Gapped boundary of (4+1)d beyond-cohomology bosonic SPT phase

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In this paper we study gapped boundary states of $\mathbb{Z}_N$ bosonic symmetry-protected topological (SPT) phases in (4+1)d, which are characterized by mixed $\mathbb{Z}_N$-gravity response, and the closely related phases protected by C_N rotation symmetry. We show that if $N \notin \{2, 4, 8, 16\}$, any symmetry-preserving boundary theory is necessarily gapless for the root SPT state. We then propose a (3+1)d $\mathbb{Z}_2$ gauge theory coupled to fermionic matter as a candidate boundary theory for $N = 2, 4, 8, 16$, where the anomalous symmetry is implemented by invertible topological defects obtained from gauging (2+1)d chiral topological superconductors. For the C_N case, we present an explicit construction for the boundary states for $N = 2, 4, 8, 16$, and argue that the construction fails for other values of N.

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I. INTRODUCTION

The concept of bulk-boundary correspondence is fundamental to the theory of topological phases. It is best understood when the bulk is an invertible symmetry-protected topological (SPT) phase, where the boundary has a 't Hooft anomaly of the symmetry group that protects the bulk SPT phase [1]. The presence of a 't Hooft anomaly puts nontrivial constraints on the low-energy dynamics, and in particular excludes a trivially gapped symmetric ground state. More generally, any low-energy theory realized in the system must have the given anomaly. Generally, three options are possible for the boundary theory: gapless, symmetry breaking, or a symmetry-preserving topologically ordered phase (when the boundary has spatial dimension D greater than 1).

The last possibility, namely the boundary forming a symmetry-enriched topological (SET) phase [2,3] with the 't Hooft anomaly, has been extensively investigated in the past few years. In particular for $D = 2$, general theories of SET phases in both bosonic and fermionic systems have been formulated [4–9]. Systematic methods to compute 't Hooft anomalies given a SET phase. It is also known that certain 't Hooft anomalies cannot be matched by any SET in $D = 2$, thus any symmetry-preserving theory must be gapless. Known examples of "symmetry-enforced gaplessness" involve continuous and anti-unitary symmetry group [10], such as a bosonic anomaly for $\mathrm{SO}(5) \times \mathbb{Z}_2^{\mathsf{T}}$ symmetry [11,12].

The focus of this paper is $D = 3$, where a full theory of SET phases is not available yet. A necessary ingredient of such a theory is a complete understanding of the structure of 't Hooft anomaly. It is well known that 't Hooft anomalies are classified by SPT phases in one dimension higher. Interestingly, in (4+1)d there is a class of bosonic SPT phases protected by unitary symmetry, that goes beyond the well-known "group-cohomology" classification. Such "beyond-cohomology" SPT phases can be understood as decorating lower-dimensional invertible topological phases on symmetry defects. The physical characterization of these beyond-cohomology SPT phases turns out to be rather subtle. An argument based on topological quantum field theory (TQFT) consideration suggests that for $N = 2$, the nontrivial phase can be characterized by the ground state having an odd $\mathbb{Z}_2$ charge when put on the $\mathbb{CP}^2$ manifold. However, a commuting-projector model Hamiltonian, unitarily equivalent to a group-cohomology SPT model in flat space, can also exhibit the same phenomenon [13]. Therefore a definite invariant of the phase requires considering the boundary anomaly (or the closely related defect decoration) [14].

We will study the boundary theory of the root $\mathbb{Z}_N$ beyond-cohomology SPT phase, using two complementary points of view. First of all, we use a theorem proven by Cordova and Ohmori to show that only when $N = 2, 4, 8, 16$, there can be Z_N symmetry-preserving TQFT boundary states. We then propose a boundary TQFT for such allowed values of N: a (3+1)d $\mathbb{Z}_2$ gauge theory with a fermionic $\mathbb{Z}_2$ charge (which will be referred to as a fermionic $\mathbb{Z}_2$ gauge theory from now on). Secondly, we provide an explicit construction of the symmetry-preserving boundary state for $N = 2, 4, 8, 16$. However, given that solvable models for $\mathbb{Z}_N$ BC SPT phases are still lacking for $N > 2$, we turn to a different but related system, that is a (4+1)d SPT phase protected by C_N rotation symmetry. Following the dimensional reduction approach [15], we show that our construction of gapped boundary topological orders for $N = 2, 4, 8$ preserves the boundary C_N symmetry, while surprisingly fails for $N = 16$. Yet, a slight modification yields a similar boundary state.

II. BOSONIC $\mathbb{Z}_N$ SPT IN (4+1)d

First we review the classification of (4+1)d bosonic SPT phases, following [16,17]. Let G be a compact unitary group. The "group-cohomology" SPT phases are classified by $\mathcal{H}^5[G, \mathrm{U}(1)]$, and the "beyond-cohomology" SPT phases are classified by $\mathcal{H}^2[G, \mathbb{Z}]$, as we will argue below. The total group of SPT phases is an extension of $\mathcal{H}^2[G, \mathbb{Z}]$ by $\mathcal{H}^5[G, \mathrm{U}(1)]$. We show that for $3 \nmid N$, the group structure is $\mathbb{Z}_N^2$, while for $3 \mid N$, it is $\mathbb{Z}_{3N} \times \mathbb{Z}_{N/3}$. Details to determine the group structure of $\mathbb{Z}_N$ SPT are given in Appendix A. For finite

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G, exactly solvable models (either in the form of a state-sum TQFT, or commuting-projector Hamiltonian) are known for such phases [1].

A. Defect decoration

To see how to construct the $\mathbb{Z}_N$ BC SPT phase, it would be instructive to start from $G = \mathrm{U}(1)$ and then break it down to $\mathbb{Z}_N$. The group-cohomology SPT phases are classified by $\mathcal{H}^5[\mathrm{U}(1), \mathrm{U}(1)] = \mathbb{Z}$, which can be characterized by the (4+1)d quantum Hall response. The beyond-cohomology phases $[\mathcal{H}^2[\mathrm{U}(1), \mathbb{Z}] = \mathcal{H}^1[\mathrm{U}(1), \mathrm{U}(1)] = \mathbb{Z}]$ can be constructed as follows: suppose the U(1) symmetry is spontaneously broken so the system is in a superfluid phase. In a (4+1)d superfluid, the vortices are codimension-2 domain walls (i.e., spatially they are surfaces). It is well known that by proliferating the vortices one can restore the U(1) symmetry and enter an insulating state. To create a nontrivial BC SPT state, the vortices are decorated by (2+1)d invertible states. Once the U(1) symmetry is broken down to $\mathbb{Z}_N$, a U(1) vortex should be viewed as a junction fusing N fundamental $\mathbb{Z}_N$ domain walls together.

The "beyond-cohomology" SPT phases can thus be understood as decorating (2+1)d nontrivial invertible states on junctions of symmetry defects [18,19]. Recall that invertible phases in (2+1)d form a $\mathbb{Z}$ group, generated by the so-called E_8 state with chiral central charge $c_- = 8$. It has the simplest edge theory being a chiral $(E_8)_1$ conformal field theory (CFT). We then consider decorating the codimension-2 trijunctions of symmetry defects, which are surfaces in 4D space, with invertible states. A trijunction is labeled by a pair of group elements $(\mathbf{g}, \mathbf{h}) \in G$. Fusing this pair of defects gives a $\mathbf{gh}$ domain wall in G. The decoration pattern is parametrized by an integer-valued function $n(\mathbf{g}, \mathbf{h}) \in \mathbb{Z}$, evaluated at each trijunction. Physically $n(\mathbf{g}, \mathbf{h}) = c_-/8$ is the E_8 state decorated on the trijunctions. In order to get a short-range entangled state, it is necessary that the decoration patterns on defect configurations, which can be locally deformed to each other are (adiabatically) equivalent. This requirement is equivalent to imposing the 2-cocycle condition on $n(\mathbf{g}, \mathbf{h})$ when we consider a junction that fuses three symmetry defects $\mathbf{g}, \mathbf{h}, \mathbf{k}$ into $\mathbf{ghk}$. In addition, the following procedure does not change the underlying phase of matter: we create a pair of invertible states labeled by $m(\mathbf{g})$ and $-m(\mathbf{g})$ on the $\mathbf{g}$ defect and move them to the adjacent trijunctions, i.e., $n(\mathbf{g}, \mathbf{h}) \to n(\mathbf{g}, \mathbf{h}) + m(\mathbf{g}) + m(\mathbf{h}) - m(\mathbf{gh})$. Thus $n(\mathbf{g}, \mathbf{h})$ is a 2-cocycle defined up to 1-coboundary and its classification is given by $\mathcal{H}^2[G, \mathbb{Z}]$.

Mathematically, we can understand such defect decoration by looking at the explicit expression of the 2-cocycle. For $G = \mathbb{Z}_N$, a 2-cocycle in $\mathcal{H}^2[\mathbb{Z}_N, \mathbb{Z}]$ can be written in this canonical form

$$n(a, b) = \frac{s}{N}(a + b - [a + b]_N), \tag{1}$$

where $a, b \in \{0, 1, \ldots, N-1\}$ denote the elements of $\mathbb{Z}_N$ additively, s takes values in $\{0, 1, \ldots, N-1\}$, and $[x]_N$ means $x \bmod N$. This explicit expression of n shows that fusing N of the fundamental $\mathbb{Z}_N$ defects should yield a (2+1)d invertible state labeled by $n(1, 1) + n(2, 1) + \cdots + n(N-1, 1) = s$.

A key question here is what (2+1)d invertible states can be decorated consistently on the U(1) vortex sheets or the junction of $\mathbb{Z}_N$ defects. Naively one might think of the E_8 states but it is not obvious that such a decoration is consistent. While we do not have a direct way to check the consistency at the level of a wavefunction, it is useful to consider the following QFT argument: Assuming that the system can be described by a relativistic field theory, then we can study the theory on a general curved manifold (in Euclidean spacetime) and its response to background U(1) gauge field. That is, we consider the partition function $\mathcal{Z}(M_5, A)$ of the theory defined on a closed 5-manifold M_5 equipped with a U(1) background gauge field A. To write down the well-defined action, it is convenient to introduce a six-dimensional manifold B_6 with $\partial B_6 = M_5$. The gauge field A is also extended to B_6. The BC SPT phase is characterized by the following topological term [20–22]:

$$\mathcal{Z}(B_6, A) = \exp\left(ik \int_{B_6} F \wedge p_1\right), \tag{2}$$

where $F = dA$ is the field strength, and p_1 is the Pontryagin class of the tangent bundle of the manifold. Then the partition function on M_5 can be written as $\mathcal{Z}(M_5, A) = \exp(2\pi ik \int_{M_5} A \wedge p_1)$. Notice that a well-defined topological term in Eq. (2) should be independent of the choice of manifold extension. In other words, the right-hand side of Eq. (2), when evaluated on any closed 6-manifold, must give 1. This requires k to be an integer [23]. On the other hand, observe that $e^{2\pi ik \int p_1}$ defines an invertible theory in (3+1)d, [24] thus when the manifold has a (2+1)d boundary, the theory reduces to a boundary gravitational Chern-Simons term with chiral central charge $24k$. Thus we identify $24k$ as the chiral central charge of the invertible theory decorated on the domain wall junctions [22]. In other words, the index s in Eq. (1) must be a multiple of 3.

This QFT argument suggests that one can only decorate minimally $c_- = 24$ invertible states (i.e., three copies of E_8 states) on vortex surfaces. However, it is worth emphasizing that we have assumed relativistic symmetry in this argument, so it is not entirely clear whether the same argument applies to gapped phases in nonrelativistic systems, such as lattice models. A related fact is that the ground-state wavefunctions of such an invertible state with c_- a multiple of 24 on any closed surfaces are completely invariant under modular transformation (no additional phase factor). Such modular invariance may be required for a consistent decoration.

B. Partition function for $\mathbb{Z}_N$ BC SPT Phase

The argument in the previous section does not fully capture the subtlety of writing down the response action of $\mathbb{Z}_N$ BC SPT phase. When $3 \nmid N$, the response action is indeed

$$\mathcal{Z}(M_5, A) = \exp\left(\frac{2\pi ik}{N} \int_{M_5} A \cup p_1\right), \quad k \in \mathbb{Z}_N, \tag{3}$$

where A is the background $\mathbb{Z}_N$ gauge field (valued in $\mathbb{Z}/N\mathbb{Z}$).

The $3 \mid N$ case requires a separate treatment. In the case of $N = 3$ it is known that the action (3) turns out to be equivalent to that of a $\mathbb{Z}_3$ group-cohomology SPT phase [25]. In fact, the $\mathbb{Z}_3$ SPT phases are classified by $\mathbb{Z}_9$ [26], where the generator is the root BC SPT phase. We discuss the partition function of the root phase in Appendix B.

In the Hamiltonian formalism, one can interpret this action as follows: Let $M_5 = M_4 \times S^1$, where M_4 is a closed 4-manifold. We also assume there is a unit $\mathbb{Z}_N$ holonomy along S^1. For $3 \nmid N$, the partition function on M_5 evaluates to $e^{\frac{6\pi i k}{N}\sigma(M_4)}$. The physical interpretation is that the ground state on a closed 4-manifold M_4 has $\mathbb{Z}_N$ charge $3k\sigma(M_4)$ mod N. However, the ground-state charge becomes ambiguous away from the pure TQFT limit. The subtlety was recently exemplified in the generalized double semion model [13], which is an exactly solvable (commuting-projector) lattice model with the same ground-state property as the $\mathbb{Z}_2$ BC SPT phase on any closed 4-manifold, but on the other hand is locally equivalent to a group-cohomology $\mathbb{Z}_2$ SPT model.

C. Constraints on the boundary states

We are interested in the existence of gapped, symmetry-preserving (3+1)d boundary conditions for the BC $\mathbb{Z}_N$ SPT phases. Recently, within the mathematical framework of TQFTs [27,28], established a necessary condition for a (3+1)d 't Hooft anomaly to be saturated by a symmetry-preserving TQFT: the corresponding (4+1)d SPT topological partition function must evaluate to 1 on $K_3 \times S^1$ (with any choice of background gauge field). Here K_3 is a closed simply-connected 4-manifold with signature 16. In other words, if one can find a gauge field configuration of the corresponding symmetry such that the partition function yields a phase factor different from 1 on $K_3 \times S^1$, then the SPT phase cannot have a symmetry-preserving TQFT boundary.

Let us use the criterion to study the $\mathbb{Z}_N$ BC SPT phase, whose partition function is given by Eq. (3). If we require the partition function to be 1 on $K_3 \times S^1$, for $3 \nmid N$ using (3) we find

$$\frac{48k}{N} \in \mathbb{Z}. \tag{4}$$

For $k = 1$, it means a symmetry-preserving TQFT boundary is possible only for

$$N = 2, 4, 8, 16. \tag{5}$$

The case of $3|N$ is more delicate. We provide an argument that the partition function on $K_3 \times S^1$ is not 1 for the root $N = 3$ BC SPT phase in the Appendix B, and hence there cannot be a symmetry-preserving boundary TQFT. Interestingly, even though the root phase does not allow symmetry-preserving gapped boundary, three copies of the root phase is equivalent to a group-cohomology SPT phase, which can have symmetric gapped boundary. Similar results can be proven for other $3|N$.

