

Anisotropic surface polaritons at isotropic-uniaxial interface: An exact algebraic solutionK. Yu. Golenitskii **Ioffe Institute, 194021 St. Petersburg, Russia and Skolkovo Institute of Science and Technology, 121205 Moscow, Russia*

(Received 12 February 2024; accepted 25 June 2024; published 9 July 2024)

Surface polaritons in an anisotropic media possess a strong dependence of the wave vector on the propagation direction, which is called the isofrequency contour. This can lead to the fact that polariton propagation is possible only in a limited range of angles in the boundary plane. Notable examples are Dyakonov surface waves at the boundary of two dielectrics and hyperbolic plasmons in a hyperbolic metamaterial. Exact closed-form solutions of the polariton dispersion equation are known only in special cases: in a weakly anisotropic medium and in an arbitrary medium for highly symmetric directions of polariton propagation. This work provides an universal exact solution in algebraic form for a surface polariton at the interface of arbitrary isotropic and uniaxial media for the case of the optic axis parallel to the boundary. As an example, it is used to analyze the shapes of isofrequency contours of surface polaritons. The work brings together previously scattered results of studies on surface polaritons of various types in uniaxial media. In addition, two cases not previously considered in the literature are analyzed in detail here. The first corresponds to the boundary of an isotropic metal and a Type I hyperbolic medium, for which the existence of a surface polariton is predicted. In the second case, “elliptic” surface polaritons at the boundary of an isotropic dielectric and an anisotropic metal-like medium are analyzed.

DOI: [10.1103/PhysRevB.110.035301](https://doi.org/10.1103/PhysRevB.110.035301)**I. INTRODUCTION**

Surface waves have attracted the attention of researchers for many years. Specifically, in physics, active research continues on surface electromagnetic waves in a variety of media, both natural and artificial. In the literature they are also called surface polaritons emphasizing the connection with a dispersive medium. Due to edge cases it is not always possible to clearly determine when a surface wave becomes a surface polariton. Therefore, both terms are used equally in the literature, often implying the same meaning.

As we know, the basic properties of waves, such as polarization and speed, in an anisotropic medium depend on the direction of propagation. In particular, the range of possible propagation directions in the boundary plane becomes an important characteristic of surface waves. This is also called the angular existence domain (AED). In the simple case AED always equals 2π when the boundary between materials is isotropic. For example, surface plasmon polaritons (SPPs) at the boundary of metal and a dielectric, if it exists, can propagate in any direction. In this work we consider the case of anisotropic boundary, and AED in some cases may be less than 2π . An illustrative example is Dyakonov surface waves (DSWs) [1] that are propagating along the interface between two transparent media at least one of them is birefringent. The characteristic AED for DSWs in highly anisotropic natural minerals (Hg_2Cl_2 , TiO_2 , YVO_4) in the optical spectrum is only a few degrees. The AED for surface polaritons may depend on the frequency due to the dispersion of dielectric constants. All other properties of surface waves, such as the

localization length, propagation length, frequency dispersion, and others, may also be highly direction dependent.

A. Early studies

Surface polaritons in anisotropic medium have been studied for a long time. Early studies examined plasmonic waves and others that have existence conditions similar to SPPs (a negative dielectric constant): surface magnetoplasmons in a magnetoactive plasma [2–5] in a semiconductor [6]; surface phonon polaritons (SPhPs) [7–10] and surface phonon plasmons [11,12]; and surface polaritons in a resonant medium [13,14] in uniaxial antiferromagnets [15] and in a general uniaxial medium [16]. See also reviews [17–19].

It is well known that a SPP in isotropic media is a purely transverse magnetic wave (TM wave). We can roughly take anisotropy of the medium into account as a perturbation leading to small changes in the wave polarization [4,20]. The conditions for the existence of a SPP with such a rough approach remain almost unchanged. Therefore, it was assumed that the negativity of at least one of the principal components of a dielectric tensor is the necessary condition for surface polariton existence in a homogeneous [21] medium. As we now know, this idea turned out to be wrong.

B. Dyakonov surface waves

One of the earliest works mentioning the existence of surface polaritons in an anisotropic dielectric is [22]. It predicts the existence of a singular surface polariton, also called an exceptional surface wave or surface Voigt wave, in two cases. One of them is interpreted as the case of two dielectric media. But the existence condition requires extremely strong anisotropy of an uniaxial medium [23], and the authors did

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not pay much attention to this. To the best of the author's knowledge, a detailed analysis of the existence of surface waves in a homogeneous anisotropic dielectric was carried out for the first time in [1] by M. I. Dyakonov, in the case of the optic axis parallel to the interface. Later it was generalized to an arbitrary oriented optic axis in [24–26]. Now we call these Dyakonov surface waves (DSWs). The obtained existence condition in [1] is less strict and in the strong anisotropy limit includes previous given in [22]. This statement was noted much later [27]. DSWs possess two remarkable properties. Unlike SPPs, the DSWs usually [28] do not propagate in the most symmetrical directions: along the optic axis or orthogonal to it. This is due to the second property that the DSW is a hybrid wave mixing two polarizations. Thus, birefringence of the medium is the crucial property required for DSWs. Other properties and features are described in more detail in the review [29].

C. Surface hyperbolic plasmons, Dyakonov plasmons, and others

A new round of research on surface polaritons in anisotropic media is associated with metamaterials and structures related to them. A metamaterial in a certain frequency range may have significantly stronger anisotropy and lower losses than natural minerals. For example, the AED of DSWs in a 2D photonic crystal based on Si and SiO₂ [30] may be a couple of times wider, especially for an optical spectrum. Hyperbolic media are one of the types of metamaterials in which Dyakonov-type surface waves continue to be actively studied [31–37]. They are named *hyperbolic* due to the shape of isofrequency surfaces for a plane wave, which is a one- or two-sheet hyperboloid [38]. This is possible when the components of the dielectric tensor in such a medium have different signs, or in other words, it has an indefinite dielectric tensor. Despite the fact that such media have been studied for a long time, the term *hyperbolic medium* has been established relatively recently. A notable example mentioned above is light-optical phonon coupling in an anisotropic crystal that may lead to hyperbolic behavior in some frequency ranges around the frequencies of optical phonons [36]. After Dyakonov's paper, already known surface polaritons were studied in more detail and new types were analyzed in hyperbolic media: Dyakonov plasmons [31,35,39,40], DSWs [41], SPP and DSW coupling on two closely spaced interfaces [42,43], and others (see the review [33]).

D. Experimental studies and suitable materials

Despite the fact that the work is devoted to a theoretical description of polaritons, it is worth mentioning experimental studies. Surface waves have been actively studied and continue to be studied, especially in the context of near-field heat transfer [44–47]. Examples of materials in which phonon polaritons of interest to us were studied are LiTaO₃ and LiNbO₃ [48], CdS [49], MgF₂ and TiO₂ [50], BaTiO₃ and PbTiO₃ [51], α -SiO₂ [8,52], α -LiIO₃ [10], uniaxial Al₂O₃ [53], 6H-SiC [54], and CaCO₃ [55]; see also reviews [56–58]. Polaritons in anisotropic van der Waals structures are also of great interest at the present time [59]. Surface plasmon

polaritons at the interface of a metal and an anisotropic dielectric have also been studied experimentally in liquid crystal [60], in CaCO₃ [61], in ZnO [62], and in *para*-sexiphenyl nanowire films [63]. The possibility to create anisotropic metal layers and, accordingly, the existence of anisotropic surface plasmons was experimentally demonstrated in [64,65]. Hyperbolic waves and Dyakonov plasmons were studied in a large number of artificial structures, for example, based on Au nanorods [66], on Ag/SiN thin-layered structure [67], on metal gratings Ag [68], on Si/ZnO:Al nanorods [69], in deep trench structures [34,70], and in nanostructured hexagonal BN [71]. Surface hyperbolic phonon polaritons in 2D films are being actively studied, for example, in a single [72,73] or twisted [74] α -MoO₃ flakes. At the same time, DSWs have not been experimentally studied in such detail as others. The first observation of DSWs was made more than 20 years after the theoretical prediction in [75] at the interface of a positive biaxial crystal (KTiOPO₄) and index-matching liquid. There are only a few other studies in which the excitation of DSWs was also observed: nematic liquid crystal and polycarbonate [76], MgF₂, and chiral sculptured thin film [77]. Guided modes enabled by DSWs have been observed in a Al₂O₃ nanosheet between a negative biaxial crystal (LiB₃O₅) and index-matching liquid [78].

In all works up to [79], it was believed that the solution to the dispersion equation of a surface wave for an arbitrary direction of propagation cannot be obtained in analytical form. The systems of equations were solved numerically. This may be inconvenient for further analysis of frequency dispersion, for example, in metamaterials. Therefore, the aim of the present work is to extend the solution for Dyakonov surface waves given in [79], where only positive dielectric permittivities were considered, to the case of arbitrary dielectric permittivities. A universal analytical solution in algebraic form is given for an arbitrary ratio of the dielectric permittivities of media.

The main part of the work is devoted to the detailed analysis of it and correspondence with previously known results. This part also provides examples of the basic isofrequency contour shapes depending on the dielectric permittivities. When analyzing combinations of different materials, it was discovered that two types of surface polaritons were not described in the literature. The first type is polaritons at the interface of an isotropic metal-like medium and Type I hyperbolic medium, which can be elliptic- or hyperbolic-like. The second is elliptic polaritons at the interface of anisotropic medium with both negative dielectric permittivities. Other papers mentioned only hyperbolic polaritons. The classification of surface polaritons according to the shapes of isofrequency contours is given in the summary.

Cumbersome transformations that may be useful are given in the Appendixes.

II. MODEL

We are looking for surface electromagnetic waves propagating along the interface of two homogeneous nonmagnetic nonchiral media. Electromagnetic properties of media are described within dielectric permittivity which may depend on the frequency of the wave ω . Let us suppose an isotropic

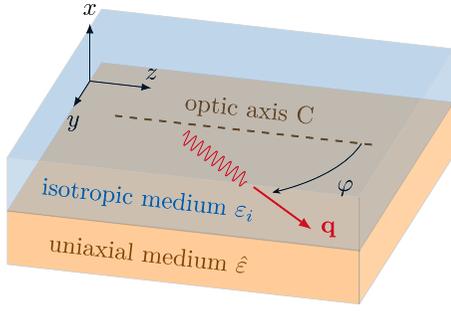


FIG. 1. A planar interface is formed between an uniaxial medium and an isotropic medium where surface polaritons propagate at angle φ to the z axis. The optic axis of a uniaxial medium coincides with the z axis and is parallel to the interface plane.

medium filling the upper half-space $x > 0$ and an anisotropic uniaxial medium filling the lower half-space $x < 0$. Let ε_i be dielectric permittivity of the isotropic medium and $\hat{\varepsilon}$ be dielectric tensor of the anisotropic medium. In principal axes

$$\hat{\varepsilon} = \begin{pmatrix} \varepsilon_o & 0 & 0 \\ 0 & \varepsilon_o & 0 \\ 0 & 0 & \varepsilon_e \end{pmatrix} \quad (1)$$

has two independent components [80].

