structures and increased peak to valley current ratio in double barrier quantum systems under perpendicular electric and magnetic bias, J. Magn. Magn. Mater. **580**, 170870 (2023).
[45] L. D. Landau, E. M. Lifshitz, and L. P. Pitaevskii, *Quantum Mechanics. Nonrelativistic Theory* (Pergamon Press, New York, 1989), Vol. 3.
[46] L. D. Landau, E. M. Lifshitz, and L. P. Pitaevskii, *Electrodynamics of Continuous Media* (Pergamon Press, New York, 1984), Vol. 8.
[47] A. B. Pashkovskii, Parity and abrupt broadening of resonance levels in triple-barrier structures, JETP Lett. **82**, 210 (2005).
[48] A. I. Baz, Y. B. Zeldovich, and A. Perelomov, *Dispersion, Reactions and Disintegrations in the Nonrelativistic Quantum Mechanics* (IPST Press, Jerusalem, 1969), Vol. 1.
[49] C. Jirauschek and T. Kubis, Modeling techniques for quantum cascade lasers, Appl. Phys. Rev. **1**, 011307 (2014).
[50] M. Tkach, J. Seti, and O. Voitsekhivska, Spectrum of electron in quantum well within the linearly-dependent effective mass model with the exact solution, Superlattices Microstruct. **109**, 905 (2017).
[51] L. A. Gonzalez-Diaz and S. Diaz-Solorzano, Bound states and scattering coefficients of self-adjoint Hamiltonians with a mass jump, J. Math. Phys. **54**, 042106 (2013).
[52] I. A. Pedrosa and I. Guedes, Wave function of the time-dependent inverted harmonic oscillator, Mod. Phys. Lett. B **16**, 637 (2002).
[53] I. A. Pedrosa, Complete exact quantum states of the generalized time-dependent harmonic oscillator, Mod. Phys. Lett. B **18**, 1267 (2004).
[54] I. A. Pedrosa, A. Rosas, and I. Guedes, Exact quantum motion of a particle trapped by oscillating fields, J. Phys. A: Math. Gen. **38**, 7757 (2005).
[55] V. F. Zaitsev and A. D. Polyanin, *Handbook of Exact Solutions for Ordinary Differential Equations*, 2nd ed. (Chapman and Hall/CRC, New York, 2002).
[56] L. Li and L. Bo-Zang, Lewis–Riesenfeld phases and Berry phases for the harmonic oscillator with time-dependent frequency and boundary conditions, Int. J. Mod. Phys. B **17**, 2045 (2003).
[57] X. Chen, E. Torrontegui, and J. G. Muga, Lewis-Riesenfeld invariants and transitionless quantum driving, Phys. Rev. A **83**, 062116 (2011).
[58] V. Bessa and I. Guedes, Wave functions of log-periodic oscillators, J. Math. Phys. **52**, 062106 (2011).
[59] I. A. Pedrosa, A. Y. Petrov, and A. Rosas, On the electrodynamics in time-dependent linear media, Eur. Phys. J. D **66**, 309 (2012).
[60] N. Ünal, Quasi-coherent states for harmonic oscillator with time-dependent parameters, J. Math. Phys. **53**, 012102 (2012).
[61] J. Guerrero and F.-R. Francisco, On the Lewis–Riesenfeld (Dodonov–Man'ko) invariant method, Phys. Scr. **90**, 074046 (2015).
[62] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.110.045438 for example of calculations of the resonance energies and resonance width.
[63] A. Balandin and K. L. Wang, Significant decrease of the lattice thermal conductivity due to phonon confinement in a free-standing semiconductor quantum well, Phys. Rev. B **58**, 1544 (1998).
[64] I. Boyko and M. Petryk, Tunneling transport in open nitride resonant tunneling structures taking into account the acoustic phonons: An variational approach, Phys. B: Condens. Matter **636**, 413862 (2022).