III. BOSONIC C_N SPT IN (4+1)d

A drawback of our discussions of the $\mathbb{Z}_N$ BC SPT phases is that it is entirely based on topological partition functions, and at the moment we do not have a concrete microscopic model for them. In this section we turn to a different but closely related symmetry C_N, the point group of N-fold rotations, and study SPT phases protected by this symmetry. The advantage of considering spatial SPT states (i.e., those protected by spatial symmetries) is that they can be classified and explicitly constructed using the block construction [15]. It is valid to relate $\mathbb{Z}_N$ SPT phases with C_N SPT phases because there is a one-to-one correspondence between these two SPT phases, known as the crystalline correspondence principle [29].

Let us elaborate on the relation between the $\mathbb{Z}_N$ SPT phases and the C_N ones. Starting from a $\mathbb{Z}_N$ SPT phase, we create a symmetry-breaking state in the following way: insert N copies of $\mathbb{Z}_N$ domain walls in a C_N symmetric configuration. This system breaks $\mathbb{Z}_N$ and C_N, but preserves the diagonal subgroup called C_N'. At the rotation center, the N-domain walls fuse together to a codimension-2 defect, which is the state that lives at the rotation center. Therefore, from a $\mathbb{Z}_N$ SPT state we can always construct a C_N SPT state. For the other direction, consider a continuum QFT with continuous spatial symmetry [i.e., SO(D) and translations, where D is the spatial dimension]. The C_N rotation can always be written as a $\mathbb{Z}_N$ internal symmetry transformation combined with the corresponding rotation in SO(D). It is then expected that the theory with the $\mathbb{Z}_N$ symmetry is in the corresponding $\mathbb{Z}_N$ SPT phase.

A. Block construction for C_N SPT

Let us carry out the block construction for C_N symmetry in (4+1)d. By the definition of SPT phases, the bulk state can be disentangled everywhere except on the rotation "axis", which is two-dimensional in 4D, and the C_N symmetry reduces to a $\mathbb{Z}_N$ internal symmetry on the rotation axis. Now there are two possibilities: a $\mathbb{Z}_N$ group-cohomology SPT phase, or a E_8 state, on which the $\mathbb{Z}_N$ does not act. The latter corresponds to the generator of the $\mathbb{Z}_N$ beyond-cohomology SPT phases. In Appendix A we compute the group structure of the C_N SPT phases, which turns out to be $\mathbb{Z}_N \times \mathbb{Z}_N$ for $3 \nmid N$, and $\mathbb{Z}_{3N} \times \mathbb{Z}_{N/3}$ for $3 \mid N$. Also note that in this construction for C_N SPT, the rotation center can be decorated with any (2+1)d invertible topological phase. Unlike the internal $\mathbb{Z}_N$ case, there is no need to impose the $c_- = 24k$ condition from the boundary gravitational Chern-Simons terms of $\mathbb{Z}_N$ topological action.

We then choose a 3D boundary perpendicular to the rotation axis, so the boundary is invariant under rotation. The rotation plane in the bulk terminates as the 1D axis on the boundary. Since there is an E_8 state on the plane, the 1D axis carries the corresponding $(E_8)_1$ chiral edge mode. In the following we fix that the rotation axis to be in the z direction.

In order to create a fully gapped (3+1)d boundary, we use the following construction: choose N half planes all terminating at the z axis, the positions of which are related to each other by C_N rotation. For example, one of them could be the plane defined by $y = 0, x \geqslant 0$, and the others are obtained by C_N rotations. On each plane we place a 2D chiral topological phase $\mathcal{B}$. Again all of them are placed in a C_N-symmetric way. At the 1D rotation center, we have N edge modes from the topological phases on the half-planes and the $(E_8)_1$ CFT from the rotation center in the (4+1)d bulk. The setup is illustrated in Fig. 1 for $N = 4$. We require that these edge modes together can be gapped out while preserving the C_N symmetry. In other words, the N blocks have a C_N-preserving gapped boundary to E_8. We can further fold the N layers into one topological phase denoted by $\mathcal{B}^{\boxtimes N}$, where the C_N symmetry becomes the $\mathbb{Z}_N$ cyclic permutation symmetry between layers. A similar

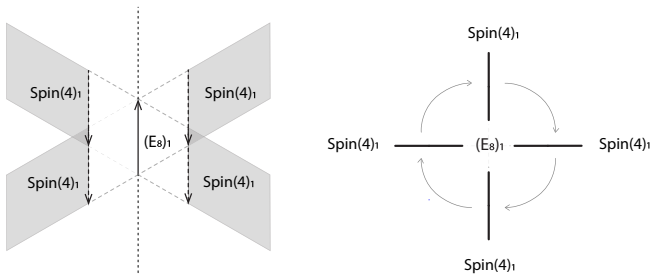

FIG. 1. Illustration of the construction for a gapped state on the boundary of (4+1)d C_N BC SPT state.

method was used in Ref. [30] to study (2+1)d topological phases enriched by reflection symmetry, which is essentially the $N = 2$ case.

To summarize, our construction requires topological phases that satisfy the following conditions:

(1) The topological phase $\mathcal{B}$ has a chiral central charge $c_- = \frac{8}{N}$.

(2) N layers of $\mathcal{B}$ can have a fully gapped edge to a E_8 state.

(3) The gapped edge preserves the $\mathbb{Z}_N$ cyclic permutation symmetry of the N layers.

We will say $\mathcal{B}$ is N *gappable* if all the conditions are satisfied.

Therefore, the construction reduces to finding a $\mathbb{Z}_N$ symmetry-preserving gapped boundary (to an E_8 state) of N copies of $\mathcal{B}$.

B. Gapped boundary conditions

Below we will study this problem using the mathematical framework of modular tensor category (MTC), also known as the anyon theory in physics literature [31]. In this formalism, a topologically ordered phase in (2+1)d is fully described in terms of the universal data of the low-energy quasiparticle excitations, i.e., the anyons. The universal data describe the fusion and braiding properties of the anyons. Alternatively, this collection of data also suffices to specify the (2+1)d TQFT associated with the topological phase. We should note that the MTC description does not fully determine the edge property, i.e., the chiral central charge c_-. In fact, it can be shown that the MTC (or the anyon theory) determines c_- mod 8. Physically the ambiguity precisely comes from stacking E_8 states in the bulk, which does not affect the anyon excitations but can change edge c_- by integer multiples of 8.

Gapped boundaries of a topological phase can also be described in this formalism [32–38]. Each gapped boundary is associated to a unique (composite) anyon object, which determines which anyons can condense on the boundary. This object is called the Lagrangian algebra, denoted by $\mathcal{A}$ below. For Abelian anyons, the condensed anyons form a Lagrangian subgroup [34]. In this case, it is relatively simple to state the condition for a set of anyons to condense: they must all be bosons, and the mutual braiding statistics between them must all be trivial. In addition, the number of anyons in the Lagrangian group must be the square root of the total number of anyons. The definition of the algebra in the general case is reviewed in Appendix E.

We can now define the notion of Witt equivalence for topological phases. Two topological phases $\mathcal{B}_1$ and $\mathcal{B}_2$ are Witt equivalent, if $\mathcal{B}_1 \boxtimes \overline{\mathcal{B}}_2$ has a fully gapped interface to an invertible state (i.e., some copies of E_8 states). Here $\boxtimes$ denotes the operation of stacking two systems, and $\overline{\mathcal{B}}_2$ is the mirror image of $\mathcal{B}_2$. In other words, there is a gapped interface between $\mathcal{B}_1$ and $\mathcal{B}_2$ as long as we are allowed to freely stack copies of E_8 states. Mathematically, two MTCs $\mathcal{B}_1$ and $\mathcal{B}_2$ are Witt equivalent if $\mathcal{B}_1 \boxtimes \overline{\mathcal{B}}_2$ has a Lagrangian algebra. The Witt equivalence classes of MTCs form an Abelian group, known as the Witt group of MTCs.

With these definitions, the first two conditions of N gappability implies that the MTC $\mathcal{B}$ has a minimal order N in the Witt group. However, it is also known that the structure of the Witt group is highly constrained [39–41]: the order of elements in the Witt group cannot be any odd integer. In fact, the only possible finite values are 2^n with $1 \leqslant n \leqslant 5$ (there are obviously elements of infinite order). From this point of view, we can immediately rule out any odd $N \geqslant 3$ in the construction.

In our problem there is also a global symmetry G, i.e., the $\mathbb{Z}_N$ group of cyclic permutations of the N layers, and the gapped boundary must preserve this global symmetry. Therefore it is necessary to incorporate global symmetry in the categorical formalism. First of all, the global symmetry can act on the anyons in nontrivial ways. In this case, the action is given by the layer permutation. There are more subtle aspects of symmetry actions in the topological phase but they do not occur in our system, so we will not go into details. For more details on symmetry-enriched topological phases in (2+1)d, see Ref. [4].

Next, we will need to understand whether a given gapped boundary, or the Lagrangian algebra, can preserve the global symmetry. Clearly, the set of condensed anyons must be invariant under the global symmetry (layer permutations in our case), otherwise the symmetry is explicitly broken. For a condensed anyon a, we define G_a as the subgroup of G that keeps a invariant. Then the condensed anyon a should carry a well-defined charge under the symmetry group G_a. Here by "charge" we mean a one-dimensional representation of G_a, i.e., a homomorphism from G_a to U(1). Different choices of these charges correspond to different types of symmetry-preserving gapped boundary conditions. Again we note that here our description is heuristic, and a more precise formulation is given in Appendix E.

To understand the physical consequence of the symmetry charges of the condensed anyons, it is useful to consider the nature of the gapped boundary. By definition, the gapped boundary is the interface between the topological phase and an invertible state, which is often taken to be the vacuum. However, in the presence of global symmetry, the invertible state could be a nontrivial SPT state. The nature of this invertible state is determined by the symmetry charges of the condensate [30,42]. Here we are mainly interested in the case where the invertible state has no nontrivial SPT order. Following Ref. [30], we study the problem by gauging the $\mathbb{Z}_N$ symmetry in the $\mathcal{B}^{\boxtimes N}$ theory, the result of which is denoted by $[\mathcal{B}^{\boxtimes N}]/\mathbb{Z}_N$. The interface then becomes one between the gauged theory $[\mathcal{B}^{\boxtimes N}]/\mathbb{Z}_N$, and a $\mathbb{Z}_N$ gauge theory obtained from gauging the $\mathbb{Z}_N$ SPT state (possibly stacked with an

E_8 state). After gauging, the Lagrangian algebra in $\mathcal{B}^{\boxtimes N}$ is "lifted" to one in $[\mathcal{B}^{\otimes N}]/\mathbb{Z}_N$; however, the lifting requires additional data, i.e., the symmetry charges of the condensed anyons. Once the Lagrangian algebra and its lifting are given, one can apply the theory of anyon condensation to determine the nature of the $\mathbb{Z}_N$ gauge theory after condensation. We leave the details of the derivations in Appendix F.

Below we apply this theory to several examples. In particular, we will consider a family of anyon theories known as Kitaev's 16-fold ways [31]. They can be described as (2+1)d fermionic topological superconductors with Chern number ν coupled to $\mathbb{Z}_2$ gauge field, or as $\mathrm{Spin}(\nu)_1$ Chern-Simons theory. We will show that $\mathrm{Spin}(\nu)_1$ is $\frac{16}{(16,\nu)}$ gappable when ν is even. However, we find that $\mathrm{Spin}(\nu)_1$ is *not* 16-gappable when ν is odd. Instead, we find a different but closely related construction for the C_{16} case.

1. Spin($2n$)$_1$

In this section we will consider $\mathrm{Spin}(2n)_1$ theories. First we review the basic properties of these MTCs. $\mathrm{Spin}(2n)_1$ has four anyons: $1, \psi, v, v' = v \times \psi$. ψ is a fermion and satisfies $\psi^2 = 1$. v can be viewed as a fermion parity flux since the braiding phase between v and ψ is $M_{v\psi} = -1$. The topological twist factor of v is $\theta_v = e^{\frac{i\pi n}{4}}$. The chiral central charge is $c_- = n$. The order of $\mathrm{Spin}(2n)_1$ in the Witt group is $r = \frac{8}{\gcd(8,n)}$.

We will show that the $\mathrm{Spin}(2n)_1$ MTCs are r gappable.

First, we construct the Lagrangian subgroup for $\mathrm{Spin}(2n)_1^{\boxtimes r}$, preserving the $\mathbb{Z}_r$ cyclic permutation symmetry. We label the anyons by a r-tuple $(a_1, a_2, \ldots, a_r)$. Since all $\mathrm{Spin}(2n)_1$ theories have total quantum dimension $\mathcal{D} = 2$, the Lagrangian subgroup has to have size 2^r. Consider all bosons of the form

$$\mathcal{A}_0 = \{(a_1, a_2, \ldots, a_r) | a_i \in \{\mathbb{1}, \psi\}\}, \tag{6}$$

with an even number of ψ's in the tuple. We will refer those as fermion bound states. There are 2^{r-1} such bosons. Then we fuse $(v, v, \ldots, v)$ with the fermion bound states, which is equivalent to replacing an even number of v's with $v' = v\psi$. Together they generate a Lagrangian subgroup consist of 2^r bosons, and it is straightforward to check that they have trivial mutual braiding statistics, thus forming a Lagrangian subgroup. Clearly this subgroup is invariant under the $\mathbb{Z}_r$ symmetry.

As described in the beginning of this section, we need to examine finer structures of the Lagrangian algebra under the symmetry action. We perform these calculations carefully for two representative values of $n = 1, 2$ in Appendix E. As discussed above, to determine the nature of the condensed phase it is necessary to know the symmetry charges of the condensed anyons. We find that a consistent choice is to have all the $\mathbb{Z}_r$-invariant anyons [i.e., $(\psi, \ldots, \psi)$ and $(v, \ldots, v)$] carry trivial charges under $\mathbb{Z}_r$. With this choice, we can apply the results in Appendix F to show that the condensation leads to an E_8 state where the $\mathbb{Z}_r$ symmetry acts trivially. Together we have established that there is $\mathbb{Z}_r$-symmetric gapped interface between $\mathrm{Spin}(2n)_1^{\boxtimes r}$ to an E_8 state where the $\mathbb{Z}_r$ symmetry acts trivially, so $\mathrm{Spin}(2n)_1$ is r gappable.

2. Spin($2n+1$)$_1$

Let us now turn to $\mathrm{Spin}(2n+1)_1$ theories. Recall that the $\mathrm{Spin}(2n+1)_1$ MTC has three types of anyons $\mathbb{1}, \sigma$, and ψ, where ψ is a fermion and σ is a non-Abelian anyon that satisfies $\sigma \times \sigma = \mathbb{1} + \psi$ and with a topological twist factor $\theta_\sigma = e^{\frac{i\pi(2n+1)}{8}}$.