As a good first approximation, we neglect absorption, dissipation, and gain in media if $|\text{Re } \varepsilon_{i,o,e}| \gg |\text{Im } \varepsilon_{i,o,e}|$. Then let $\text{Im } \varepsilon_i = \text{Im } \varepsilon_o = \text{Im } \varepsilon_e = 0$ at ω . If we do not neglect losses, then a surface polariton would have a finite propagation length or finite lifetime. To find them, one can use the solution obtained as a good first approximation. One should be careful, because such an approximation is unacceptable in ε -near-zero materials in which it is assumed $\text{Re } \varepsilon_{i,o,e} \approx 0$. Also, in an anisotropic medium, this may change the AED of the surface polariton.

A. Dispersion equation

Let us choose the coordinate system (Fig. 1) in such a way that the plane $x = 0$ coincides with the interface between media. An optic axis C is parallel to the interface and directed along the z axis. That is, the coordinate axes coincide with the principal axes of (1). Let $\mathbf{q} = (0, q_y, q_z)$ be the wave vector of a surface polariton making angle φ with C . It is then determined from the dispersion equation. In a general case, a surface wave should be a linear combination of all eigenwaves, with some specific exceptions [81]. They all have the same \mathbf{q} , but possibly different decay with distance from the interface. As usual, eigenwaves mean solutions to Maxwell's equations with constitutive relations

$$\begin{aligned} \nabla \times \mathbf{E} &= \frac{i\omega}{c} \mathbf{B}, & \nabla \times \mathbf{H} &= -\frac{i\omega}{c} \mathbf{D}, \\ \nabla \cdot \mathbf{D} &= 0, & \nabla \cdot \mathbf{B} &= 0, \\ D_i(\omega) &= \hat{\varepsilon}_{ik}(\omega) E_k(\omega), & B_i &= H_i, \end{aligned}$$

in the form of a monochromatic plane wave, such that $\mathbf{E}, \mathbf{D}, \mathbf{H} \propto \exp(iq_y y + iq_z z - i\omega t \pm \kappa x)$. In the isotropic medium we can freely choose the combination of two waves with orthogonal polarization and transverse electric (TE) and

transverse magnetic wave (TM wave), for example. There is no such freedom in the uniaxial medium, and the general solution is a combination of an ordinary and an extraordinary wave. There is an exception to this that will be specifically mentioned.

Although there are many works [1,10,82–85] that study surface waves in this case and even for an arbitrary optic axis direction [24,26,35,86], nowhere previously [79] was an analytical solution obtained in a simple and elegant 1-parametric form. Here we use the similar method proposed in [79] for DSWs to obtain an analytical solution for any possible combination of signs $\varepsilon_i, \varepsilon_o, \varepsilon_e$ [87].

From now on, all dimensional quantities are expressed in units of the wave vector in vacuum $k_0 = \omega/c$. Using standard boundary conditions for the continuity of tangential components of the electric $E_{y,z}$ and magnetic fields $B_{y,z}$, we obtain the dispersion equation

$$\begin{aligned} [\kappa_i(n_z^2 - \varepsilon_o) + \kappa_o(n_z^2 - \varepsilon_i)] \times [\kappa_i \varepsilon_i(n_z^2 - \varepsilon_o) + \kappa_e \varepsilon_o(n_z^2 - \varepsilon_i)] \\ = n_y^2 n_z^2 (\varepsilon_i - \varepsilon_o)^2 \end{aligned} \quad (2)$$

to which we need to add the relations for different eigenwaves

$$\kappa_i^2 = n_y^2 + n_z^2 - \varepsilon_i, \quad (3)$$

$$\kappa_o^2 = n_y^2 + n_z^2 - \varepsilon_o, \quad (4)$$

$$\kappa_e^2 = n_y^2 + \frac{\varepsilon_e}{\varepsilon_o} n_z^2 - \varepsilon_e, \quad (5)$$

where κ_i = the TE and TM wave decay constant, κ_o = the ordinary wave decay constant, κ_e = the extraordinary wave decay constant, and $n_y = q_y/k_0$, $n_z = q_z/k_0$. The system of four equations (2)–(5) contains five unknowns $n_y, n_z, \kappa_i, \kappa_o, \kappa_e$ that need to be found. One also needs to remember that $n_y = n_z \tan \varphi$. The detailed derivation of dispersion equation (2) and expressions for field components in a surface wave are given in Appendix A.

It should be noted that (2) differs from the previously derived [Eq. (8) in 1] and [Eq. (12) in 29], but is similar to the dispersion equation for the eigenmodes in an anisotropic cylindrical wave guide [Eq. (4) in 88]. In Appendix B, dispersion equation (2) is proven to be equivalent to the equation obtained by Dyakonov [Eq. (9) in 1].

B. Algebraic solution

We say that a surface polariton exists at angle φ if there is a solution to the system (2)–(5) that satisfies the conditions $\kappa_i, \kappa_o, \kappa_e > 0$ and n_y, n_z is real. According to symmetry, the surface polariton also exists for angles $-\varphi, \pi + \varphi, \pi - \varphi$. Often the dispersion equation (5) is solved numerically for a chosen direction making an angle φ with the optic axis. But, in fact, it is possible to write the exact solution in algebraic form [79] using an additional variable,

$$\kappa_o(s) = \sqrt{-\frac{P_2(s)}{P_4(s)}}, \quad \kappa_e(s) = s \kappa_o(s), \quad (6)$$

$$\kappa_i(s) = \sqrt{\kappa_o^2(s) + \varepsilon_o - \varepsilon_i}, \quad (7)$$

$$n_y(s) = \sqrt{\frac{(\varepsilon_e - \varepsilon_o s^2)\kappa_o^2(s)}{\varepsilon_e - \varepsilon_o}}, \quad (8)$$

$$n_z(s) = \sqrt{\frac{(s^2 - 1)\varepsilon_o\kappa_o^2(s)}{\varepsilon_e - \varepsilon_o} + \varepsilon_o}, \quad (9)$$

$$q(s) = \sqrt{n_y(s)^2 + n_z(s)^2}, \quad (10)$$

$$\varphi(s) = \arctan \frac{n_y(s)}{n_z(s)}, \quad (11)$$

where $P_2(s)$ and $P_4(s)$ are polynomials of degree 2 and degree 4 defined as

$$P_2(s) = (\varepsilon_i - \varepsilon_o)(\varepsilon_e + \varepsilon_o s^2), \quad (12)$$

$$P_4(s) = (\varepsilon_i + \varepsilon_o s)(1 + s) \times [\varepsilon_i - 2\varepsilon_e + (\varepsilon_i - \varepsilon_o)s + \varepsilon_o s^2]. \quad (13)$$

A detailed derivation of the solution (6) is given in Appendix C. If the value s is known, then all parameters of the surface polariton are determined uniquely. In fact, the range of permissible values of s is limited and corresponds to the surface wave solution. This range is determined only by the dielectric permittivities ε_i , ε_o , ε_e .

The next section is devoted to the analysis of permissible values of s in different media, extending the results obtained in [79] for DSWs.

III. ANALYSIS OF ALGEBRAIC SOLUTION

Let us consider the general case when the signs ε_i , ε_o , and ε_e are arbitrary. We divide all combinations of signs into three groups: *negative permittivity group*, *positive permittivity group*, and *hyperbolic group*. In fact, the division is quite arbitrary, because negative permittivity and hyperbolic groups have something in common with each other. This will become clear later.

The number of cases that need to be analyzed is slightly reduced. It follows from (2) that there are no solutions if $\varepsilon_i, \varepsilon_o < 0$ regardless of ε_e .

A. Negative permittivity materials: SPPs and SPhPs

The *negative ε group* includes such cases where one medium has all negative components of the dielectric tensor, and other has all positive components of the dielectric tensor. From (2) it follows that there is no solution if all $\varepsilon_i, \varepsilon_o, \varepsilon_e < 0$. Therefore there are four different cases of partnering media: anisotropic metal-like medium ($\varepsilon_e < \varepsilon_o < 0$ or $\varepsilon_o < \varepsilon_e < 0$) and isotropic dielectric $\varepsilon_i > 0$, isotropic metal-like medium $\varepsilon_i < 0$, and uniaxial dielectric medium, positive ($\varepsilon_o < \varepsilon_e$) or negative ($\varepsilon_e < \varepsilon_o$). It is clear that in the limit of weak anisotropy ($\varepsilon_o \approx \varepsilon_e$), surface polaritons in this group include surface plasmon polaritons (SPPs), surface phonon polaritons (SPhPs), and other polaritons in isotropic media with negative permittivity [89].

1. Isotropic metal-like medium and uniaxial dielectric

Surface polaritons at the boundary of a metal or medium having negative permittivity at some frequency and an uniaxial dielectric have been considered earlier. Surface plasmon (potential wave) dispersion equations in the quasistatic approximation ($q \gg \omega/c$) have been obtained in [2,4]. In [83,84,90,91] authors numerically analyzed dispersion of SPPs for an arbitrary propagation direction. Also in [84], surface polariton AED has been obtained analytically. The paper [85] pays attention to the special case when the SPP propagation length increases significantly for some directions. It happens due to a large penetration length ($\kappa_e \rightarrow 0$) in the uniaxial dielectric, physically meaning that the polariton energy is transferred mainly in the dielectric without significant losses. The case of an arbitrary oriented optic axis has been analyzed in [86] using an iterative method.

Let

$$\eta = \frac{\varepsilon_e}{\varepsilon_o} - 1 \quad (14)$$

be the relative anisotropy strength, and

$$\chi = -\frac{\varepsilon_i}{\varepsilon_o} \quad (15)$$

be the relative permittivity of the isotropic medium. More general $\eta = \eta(\omega)$ and $\chi = \chi(\omega)$. Then $\eta > 0$ corresponds to positive birefringence and $-1 < \eta < 0$ to negative birefringence. In this section $\chi > 0$. Its value is important and allows us to make several statements about the existence of surface polaritons. If $\chi > 1$ and anisotropy is weak $\eta \approx 0$, then a surface polariton propagating perpendicular to the optic axis $\varphi = \pi/2$ exists. This is clear from the well-known dispersion law for SPP in isotropic media,

$$q_{\text{SPP}}^2 = \frac{\varepsilon_i \varepsilon_o}{\varepsilon_i + \varepsilon_o} = \frac{\varepsilon_o \chi}{\chi - 1}. \quad (16)$$

More details are given in Appendix D.