We first enumerate all bosons in $[\mathrm{Spin}(2n+1)_1]^{\boxtimes 16}$. There are 2^{15} Abelian bosons, which are bound states with an even number of fermions. There is also a non-Abelian boson $(\sigma, \ldots, \sigma)$, with quantum dimension $(\sqrt{2})^{16} = 256$. We can form the following Lagrangian algebra:

$$\mathcal{A} = \sum_{a \in \mathcal{A}_0} a + 128(\sigma, \sigma, \ldots, \sigma). \tag{7}$$

Unfortunately, due to the large multiplicity 128, we are not able to obtain an explicit structure of the condensate. Therefore, we adopt a different approach here. We condense only the Abelian subset $\mathcal{A}_0$ of $\mathcal{A}$ to obtain a $\mathbb{Z}_2$ toric code (TC) phase. From there, we try to construct a C_{16} symmetry-preserving gapped boundary of the $\mathbb{Z}_2$ TC. We will show that in fact it is not possible to condense $\mathcal{A}$ in $[\mathrm{Spin}(2n+1)_1]^{\boxtimes 16}$ without breaking the $\mathbb{Z}_{16}$ symmetry. This can be easily proven using the anyon condensation theory, see Appendix F. Below we provide a more physical argument, which will also suggest a way to fix the problem.

We first condense the Abelian subgroup $\mathcal{A}_0$ of $\mathcal{A}$, resulting in a $\mathbb{Z}_2$ toric code (TC) phase. It is easy to see the only deconfined anyons are the Abelian anyons, and $(\sigma, \ldots, \sigma)$. All the Abelian bosons are already condensed, and all the (Abelian) fermions are identified and become the same ψ of the toric code phase. While $(\sigma, \ldots, \sigma)$ is deconfined, it is invariant under fusion with any of the condensed bosons, so it must split into direct sums of e and m.

In the following denote g as the cyclic permutation of the 16 layers. Namely, g generates the $\mathbb{Z}_{16}$ symmetry group. The $\mathbb{Z}_{16}$ symmetry should be preserved by the condensation, so the $\mathbb{Z}_2$ TC is enriched by the $\mathbb{Z}_{16}$ symmetry. Following the general classification [4], first we need to know how the generator g permutes anyon types. Given that the permutation must preserve the fusion and braiding properties of anyons, there are two possibilities for the $\mathbb{Z}_2$ TC: either g does not permute, or g swaps e and m. In the latter case, there is no symmetric gapped boundary. This is because the only Lagrangian subgroups for the $\mathbb{Z}_2$ TC are $\mathbb{1} + e$ and $\mathbb{1} + m$ and neither of them is invariant under $e \leftrightarrow m$. Therefore, a symmetric gapped boundary requires that g does not permute e and m anyons.

To further determine the action of g on anyon ψ, we take a slightly different approach. The $\mathbb{Z}_2$ toric code can be obtained by gauging the 2D fermion parity of 16 copies of $p_x + ip_y$ superconductors. In the block construction, we place the 16 copies in a C_{16}-invariant configuration, all terminating at the rotation axis. Then we mirror-fold them into a stack of $p_x + ip_y$ superconductors. The g symmetry again permutes the 16 layers cyclically. In this construction, the e and m anyons correspond to a fermion parity flux. The fermion parity flux binds 16 Majorana zero modes γ_i for $i = 1, 2, \ldots, 16$, one from each layer. The local fermion parity of the flux is thus $P = \prod_i \gamma_i$. After gauging, the fermion parity flux with

even and odd local fermion parity ($P = \pm 1$) become the e and m anyons.

Now under the g action, $\gamma_i \to \gamma_{[i+1]_{16}}$, so we find

$$P = \prod_i \gamma_i \to \gamma_2 \cdots \gamma_{16}\gamma_1 = -P. \tag{8}$$

Therefore, the g symmetry flips the fermion parity of the flux, which becomes the $e \leftrightarrow m$ symmetry after gauging. As a result, there is no symmetric gapped boundary.

However, we can fix this issue by modifying the transformation of the Majorana modes to the following form:

$$\gamma_i \to \gamma_{i+1}, \quad i = 1, \ldots, 15, \quad \gamma_{16} \to -\gamma_1. \tag{9}$$

Under this transformation, the local parity P remains unchanged. However, it then follows that g^{16} acts as $\gamma_i \to -\gamma_i$ for all i, i.e., $g^{16} = (-1)^{N_f}$, so the fermion should transform projectively under g.

We thus conclude that the ψ anyon should have $g^{16} = -1$ in order to avoid the g symmetry swapping e and m. Since $\psi = e \times m$, one of them must have $g^{16} = 1$, which can then condense to obtain a symmetric gapped boundary. We have thus shown that there should exist a C_{16}-symmetric gapped boundary of the $\mathbb{Z}_2$ toric code phase to an invertible state.

Notice that in the original construction, we have 16 layers of Ising theories, and the C_{16} rotation acts as the cyclic permutation. One can show that there is no nontrivial symmetry fractionalization for this symmetry. It is natural to postulate that after condensing $\mathcal{A}_0$ one is led precisely to the toric code with $g^{16} = 1$ on ψ. We can then conclude that the 16 layers of Ising topological orders admit a C_{16}-symmetric gapped boundary at the rotation center to an invertible state.

However, we are not able to precisely determine the nature of the invertible state besides its chiral central charge $c_- = 8$. So this (3+1)d fermionic $\mathbb{Z}_2$ gauge theory may exist on the boundary of the root C_{16} BC SPT state, possibly stacked with another (in-cohomology) SPT state.

IV. (3+1)d BOUNDARY TQFT

Reference [14] constructed a fermionic $\mathbb{Z}_2$ gauge theory on the boundary of the $\mathbb{Z}_2$ BC SPT state. Here a fermionic $\mathbb{Z}_2$ gauge theory refers to a $\mathbb{Z}_2$ gauge theory with fermionic $\mathbb{Z}_2$ gauge charges. We will argue that the same theory can realize the anomaly for $\mathbb{Z}_N$ BC SPT state when $N = 4, 8, 16$ as well. Recall the domain wall decoration construction for $\mathbb{Z}_N$ BC SPT in Sec. II A, the codimension-2 trijunctions in the bulk are decorated by minimally three copies of E_8 states. Thus the truncated domain walls on the (3+1)d boundary should host the E_8 on the junctions as well. We will show that a fermionic $\mathbb{Z}_2$ gauge theory in (3+1)d has an anomalous $\mathbb{Z}_{16}$ (0-form) global symmetry [43]. More precisely, the (3+1)d boundary TQFT is a fermionic $\mathbb{Z}_2$ gauge theory with E_8 defects.

We now construct the codimension-1 invertible topological defects that implement the $\mathbb{Z}_{16}$ 0-form symmetry. We insert a (2+1)d chiral topological superconductor (TSC) of Chern number ν (equivalent to ν copies of $p + ip$ superconductors) into the fermionic theory before gauging, and then couple the system to a $\mathbb{Z}_2$ gauge field. This way we obtain an invertible topological defect of codimension-1 in the fermionic $\mathbb{Z}_2$ gauge theory [44], which defines a 0-form symmetry. From

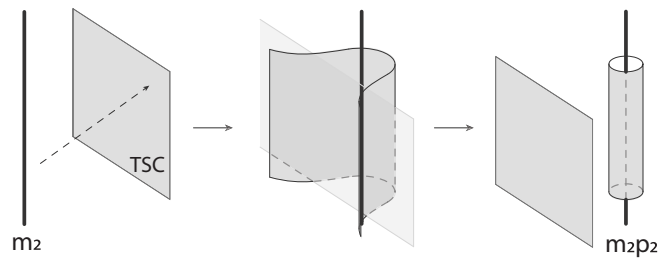

FIG. 2. Illustration of the 0-form symmetry action on a flux loop.

this construction naively it seems that the defect is labeled by the integer ν and fusion of two defects of ν_1 and ν_2 results in a defect of $\nu_1 + \nu_2$. However, we will argue that ν is defined mod 16. This is because before gauging, a $\nu = 16$ TSC is topologically equivalent to a E_8 state stacked with completely trivial gapped fermions. Since this equivalence can be generated by adiabatic evolution with a gapped local Hamiltonian preserving fermion parity, it is expected that the equivalence is preserved after gauging. In other words, a $\nu = 16$ defect is equivalent to a E_8 state. Since the E_8 state is purely bosonic and decoupled from the $\mathbb{Z}_2$ gauge theory, when viewed as a topological defect it can only act on the $\mathbb{Z}_2$ gauge theory trivially. Therefore the faithful symmetry group generated by these TSC defects is $\mathbb{Z}_{16}$. However, the fact that 16 $\nu = 1$ defects fuse to a E_8 state suggests that the $\mathbb{Z}_{16}$ symmetry is anomalous. Indeed this is exactly what should happen on the boundary of a $\mathbb{Z}_{16}$ BC SPT state [45].

Let us examine how the symmetry acts on various objects in the theory. First we review the low-energy excitations of the $\mathbb{Z}_2$ gauge theory. There are two elementary types of excitations: a fermionic $\mathbb{Z}_2$ particle, and a $\mathbb{Z}_2$ flux loop (denoted by m_2 below, where the subscript 2 is the codimension). In addition, it is useful to introduce an invertible line defect as follow: We can think of the theory as a system of fermions in a gapped trivial state coupled to a $\mathbb{Z}_2$ gauge field. We insert a Majorana chain in the ungauged fermion system, and then gauge the $\mathbb{Z}_2$ fermion parity. The Majorana chain then becomes an invertible topological defect of codimension 2, which will be denoted by p_2. It is evident that the TSC defect does not act on the fermionic particle, and the nontrivial action only happens on the flux loops. To see what is going on when a flux loop m_2 passes through the domain wall, we note that the process is equivalent to wrapping the topological superconductor around the flux loop. It is a well-known fact that when ν is odd, the topological superconductor when wrapped on a cylinder with antiperiodic boundary condition for the fermions (i.e., with a π flux threading the cylinder) is equivalent to a Majorana chain [46]. See Fig. 2 for an illustration of the symmetry action. Thus we find that $m_2 \to m_2 p_2$ when passing through a TSC defect with an odd ν [43].

When ν is even, the symmetry does not change the type of the m_2 loop. Instead, let us consider a Hopf link of two flux loops, and pass the link through the domain wall. During this process, the worldlines of the four intersection points of the link with the domain wall precisely trace out the Hopf link, which correspond to a full braiding between two of them and result in a phase factor $\pm e^{\frac{i\pi\nu}{4}}$. Here the sign ambiguity $\pm$ comes from possible fermions attached to the flux loops.

Notice that this characterization only applies to $\nu \equiv 2 \bmod 4$. More generally, we can consider a "three-loop braiding" process [47], where two flux loops are linked to a base loop of the defect. The exchange statistics of the two flux loops is $e^{\frac{i\pi\nu}{8}}$, which can distinguish all different ν mod 16.

Now we describe the boundary TQFT for C_N BC SPT states. Starting from the block construction of the boundary state in Sec. III, we can now construct a $\mathbb{Z}_2$ gauge theory in the following way: fill the (3+1)d boundary with a fermionic $\mathbb{Z}_2$ gauge theory where the C_r symmetry acts trivially (besides the coordinate transformation). The (2+1)d intrinsic topological orders [i.e., $\mathrm{Spin}(2n)_1$ layers] need to be transformed into invertible codimension-1 defects in the $\mathbb{Z}_2$ gauge theory such that boundary theory is a (3+1)d TQFT. To achieve this, on each of the $\mathrm{Spin}(2n)_1$ layer, we drive a condensation of the bound state of the emergent fermion in the $\mathbb{Z}_2$ gauge theory and the ψ in $\mathrm{Spin}(2n)_1$. In other words, the ψ in $\mathrm{Spin}(2n)_1$ layers are all identified with the fermion in the $\mathbb{Z}_2$ gauge theory. Consequently, v or v' anyons are attached to the $\mathbb{Z}_2$ flux lines. There are no separate anyons confined on the $\mathrm{Spin}(2n)_1$ layers anymore, so these layers become invertible defects embedded in the fermionic $\mathbb{Z}_2$ gauge theory. These defects are precisely the TSC defects introduced in the previous paragraph, since the $\mathrm{Spin}(2n)_1$ can be thought of as coupling a TSC of Chern number $2n$ to a $\mathbb{Z}_2$ gauge field, and what we just did is to "Higgs" the emergent $\mathbb{Z}_2$ gauge field in $\mathrm{Spin}(2n)_1$ with that of the (3+1)d gauge theory. Notice that the kind of condensation transitions on the $\mathrm{Spin}(2n)_1$ layers can be driven by interactions that preserve the C_r symmetry, and it is expected that there is no spontaneous symmetry breaking. As a result, the new $\mathbb{Z}_2$ gauge theory obtained this way still has the C_r symmetry with the same anomaly.

V. DISCUSSIONS AND CONCLUSIONS

In this paper we have studied symmetry-preserving gapped boundary states for (4+1)d BC SPT phases protected by $\mathbb{Z}_N$ and C_N symmetries. We show that for $N \notin \{2, 4, 8, 16\}$, no such boundary states exist for the root $\mathbb{Z}_N$ SPT phases. We then propose that for $N = 2, 4, 8, 16$ a candidate boundary topological order is a fermionic $\mathbb{Z}_2$ gauge theory, where the anomalous symmetry is generated by topological superconductor defects. We provide explicit constructions of the boundary theory for C_N SPT phases for $N = 2, 4, 8, 16$.

One immediate question left open from our analysis is the 32 gappability of $\mathrm{Spin}(2n+1)_{2n+1}$. Given that the simplest of the series, $\mathrm{SU}(2)_6$, already has seven anyon types, it is challenging to classify the Lagrangian algebras in $\mathrm{SU}(2)_6^{\boxtimes 32}$. We conjecture that $\mathrm{Spin}(2n+1)_{2n+1}$ is not 32 gappable.

An interesting question for future works is to construct possible gapless boundary theories for general N, or even the U(1) symmetry group. For $N = 2$, a gapless boundary theory was constructed in Ref. [48].

It will also be interesting to clarify the relation between the bosonic $\mathbb{Z}_N$ (or C_N) SPT phases and the fermionic ones. The fermionic phases can be realized by noninteracting fermions and the natural boundary states are Weyl fermions. One can imagine that certain fermionic phases are actually adiabatically connected to a bosonic one with trivial gapped fermions. In fact, this provides a possible route to construct gapless boundary states for many values of N if the bosonic phase can be "embedded" into a noninteracting fermionic one. This is the case for all odd N, so a possible boundary theory is obtained by gauging fermion parity in a (3+1)d Weyl fermion.

We have discussed the 0-form symmetry in a fermionic $\mathbb{Z}_2$ gauge theory in (3+1)d. The full symmetry group includes $\mathbb{Z}_2$ 1-form and 2-form symmetries, and together with the $\mathbb{Z}_{16}$ 0-form symmetry they are expected to form a 3-group [49]. The 1- and 2-form symmetries and their anomalies are analyzed in Ref. [50]. It is important to fully understand the structure and the anomaly of the 3-group. In addition, one can also consider noninvertible defects to get an even richer structure (conjecturally a fusion 3-category).