Let us look at the exact solution (6). The range of s is determined in this case from the condition that the radical expressions (8) and (9) are positive and finite. When solving inequalities, it turns out to be convenient to introduce special values

$$s_1 = \begin{cases} \sqrt{1 + \eta - \eta\chi}, & \text{if } 1 + \eta - \eta\chi > 0 \\ 0, & \text{otherwise} \end{cases}, \quad (17)$$

$$s_2 = \sqrt{1 + \eta}, \quad (18)$$

$$s_3 = \chi. \quad (19)$$

These values correspond to zeros or poles of any rational expression under radicals (6)–(10): $n_z \rightarrow 0$ as $s \rightarrow s_1 > 0$ (if $s = 0$ then $\kappa_e = 0$); $n_y \rightarrow 0$ as $s \rightarrow s_2$; and $P_4(s) \rightarrow 0$ as $s \rightarrow s_3$. Condition (17) may seem artificial, but it has a clear physical sense. The extraordinary wave becomes a TM wave for $\varphi = \pi/2$ and has only E_y, E_x, H_z components. It means that surface polariton wave vector q is determined by ε_o and ε_i only (16). But at the same time, for the existence of a surface wave, q_{SPP} must be greater than the wave vector of a homogeneous extraordinary wave $q_{\text{SPP}}^2 > \varepsilon_e$, which implies (17).

One may note that $s_1 < s_2$ if $\eta > 0$, and $s_1 > s_2$ if $\eta < 0$. The range of s differs slightly for these cases. Let us consider both cases in more detail.

Positive birefringent dielectric, $\eta > 0$.

$$s_1 < s < \min(s_2, s_3).$$

Using (17)–(19) we obtain a necessary and sufficient condition for the existence of a surface polariton

$$\chi > 1 \Leftrightarrow \varepsilon_o + \varepsilon_i < 0.$$

One can already notice several features. If $s_1 = 0$, then $\kappa_e = 0$ for some angle $\varphi(0) = \varphi_e$ and

$$\tan^2 \varphi_e = \frac{(1 + \eta)^3(1 + \chi)}{(1 + \eta + \chi)(\eta\chi - 1 - \eta)}. \quad (20)$$

It means that surface polariton is weakly localized in dielectric for directions $\varphi \approx \varphi_e$ which is noted in [84,85]. φ_e exists if η, χ fall into domain I + II in Fig. 2(a).

If $\min(s_2, s_3) = s_3$ and $\kappa_o, \kappa_e, \kappa_i, q \rightarrow \infty$ as $s \rightarrow s_3$. This limit corresponds to the quasistatic approximation, when $\kappa_o, \kappa_e, \kappa_i, q \gg k_0$, and we can neglect retardation. The corresponding angle φ_{QS} at which this occurs is determined by

$$\tan^2 \varphi_{QS} = \frac{1 + \eta - \chi^2}{\chi^2 - 1}. \quad (21)$$

Angle φ_{QS} [92] exists only if $1 < \chi < \sqrt{1 + \eta}$, corresponding to domain II + III in Fig. 2(a). The analytical expression (21) was first obtained in [4]. It should be understood that $q \rightarrow \infty$ as $\varphi \rightarrow \varphi_{QS}$ only in the absence of any losses ($\text{Im } \varepsilon_{o,e,i} = 0$). Therefore, in real materials $\text{Re } q$ is finite, but possibly large.

Negative birefringent dielectric, $\eta < 0$.

$$s_2 < s < \min(s_1, s_3).$$

The existence condition in this case includes part of the region $\chi < 1$ and is written as

$$\chi > \sqrt{1 + \eta}.$$

If $\chi < 1$ then angle φ_{QS} exists in the same way as in the case of $\eta > 0$. This domain is marked V in Fig. 2(a).

When $\eta \neq 1$ and $\chi > 1$ (domains I–IV) singular surface polariton (SSP) exists, which is propagating at an angle

$$\tan^2 \varphi_{SSP} = \frac{(2 + \eta)^2(1 + \chi)}{4(1 + \eta + \chi)(\chi - 1)}. \quad (22)$$

At this propagation angle $\kappa_e = \kappa_o$, and the polarization vectors of the ordinary and extraordinary wave are proportional to each other. This leads to the fact that the independent solutions of the Maxwell's equations are not $\mathbf{A}_o \exp(-\kappa_o x)$ and $\mathbf{A}_e \exp(-\kappa_e x)$, but $\mathbf{A}_1 \exp(-\kappa x)$ and $\mathbf{A}_2 x \exp(-\kappa x)$ where $\kappa = \kappa_o = \kappa_e$ [93]. This consideration for surface waves called SSPs were discussed in [22], but only in the case of isotropic dielectric. The case of isotropic metal and anisotropic dielectric was addressed in [94], and the authors called this the SPP-Voigt wave.

It should be noted that $\eta = 0$ and $s_1 = s_2 = 1$ for the case of isotropic media. Accordingly, the range of s degenerates into a single point. Having carefully calculated the limit in this case, we obtain the dispersion of an isotropic SPP (16).

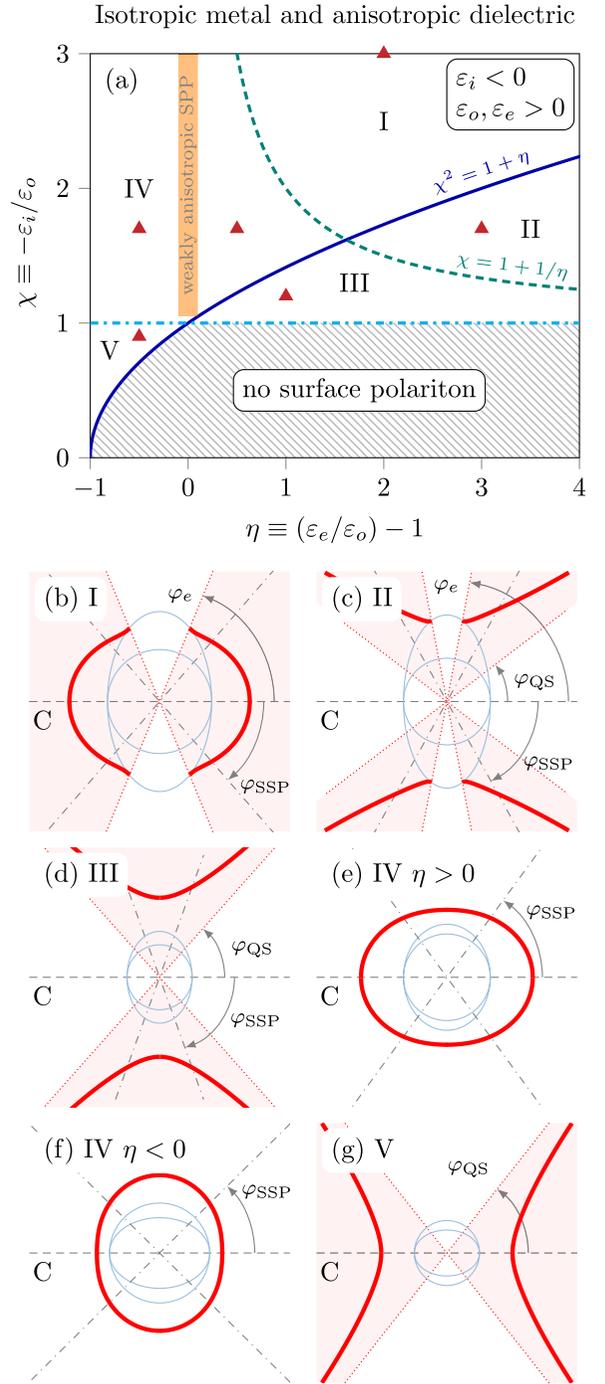


FIG. 2. Types of surface polaritons and its existence domains at the interface of a metal-like medium $\varepsilon_i < 0$ and an uniaxial dielectric $\varepsilon_o, \varepsilon_e > 0$. (a) Surface polariton existence domains in material parameter space. Vertical orange rectangle shows the domain corresponding to the usual surface plasmon polariton (SPP) in a weakly anisotropic medium. The shaded region is the domain of parameters in which there is no surface polariton at any propagation angle φ . (b)–(g) Isofrequency contours $q(\varphi)$ of a surface polariton for selected parameters (triangles) from different domains. The thick red curve is the isofrequency contour. The red shaded region is the angular existence domain. The optic axis C is horizontal. Thin blue lines are the light cones for ordinary and extraordinary waves. Angle φ_{SSP} (22) shows the angle at which singular polariton propagates. Angles φ_e and φ_{QS} are defined by (20) and (21).

All analysis results are shown in Fig. 2. It shows all domains of material parameters in which a surface polariton exists and examples of isofrequency contours $q(\varphi)$ for different material parameters. This should be understood as follows. η and χ change with frequency ω due to frequency dispersion, and accordingly the point (η, χ) in the plot [Fig. 2(a)] falls into different domains depending on ω . This leads to a change in the shape of the polariton isofrequency contour and AED. The combined existence condition can be written as

$$\chi > \min(1, \sqrt{1 + \eta}).$$

It is equivalent to the existence condition given in [84]. The case of isotropic media falls into domain IV. In this domain of parameters the contour is ellipse-like and is most similar to a circle as for an isotropic SPP. AED in IV is a full plane, $\varphi \in [0, 2\pi)$. In domain I the angle φ_e exists, and the surface polariton contour looks like the trimmed ellipse. If material parameters fall into domains II, III, and V, then the contour is hyperbola-like curve due to the existence of angle φ_{QS} . Depending on the type of isofrequency curve, we call surface polaritons *elliptic-like* or *hyperbolic-like*. It should be noted that only in domain II does the surface polariton not exist for highly symmetric directions, along the optic axis, and/or perpendicular to the optic axis.

2. Isotropic dielectric and anisotropic metal-like medium

There is not much variety of surface polaritons in this case. In the first works [2,14,16] existence conditions of a surface polariton were analyzed in the quasistatic approximation and without it for directions of high symmetry. The case of an optic axis perpendicular to the boundary is analyzed in [95]. SPP in anisotropic metals with anisotropic plasma frequency were studied in [96]. A detailed analysis of different types of surface polaritons in this case, and its AED, was carried out in [97] and in an unpublished paper [98]. It is also worth noting the work [99] in which similar results were obtained for a metal metasurface designed as diffraction grating.

It is possible to use previously defined η and χ in (14) and (15) to describe the domain of material parameters where surface polaritons exist. The surface polariton existence condition in this case [97] is

$$0 < \chi < \max(1, \sqrt{1 + \eta}).$$

The range of the parameter s again depends on the sign of the anisotropy strength η .

Positive anisotropy, $\eta > 0$:

$$\max(s_1, s_3) \leq s \leq s_2. \quad (23)$$

Negative anisotropy, $\eta < 0$:

$$\max(s_2, s_3) \leq s \leq s_1. \quad (24)$$

Limit values $s_{1,2,3}$ are the same as in (17)–(19). Similar to the previous case $s_1 = s_2$ in the limit of isotropic metal $\eta = 0$. Carefully calculating the polariton wave vector in this limit, we obtain an isotropic SPP (16).

All possible types of surface polaritons and material parameters domains are shown in Fig. 3. A surface polariton has *hyperbolic-like* isofrequency contour for parameters from domains I and III. The boundaries of AED are determined by the

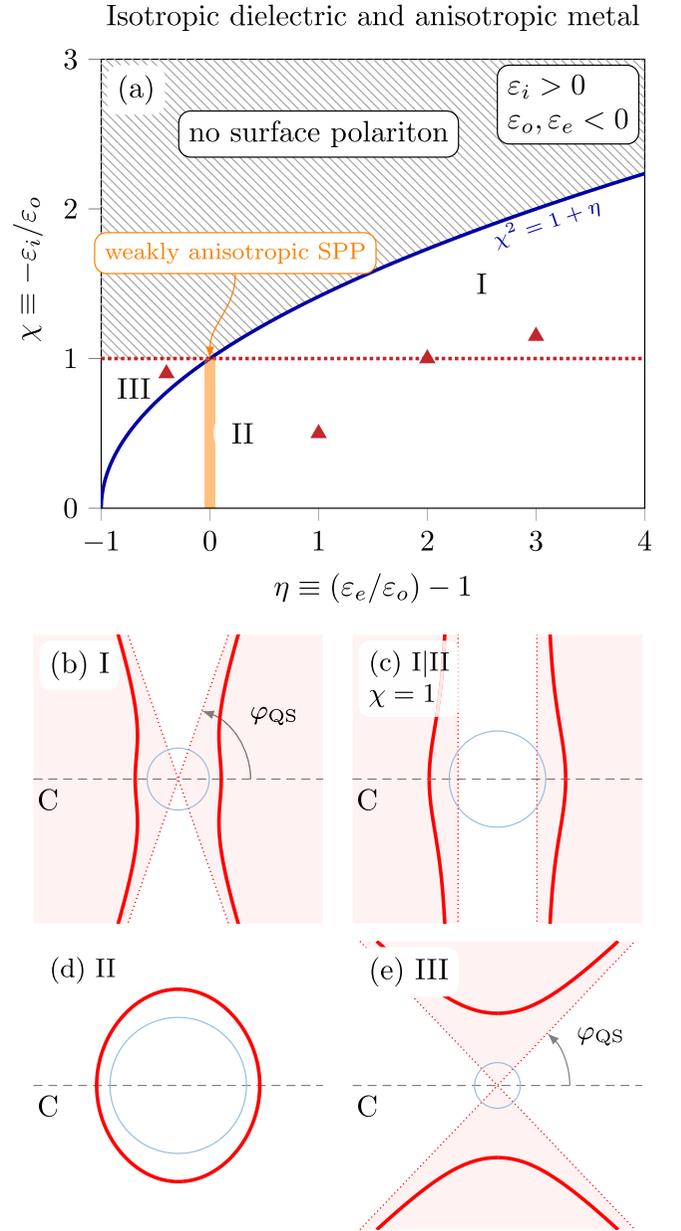


FIG. 3. Types of surface polaritons and its existence domains at the interface of an isotropic dielectric $\varepsilon_i > 0$ and an uniaxial metal-like medium $\varepsilon_o, \varepsilon_e < 0$. (a) Surface polariton existence domains in material parameter space. (b)–(e) Isofrequency contours $q(\varphi)$ of a surface polariton for selected parameters (triangles) from different domains. Thin blue line is the light cone in isotropic medium. Other notations used are the same as in Fig. 2. Angle φ_{QS} is defined by (21).

same angle φ_{QS} (21). It can also be noted that the isofrequency contour has additional inflection points in domain I. In domain II the surface polariton is *elliptic-like*.

B. Positive permittivity materials: DSWS

The next group of materials is the *positive permittivity group*. A surface polariton, known as DSWS, at the interface of two media with positive components of dielectric exists only if $0 < \varepsilon_o < \varepsilon_i < \varepsilon_e$ [1]. That is, an uniaxial medium has to

be positive with an additional condition to isotropic dielectric permittivity ε_i . In our notation the DSW existence condition is written as

$$-1 - \eta < \chi < -1.$$

This can be seen as an extension of the case (Sec. III A 1) to the region $\chi < 0$. A complete analytical solution to this case is discussed in [79]. Let us repeat it briefly for completeness. The parameter s varies within

$$0 < s < s_4,$$

where s_4 is determined from $\kappa_i = 0$ and equals

$$s_4 = \frac{\chi + \sqrt{(2 + \chi)^2 + 4\eta}}{2}. \quad (25)$$

Let φ_i be the angle at which $\kappa_i = 0$. AED is determined by the angles $\varphi_i < \varphi < \varphi_e$ and the mirrored domains relative to optic axis and normal to it. Angle φ_e is defined above (20), and φ_i in our notation

$$\sin^2 \varphi_i = \frac{(-1 - \chi)[2 + \chi + \sqrt{(2 + \chi)^2 + 4\eta}]}{2\eta}. \quad (26)$$

Let us recall that before DSWs, the possibility of SSP existence was mentioned in [22]. It is now often called the Dyakonov-Voigt surface wave (DVSW). The domain of material parameters for which it exists lies within the domain of DSWs $-\eta/2 < \chi < -1$. This was also noted in [27]. Figure 4 shows the existence domain of DSWs (I and II) and DVSW (II).

Unlike surface polaritons discussed above, DSWs are weakly localized at the interface at any propagation angle. This can be understood by the proximity of the isofrequency contour [Figs. 4(b) and 4(c)] to the light cones. The field distributions fall away from the interface $\propto \exp(-|x|/x_{\text{DSW}})$, where x_{DSW} is the characteristic localization length. The estimated value $x_{\text{DSW}} \approx 2\sqrt{2}\eta^{-3/2}\lambda_i$ in the limit of weak anisotropy $\eta \ll 1$, where λ_i is the wavelength in isotropic medium λ_i . For example, $\eta \approx 0.8$ in strongly anisotropic Hg_2Cl_2 for visible wavelengths and the corresponding length $x_{\text{DSW}} \approx 10\lambda_i$. In more common natural anisotropic minerals $x_{\text{DSW}} \approx 10\text{--}100\lambda_i$.

C. Hyperbolic medium

In the last *hyperbolic group* we include cases in which the uniaxial medium has different signs of the dielectric tensor components. An indefinite medium or hyperbolic medium [38,100], as we now call them, has been known for a long time in the infrared. The contribution of optical phonons to the dielectric permittivity of a uniaxial crystal can be highly anisotropic, leading to hyperbolic dispersion of electromagnetic waves in the bulk. Hyperbolic dispersion means that the light cone of an extraordinary wave is a one-sheet or two-sheet hyperboloid, instead of an ellipsoid. Therefore, researchers were also interested in the dispersion of surface polaritons in such a hyperbolic medium. The analysis of surface polaritons was first done in the quasistatic approximation in [4,16], and then revisited much later [101] in structured hyperbolic metamaterials. For a long time, the exact solution was known only for directions of high symmetry [14], and in the form of a

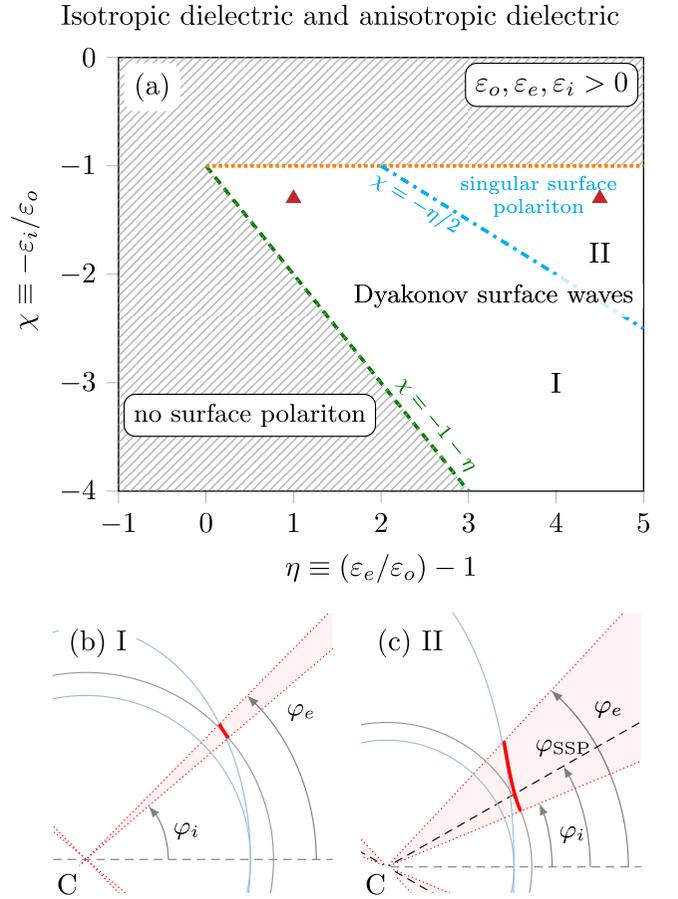


FIG. 4. Dyakonov surface waves (DSWs) and singular surface polariton existence domains at the interface of an isotropic $\varepsilon_i > 0$ and an uniaxial dielectric $\varepsilon_o, \varepsilon_e > 0$. (a) Its existence domains in material parameter space. The shaded region is the domain of parameters in which there is no surface polariton at any propagation angle φ . (b)–(c) Isofrequency contours $q(\varphi)$ of DSW for selected parameters (triangles) for a “weakly” and strongly anisotropic medium. Thin blue lines are the light cones for ordinary, extraordinary, TE, and TM waves. Angle φ_{SSP} (22) shows the angle at which a singular polariton propagates. Angles φ_e and φ_i are defined by (20) and (26). Other notations used are the same as in Fig. 2.

singular surface polariton [22]. Researchers’ interest in surface polaritons in already known hyperbolic media returned after a long time. In [31] it was proposed to exploit hyperbolic metamaterials for extending AED of DSWs. The resulting surface polariton is called a Dyakonov plasmon because it “combines” the properties of both SPPs and DSWs.