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APPENDIX A: GROUP STRUCTURE OF C_N SPT PHASES

In the following we denote the root BC SPT phase (with a single E_8 state on the rotation center) as x. Similarly, denote the root group-cohomology SPT phase (with the root group-cohomology 2D $\mathbb{Z}_N$ SPT state on the center) by y. Stacking of phases is denoted additively and 0 represents the trivial phase. We then have $Ny = 0$.

To determine the group structure of C_N SPT phases, we observe that in the block construction [15], a state $|\psi\rangle$ with N copies of E_8 states, all parallel to the rotation center and arranged in a C_N-symmetric configuration is actually adiabatically connected to a trivial state. On the (3+1)d boundary, we have N copies of $(E_8)_1$ CFTs, where the C_N acts as $\mathbb{Z}_N$ cyclic permutations. Apparently, it has the same mixed-gravitational anomaly as the boundary of the Nx phase. On the other hand, it can also have a pure $\mathbb{Z}_N$ anomaly. Recall that 't Hooft anomalies of a $\mathbb{Z}_N$ symmetry are classified by $\mathcal{H}^3[\mathbb{Z}_N, \mathrm{U}(1)] = \mathbb{Z}_N$, so they can be labeled by an integer $\omega \in \mathbb{Z}/N\mathbb{Z}$. It is known that the $\mathbb{Z}_N$ anomaly ω_N for N copies of $(E_8)_1$ CFTs is given by [51]

$$\omega_N = \begin{cases} 0 & 3 \nmid N \\ \frac{N}{3} & 3 \mid N \end{cases}. \tag{A1}$$

Therefore, the triviality of $|\psi\rangle$ implies $Nx + \omega_N y = 0 \bmod N$.

For $3 \nmid N$, we have $Nx = 0$, so the group is $\mathbb{Z}_N^2$.

For $3 \mid N$, we find $Nx + \frac{N}{3}y = 0 \bmod N$, so it follows that x generates a $\mathbb{Z}_{3N}$ subgroup. The group structure is $\mathbb{Z}_{3N} \times \mathbb{Z}_{N/3}$, where $\mathbb{Z}_{N/3}$ is generated by $3x + y$.

Special cases of the classification (for $N = 2, 3, 4, 8$) have been obtained in Ref. [26].

Let us now consider the boundary states for the $N = 3^n$ SPTs. Since it is well known that the group-cohomology SPT phases admit topological boundary theories, for this purpose we mod out the group-cohomology phases from the classification. The remaining group $\mathbb{Z}_N$ is generated by the root

BC phase x. For the kx state, there are k copies of chiral $(E_8)_1$ CFTs at the rotation axis on the (3+1)d boundary. To apply our construction, the (2+1)d MTC needs to have chiral central charge $c_- = \frac{8k}{N}$. Let us write $k = b \cdot 3^a$, where b is the coprime of 3^n and $0 \leqslant a < n$. Therefore, the MTC must have order 3^{n-a} in the Witt group, which is impossible. So we conclude that our construction does not work for these SPTs.

APPENDIX B: U(1) AND $\mathbb{Z}_N$ SPT PHASES IN (4+1)d

In this Appendix, we discuss in details the computation of the partition functions of $\mathbb{Z}_N$ SPT phases coupled to background gauge field, in particular for the BC phases. For $3 \nmid N$, the action given in Eq. (3) correctly describes the BC SPT phases, and evaluating the partition function on $K_3 \times S^1$ yields. However, when $3|N$, Eq. (3) does not apply anymore. For example, for $N = 3$, it is known that for a $\mathbb{Z}/3\mathbb{Z}$ gauge field A, $\int A \cup p_1 \equiv 0 \bmod 3$ on any closed 5-manifold [52].

The cobordism classification gives a $\mathbb{Z}_9$ classification for $\mathbb{Z}_3$ SPT phases in (4+1)d, where the generator is the root BC SPT phase. This is consistent with the C_3 analysis done in Appendix A. We now discuss how to compute the partition function on $K_3 \times S^1$ for the $\mathbb{Z}_3$ root BC phase.

In the following we write the partition function as $\mathcal{Z} = e^{2\pi i S}$, where S is the action.

1. Topological responses

To do this, we start from SPT phases with U(1) symmetry, and then break the symmetry down to $\mathbb{Z}_N$. For a background U(1) gauge field $\tilde{A}$, one can write down the following two topological terms in 5d:

$$S_1 = \int_{B_6} c_1^3, \quad S_2 = \int_{B_6} c_1 \wedge p_1. \tag{B1}$$

Here B_6 is a six-dimensional extension of the 5-manifold, and $c_1 = \frac{\tilde{F}}{2\pi}$ is the first Chern class. Both terms are well defined since on closed six-manifolds they are quantized to integers.

And yet, the following combination is also integral on a closed six-manifold B_6 based on the Hirzebruch signature theorem [53],

$$S_3 = \int_{B_6} \frac{1}{3}\left(c_1^3 - c_1 \wedge p_1\right). \tag{B2}$$

As a result, there is a well-defined 5d topological term when S_3 is defined on a 6d manifold with boundary. We will schematically write it as

$$S_3 = \int_{M_5} \frac{1}{3}\left(\frac{\tilde{A} \wedge \tilde{F} \wedge \tilde{F}}{(2\pi)^2} - \tilde{A} \wedge p_1\right). \tag{B3}$$

Let us first consider dimensional reduction of both terms S_1 and S_2 on $S^2 \times M_3$, where M_3 is a three manifold, with a unit flux through S^2 (i.e., $\int_{S^2} \frac{\tilde{F}}{2\pi} = 1$). For S_1, one finds a 3d Chern-Simons term at level 6, i.e., a bosonic U(1) SPT phase with Hall conductance $\sigma_H = 6$. Notice that the most fundamental bosonic U(1) SPT phase has $\sigma_H = 2$. For S_2, as discussed in Sec. II A the dimensional reduction gives an invertible bosonic topological phase with chiral central charge $c_- = 24$.

For S_2, it is also useful to consider dimensional reduction on $S^1 \times M_4$, which can be interpreted as quantization of the theory on a closed spatial manifold M_4. We find that the state on M_4 carries a U(1) charge $p_1(M_4) = 3\sigma(M_4)$, where $\sigma(M_4)$ is the signature of the manifold. On the other hand, S_1 the action does not contribute to the ground-state charge on M_4.

Combining S_1 and S_2, S_3 when dimensionally reduced on $S^2 \times M_3$ yields a (2+1)d invertible phase with $\sigma_H = 2$ and $c_- = -8$, while carries U(1) charge $-\sigma(M_4)$ upon quantization on M_4. We can choose S_3 and S_1 (or S_2) as representing the two generators of the U(1) SPT phases in (4+1)d.

2. Defect decorations

This dimensional reduction procedure is closely related to the decorated defect picture explained in Sec. II A. Here, the topological defect to consider is a fundamental vortex, which has spatial dimension 2, i.e., a vortex sheet. Note that a vortex sheet is only available when the U(1) symmetry is spontaneously broken. Naively, it might appear that one cannot discuss any SPT decoration protected by the original U(1) symmetry. However, one can combine U(1) with a spatial SO(2) rotation to get a new U(1)′ symmetry, which is respected by the vortex system. We can then ask what kind of (2+1)d U(1)′ SRE phase is decorated on the domain wall. It is well known that such SRE phases are labeled by a pair of integers $[\sigma_H/2, c_-/8]$, where σ_H is the Hall conductance and c_- is the chiral central charge ($c_-/8$ gives the number of E_8 states).

To make connection with the topological responses in the previous section, observe that the "boundary" of a vortex sheet is a U(1) vortex in the symmetry-breaking phase. If one restores the U(1) symmetry [and the SO(2) spatial symmetry as well], the vortex sheet boundary should become the U(1) monopole, and the U(1)′ symmetry is identified with the original U(1). Therefore, the decoration on the vortex sheet should be identified with the theory obtained from the compactification on S^2. We thus have the following identification: $S_3 \sim [1, -1]$, $S_1 \sim [3, 0]$, $S_2 \sim [0, 3]$. A general topological term $aS_1 + bS_3$ should give decoration $[3a + b, -b]$. In other words, there appears to be a constraint

$$\frac{\sigma_H}{2} + \frac{c_-}{8} \equiv 0 \,(\mathrm{mod}\ 3). \tag{B4}$$

When the symmetry is broken down to $\mathbb{Z}_N$, the decoration on the vortex sheet becomes the one at the junction of N fundamental $\mathbb{Z}_N$ defects (each carrying $2\pi/N$ flux). Equivalently, the decoration is the fusion outcome of N such defects as indicated in Eq. (1). In the symmetry-breaking phase, the defects become domain walls, then the configuration of N domain walls meeting at the fusion junction preserves the combined symmetry $\mathbb{Z}_N'$ of the $\mathbb{Z}_N$ and C_N transformation. Then one can ask what kind of (2+1)d $\mathbb{Z}_N'$ SPT phase is decorated on the junction. Recall the discussion in Sec. II A, such SPT phases are classified by $[k, c_-/8]$, where $k \in \mathbb{Z}/N\mathbb{Z}$ labels the group-cohomology $\mathbb{Z}_N$ SPT phases.

Notice that there is an additional equivalence relation: one can attach the trivial configuration, N copies of E_8 states, to N of $\mathbb{Z}_N$ codimension-1 domain walls in a C_N (thus also $\mathbb{Z}_N'$) symmetric way, increasing $c_-/8$ by N. However, as shown in Appendix A, when $3 \mid N$ the index k is also modified to $k +$

$N/3$. So for $3 \mid N$ we find the important equivalence relation

$$[k, c_-/8] \sim [k + N/3, c_-/8 + N]. \tag{B5}$$

As a result, we have $[0, N] \sim [-N/3, 0]$, and hence [0,1] generates a $\mathbb{Z}_{3N}$ subgroup. The other $\mathbb{Z}_{N/3}$ subgroup is generated by [1,3].

For $3 \nmid N$ we instead have $[k, c_-/8] \sim [k, c_-/8 + N]$, and the classification is $\mathbb{Z}_N \times \mathbb{Z}_N$, with the two generators being $[k = 1, c_- = 0]$ and $[k = 0, c_-/8 = 1]$.

If the phase can be connected to a U(1) SPT phase, then we have $k = \sigma_H/2 \bmod N$ satisfying the constraint Eq. (B4) for U(1) SPT phase. Therefore, when $3 \mid N$ certain $\mathbb{Z}_N$ group-cohomology SPT states, e.g., $[k = 1, c_- = 0]$ for $N = 3$, cannot be lifted to a U(1) SPT phase. The topological response for such phases cannot be described by S_1, instead we define S_1' as the topological response for the $[k = 1, c_- = 0]$ phase. Note that for $3 \nmid N$, one can easily show that all $\mathbb{Z}_N$ SPT phases can be embedded to U(1) SPTs.

From this analysis, the correspondence between the C_N and $\mathbb{Z}_N$ classifications can also be made clear. In fact, the decoration $[\sigma_H/2, c_-/8]$ in the $\mathbb{Z}_N$ case is precisely the SRE state placed on the rotation center in the C_N block construction.

3. Computing $Z(K_3 \times S^1)$

For $\mathbb{Z}_N$ symmetry, if there is a nontrivial holonomy $e^{2\pi i/N}$ along S^1, $Z(K_3 \times S^1)$ is given by $e^{2\pi i q(K_3)/N}$, where $q(K_3)$ is the $\mathbb{Z}_N$ charge of the state on K_3. The group-cohomology SPT phases always give $q(K_3) = 0$ (at least within TQFT), thus only the BC SPT phases contribute nontrivially to the charge $q(K_3)$.

For $3 \nmid N$, we can always embed the $\mathbb{Z}_N$ SPT phase to a U(1) SPT phase, and it follows from the dimensional reduction that $q(K_3) = -\sigma(K_3)\frac{c_-}{8} = -2c_- \bmod N$. When $N = 3$, $q(K_3) = -\sigma(K_3) = -16 \equiv 2 \bmod 3$, corresponding to the $\mathbb{Z}_{3N}$ subgroup generated by [0,1]. Thus the partition function on $K_3 \times S^1$ is $e^{4\pi i/3} \neq 1$.

Now let us consider $N = 3^p$ symmetry, with $p > 1$. Again, since group-cohomology SPT phases do not contribute to $q(K_3)$, we can mod out the $\mathbb{Z}_{3N} \times \mathbb{Z}_{N/3}$ group by the $\mathbb{Z}_N$ group generated by the group-cohomology SPTs, leaving only the $\mathbb{Z}_N$ BC SPTs. Essentially, the quotient is given by the mapping $[k, c_-/8] \Rightarrow c_-/8 \bmod N$. For the purpose of computing $q(K_3)$, one can use S_3 [with the symmetry reduced from U(1) to $\mathbb{Z}_N$] for the generator of this quotient group, which gives $q(K_3) = -\sigma(K_3) = -16$. Note that the generator of $Z_{N/3}$ corresponds to 3 after quotient, so the partition function on $K_3 \times S^1$ is $e^{6\pi i q(K_3)/N} = e^{-32\pi i/3^{p-1}} \neq 1$ as long as $p > 1$.

In [26], it was claimed that the response action for the root BC phase should be the Postnikov square: $\beta_{(9,3)}(\beta_{(3,3)}A \cup \beta_{(3,3)}A)$, where A is the $\mathbb{Z}_3$ gauge field and $\beta_{(n,m)}$ is the Bockstein homomorphism associated with the extension $\mathbb{Z}_n \xrightarrow{\cdot m} \mathbb{Z}_{nm} \to \mathbb{Z}_m$. It will be interesting to understand the relation between the various forms of actions.

APPENDIX C: WITT GROUP OF MTCS

In this section we will review the definition of the Witt group, as well as some known facts about it.

First we define the notion of Witt equivalence between two (2+1)d topological phases. Two topological phases $\mathcal{B}_1$ and $\mathcal{B}_2$ are Witt equivalent, if $\mathcal{B}_1 \boxtimes \overline{\mathcal{B}}_2$ has a fully gapped interface to an invertible state (i.e., some copies of E_8 states). Here $\overline{\mathcal{B}}_2$ is the mirror image of $\mathcal{B}_2$. In other words, there is a gapped interface between $\mathcal{B}_1$ and $\mathcal{B}_2$ as long as we are allowed to freely stack copies of E_8 states. Mathematically, two MTCs $\mathcal{B}_1$ and $\mathcal{B}_2$ are Witt equivalent if $\mathcal{B}_1 \boxtimes \overline{\mathcal{B}}_2$ is a quantum double (Drinfeld center of some fusion category).

Below we review known results about the torsion subgroup of the Witt group, particularly for Abelian MTCs. We adopt notations in Ref. [54] for MTCs. In particularly, $\mathbb{Z}_N^{(k)}$ refers to an Abelian topological order with N Abelian anyons labeled by $[a]_N$, where $a \in \{0, 1, \ldots, N-1\}$, and $[\cdot]_N$ denotes mod N. The fusion rules are given by addition: $[a]_N \times [b] = [a+b]_N$. The topological twist factor $\theta_{[a]_N} = e^{i\frac{\pi k}{N}a^2}$. Notice that for odd N, k must be an integer. While for even N, k can be an integer or half-integer, but only half-integer values give modular theories. $\mathbb{Z}_{2n}^{(1)}$ represents the topological order of the familiar $\mathrm{U}(1)_{2n}$ Chern-Simons theories.