It is known that there are two types of hyperbolic media [100]. In a Type I hyperbolic medium two components of dielectric tensor are positive $\varepsilon_o > 0$, and one is negative $\varepsilon_e < 0$. In a Type II hyperbolic medium two components are negative $\varepsilon_o < 0$, one is positive $\varepsilon_e > 0$. Let us consider these cases separately.

1. Type I hyperbolic medium

To the best of author’s knowledge, this case is the least well described in the literature and may not have been studied experimentally. Since $\varepsilon_o > 0$, the existence of a surface polariton

at the boundary is possible not only with an isotropic dielectric (e.g., air), but also with a metal-like medium. In our notation $\chi < 0$ corresponds to the case of a dielectric and $\chi < 0$ to a “metal.”

The latter case means, in a sense, the coupling of SPP in a metal and a surface phonon polariton (SPhP) in a hyperbolic medium. It should be distinguished from the case of SPP-SPhP coupling in the same medium [62,102]. The first description of the surface polariton at the interface of a metal and a hyperbolic medium is given in the quasistatic approximation [11,12]. Much later and independently, the possibility of the existence of the surface waves in a planar anisotropic waveguide with perfectly conducting walls was noted in [103]. It is possible that hybrid SPPs in graphene-coupled anisotropic van der Waals material [104] refers specifically to the case under discussion in the limit of a thin metal layer.

The case of a boundary with a dielectric was first considered in [22] (free surface LiTaO₃), in the form of a singular surface polariton propagating at fixed angle φ_{SSP} . A more detailed analysis for an arbitrary propagation angles was carried out in [32,40,41] by numerically solving the dispersion equation.

The only necessary and sufficient condition for surface polariton existence in this case is $\varepsilon_i < \varepsilon_o$ or in our notation

$$\chi > -1. \quad (27)$$

The range of acceptable values s in the exact solution (6) in the case under consideration is not as simple as the previous ones. To describe the range of possible values of s in this case, another special value is needed:

$$s_5 = -1 - \eta. \quad (28)$$

First, it should be noted that there is an exact solution that has a very simple form. If it turns out that $\varepsilon_i = \varepsilon_e$, and therefore $\chi = -1 - \eta$, then dispersion equation (2) can be simplified to the form

$$\varepsilon_i k_o + \varepsilon_o k_e = 0. \quad (29)$$

The solution to it is

$$n_y^2 \frac{\eta + 2}{\eta + 1} + n_z^2 = \varepsilon_o. \quad (30)$$

The shape of isofrequency contour is an ellipse if $\eta < -2$, and a hyperbola otherwise. In terms of s the range of possible values degenerates to a point $s = s_1 = s_3 = s_4 = s_5$. This is similar to the isotropic case $\varepsilon_o = \varepsilon_e$. It can also be noted that $P_2(s)$ and $P_4(s)$ have a common divisor when $\varepsilon_i = \varepsilon_e$.

In all other cases, the range of s is not degenerated. For $\chi > -1 - \eta$ it is defined by

$$s_5 < s < \min(s_1, s_3),$$

and in the region $\chi < -1 - \eta$ by

$$\max(s_3, s_1) < s < s_5.$$

There are also many types of isofrequency contours of surface polariton (Fig. 5), as in the case of a metal and an anisotropic dielectric (Sec. III A 1). They all have one feature in common. The propagation of polariton along the optic axis

Isotropic medium and Type I hyperbolic medium

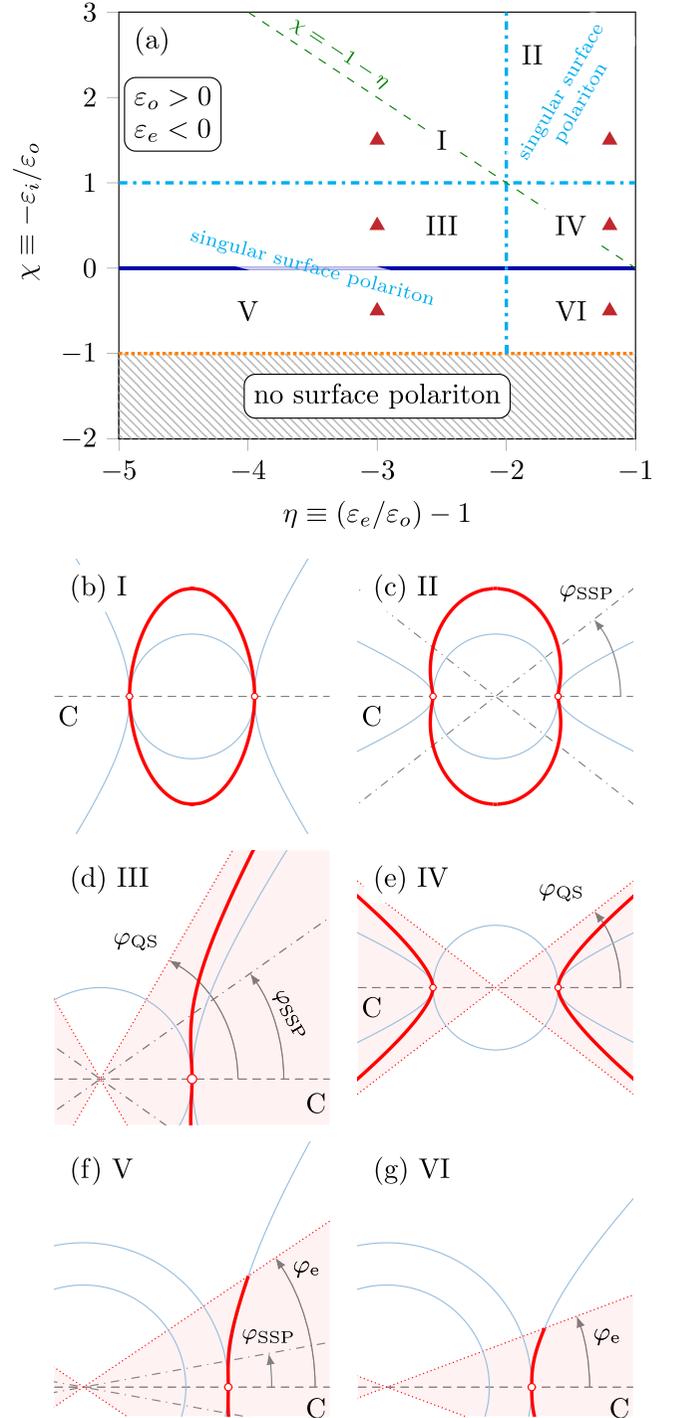


FIG. 5. Types of surface polaritons and their existence domains at the interface of an isotropic medium ε_i and a Type I hyperbolic medium $\varepsilon_o > 0$, $\varepsilon_e < 0$. (a) Surface polariton and singular surface polariton existence domains in material parameter space. (b)–(g) Isofrequency contours $q(\varphi)$ of surface polariton for selected parameters (triangles) from different domains. White point at the contour shows that surface polariton cannot propagate along the optic axis. Thin blue lines are the light cones for ordinary, extraordinary, TE, and TM, waves. Angle φ_{SSP} (22) shows the angle at which singular polariton propagates. Angles φ_e and φ_{QS} are defined by (20) and (21). Other notations used are the same as in Fig. 2.

is prohibited, but at an any small angle $\varphi \neq 0$ it is already possible.

As far as the author knows, only the case of the boundary between a dielectric and a hyperbolic medium has been considered previously. In the notation used, this means $\chi < 0$. These are domains V and VI in Fig. 5(a). This case corresponds, for example, to a surface polariton at the free boundary ($\varepsilon_i = 1$) of hyperbolic medium. The AED of a surface polariton in this case is limited by angle φ_e (20) [32,40]. Before these works, the existence of a SSP in domain V was predicted [22]. It propagates at angle φ_{SSP} (22). The isofrequency contour shape is more like an arc than an ellipse or a hyperbola. This is reminiscent of DSWs. The polariton in this case is also weakly localized. This can be understood by the proximity of the isofrequency contour [Figs. 5(f) and 5(g)] to the light cone of extraordinary waves.

The region $\chi > 0$ (domains I–IV) corresponds to the boundary with an isotropic metal-like medium and is analyzed in this paper without the quasistatic approximation. In domains I and II isofrequency contours (Fig. 5) of the surface polariton is similar to an ellipse, and we call it *elliptic-like*. AED is the entire plane except $\varphi = 0$ and $\varphi = \pi$. In domains III and IV the surface polariton is *hyperbolic-like*, as can be seen from the shape of the isofrequency contour [Figs. 5(d) and 5(e)]. AED in this case is limited by the angle φ_{QS} (21). The angle φ_{QS} was obtained in the quasistatic approximation in [11]. Compared to the previous case, the polariton is well localized at the boundary for any propagation direction not near the optic axis. Also in domains II and III there is SSP at angle φ_{SSP} (22). It is worth recalling that there is a simple shape (ellipse or hyperbola) of the isofrequency contour (30) in I and IV if $\varepsilon_i = \varepsilon_e$, or $\chi = -1 - \eta$ in our terms.

2. Type II hyperbolic medium

In a Type II hyperbolic medium $\varepsilon_o < 0$ and $\varepsilon_e > 0$. As mentioned at the beginning, there is no surface polariton if $\varepsilon_i < 0$ and $\varepsilon_o < 0$. Therefore, we need to consider only the case $\varepsilon_i > 0$ corresponding to $\chi > 0$. The properties of surface polaritons in this case are known and well described in the literature [31,35,40,41].

The necessary condition for existence in this case is quite simple, $\varepsilon_i < |\varepsilon_o|$, which is

$$0 < \chi < 1$$

in our notation. The range of possible values s is defined by

$$\min(s_1, s_3) < s < \max(s_1, s_3), \quad (31)$$

and it always includes the bound $s_1 \neq 0$. If $s_1 = s_3$, the range degenerates to a point and the solution is given by Eq. (30), the same as in the case of a Type I hyperbolic medium. The corresponding relation $\chi = -1 - \eta$ is shown in Fig. 6(a). Since $-2 < \eta < -1$, the shape of an isofrequency contour is a hyperbola.