The Witt group of all Abelian MTCs, denoted by $\mathcal{W}_{\mathrm{pt}}$ following the notation in Ref. [41], has the following decomposition:

$$\mathcal{W}_{\mathrm{pt}} = \bigoplus_{p \text{ prime}} \mathcal{W}_{\mathrm{pt}}(p). \tag{C1}$$

For each prime p, the p-subgroup $\mathcal{W}_{\mathrm{pt}}(p)$ is given by

- $p = 2$ $\mathcal{W}_{\mathrm{pt}}(2) = \mathbb{Z}_8 \times \mathbb{Z}_2$. Here $\mathbb{Z}_8$ is generated by the semion theory $\mathbb{Z}_2^{(1/2)}$, and $\mathbb{Z}_2$ is generated by $\mathbb{Z}_2^{(1/2)} \times \overline{\mathbb{Z}_4^{(1/2)}}$.
- $p \equiv 1 \pmod 4$ $\mathcal{W}_{\mathrm{pt}}(p) = \mathbb{Z}_2 \times \mathbb{Z}_2$. One generator can be chosen as $\mathbb{Z}_p^{(1)}$, and the other $\mathbb{Z}_p^{(k)}$ where k is a quadratic nonresidue mod p.
- $p \equiv 3 \pmod 4$ $\mathcal{W}_{\mathrm{pt}}(p) = \mathbb{Z}_4$. The generator could be any $\mathbb{Z}_p^{(n)}$ theory for $1 \leqslant n < p$.

Another important example is Kitaev's 16-fold way: the Ising MTC generates a $\mathbb{Z}_{16}$ group, which contains order-2,4,8 subgroups.

There also exists infinitely many order-32 elements in the Witt group. They are represented by the $\mathrm{Spin}(2n+1)_{2n+1}$ Chern-Simons theories, where $n \geqslant 1$. The simplest one of them is $\mathrm{Spin}(3)_3 \simeq \mathrm{SU}(2)_6$. It is the "square root" of an Ising Witt class: two copies of $\mathrm{Spin}(2n+1)_{2n+1}$ is Witt equivalent to $\mathrm{Spin}((2n+1)^2)_1$.

We will now examine the N gappability of the examples mentioned above.

1. $\mathbb{Z}_p^{(n)}$

We start from the $\mathbb{Z}_p^{(n)}$ theories, where p is an odd prime.

First, we consider $p \equiv 3 \bmod 4$, and $\mathbb{Z}_p^{(n)}$ MTCs have order 4 in the Witt group. We now show that they are not 4 gappable.

Label anyon in four copies of $\mathbb{Z}_p^{(n)}$ by $a = (a_1, a_2, a_3, a_4)$, where $a_i \in \{0, 1, \ldots, p-1\}$. The most general form of a cyclic permutation is generated by

$$g(a) = (s_1 a_4, s_2 a_1, s_3 a_2, s_4 a_3), \tag{C2}$$

where $s_i = \pm 1$. Basically, this is a "bare" permutation that takes a to (a_4, a_1, a_2, a_3), combined with a topological symmetry of each of the $\mathbb{Z}_p^{(n)}$ layer. It is known that the only nontrivial topological symmetry of the $\mathbb{Z}_p^{(n)}$ MTC is the charge conjugation $a \to -a$.

In our setup, we require g^4 is the identity. Under repeated actions of g we have

$$\begin{aligned}(a_1, a_2, a_3, a_4) &\to (s_1 a_4, s_2 a_1, s_3 a_2, s_4 a_3)\\ &\to (s_1 s_4 a_3, s_2 s_1 a_4, s_3 s_2 a_1, s_4 s_3 a_2)\\ &\to (s_1 s_4 s_3 a_2, s_2 s_1 s_4 a_3, s_3 s_2 s_1 a_4, s_4 s_3 s_2 a_1)\\ &\to s_1 s_2 s_4 s_4 (a_1, a_2, a_3, a_4). \end{aligned} \tag{C3}$$

So we must have $s_1 s_2 s_3 s_4 = 1$, otherwise g^4 is the global charge conjugation, which acts nontrivially in $\mathbb{Z}_p$ theories.

If a is in the Lagrangian subgroup and the subgroup preserves the $\mathbb{Z}_4$ symmetry, then $a, g(a), g^2(a), g^3(a)$ must form a condensable subset. Then they must all be bosons, which gives

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 \equiv 0 \bmod p. \tag{C4}$$

In addition, they must have trivial mutual braiding statistics, which lead to

$$\begin{aligned} s_2 a_1 a_2 + s_3 a_2 a_3 + s_4 a_3 a_4 + s_1 a_4 a_1 &\equiv 0 \bmod p,\\ (s_1 s_4 + s_2 s_3) a_1 a_3 + (s_1 s_2 + s_3 s_4) a_2 a_4 &\equiv 0 \bmod p. \end{aligned} \tag{C5}$$

The last equation simplifies to $2(s_1 s_4 a_1 a_3 + s_1 s_2 a_2 a_4) \equiv 0 \bmod p$, and since 2 is invertible mod p, we have

$$(s_1 s_4 a_1 a_3 + s_1 s_2 a_2 a_4) \equiv 0 \bmod p. \tag{C6}$$

Since $s_1 s_2 s_3 s_4 = 1$, we can represent $s_1 = c_4 c_1$, $s_2 = c_1 c_2$, $s_3 = c_2 c_3$, $s_4 = c_3 c_4$, and define $b_i = c_i a_i$. The equations are simplified to

$$\begin{aligned} b_1^2 + b_2^2 + b_3^2 + b_4^2 &\equiv 0 \bmod p,\\ b_1 b_2 + b_2 b_3 + b_3 b_4 + b_4 b_1 &\equiv 0 \bmod p,\\ b_1 b_3 + b_2 b_4 &\equiv 0 \bmod p. \end{aligned} \tag{C7}$$

Together they imply $(b_1 + b_2 + b_3 + b_4)^2 \equiv 0 \bmod p$. Then the second relation leads to

$$\begin{aligned} b_1 b_2 + b_2 b_3 + b_3 b_4 + b_4 b_1 &= (b_2 + b_4)(b_1 + b_3)\\ &\equiv -(b_1 + b_3)^2 \equiv 0 \bmod p, \end{aligned}$$

from which we conclude that $b_3 \equiv -b_1, b_4 \equiv -b_2 \bmod p$. The last relation in Eq. (C7) then implies $-2(b_1^2 + b_2^2) \equiv 0 \bmod p$. For p odd, we then have $b_1^2 + b_2^2 \equiv 0 \bmod p$.

One can show that there is no solution for $p \equiv 3 \pmod 4$. To show this is the case, we write $b_1^2 \equiv q$, so $b_2^2 \equiv -q$. In other words, both q and $-q$ are quadratic residues of p. Compute the Legendre symbols,

$$\left(\frac{-q}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}}\left(\frac{q}{p}\right) = -\left(\frac{q}{p}\right). \tag{C8}$$

So it is impossible to have both q and $-q$ being quadratic residues when $p \equiv 3 \pmod 4$.

Let us now turn to $p \equiv 1 \pmod 4$, and the $\mathbb{Z}_p^{(n)}$ theories have order 2 in the Witt group. We will show that they are not 2-gappable. Using the same argument, (a_1, a_2) and $(s_1 a_2, s_2 a_1)$ should form a condensable subgroup, which requires

$$a_1^2 + a_2^2 \equiv 0 \bmod p, \quad (s_1 + s_2) a_1 a_2 \equiv 0 \bmod p. \tag{C9}$$

Thus we need to have $s_1 = -s_2$. However, under this permutation $(a_1, a_2) \to (s_1 a_2, s_2 a_1) \to s_1 s_2 (a_1, a_2)$, if $s_1 s_2 = -1$ then g^2 is equal to the charge conjugation.

2. $\mathbb{Z}_2^{(\frac{1}{2})}$

$\mathbb{Z}_2^{(\frac{1}{2})}$ has order 8 in the Witt group. A Lagrangian subgroup in eight copies of $\mathbb{Z}_2^{(\frac{1}{2})}$ should have dimension 16, so generated by four bosons.

We first find all bosons, such that the its image under $\mathbb{Z}_8$ form a condensable subgroup. It turns out that there are 8 such bosons, and they form $\mathbb{Z}_2^3$ group generated by (1,1,0,0,1,1,0,0),(0,1,1,0,0,1,1,0),(0,0,1,1,0,0,1,1). So it is impossible for find a Lagrangian subgroup invariant under $\mathbb{Z}_8$. Interestingly, if we actually condense this $\mathbb{Z}_2^3$ subgroup, we find a $\mathbb{Z}_2$ toric code, and the $\mathbb{Z}_8$ generator acts as electromagnetic duality in this theory. Therefore, we cannot further condense bosons without breaking the $\mathbb{Z}_8$ symmetry.

APPENDIX D: LATTICE MODEL REALIZATION

Given the subtleties in the TQFT classification of SPT phases, it is highly desirable to find microscopic constructions of the nontrivial states. For the $N = 2$ case, an exactly solvable model was given in [14] (see also [55] for an alternative construction, which applies to $N = 4$ as well.). The key observation there is the following: because $p_1 \equiv w_2^2 \pmod 2$, we can interpret the action in Eq. (3) as $\mathbb{Z}_2$ domain walls being decorated with the "w_2^2" SPT states, which are realized as the ground state of the so-called three-fermion Walker-Wang (WW) model [56]. The key ingredient in this construction is a quantum cellular automata (QCA), or a locality-preserving unitary, that disentangles the 3-fermion WW state. In addition, the QCA exactly squares to 1. With such a QCA, a wavefunction of a consistent, equal-weight superposition of decorated domain wall states, as well as a commuting projector parent Hamiltonian, can be written down.

It is not clear whether similar constructions can be generalized to other $N > 2$ cases. If we simply generalize the construction in [14], according to the action in (3), the $\mathbb{Z}_N$ domain wall is decorated by a (3+1)d gapped state, partition function of which is given by $e^{\frac{2\pi i}{N}\int p_1}$. In addition, the disentangling QCA for this state must have order N. The constraints on QCAs could be seen from the Witt group of (2+1)d modular tensor categories (MTC), which are mathematical theories describing the universal bulk properties of topological phases. While a complete topological classification of QCAs in (3+1)d is still unknown, there is a growing body of evidences [57–59] suggesting that they are classified by the Witt group of (2+1)d MTCs. We review the definition of Witt group in Appendix C. Conjecturally, a QCA that disentangles a WW model with the input MTC in a nontrivial Witt class is topologically nontrivial. The partition function for the WW model is $e^{\frac{2\pi i c_-}{24}\cdot p_1}$, where c_- is the chiral central charge of the input MTC. Assuming that this conjectured classification of

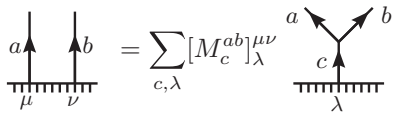

FIG. 3. A diagrammatic representation of the M symbol.

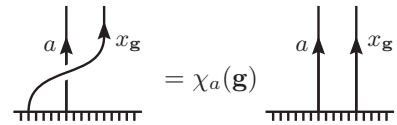

FIG. 4. A diagrammatic representation of the χ symbol.

QCA is correct, we conclude that in order to generalize the construction to $\mathbb{Z}_N$, we would need to find a topological phase with chiral central charge $\frac{48}{N}$ mod 8 and with order N in the Witt group.

However, as already mentioned in Sec. III B, by the constraint of the order of elements in the Witt group. The only possible finite values for N are 2^n with $1 \leqslant n \leqslant 5$ (there are obviously elements of infinite order). This immediately rules out any odd $N > 3$ in the construction.

Even for N that divides 48, when there exist order-N elements in the Witt group, there is a further constraint. If the MTC has an order N element in the Witt group, the corresponding QCA conjecturally is also of order N, meaning that Nth power of the QCA is a finite-depth local unitary circuit. However, for the construction to work, the N-th power needs to be exactly 1. So far this has only been done for the 3-fermion QCA with $N = 2$, and to the best of our knowledge, no other known examples of QCA satisfy this property.

APPENDIX E: ALGEBRAIC DESCRIPTION OF SYMMETRY-PRESERVING ANYON CONDENSATION

We review the algebraic theory of gapped boundaries of a two-dimensional topological phase [32,33,35–38], closely following the formulation in [38] and [42]. We extensively use the language of unitary modular tensor category (UMTC) for (2+1)d topological phases. A brief summary of UMTC in this context can be found in the Appendix of [42].

A gapped boundary corresponds to a Lagrangian algebra of the bulk MTC. Physically the Lagrangian algebra indicates which bulk anyons are condensed on the boundary [34,35].

The mathematical theory of the gapped boundary takes into account the local process of annihilating a condensable anyon a on the boundary. Similar to fusion/splitting spaces, we associate a vector space for local operators that annihilate a, denoted as V^a, with basis vector $|a;\mu\rangle$. The dimension of this vector space is the "multiplicity" n_a of a in the Lagrangian algebra. Obviously we must have $n_1 = 1$.

Diagrammatically, the condensation process is represented by an anyon line terminating on a wall representing the boundary. We also attach a label at the termination point, which represents the state of the boundary condensation space. When $n_a = 1$ it can be suppressed.

An important property of the algebra is the following "M symbol",

$$|a;\mu\rangle|b;\nu\rangle = \sum_{c,\lambda} \left[M_c^{ab}\right]_\lambda^{\mu\nu} |c;\lambda\rangle. \tag{E1}$$

The definition is illustrated diagrammatically in Fig. 3.

Next we impose consistency conditions on the M symbols. We can apply M moves to three anyon lines terminating a, b, c on the boundary, but in different orders, which leads to a variation of the pentagon equation,

$$\sum_{e,\sigma} \left[M_e^{ab}\right]_\sigma^{\mu\nu} \left[M_d^{ec}\right]_\delta^{\sigma\lambda} \left[F_d^{abc}\right]_{ef} = \sum_\psi \left[M_d^{af}\right]_\delta^{\mu\psi} \left[M_f^{bc}\right]_\psi^{\nu\lambda}. \tag{E2}$$

The M symbols also have gauge degrees of freedom, originating from the basis transformation of the boundary condensation space V^a: $\widetilde{|a;\mu\rangle} = \Gamma^a_{\mu\nu}|a;\nu\rangle$, where $\Gamma^a_{\mu\nu}$ is a unitary transformation. The M symbol becomes

$$\left[\tilde{M}_c^{ab}\right]_\lambda^{\mu\nu} = \sum_{\mu',\nu',\lambda'} \Gamma^a_{\mu\mu'} \Gamma^b_{\nu\nu'} \left[M_c^{ab}\right]_{\lambda'}^{\mu'\nu'} [\Gamma^c]^{-1}_{\lambda'\lambda}.. \tag{E3}$$

M symbols are affected by the gauge transformation of bulk fusion space as well.

It is convenient to fix the gauge for the following symbols:

$$\left[M_a^{1a}\right]_\nu^\mu = \left[M_a^{a1}\right]_\nu^\mu = \delta_{\mu\nu}. \tag{E4}$$

Braiding puts further constraints on the M symbols. Since the anyons condense on the boundary, it should not matter in which order the anyon lines terminate on the boundary,

$$\left[M_c^{ba}\right]_\lambda^{\nu\mu} R_c^{ab} = \left[M_c^{ab}\right]_\lambda^{\mu\nu}. \tag{E5}$$

There is a similar condition for the inverse braiding.

It was shown in Ref. [38] that these conditions are equivalent to the mathematical definition of a commutative, connected and separable Frobenius algebra $\mathcal{A} = \bigoplus_a n_a a$ in a braided tensor category, with the algebra morphism $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ precisely given by the M symbol.

1. Symmetry-preserving condensation

We now give a precise definition of anyon condensation that preserves the global symmetry [60]. This builds on top of the algebraic theory of (2+1)d symmetry-enriched topological phases, known as the G-graded braided tensor category, reviewed in [42]. For a more complete account see [4]. A key fact that we will use is that in a topological phase enriched by symmetry group G, the symmetry action on anyons is fully specified by the following data: $\rho_{\mathbf{g}}, U_{\mathbf{g}}(a,b;c)$ and $\eta_a(\mathbf{g},\mathbf{h})$. Here $\rho_{\mathbf{g}}$ denotes a permutation of anyon labels, and in the following we write ${}^{\mathbf{g}}a \equiv \rho_{\mathbf{g}}(a)$. $U_{\mathbf{g}}(a,b;c)$ are unitary transformations acting on the fusion spaces. $\eta_a(\mathbf{g},\mathbf{h})$ are phase factors that describe projective symmetry transformations on individual anyons. ρ, U and η need to satisfy consistency conditions given in [4]. Again we refer to [4] for their definitions and properties. In the present case of the $\mathcal{B}^{\boxtimes N}$ MTC, ρ is the $\mathbb{Z}_N$ cyclic permutation, and U and η are both 1 if we use the natural gauge choice where the F and R symbols of $\mathcal{B}^{\boxtimes N}$ are simply given by the Cartesian products of those of $\mathcal{B}$.

In the following $a, b, c, \ldots$ denote anyons in the condensate, unless otherwise specified. We assume $n_a = 1$ whenever a belongs to the condensate, so we omit the index for the

boundary condensation space. Since the boundary is fully gapped and symmetric, we can posit that for each $\mathbf{g}$ there exists at least one $\mathbf{g}$ defect that can be absorbed without creating any additional excitations on the boundary.

Here $\chi_a(\mathbf{g})$ is a phase factor. Physically, $\chi_a(\mathbf{g})$ encodes the $\mathbf{g}$ action on the condensed anyon, illustrated in Fig. 4. When there is more than one condensation channel, χ should be replaced by a unitary transformation acting on the condensation space.

If we slide a vertex which splits a $\mathbf{gh}$ defect to $\mathbf{g}$ and $\mathbf{h}$ defects over a boundary vertex, we find

$$\eta_a(\mathbf{g},\mathbf{h}) = \frac{\chi_a(\mathbf{gh})}{\chi_{\bar{\mathbf{g}}a}(\mathbf{h})\chi_a(\mathbf{g})}. \tag{E6}$$

We can also consider fusion of condensable anyons on a boundary. For a, b, c in the condensate, sliding a $\mathbf{g}$ line over the diagrammatic equation that defines M symbol, one finds

$$M_{\bar{\mathbf{g}}c}^{\bar{\mathbf{g}}a,\bar{\mathbf{g}}b} U_{\mathbf{g}}(a,b;c) = M_c^{ab}\frac{\chi_a(\mathbf{g})\chi_b(\mathbf{g})}{\chi_c(\mathbf{g})}. \tag{E7}$$

We believe that these two conditions Eqs. (E6) and (E7) are sufficient and necessary for the condensation to preserve symmetry. Mathematically, $\chi_a(\mathbf{g})$ defines an algebraic isomorphism for each $\mathbf{g}$. The consistency conditions guarantee that one has a G-equivariant algebra structure on $\mathcal{L}$ [60].

Note that for a $\mathbf{g}$-invariant anyon a, $\chi_a(\mathbf{g})$ can be interpreted as the $\mathbf{g}$ charge carried by a. In particular, it means that $\eta_a(\mathbf{g},\mathbf{h}) = \frac{\chi_a(\mathbf{gh})}{\chi_a(\mathbf{g})\chi_a(\mathbf{h})}$. Below, in all our examples the bulk has no symmetry fractionalization so we make the canonical choice that $\eta_a(\mathbf{g},\mathbf{h}) = 1$.

2. Condensation in Spin$(2n)_1$ layers

Below we focus on the cases of $\mathrm{Spin}(2n)_1^{\boxtimes r}$. We will only study the $n = 1, 2$ cases. $n = 4$ has been treated in Ref. [30], and the other values of n are similar. We start from a few general results that apply to all n. As before, anyons are denoted by a r tuple.

Denote the Lagrangian subgroup as $\mathcal{A}$ for the system, which has two types of group elements. The subgroup $\mathcal{A}_0$ consists of all bosons $\{\mathbf{a}_0, \mathbf{b}_0, \dots\} \in \mathcal{A}_0$ made of an even number of fermions while the rest of the group elements are expressed as $\mathbf{v} + \mathbf{a}_0$, where $\mathbf{v}$ is the fermion parity flux.

We observe that $F^{\mathbf{a}_0,\mathbf{b}_0,\mathbf{c}_0} = 1$. So $M^{\mathbf{a}_0,\mathbf{b}_0}$ forms a 2-cocycle over $\mathcal{A}_0$. We will postulate that $M^{\mathbf{a}_0,\mathbf{b}_0} = 1$ in our solutions.

To simplify Eq. (E2), we make the following gauge choices: By using the $\Gamma^{\mathbf{v}+\mathbf{a}_0}$ gauge freedoms, we set $M^{\mathbf{v},\mathbf{a}_0} = 1$. Then using $\Gamma^{\mathbf{v}}$ we set $M^{\mathbf{v},\mathbf{v}} = 1$. From the consistency equations we find the following expressions:

$$\begin{aligned} M^{\mathbf{v}+\mathbf{a}_0,\mathbf{b}_0} &= F^{\mathbf{v},\mathbf{a}_0,\mathbf{b}_0}, \\ M^{\mathbf{a}_0,\mathbf{v}+\mathbf{b}_0} &= \frac{M^{\mathbf{a}_0+\mathbf{b}_0,\mathbf{v}}}{M^{\mathbf{b}_0,\mathbf{v}}}, \\ M^{\mathbf{v}+\mathbf{a}_0,\mathbf{v}+\mathbf{b}_0} &= M^{\mathbf{a}_0,\mathbf{v}}. \end{aligned} \tag{E8}$$

Now $M^{\mathbf{a}_0,\mathbf{v}}$ has to satisfy

$$\begin{aligned} &M^{2\mathbf{v}+\mathbf{a}_0+\mathbf{b}_0+\mathbf{c}_0,\mathbf{v}} M^{\mathbf{b}_0,\mathbf{v}} F^{\mathbf{v}+\mathbf{a}_0,\mathbf{v}+\mathbf{b}_0,\mathbf{v}+\mathbf{c}_0} \\ &= M^{\mathbf{a}_0,\mathbf{v}} M^{\mathbf{c}_0,\mathbf{v}} F^{\mathbf{v},\mathbf{a}_0,2\mathbf{v}+\mathbf{b}_0+\mathbf{c}_0}. \end{aligned} \tag{E9}$$

We can write a general solution of the consistency equations $M^{\mathbf{a},\mathbf{b}} = M_0^{\mathbf{a},\mathbf{b}}\omega^{\mathbf{a},\mathbf{b}}$, consisting of a special solution $M_0^{\mathbf{a},\mathbf{b}}$ of Eq. (E9) and a group 2-cocycle $\omega^{\mathbf{a},\mathbf{b}}$ over $\mathcal{A}$. Then ω will be fixed through Eq. (E5).

3. $\mathbb{Z}_4$-symmetric Lagrangian algebra of Spin$(4)_1^{\boxtimes 4}$

We denote the anyons in $\mathrm{Spin}(4)_1^{\boxtimes 4}$ by $\mathbf{a} = (\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4)$, where $\vec{a}_i = (a_i^1\ a_i^2)$ are defined mod 2. The F and R symbols are given by

$$\begin{aligned} F^{\mathbf{a},\mathbf{b},\mathbf{c}} &= \exp\left(\frac{i\pi}{2}\sum_{i=1}^{4}\vec{a}_i\cdot(\vec{b}_i+\vec{c}_i-[\vec{b}_i+\vec{c}_i]_2)\right), \\ R^{\mathbf{a},\mathbf{b}} &= \exp\left[\frac{i\pi}{2}\sum_{i=1}^{4}\vec{a}_i\cdot\vec{b}_i\right]. \end{aligned} \tag{E10}$$

The $\mathbb{Z}_4$ symmetry generator $\mathbf{g}$ acts on the anyons in the following way:

$$\rho : (\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4) \to (\vec{a}_2, \vec{a}_3, \vec{a}_4, \vec{a}_1). \tag{E11}$$

It is obvious that F and R symbols are invariant under ρ,

$$F^{\rho(\mathbf{a}),\rho(\mathbf{b}),\rho(\mathbf{c})} = F^{\mathbf{a},\mathbf{b},\mathbf{c}}, \quad R^{\rho(\mathbf{a}),\rho(\mathbf{b})} = R^{\mathbf{a},\mathbf{b}}. \tag{E12}$$

Therefore the corresponding U symbols are all 1. We can then set all the η symbols to 1 as well.

It will also be convenient to pick a set of generators for the Lagrangian subgroup. We will use the following set:

$$\begin{aligned} \mathbf{v}_1 &= (1\ 0, 1\ 0, 1\ 0, 1\ 0), \\ \mathbf{v}_2 &= (1\ 1, 1\ 1, 0\ 0, 0\ 0), \\ \mathbf{v}_3 &= (0\ 0, 1\ 1, 1\ 1, 0\ 0), \\ \mathbf{v}_4 &= (0\ 0, 0\ 0, 1\ 1, 1\ 1). \end{aligned} \tag{E13}$$

Together they form a $\mathcal{A} = \mathbb{Z}_2^4$ group. Then any group element $\mathbf{a}$ can be expanded in terms of the generators: $\mathbf{a} = \sum_{i=1}^4 \tilde{a}_i\mathbf{v}_i$.

Following the procedure in Appendix E 2, we find the special solution $M_0^{\mathbf{a},\mathbf{b}}$,

$$M_0^{\mathbf{a}_0,\mathbf{v}} = e^{\frac{i\pi}{2}\mathbf{v}\cdot\mathbf{a}_0}, \tag{E14}$$

and the group 2-cocycle $\omega^{\mathbf{a},\mathbf{b}}$:

$$\omega^{\mathbf{a},\mathbf{b}} = e^{i\pi(\tilde{a}_2\tilde{b}_3+\tilde{a}_3\tilde{b}_4)}. \tag{E15}$$

We find the following solution of χ,

$$\chi_{\mathbf{a}}(\mathbf{g}) = e^{i\pi(1-\tilde{a}_4)(\tilde{a}_2+\tilde{a}_3)}, \tag{E16}$$

and

$$\chi_{\mathbf{a}}(\mathbf{g}^2) = e^{i\pi[(1-\tilde{a}_4)\tilde{a}_3+(1-\tilde{a}_3)\tilde{a}_2]}. \tag{E17}$$

One can easily verify that all $\mathbf{g}$-invariant anyons have $\chi = 1$. The same is true for all $\mathbf{g}^2$-invariant anyons.

4. $\mathbb{Z}_8$-symmetric Lagrangian algebra of U$(1)_4^{\boxtimes 8}$

We denote the anyons in $\mathrm{U}(1)_4^{\boxtimes 8}$ by a 8-tuple $\mathbf{a} = (a_1, a_2, \dots, a_8)$, where $a_i \in \{0, 1, 2, 3\}$ are defined mod 4. Since the theory is Abelian, fusion will be denoted additively.

We will use shorthand notations for the F and R symbols $(F^{\mathbf{a},\mathbf{b},\mathbf{c}}_{\mathbf{a}+\mathbf{b}+\mathbf{c}})_{\mathbf{a}+\mathbf{b},\mathbf{b}+\mathbf{c}} \equiv F^{\mathbf{a},\mathbf{b},\mathbf{c}}$, $R^{\mathbf{a},\mathbf{b}}_{\mathbf{a}+\mathbf{b}} = R^{\mathbf{a},\mathbf{b}}$. They are given by

$$F^{\mathbf{a},\mathbf{b},\mathbf{c}} = \exp\left(i\pi\,\mathbf{a}\cdot\frac{\mathbf{b}+\mathbf{c}-[\mathbf{b}+\mathbf{c}]_4}{4}\right),$$
$$R^{\mathbf{a},\mathbf{b}} = \exp\left(\frac{i\pi}{4}\mathbf{a}\cdot\mathbf{b}\right). \tag{E18}$$

The $\mathbb{Z}_8$ symmetry generator $\mathbf{g}$ acts on the anyons in the following way:

$$\rho : (a_1, a_2, \ldots, a_8) \to (a_2, a_3, \ldots, a_1). \tag{E19}$$

It is obvious that F and R symbols are invariant under ρ,

$$F^{\rho(\mathbf{a}),\rho(\mathbf{b}),\rho(\mathbf{c})} = F^{\mathbf{a},\mathbf{b},\mathbf{c}}, \quad R^{\rho(\mathbf{a}),\rho(\mathbf{b})} = R^{a,b}. \tag{E20}$$

Therefore the corresponding U symbols are all 1. We can then set all the η symbols to 1 as well.

The Lagrangian subgroup is generated by the following anyons:

$$\begin{aligned}
\mathbf{v}_1 &= (1,1,1,1,1,1,1,1),\\
\mathbf{v}_2 &= (2,2,0,0,0,0,0,0),\\
\mathbf{v}_3 &= (0,2,2,0,0,0,0,0),\\
\mathbf{v}_4 &= (0,0,2,2,0,0,0,0),\\
\mathbf{v}_5 &= (0,0,0,2,2,0,0,0),\\
\mathbf{v}_6 &= (0,0,0,0,2,2,0,0),\\
\mathbf{v}_7 &= (0,0,0,0,0,2,2,0).
\end{aligned} \tag{E21}$$

Together they generate a $\mathcal{A} = \mathbb{Z}_2^6 \times \mathbb{Z}_4$ group. We find the solution to be

$$M_0^{\mathbf{a}_0,\mathbf{v}} = e^{\frac{i\pi}{4}\mathbf{v}\cdot\mathbf{a}_0}, \tag{E22}$$

and

$$\omega^{\mathbf{a},\mathbf{b}} = e^{i\pi(\tilde{a}_2\tilde{b}_3+\tilde{a}_3\tilde{b}_4+\tilde{a}_4\tilde{b}_5+\tilde{a}_5\tilde{b}_6+\tilde{a}_6\tilde{b}_7)}. \tag{E23}$$

The solution for χ is given by

$$\chi_{\mathbf{a}}(\mathbf{g}) = e^{i\pi(1-\tilde{a}_7)(\tilde{a}_2+\tilde{a}_4+\tilde{a}_6)}. \tag{E24}$$

In this case, the only $\mathbf{g}$-invariant anyons are generated by $\mathbf{v}_1$, so obviously $\chi_{\mathbf{v}_1}(\mathbf{g}) = 1$. The same is true for other symmetry transformations.