There are two domains of parameters in which the isofrequency contour has a different shape [Fig. 6(a)]. They are separated by the relation $\varepsilon_i = |\varepsilon_o|\varepsilon_e/(|\varepsilon_o| + \varepsilon_e)$, which is written as $\chi = 1 + 1/\eta$ in the notation used. AED of surface polariton in domain I [Fig. 6(b)] is limited only by the angle φ_{QS} (21). But in domain II there is no surface polariton

Isotropic medium and Type II hyperbolic medium

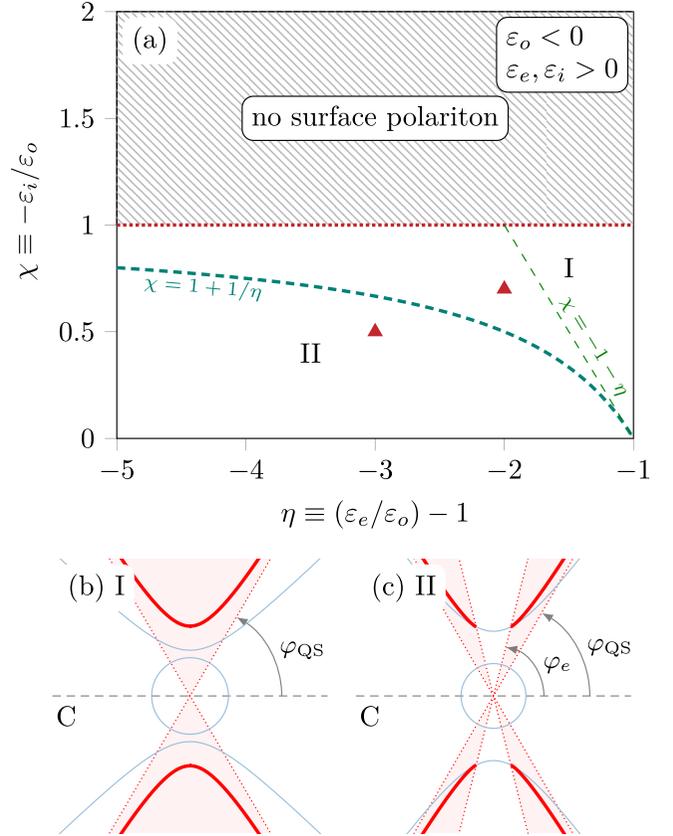


FIG. 6. Types of surface polaritons and their existence domains at the interface of an isotropic dielectric $\varepsilon_i > 0$ and a Type II hyperbolic medium $\varepsilon_o < 0$, $\varepsilon_e > 0$. (a) Surface polariton existence domain in material parameter space. (b)–(c) Isofrequency contours $q(\varphi)$ of surface polaritons for selected parameters (triangles) from different domains. Thin blue lines are the light cones for extraordinary, TE, and TM waves. Angles φ_e and φ_{QS} are defined by (20) and (21). Other notations used are the same as in Fig. 2.

propagating perpendicular to the optic axis $\phi = \pi/2$, and AED is more narrow $\varphi_{QS} < \varphi < \varphi_e$ [Fig. 6(c)]. The isofrequency contour of a polariton starts from the light cone of extraordinary waves at the angle φ_e (20). In both cases surface polariton is called *hyperbolic-like*.

D. Summary

The results of the analysis are summarized in Table I. Each line of the table shows the main features of the surface polariton, based on its isofrequency contour shape and the conditions for its existence. The first three columns are self-explanatory. The fourth column describes what curve an isofrequency contour resembles. “0–hyperbola” means that the contour looks like a hyperbola with foci on the optic axis, similar to a “ $\pi/2$ –hyperbola,” but the foci are on the perpendicular to the optic axis. A finite or infinite arc means whether the wave vector q of the surface polariton is limited or not at any propagation angle φ . The parameter s range corresponds to the permissible values of s in the algebraic solution (6)–(11). The AED column describes the range of

TABLE I. List of surface polaritons isofrequency contour shapes and its basic properties.

Sign			Contour shape	s range in Eqs. (6)–(11)	AED ^a	Existence condition	Refs. ^b	SSP ^c
ε_i	ε_o	ε_e						
–	–	±	—	—	—	—	—	—
–	+	+	“Ellipse”	$[s_1, s_2] \cup [s_2, s_1]^d$	$\left[0, \frac{\pi}{2}\right]$	$\varepsilon_e < \varepsilon_o < \varepsilon_i $ or $\sqrt{\varepsilon_o \varepsilon_e} < \varepsilon_i < \frac{\varepsilon_e}{\eta};^e$	[83–86]	yes [94]
			“Elliptic” arc	$(0, s_2]$	$[0, \varphi_e)$	$\max\left(\frac{\varepsilon_e}{\eta}, \sqrt{\varepsilon_o \varepsilon_e}\right) < \varepsilon_i $	[83–86]	yes [94]
			“0–hyperbola”	$[s_2, s_3)$	$[0, \varphi_{QS})$	$\sqrt{\varepsilon_o \varepsilon_e} < \varepsilon_i < \varepsilon_o$	[84]	no [94]
			“ $\pi/2$ –hyperbola”	$[s_1, s_3)$	$\left(\varphi_{QS}, \frac{\pi}{2}\right]$	$\varepsilon_o < \varepsilon_i < \min\left(\frac{\varepsilon_e}{\eta}, \sqrt{\varepsilon_o \varepsilon_e}\right)$	[84]	yes [94]
			“ $\pi/2$ –hyperbolic” Infinite arc	$(0, s_3)$	$(\varphi_{QS}, \varphi_e)$	$\frac{\varepsilon_e}{\eta} < \varepsilon_i < \sqrt{\varepsilon_o \varepsilon_e}$	[84]	yes [94]
+	–	–	“ellipse”	$[s_1, s_2] \cup [s_2, s_1]^d$	$\left[0, \frac{\pi}{2}\right]$	$\varepsilon_i < \min(\varepsilon_o , \sqrt{ \varepsilon_o \varepsilon_e })$	This work	no
			“0–hyperbola”	$(s_3, s_2]$	$[0, \varphi_{QS})$	$ \varepsilon_o < \varepsilon_i < \sqrt{ \varepsilon_o \varepsilon_e }$	[97]	no
			“ $\pi/2$ –hyperbola”	$(s_3, s_1]$	$\left(\varphi_{QS}, \frac{\pi}{2}\right]$	$\sqrt{ \varepsilon_o \varepsilon_e } < \varepsilon_i < \varepsilon_o $	[97]	no
+	+	+	arc	$(0, s_4)$	(φ_i, φ_e)	$\varepsilon_o < \varepsilon_i < \varepsilon_e$	[1,79]	yes [22], if $\varepsilon_e > 3\varepsilon_o$
–	+	–	“Ellipse”	$[s_1, s_5] \cup (s_5, s_1]^f$	$\left(0, \frac{\pi}{2}\right]$	$0 < \varepsilon_o < \varepsilon_i $	This work	yes, if $\varepsilon_o > \varepsilon_e $
			“0–hyperbola”	$(s_3, s_5) \cup (s_5, s_3)^f$	$(0, \varphi_{QS})$	$ \varepsilon_i < \varepsilon_o$	This work	yes, if $\varepsilon_o < \varepsilon_e $
+	+	–	“0–hyperbolic” finite arc	$(0, s_5)$	$(0, \varphi_e)$	$0 < \varepsilon_i < \varepsilon_o$	[32,40]	yes [22], if $\varepsilon_o < \varepsilon_e $
+	–	+	“ $\pi/2$ –hyperbola”	$[s_1, s_3) \cup (s_3, s_1]^f$	$\left(\varphi_{QS}, \frac{\pi}{2}\right]$	$\frac{\varepsilon_e}{ \eta } < \varepsilon_i < \varepsilon_o $	[31,40,41]	no
			“ $\pi/2$ –hyperbolic” infinite arc	$(0, s_3)$	$(\varphi_{QS}, \varphi_e)$	$0 < \varepsilon_i < \frac{\varepsilon_e}{ \eta }$	[31,40,41]	no

$\eta = \varepsilon_e/\varepsilon_o - 1$; s_1, s_2, s_3, s_4, s_5 are defined by Eqs. (17)–(19), (25), (28).

^aAngular existence domain (quadrant I only).

^bIn all references below the dispersion equation (2) of surface polariton is solved numerically. An analytical solution in particular cases is given in [1,79,85,86].

^cSingular surface polariton.

^d $s = s_1 = s_2$ corresponds to the isotropic case $\varepsilon_o = \varepsilon_e$.

^eThe last inequality can be read as $\varepsilon_e < \varepsilon_o|\varepsilon_i|(|\varepsilon_i| - \varepsilon_o)^{-1}$, simply meaning that the polariton wave vector is greater than the greatest possible wave vector for an extraordinary wave.

^f $s = s_1 = s_5 = s_3$ corresponds to the case $\varepsilon_e = \varepsilon_i$ (30).

angles φ relative to the optic axis along which surface polariton propagation is possible. The existence condition column describes restrictions on dielectric permittivities of the media under which the corresponding isofrequency contour shape is implemented. The last column indicates whether the singular surface polariton [22,27,94,106] exists and an additional condition if it does. Recall that the angle at which it propagates is equal to φ_{SSP} (22).

Table II shows examples of pairs of materials in which the surface polaritons discussed in this work were experimentally studied. It mentions only a part of the works listed in the introduction, which seemed the most interesting for further

research. The fourth column indicates which the parameter domain (η, χ) that the pair of materials from the fifth and sixth columns falls into. The fifth and sixth columns indicate the name of the material and the range of dielectric permittivity (or dielectric tensor in principal axis) for the wavelength range in which they were studied. As can be seen, experimental methods make it possible to study surface polaritons in a wide range of wavelengths. However, it should be noted that in many works only the most symmetrical directions were considered to simplify further analysis. More examples of potential materials could be found in the reviews [29,56–58] and in the collection [105].

TABLE II. List of experimental works, media, and wavelengths by type of surface polaritons.