APPENDIX F: GAUGING $\mathbb{Z}_N$ CYCLIC PERMUTATION SYMMETRY

We describe the $\mathbb{Z}_N$ gauging of $\mathcal{B}^{\boxtimes N}$, where the $\mathbb{Z}_N$ generator g acts as

$$(a_1, a_2, \ldots, a_N) \to (a_N, a_1, \ldots, a_{N-1}). \tag{F1}$$

We will denote the gauged theory by $\mathcal{U}$. Anyons in $\mathcal{U}$ will be labeled by (x_{g^r}, χ), where g^r is the symmetry flux, and χ is an irreducible representation of the stabilizer group. Here since G is Abelian we can view χ as a group homomorphism from the stabilizer group to U(1). For example, if $r = 0$, then x runs through all orbits of anyons $[\mathbf{b}]$ in $\mathcal{B}^{\boxtimes N}$ under G, where $\mathbf{b} = (b_1, b_2, \ldots, b_N)$ is a representative element of the orbit. Its stabilizer group, i.e., the subgroup of G that keeps the anyon $\mathbf{b}$ invariant, is denoted by $G_{\mathbf{b}}$. When the stabilizer group is strictly smaller than G, we call $([\mathbf{b}]_1, \chi_{\mathbf{b}})$ a superposition anyon. In general, the label x can actually be chosen as an element in $\mathcal{B}^{\boxtimes(N,r)}$.

Among all (x_{g^r}, χ) there is one with the minimal quantum dimension, which we will call $(\mathbb{1}_{g^r}, \chi)$ (for $r = 0$, $(\mathbb{1}_1, 1)$ is the identity anyon). We have the following fusion rule:

$$(a_1, a_2, \ldots, a_{(N,r)}, \mathbb{1}, \ldots, \mathbb{1}) \times (\mathbb{1}_g, \chi) = (\mathbf{a}_g, \chi). \tag{F2}$$

Here $\mathbf{a} = (a_1, a_2, \ldots, a_{(N,r)})$. Therefore $d_{(\mathbf{a}_{g^r},\chi)} = d_{\mathbf{a}} d_{(\mathbb{1}_{g^r},1)}$.

To compute $d_{(\mathbb{1}_{g^r},1)}$, we use the fact that the total quantum dimension squared of all g^r defects must be equal to $D^2_{\mathcal{B}}$ [4]. That is,

$$\sum_{\mathbf{a}\in\mathcal{B}^{\boxtimes(N,r)}} d^2_{\mathbf{a}} d^2_{\mathbb{1}_{g^r}} = D^{2(N,r)}_{\mathcal{B}} d^2_{\mathbb{1}_{g^r}} = D^{2N}_{\mathcal{B}}.$$

Thus $d_{\mathbb{1}_{g^r}} = D^{N-(N,r)}_{\mathcal{B}}$.

We also have the topological twist factors for the fluxes,

$$\theta_{(\mathbf{a}_{g^r},\chi)} = \theta_{\mathbf{a}_{g^r}}\chi(g^r). \tag{F3}$$

Here $\theta_{\mathbf{a}_{g^r}}$ is the topological twist factor of the $\mathbf{a}_{g^r}$ defect. It can be chosen to be $\theta_{\mathbf{a}_{g^r}} = \theta_{\mathbf{a}}^{\frac{(N,r)}{N}}$, but we do not need its value in this section.

Now we consider the S matrix elements between fluxes and anyons. The entries of S matrix could be computed from the topological spins and the fusion coefficients,

$$S_{x,y} = \frac{1}{D}\sum_z N_z^{xy} d_z \frac{\theta_x\theta_y}{\theta_z}. \tag{F4}$$

We will use this to derive an important property of $S^{\mathcal{U}}_{(\mathbf{a}_{g^r},\chi_a),([\mathbf{b}]_1,\chi_b)}$, where $[\mathbf{b}]_1$ denotes a superposition anyon (i.e., $G_{\mathbf{b}}$ is smaller than G). First we consider the fusion between a superposition anyon $([\mathbf{b}]_1, \chi_b)$ and a bare charge $(\mathbb{1}_1, \chi)$, where χ is a 1D rep. of $\mathbb{Z}_N$. Following Ref. [4], we have

$$([\mathbf{b}]_1, \chi_b) \times (\mathbb{1}_1, \chi) = ([\mathbf{b}]_1, \chi_b \cdot \chi|_{G_{\mathbf{b}}}). \tag{F5}$$

Here $\chi|_{G_{\mathbf{b}}}$ is the restriction of χ to $G_{\mathbf{b}}$.

Suppose $\chi|_{G_{\mathbf{b}}}$ is trivial. For such a χ, using (F5) we get

$$([\mathbf{b}]_1, \chi_b) \times (\mathbb{1}_1, \chi) = ([\mathbf{b}]_1, \chi_b), \tag{F6}$$

which implies that

$$(\mathbf{a}_{g^r}, \chi_a) \times ([\mathbf{b}]_1, \chi_b) \times (\mathbb{1}_1, \chi) = (\mathbf{a}_{g^r}, \chi_a) \times ([\mathbf{b}]_1, \chi_b). \tag{F7}$$

Let us define Λ_b as the group of χ's, which restrict to identity on $G_{\mathbf{b}}$. It is easy to see that Λ_b is all the irreps on G/G_b. Then Eq. (F7) implies

$$(\mathbf{a}_{g^r}, \chi_a) \times ([\mathbf{b}]_1, \chi_b) = \sum_{\mathbf{c}_{g^r}} N^{\mathbf{c}_{g^r}}_{\mathbf{a}_{g^r},[\mathbf{b}]_1} \sum_{\chi\in\Lambda_b} (\mathbf{c}_{g^r}, \chi_c\chi). \tag{F8}$$

While $N^{\mathbf{c}_{g^r}}_{\mathbf{a}_{g^r},[\mathbf{b}]_1}$ and χ_c are undetermined, they are not needed for our purpose.

Using the definition (F4), we obtain

$$S^{\mathcal{U}}_{(\mathbf{a}_{g^r},\chi_a),([\mathbf{b}]_1,\chi_b)} = \frac{\theta_{(\mathbf{a}_{g^r},\chi_a)}\theta_{([\mathbf{b}]_1,\chi_b)}}{D_{\mathcal{U}}} \sum_{\mathbf{c}_{g^r}} N^{\mathbf{c}_{g^r}}_{\mathbf{a}_{g^r},[\mathbf{b}]_1} d_{(\mathbf{c}_{g^r},\chi_c)} \sum_{\chi\in\Lambda_b} \theta^{-1}_{(\mathbf{c}_{g^r},\chi_c\chi)}. \tag{F9}$$

Let us consider the sum over $\chi \in \Lambda_b$, which can be simplified further to

$$\sum_{\chi\in\Lambda_b} \theta^{-1}_{(\mathbf{c}_{g^r},\chi_c\chi)} = \theta^{-1}_{(\mathbf{c}_{g^r},\chi_c)} \sum_{\chi\in\Lambda_b} \chi^{-1}(g^r). \tag{F10}$$

First, we assume $g^r \in G_b$. By the definition of Λ_b, $\chi(g^r) = 1$ so the sum evaluates to $|\Lambda_b|$. Then if $g^r \notin G_b$, the Schur's orthogonality theorem applied to G/G_b shows that the sum should be 0. To summarize, we have shown that

$$S^{\mathcal{U}}_{(\mathbf{a}_{g^r},\chi_a),([\mathbf{b}]_1,\chi_b)} \propto \delta_{g^r\in G_{\mathbf{b}}}. \tag{F11}$$

Let us now consider what happens when $g^r \in G_{\mathbf{b}}$. It implies that b has a period (N, r), i.e., write $b = (b_1, b_2, \ldots, b_N)$, then $b_i = b_{i+(N,r) \bmod N}$. Thus for this calculation, we can group (N, r) consecutive layers into one "layer", with total number of layers $\tilde{N} = \frac{N}{(N,r)}$. It is also convenient to define $\tilde{\mathbf{b}} = (b_1, b_2, \ldots, b_{(N,r)}) \in \tilde{\mathcal{B}} = \mathcal{B}^{\boxtimes(N,r)}$. Therefore, the problem reduces to computing the S matrix between a $(\mathbf{a}_{g^r}, 1)$ defect, and $[\mathbf{b}]_1 = (\tilde{\mathbf{b}}, \ldots, \tilde{\mathbf{b}})$. We will account for the charges later.

Now we use a geometric picture to compute the S matrix. A cylinder with a $\mathbf{a}_{g^r}$ defect line can be viewed as a $\tilde{N}$-layer cylinder, with a g^r cyclic permutation branch cut, which is topologically equivalent to a single-"layer" cylinder with the topological order described by $\tilde{\mathcal{B}}$. On the cylinder there is a Wilson line of the $\mathbf{a}$ anyon. We then compute the eigenvalue of the Wilson loop of $([\mathbf{b}]_1, 1)$ along the other noncontractible cycle in two ways. Note that in the alternative picture, the Wilson loop of $([\mathbf{b}_1, 1])$ becomes a Wilson loop of $\tilde{\mathbf{b}}$. We thus have

$$\frac{S^{\mathcal{U}}_{(\mathbf{a}_{g^r},1),([\mathbf{b}]_1,1)}}{S^{\mathcal{U}}_{(\mathbf{a}_{g^r},1),(\mathbb{1}_1,1)}} = \frac{S^{\tilde{\mathcal{B}}}_{\mathbf{a}\tilde{\mathbf{b}}}}{S^{\tilde{\mathcal{B}}}_{\mathbf{a}\mathbb{1}}} \tag{F12}$$

where $S^{\mathcal{U}}_{(\mathbf{a}_{g^r},1),(\mathbb{1}_1,1)} = \frac{d_{(\mathbf{a}_{g^r},1)}}{D_{\mathcal{U}}}$, $S^{\mathcal{B}}_{a\mathbb{1}} = \frac{d_{\mathbf{a}}}{D_{\mathcal{B}}} = \frac{d_{\mathbf{a}}}{D^{(N,r)}_{\mathcal{B}}}$. Using $d_{(\mathbf{a}_{g^r},1)} = D^{N-(N,r)}_{\mathcal{B}} d_{\mathbf{a}}$ and $D_{\mathcal{U}} = ND^N_{\mathcal{B}}$, we have

$$\begin{aligned} S^{\mathcal{U}}_{(\mathbf{a}_{g^r},1),([\mathbf{b}]_1,1)} &= \frac{d_{\mathbf{a}} D_B^{N-(N,r)}}{ND^N_{\mathcal{B}}} \frac{D^{(N,r)}_{\mathcal{B}}}{d_{\mathbf{a}}} S^{\tilde{\mathcal{B}}}_{\mathbf{a}\tilde{\mathbf{b}}} \\ &= \frac{1}{N} S^{\tilde{\mathcal{B}}}_{\mathbf{a}\tilde{\mathbf{b}}}. \end{aligned} \tag{F13}$$

Including the contribution from the charge χ_b, we find

$$S^{\mathcal{U}}_{(\mathbf{a}_{g^r},\chi_a),([\mathbf{b}]_1,\chi_b)} = \frac{1}{N}\chi_b(g^r) S^{\tilde{\mathcal{B}}}_{\mathbf{a}\tilde{\mathbf{b}}} \delta_{g^r\in G_{\mathbf{b}}}. \tag{F14}$$

Now we study gapped boundaries of $\mathcal{B}^{\boxtimes N}$ from the gauging perspective. Suppose we have a Lagrangian algebra $\mathcal{A}_0$ of $\mathcal{B}^{\boxtimes N}$. It can be "lifted" to a condensable algebra $\mathcal{A}$ in the gauged theory $\mathcal{U}$. Namely, we can first break $\mathcal{A}_0$ into orbits under G, and each orbit becomes a single anyon after gauging.

$$\mathcal{A}_0 = \sum_{[\mathbf{a}]} n_{\mathbf{a}} \sum_{\mathbf{b}\in[\mathbf{a}]} \mathbf{b}. \tag{F15}$$

Here $[\mathbf{a}]$ is an orbit under G, with $\mathbf{a}$ being a representative element. After gauging, it becomes

$$\mathcal{A} = \sum_{[\mathbf{a}]} n_{\mathbf{a}}([\mathbf{a}]_1, \chi_{\mathbf{a}}). \tag{F16}$$

Roughly speaking, $\chi_{\mathbf{a}}$ is the symmetry charge carried by the $\mathbf{a}$ anyon when it condenses. The assignment of $\chi_{\mathbf{a}}$ is not arbitrary. In fact, because U and η are all 1 in our example, the $\chi_{\mathbf{a}}$'s are precisely those defined in Appendix E 1.

By definition, condensing $\mathcal{A}_0$ in $\mathcal{B}^{\boxtimes N}$ results in a trivial theory (the Vec MTC). If the condensation preserves the $\mathbb{Z}_N$ symmetry, condensing $\mathcal{A}$ in $\mathcal{U}$ should result in a deconfined $\mathbb{Z}_N$ gauge theory, denoted by $\mathcal{D}$. In the same convention, anyons in $\mathcal{D}$ will be denoted by (g^r, χ). The vacuum is $\mathbb{1} \equiv (1, 1)$. Note that depending on the choice of the charges $\chi_{\mathbf{a}}$'s, the $\mathbb{Z}_N$ gauge theory may be twisted. We are in particular interested in whether there exists a choice of χ's such that the $\mathbb{Z}_N$ gauge theory is not twisted.

We briefly recall a few basic facts about anyon condensation from a MTC $\mathcal{U}$ to $\mathcal{D}$. It is useful to think of the anyon condensation as defining a gapped interface between the theories $\mathcal{U}$ and $\mathcal{D}$. To fully describe this process, it is also necessary to define the theory of (possibly confined) excitations on the interface, called $\mathcal{T}$. $\mathcal{D}$ is the "deconfined" subcategory of $\mathcal{T}$.

The relations between the three theories are encoded in the restriction map r and the lifting map l. An anyon α in the $\mathcal{U}$ theory can be "restricted" to the $\mathcal{T}$ theory,

$$r(\alpha) = \sum_{t\in\mathcal{T}} n_{\alpha,t} t, \tag{F17}$$

where the $n_{\alpha,t}$'s are non-negative integers. On the other hand, a particle in the $\mathcal{T}$ can be lifted back to the $\mathcal{U}$ theory:

$$l(t) = \sum_{\alpha\in\mathcal{U}} n_{\alpha,t}\alpha. \tag{F18}$$

Clearly $l(\mathbb{1})$ is the Lagrangian algebra $\mathcal{A}$. We are mostly interested in $n_{\alpha,t}$'s when $t \in \mathcal{D}$.

The integers $n_{\alpha,t}$ should satisfy various constraints. We will only need the following condition: the restriction/lifting maps should commute with the modular matrices of $\mathcal{U}$ and $\mathcal{D}$. More specifically, for $\alpha \in \mathcal{U}$ and $t \in \mathcal{D}$, we have

$$\sum_{\beta\in\mathcal{U}} S_{\alpha\beta} n_{\beta,t} = \sum_{s\in\mathcal{D}} n_{\alpha,s} S_{st}. \tag{F19}$$

Secondly, for a given $t \in \mathcal{U}$, all α's with $n_{\alpha,t} \neq 0$ share the same topological twist factor and $\theta_\alpha = \theta_t$.

We will denote

$$l(g, 1) = \sum_{a\in\mathcal{B}} w_g(a)(a_g, 1), \tag{F20}$$

where $w_g(a)$ is a non-negative integer. We shall use (F19) to constrain $w_g(a)$, which can then determine the topological twist factor of $(g, 1)$.

Setting $\alpha = (a_g, 1)$, $t = \mathbb{1}$ in (F19), we have the left-hand side given by

$$\sum_{([\mathbf{b}]_1,\chi_b)\in\mathcal{A}} S^{\mathcal{U}}_{(a_g,1),([\mathbf{b}]_1,\chi_{\mathbf{b}})} n_{\mathbf{b}} = \frac{1}{N} \sum_{\mathbf{b}=g(\mathbf{b})} \chi_{\mathbf{b}} S^{\mathcal{B}}_{ab} n_{\mathbf{b}}. \tag{F21}$$

The right-hand side is $\frac{1}{N} w_g(a)$. So we find

$$\sum_{\mathbf{b}=g(\mathbf{b})} \chi_{\mathbf{b}} S^{\mathcal{B}}_{ab} n_{\mathbf{b}} = w_g(a). \tag{F22}$$

Here $\mathbf{b} = g(\mathbf{b})$ means $\mathbf{b} = (b, b, \ldots, b)$. Similar relations can be derived for other g^r. Given that the S matrix elements are generally not integers, it is a nontrivial consistency check that the sum on the left-hand side yields a non-negative integer.

Let us now apply the relation to $\mathcal{B} = \mathrm{Spin}(2n)_1$ theory. In this case, b runs over all anyons in $\mathcal{B}$ and with $n_{\mathbf{b}} = 1$ we have

$$\sum_{b\in\mathcal{B}} S^{\mathcal{B}}_{ab}\chi_{\mathbf{b}} = w_g(a). \tag{F23}$$

We have shown in that $\chi_{\mathbf{b}} = 1$ for all $\mathbf{b} = g(\mathbf{b})$ is allowed for $n = 1, 2$, and 4, and in fact for all other values of n. With this choice, the unitarity of S matrix implies that $w_g(a) = 2\delta_{a,\mathbb{1}}$. Therefore, $\theta_{(g,1)} = \theta_{(\mathbb{1}_g,1)} = 1$ and the $\mathbb{Z}_N$ gauge theory is not twisted.

In the case of the Ising theory with $N = 16$, the left-hand side becomes

$$\begin{aligned} \tfrac{1}{2}(1 + 128\sqrt{2}\chi_\sigma + \chi_\psi) &= w_g(\mathbb{1}), \\ \tfrac{1}{2}(1 - 128\sqrt{2}\chi_\sigma + \chi_\psi) &= w_g(\psi), \\ \tfrac{1}{\sqrt{2}}(1 - \chi_\psi) &= w_g(\sigma). \end{aligned} \tag{F24}$$

Regardless of the choice of χ_σ and χ_ψ, $w_g(\mathbb{1})$ and $w_g(\psi)$ cannot be an integer, and the only possible integer value for $w_g(\sigma)$ is 0. We conclude that such a symmetry-preserving condensation in $\mathrm{Ising}^{\boxtimes 16}$ does not exist. This confirms the more heuristic argument in Sec. III B 2.

[1] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Symmetry protected topological orders and the group cohomology of their symmetry group, Phys. Rev. B **87**, 155114 (2013).

[2] A. Vishwanath and T. Senthil, Physics of three-dimensional bosonic topological insulators: Surface-deconfined criticality and quantized magnetoelectric effect, Phys. Rev. X **3**, 011016 (2013).

[3] X. Chen, F. J. Burnell, A. Vishwanath, and L. Fidkowski, Anomalous symmetry fractionalization and surface topological order, Phys. Rev. X **5**, 041013 (2015).

[4] M. Barkeshli, P. Bonderson, M. Cheng, and Z. Wang, Symmetry fractionalization, defects, and gauging of topological phases, Phys. Rev. B **100**, 115147 (2019).

[5] J. C.Y. Teo, T. L. Hughes, and E. Fradkin, Theory of twist liquids: Gauging an anyonic symmetry, Ann. Phys. (NY) **360**, 349 (2015).

[6] N. Tarantino, N. H. Lindner, and L. Fidkowski, Symmetry fractionalization and twist defects, New J. Phys. **18**, 035006 (2016).

[7] D. Bulmash and M. Barkeshli, Fermionic symmetry fractionalization in $(2+1)$ dimensions, Phys. Rev. B **105**, 125114 (2022).

[8] D. Bulmash and M. Barkeshli, Anomaly cascade in $(2+1)$-dimensional fermionic topological phases, Phys. Rev. B **105**, 155126 (2022).

[9] D. Aasen, P. Bonderson, and C. Knapp, Characterization and classification of fermionic symmetry enriched topological phases, arXiv:2109.10911.

[10] There are also examples involving finite higher-form symmetries, see e.g., [27].

[11] C. Wang, A. Nahum, M. A. Metlitski, C. Xu, and T. Senthil, Deconfined quantum critical points: Symmetries and dualities, Phys. Rev. X **7**, 031051 (2017).

[12] S. D. Geraedts, C. Repellin, C. Wang, R. S. K. Mong, T. Senthil, and N. Regnault, Emergent particle-hole symmetry in spinful bosonic quantum Hall systems, Phys. Rev. B **96**, 075148 (2017).

[13] L. Fidkowski, J. Haah, M. B. Hastings, and N. Tantivasadakarn, Disentangling the generalized double semion model, Commun. Math. Phys. **380**, 1151 (2020).

[14] L. Fidkowski, J. Haah, and M. B. Hastings, An exactly solvable model for a $4+1D$ beyond-cohomology symmetry protected topological phase, Phys. Rev. B **101**, 155124 (2020).

[15] H. Song, S.-J. Huang, L. Fu, and M. Hermele, Topological phases protected by point group symmetry, Phys. Rev. X **7**, 011020 (2017).

[16] C. Z. Xiong, Minimalist approach to the classification of symmetry protected topological phases, J. Phys. A **51**, 445001 (2018).

[17] D. Gaiotto and T. Johnson-Freyd, Symmetry protected topological phases and generalized cohomology, J. High Energy Phys. 05 (2019) 007.

[18] X. Chen, Y.-M. Lu, and A. Vishwanath, Symmetry-protected topological phases from decorated domain walls, Nat. Commun. **5**, 3507 (2014).

[19] Q.-R. Wang, S.-Q. Ning, and M. Cheng, Domain wall decorations, anomalies and spectral sequences in bosonic topological phases, arXiv:2104.13233.

[20] J. C. Wang, Z.-C. Gu, and X.-G. Wen, Field theory representation of gauge-gravity symmetry-protected topological invariants, group cohomology and beyond, Phys. Rev. Lett. **114**, 031601 (2015).

[21] M. F. Lapa, C.-M. Jian, P. Ye, and T. L. Hughes, Topological electromagnetic responses of bosonic quantum Hall, topological insulator, and chiral semimetal phases in all dimensions, Phys. Rev. B **95**, 035149 (2017).

[22] X.-G. Wen, Construction of bosonic symmetry-protected-trivial states and their topological invariants via $g \times so(\infty)$ nonlinear σ models, Phys. Rev. B **91**, 205101 (2015).

[23] This is because $\int \frac{F}{2\pi} \wedge p_1$ is always an integer.

[24] For a closed 4-manifold, $\int_{M_4} p_1 = 3\sigma(M_4)$ where $\sigma(M_4)$ is the signature of the 4-manifold. The partition function $e^{2\pi i k \int p_1} = e^{6\pi i k\sigma} = e^{\frac{2\pi i}{8}\cdot 24k\cdot\sigma}$,.

[25] We thank C.-M. Jian for crucial discussions on this point.

[26] Z. Wan and J. Wang, Higher anomalies, higher symmetries, and cobordisms I: Classification of higher-symmetry-protected topological states and their boundary fermionic/bosonic anomalies via a generalized cobordism theory, Ann. Math. Sci. Appl. **4**, 107 (2019).

[27] C. Cordova and K. Ohmori, Anomaly obstructions to symmetry preserving gapped phases, arXiv:1910.04962.

[28] C. Córdova and K. Ohmori, Anomaly constraints on gapped phases with discrete chiral symmetry, Phys. Rev. D **102**, 025011 (2020).

[29] R. Thorngren and D. V. Else, Gauging spatial symmetries and the classification of topological crystalline phases, Phys. Rev. X **8**, 011040 (2018).

[30] Y. Qi, C.-M. Jian, and C. Wang, Folding approach to topological order enriched by mirror symmetry, Phys. Rev. B **99**, 085128 (2019).

[31] A. Kitaev, Anyons in an exactly solved model and beyond, Ann. Phys. (NY) **321**, 2 (2006).

[32] A. Kitaev and L. Kong, Models for gapped boundaries and domain walls, Commun. Math. Phys. **313**, 351 (2012).

[33] I. S. Eliens, J. C. Romers, and F. A. Bais, Diagrammatics for Bose condensation in anyon theories, Phys. Rev. B **90**, 195130 (2014).

[34] M. Levin, Protected edge modes without symmetry, Phys. Rev. X **3**, 021009 (2013).

[35] L. Kong, Anyon condensation and tensor categories, Nucl. Phys. B **886**, 436 (2014).

[36] T. Lan, J. C. Wang, and X.-G. Wen, Gapped domain walls, gapped boundaries, and topological degeneracy, Phys. Rev. Lett. **114**, 076402 (2015).

[37] T. Neupert, H. He, C. von Keyserlingk, G. Sierra, and B. A. Bernevig, Boson condensation in topologically ordered quantum liquids, Phys. Rev. B **93**, 115103 (2016).

[38] I. Cong, M. Cheng, and Z. Wang, Hamiltonian and algebraic theories of gapped boundaries in topological phases of matter, Commun. Math. Phys. **355**, 645 (2017),.

[39] V. Drinfeld, S. Gelaki, D. Nikshych, and V. Ostrik, On braided fusion categories I, arXiv:0906.0620.

[40] A. Davydov, M. Mueger, D. Nikshych, and V. Ostrik, The Witt group of non-degenerate braided fusion categories, arXiv:1009.2117.

[41] A. Davydov, D. Nikshych, and V. Ostrik, On the structure of the witt group of braided fusion categories, Sel. Math. New Ser. **19**, 237 (2013).

[42] M. Cheng and D. J. Williamson, Relative anomaly in $(1+1)$d rational conformal field theory, Phys. Rev. Res. **2**, 043044 (2020).

[43] T. Johnson-Freyd, (3+1)D topological orders with only a $\mathbb{Z}_2$-charged particle, arXiv:2011.11165.

[44] Cordova-Ohmori's theorem is proved for spin (i.e., fermionic) TQFTs. However, upon a close examination of the proof, one finds that the proof only requires a spin manifold, such as K_3, in order to apply the Wall's theorem, but the argument does not require the theory being fermionic/spin. Thus the theorem applies to bosonic systems as well.

[45] One subtlety is that because of the chiral central charge, strictly speaking the defect is not completely topological, in the same sense as the (2+1)d Chern-Simons theory is not a topological field theory due to the framing anomaly. For example, the partition function with defect insertions depend on the geometry. However, since the dependence comes in as a phase factor, it does not affect the symmetry action.

[46] N. Read and D. Green, Paired states of fermions in two-dimensions with breaking of parity and time reversal symmetries, and the fractional quantum Hall effect, Phys. Rev. B **61**, 10267 (2000).

[47] C. Wang and M. Levin, Braiding statistics of loop excitations in three dimensions, Phys. Rev. Lett. **113**, 080403 (2014).

[48] S.-J. Huang, 4d beyond-cohomology topological phase protected by C_2 symmetry and its boundary theories, Phys. Rev. Res. **2**, 033236 (2020).

[49] M. Barkeshli, Y.-A. Chen, S.-J. Huang, R. Kobayashi, N. Tantivasadakarn, and G. Zhu, Codimension-2 defects and higher symmetries in (3+1)D topological phases, SciPost Phys. **14**, 065 (2023).

[50] M. Barkeshli, Y.-A. Chen, P.-S. Hsin, and R. Kobayashi, Higher-group symmetry in finite gauge theory and stabilizer codes, SciPost Phys. **16**, 089 (2024).

[51] M. Bischoff, A remark about the anomalies of cyclic holomorphic permutation orbifolds, Int. J. Math. **31**, 2050080 (2020).

[52] Y. Tomonaga, Some properties of pontryagin classes mod 3, Trans. Am. Math. Soc. **117**, 37 (1965).

[53] Naively, 16 $\nu = 1$ defects fuse to a E_8 state with $c_- = 8$, but fusing 16 $\mathbb{Z}_{16}$ defects in the bulk leaves an invertible state with $c_- = 24k$. However, it should be kept in mind that the chiral central charge of the defect in the (3+1)d theory is only defined mod 8, so more precisely 16 $\nu = 1$ defects fuse to an invertible state with $c_- = 8(16m+1)$, where m is an integer. By choosing e.g., $m = 2$, we have $k = 11$ in the bulk.

[54] P. H. Bonderson, Non-Abelian anyons and interferometry, Ph.D. thesis, Caltech, 2012, doi:10.7907/5NDZ-W890.

[55] Y.-A. Chen and P.-S. Hsin, Exactly solvable lattice Hamiltonians and gravitational anomalies, SciPost Phys. **14**, 089 (2023).

[56] F. J. Burnell, X. Chen, L. Fidkowski, and A. Vishwanath, Exactly soluble model of a three-dimensional symmetry-protected topological phase of bosons with surface topological order, Phys. Rev. B **90**, 245122 (2014).

[57] J. Haah, L. Fidkowski, and M. B. Hastings, Nontrivial quantum cellular automata in higher dimensions, Commun. Math. Phys. **398**, 469 (2023).

[58] J. Haah, Clifford quantum cellular automata: Trivial group in 2D and Witt group in 3D, J. Math. Phys. **62**, 092202 (2021).

[59] W. Shirley, Y.-A. Chen, A. Dua, T. D. Ellison, N. Tantivasadakarn, and D. J. Williamson, Three-dimensional quantum cellular automata from chiral semion surface topological order and beyond, PRX Quantum **3**, 030326 (2022).

[60] M. Bischoff, C. Jones, Y.-M. Lu, and D. Penneys, Spontaneous symmetry breaking from anyon condensation, J. High Energy Phys. 02 (2019) 062.