Sign			(η, χ) domain	"Isotropic" medium, ε_i	Anisotropic medium, $\hat{\varepsilon}^a$	Wavelength	Ref.
ε_i	ε_o	ε_e					
+	+	+	Slightly biaxial, I in Fig. 4(a)	Index-matching liquid 3.15–3.20	KTiOPO ₄ (3.10, 3.14, 3.48)	632.8 nm	[75]
			I in Fig. 4(a)	polycarbonate 2.49	NLC ^b 5CB (2.38, 2.38, 2.85)	632.8 nm	[76]
			I in Fig. 4(a)	MgF ₂ (1.89, 1.89, 1.93)	CSTF ^c ZnSe ?	633 nm	[77]
–	+	+	II in Fig. 2(a)	Au –11.7 + <i>i</i> 0.73	SiO ₂ film 2.15 nm and NLC ^b K15 (2.30, 2.30, 2.95)	632.8 nm	[60]
			II in Fig. 2(a)	Al –46.40 + <i>i</i> 16.88	CaCO ₃ (2.21, 2.21, 2.75)	632.8 nm	[61]
			IV in Fig. 2(a)	Ag $\approx -8.66 + i0.61^d$	<i>p</i> -6P ^e nanowires $\approx (4.6, 4.6, 6.6)$ – (3.2, 3.2, 3.9)	495–680 nm	[63]
+	–	–	Not applicable, biaxial	Air 1	Nanostructured Au film (–1.3, –2.5, –3.9)– (–5.0, –7.9, –11.4) ^f	540–680 nm	[64]
			I, II in 3(a)	Air	Ag grating (depends on ω)	530–700 nm	[68]
+	–	+	I, II in Fig. 6(a)	Air	deep-trench AZO ^g (–1, –1, 1.5)– (–10, –10, 1.5) ^f	4–14 μ m	[34,70]
			II in Fig. 6(a)	Air	CaCO ₃ (–3.7, –3.7, 2.3) ^f	6.8–7 μ m	[55]
+	+	–	Slightly biaxial, V in Fig. 5(a)	Air	Nanostructured hBN ^h (2.1, 3.7, –15.2 + <i>i</i> 0.6)	7–7.1 μ m	[71]
+	\pm	\pm	Figs. 3(a), 5(a)	Air	Doped hexagonal ZnO (depends on ω) ⁱ	16–25 μ m	[62]
			Figs. 3(a), 5(a), 6(a)	Air	Al ₂ O ₃ ^j	13–28.5 μ m	[53]
			Figs. 3(a), 5(a), 6(a)	Air	TiO ₂ , MgF ₂ ^j	16–55 μ m	[50]
			Figs. 3(a), 5(a), 6(a)	Air	α -SiO ₂ ^j	8.3–28.5 μ m	[8,52]
			Figs. 3(a), 5(a), 6(a)	Air	α -LiIO ₃ ^j	11.7–13.3 μ m	[10]

^aIn principal axes. If two are specified, then this is the range of change in the wavelength range.

^bNematic liquid crystal.

^cChiral sculptured thin film.

^dDiffers from data in other works [105].

^e*para*-sexiphenyl.

^fWithout imaginary part.

^gAl-doped ZnO.

^hHexagonal boron nitride.

ⁱThe authors assumed that the following three terms make the main contribution: static high-frequency ε_∞ , optical phonons $(\omega_L^2 - \omega_T^2)/(\omega_T^2 - \omega^2 - i\omega\gamma_{ph})$, free carriers $\omega_p^2/(\omega^2 - i\gamma_p\omega)$. All terms are assumed to be uniaxially anisotropic.

^jMultiple Lorentz oscillators model.

IV. CONCLUSION

For a long time it was believed that dispersion equations for surface polaritons in an anisotropic medium can only be solved numerically, with the exception of weakly anisotropic media and highly symmetric propagation directions. The paper [79] was the first to show the existence of an exact solution using a computer algebra system, but the analysis was done only for Dyakonov surface waves. In this work we extend this result to all types of surface polaritons propagating at the

boundary of an isotropic and uniaxial medium in the case of the optic axis parallel to the boundary. The complete solution in algebraic form (6)–(11) is written for arbitrary ratios between the dielectric permittivities of media, neglecting losses. In particular, the obtained solution may be useful in analyzing the properties of surface polaritons in metamaterials, where the weak anisotropy approximation is not applicable. Some cases are considered in this work that have not been discussed in detail previously in the literature. The existence of surface

polaritons at the interface of a metal-like medium and a Type I hyperbolic medium is predicted. ‘‘Elliptic’’ surface polaritons at the interface of an anisotropic metal-like medium and an isotropic dielectric are also described in detail; only ‘‘hyperbolic’’ ones were considered in the literature.

The dependence of the wave vector of the surface polariton on the propagation angle relative to the optic axis has been analyzed. As a result, possible shapes of the isofrequency contour have been obtained depending on the relations between the dielectric permittivities. In general, a surface polariton is shown to be one of three types, depending on the shape of the contour: *elliptic-like*, *hyperbolic-like*, and *arc-like* (Dyakonov surface waves). The contour shape changes with frequency if there is a frequency dispersion of the permittivity of media. The results obtained can be used in the analysis of any types of surface polaritons, regardless of their origin (phonons, plasmons, etc.), at the boundary of a uniaxial and isotropic medium, described only by the dielectric permittivity tensor.

ACKNOWLEDGMENTS

The work was supported by the Theoretical Physics and Mathematics Advancement Foundation ‘‘BASIS’’ and by the ‘‘Clover’’ photonics initiative between Skoltech, Moscow Institute of Physics and Technology, and ITMO University. The author thanks N. S. Averkiev and N. A. Gippius for fruitful discussions.

APPENDIX A: FIELD COMPONENTS AND DISPERSION EQUATION

Following the statement of the problem we are looking for a wave propagating along the boundary between two media. If we exclude purely electric waves ($\mathbf{H} = 0$), then in any anisotropic linear medium there are two eigenwaves that differ in polarization. Typically, in media without spatial dispersion, longitudinal purely electric waves exist only at certain frequencies. We do not consider these cases, assuming that there are no longitudinal waves at frequency ω . Thus, the general solution is the sum of two eigenwaves in each medium. In an anisotropic medium we have to choose an ordinary and an extraordinary wave with the exception of directions called singular axes [22,93,106,107]. Let us choose an eigenvector for the ordinary wave as

$$\mathbf{E}_o = \begin{pmatrix} q_y \\ i\kappa_o \\ 0 \end{pmatrix}, \quad \mathbf{H}_o = k_0^{-1} \begin{pmatrix} -i\kappa_o q_z \\ q_y q_z \\ -(\varepsilon_o k_0^2 - q_z^2) \end{pmatrix}, \quad (\text{A1})$$

and for the extraordinary wave as

$$\mathbf{E}_e = \begin{pmatrix} -i\kappa_e q_z \\ q_y q_z \\ -(\varepsilon_o k_0^2 - q_z^2) \end{pmatrix}, \quad \mathbf{H}_e = \varepsilon_o k_0 \begin{pmatrix} -q_y \\ -i\kappa_e \\ 0 \end{pmatrix}, \quad (\text{A2})$$

where (q_y, q_z) is the in-plane wave vector, κ_o and κ_e are decrements of field decay from the boundary, and $k_0 = \omega/c$ is the wave vector in vacuum. The expressions are obtained from solving the Fresnel equation system [108].

The form of eigenvectors (A1) and (A2) is especially useful. If we put $\varepsilon_o = \varepsilon_e = \varepsilon_i$ as in an isotropic medium, they still

form a complete basis, like TE and TM waves. The transition from one basis to another is given as $\mathbf{E}_{\text{TE}} \propto (-i\kappa_i q_z \mathbf{E}_o - q_y \mathbf{E}_e)$ and $\mathbf{H}_{\text{TM}} \propto (-i\kappa_i q_z \varepsilon_i^{-1} \mathbf{H}_e + q_y k_0^2 \mathbf{H}_o)$, and the other components can be derived from Maxwell’s equations. The relations between wave vector components are given by (3)–(5).

Using (A1) and (A2), the field distributions for a surface wave are written in an anisotropic medium $x < 0$ as

$$\begin{aligned} \mathbf{E}_{x<0} &= (a_1 \mathbf{E}_o e^{\kappa_o x} + a_2 \mathbf{E}_e e^{\kappa_e x}) e^{iq_y y + iq_z z - i\omega t}, \\ \mathbf{H}_{x<0} &= (a_1 \mathbf{H}_o e^{\kappa_o x} + a_2 \mathbf{H}_e e^{\kappa_e x}) e^{iq_y y + iq_z z - i\omega t}, \end{aligned}$$

and in an isotropic medium $x > 0$ as

$$\begin{aligned} \mathbf{E}_{x>0} &= (b_1 \mathbf{E}_o^{(i)} + b_2 \mathbf{E}_e^{(i)}) e^{-\kappa_i x} e^{iq_y y + iq_z z - i\omega t}, \\ \mathbf{H}_{x>0} &= (b_1 \mathbf{H}_o^{(i)} + b_2 \mathbf{H}_e^{(i)}) e^{-\kappa_i x} e^{iq_y y + iq_z z - i\omega t}, \end{aligned}$$

where the superscript (i) means that we put $\varepsilon_o = \varepsilon_e = \varepsilon_i$ in (A1) and (A2).

In the singular case we have $q_z^2 = \varepsilon_o k_0^2$ and accordingly $\kappa_o^2 = \kappa_e^2 = q_y^2$ from (4) and (5). Where it immediately follows $\mathbf{E}_e \propto \mathbf{E}_o$ and $\mathbf{H}_e \propto \mathbf{H}_o$ means that the chosen basis is not complete. In this case we write the general solution [22,106], correctly calculating \mathbf{H} , as

$$\begin{aligned} \mathbf{E}_{x<0}^{(\text{SSP})} &= \begin{pmatrix} q_y \\ iq_y \\ 0 \end{pmatrix} (a_1 + a_2 x) e^{q_y x} e^{iq_y y + iq_z z - i\omega t}, \\ \mathbf{H}_{x<0}^{(\text{SSP})} &= \begin{pmatrix} -iq_y q_z (a_1 + a_2 x) \\ q_y q_z (a_1 + a_2 x) \\ a_2 \end{pmatrix} e^{q_y x} e^{iq_y y + iq_z z - i\omega t}. \end{aligned}$$

Even though the singular solution has a different form, this does not affect the dispersion equation. The exact solution for singular surface polariton is given in [22,109].

Now using the boundary conditions of continuity of the tangential components $E_{y,z}$ and $H_{y,z}$, we obtain the linear system

$$\underbrace{\begin{pmatrix} i\kappa_o & q_y q_z & i\kappa_i & -q_y q_z \\ q_y q_z & -i\kappa_e \varepsilon_o k_0^2 & -q_y q_z & -i\kappa_i \varepsilon_i k_0^2 \\ 0 & g_o & 0 & -g_i \\ g_o & 0 & -g_i & 0 \end{pmatrix}}_M \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = 0, \quad (\text{A3})$$

where $g_o = q_z^2 - \varepsilon_o k_0^2$ and $g_i = q_z^2 - \varepsilon_i k_0^2$. A nontrivial solution exists if $\det M = 0$. The matrix M can be viewed as a block matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with 2×2 blocks. It is clear that $CD = DC$. Then $\det M$ can be easily calculated as

$$\begin{aligned} \det M &= \det(AD - BC) \\ &= \det \begin{pmatrix} q_y q_z (g_o - g_i) & -i(\kappa_i g_o + \kappa_o g_i) \\ ik_0^2 (\varepsilon_i \kappa_i g_o + \varepsilon_o \kappa_e g_i) & q_y q_z (g_o - g_i) \end{pmatrix}. \end{aligned}$$

Calculating the last determinant yields the dispersion equation (2). The ratios of the contributions of different

polarizations can be calculated from (A3), putting, for example, $a_2 = 1$.

APPENDIX B: DISPERSION EQUATION SIMPLIFICATION

The dispersion equation (2) has two remarkable properties that will be useful. It is easy to see that substituting $n_z^2 = \varepsilon_i$ turns it into an identity. However, such a substitution does not help to obtain a solution for n_y . The chosen polarizations of waves (A1)–(A2) in the isotropic medium are parallel in this case. The wave with $E_z \neq 0$ is lost. Despite this, the simplified dispersion equation obtained below allows us to obtain n_y and accordingly the surface polariton wave vector q_s , although it will not be possible to determine all field components. Similarly, if we put $n_z^2 = \varepsilon_o$ then (2) becomes $\kappa_i \kappa_e = n_y^2$. We can prove that this is an identity if both $\kappa_i, \kappa_e > 0$.

Let us explicitly take out $(n_z^2 - \varepsilon_i)(n_z^2 - \varepsilon_o)$ in (2). Let $f_1 = n_z^2 - \varepsilon_i$ and $f_2 = n_z^2 - \varepsilon_o$. Then (2) becomes

$$(\kappa_i f_2 + \kappa_o f_1)(\kappa_i \varepsilon_i f_2 + \kappa_e \varepsilon_o f_1) - n_y^2 n_z^2 (\varepsilon_i - \varepsilon_o)^2 = 0. \quad (\text{B1})$$

Let us transform the last term:

$$\begin{aligned} n_y^2 n_z^2 (\varepsilon_i - \varepsilon_o)^2 &= n_y^2 n_z^2 (f_2 - f_1)^2 \\ &= n_y^2 [(f_1 + \varepsilon_i)(f_2^2 - f_1 f_2) \\ &\quad + (f_2 + \varepsilon_o)(-f_1 f_2 + f_1^2)] \\ &= -n_y^2 (\varepsilon_i + \varepsilon_o) f_1 f_2 + n_y^2 \varepsilon_i f_2^2 + n_y^2 \varepsilon_o f_1^2. \end{aligned}$$

Combining terms $\propto f_1^2$ and $\propto f_2^2$ with the first term in Eq. (B1) yields

$$\begin{aligned} (\kappa_i^2 - n_y^2) \varepsilon_i f_2^2 + \varepsilon_o (\kappa_o \kappa_e - n_y^2) f_1^2 + \mathcal{A} f_1 f_2 &= 0, \\ \mathcal{A} = \kappa_i (\varepsilon_i \kappa_o + \varepsilon_o \kappa_e) + n_y^2 (\varepsilon_i + \varepsilon_o). \end{aligned}$$

It follows from (3) that

$$(\kappa_i^2 - n_y^2) \varepsilon_i = \varepsilon_i f_1.$$

Combining (4) and (5) yields

$$\varepsilon_o (\kappa_o \kappa_e - n_y^2) = \kappa_o \varepsilon_o (\kappa_e - \kappa_o) + \varepsilon_o f_2 = \frac{\varepsilon_e - \varepsilon_o}{\kappa_e + \kappa_o} \kappa_o f_2 + \varepsilon_o f_2.$$

Using the obtained relations, Eq. (B1) can be rewritten as

$$\begin{aligned} \left(\varepsilon_i [\kappa_i \kappa_o + n_y^2 + f_2] + \varepsilon_o [\kappa_i \kappa_e + n_y^2 + f_1] \right. \\ \left. + \frac{\varepsilon_e - \varepsilon_o}{\kappa_e + \kappa_o} \kappa_o f_1 \right) f_1 f_2 = 0. \end{aligned} \quad (\text{B2})$$

The first square bracket is $\kappa_i \kappa_o + n_y^2 + f_2 = \kappa_o (\kappa_i + \kappa_o)$. The second square bracket is $\kappa_i \kappa_e + n_y^2 + f_1 = \kappa_i (\kappa_e + \kappa_i)$. It follows from (4) and (5) that $\varepsilon_o \kappa_e^2 - \varepsilon_e \kappa_o^2 = (\varepsilon_o - \varepsilon_e) n_y^2$. Using $f_1 = \kappa_i^2 - n_y^2$ and rearranging the remaining terms we obtain

$$\begin{aligned} \varepsilon_o \kappa_i (\kappa_e + \kappa_i) (\kappa_e + \kappa_o) + (\varepsilon_e - \varepsilon_o) \kappa_o \kappa_i^2 + \kappa_o (\varepsilon_o \kappa_e^2 - \varepsilon_e \kappa_o^2) \\ = [\varepsilon_o \kappa_e (\kappa_e + \kappa_i) - \varepsilon_e \kappa_o (\kappa_o - \kappa_i)] (\kappa_o + \kappa_i). \end{aligned} \quad (\text{B3})$$

After collecting all pieces we obtain that Eq. (B1) is equivalent to

$$\left[(\varepsilon_i \kappa_o + \varepsilon_o \kappa_e) (\kappa_e + \kappa_i) + (\varepsilon_i - \varepsilon_e) \kappa_o (\kappa_o - \kappa_i) \right] \frac{\kappa_i + \kappa_o}{\kappa_e + \kappa_o} f_1 f_2 = 0. \quad (\text{B4})$$

Equality of the expression in parentheses to zero leads to solutions corresponding to the surface polaritons. The rest respond to non-physical solutions. If we multiply the expression in square brackets by $(\kappa_o + \kappa_i)$, then the dispersion equation takes form [Eq. (9) in 1]

$$(\kappa_i + \kappa_e) (\kappa_i + \kappa_o) (\varepsilon_i \kappa_o + \varepsilon_o \kappa_e) = (\varepsilon_e - \varepsilon_i) (\varepsilon_i - \varepsilon_o) \kappa_o. \quad (\text{B5})$$

On the other hand, if we leave n_y^2 and substitute $\varepsilon_o \kappa_e^2$ in terms of κ_o^2 , then Eq. (B3) has the form [Eq. (5) in 26]

$$\varepsilon_i \kappa_o^2 + \kappa_e (\varepsilon_i \kappa_o + \varepsilon_o \kappa_i) + \varepsilon_e \kappa_i \kappa_o = (\varepsilon_e - \varepsilon_o) n_y^2. \quad (\text{B6})$$

APPENDIX C: ALGEBRAIC SOLUTION DERIVATION

Let us start from dispersion equation (B4) considering that $f_1 \neq 0$ and $f_2 \neq 0$:

$$(\varepsilon_i \kappa_o + \varepsilon_o \kappa_e) (\kappa_e + \kappa_i) + (\varepsilon_i - \varepsilon_e) \kappa_o (\kappa_o - \kappa_i) = 0. \quad (\text{C1})$$

One can notice that (C1) is a quadratic form of variable $\kappa_i, \kappa_o, \kappa_e$, and it has only linear κ_i terms. We can isolate κ_i terms

$$\kappa_i (\varepsilon_e \kappa_o + \varepsilon_o \kappa_e) = \varepsilon_i \kappa_o (\kappa_o + \kappa_e) + (\varepsilon_o \kappa_e^2 - \varepsilon_e \kappa_o^2).$$

If we want to square it, then both sides must have the same sign. In the set of equation for squares then additional roots may appear. On the other hand, we can express κ_i^2 through κ_o^2 . The resulting equation includes only the second and fourth powers of κ_o and κ_e which are combined into two homogeneous polynomials $P_2(\kappa_o, \kappa_e)$ and $P_4(\kappa_o, \kappa_e)$ of degrees 2 and 4, respectively [79],

$$P_2(\kappa_o, \kappa_e) + P_4(\kappa_o, \kappa_e) = 0, \quad (\text{C2})$$

where

$$\begin{aligned} P_2(\kappa_o, \kappa_e) &= (\varepsilon_i - \varepsilon_o) (\varepsilon_e \kappa_o + \varepsilon_o \kappa_e)^2, \\ P_4(\kappa_o, \kappa_e) &= (\kappa_o + \kappa_e) (\varepsilon_i \kappa_o + \varepsilon_o \kappa_e) \\ &\quad \times [(\varepsilon_i - \varepsilon_o) \kappa_o \kappa_e + (\varepsilon_i - 2\varepsilon_e) \kappa_o^2 + \varepsilon_o \kappa_e^2]. \end{aligned}$$

By definition, a polynomial $P_n(x, y)$ is called a homogeneous polynomial of the degree n if $P_n(\lambda x, \lambda y) = \lambda^n P_n(x, y)$ for any λ .

Now, Eq. (C2) can be easily solved for κ_e or κ_o , e.g., by introducing the parameter $s = \kappa_e / \kappa_o$. In our case it should be $s > 0$ because $\kappa_e, \kappa_o > 0$ for a surface polariton. Substituting κ_e in (C2) and canceling by κ_o^2 , we obtain

$$P_2(1, s) + \kappa_o^2 P_4(1, s) = 0.$$

Let us put $P_2(s) \equiv P_2(1, s)$ and $P_4(s) \equiv P_4(1, s)$. Solving the equation for κ_o we obtain (6). Introducing an inverse relation $\bar{s} = \kappa_o / \kappa_e = s^{-1}$ makes more sense in some cases. If $s \rightarrow 0$, then $\kappa_e \rightarrow 0$ too, regardless of κ_o and κ_i . This means that the surface polariton is becoming less localized due to the long decay length for the extraordinary component $\propto 1/\kappa_e$.

On the other hand, if $\bar{s} \rightarrow 0$ then $\kappa_o \rightarrow 0$, and polariton is becoming less localized due to the increasing decay length for the ordinary component $\propto 1/\kappa_o$.

APPENDIX D: DIRECTIONS OF HIGH SYMMETRY

For highly symmetric directions, along and perpendicular to the optic axis, dispersion equation (2) is significantly simplified. This is explained by the fact that along these directions the ordinary and extraordinary waves are not mixed by the boundary conditions. If a surface polariton propagates along the optic axis, then $n_y = 0$ and accordingly $\kappa_e^2 = (\varepsilon_e/\varepsilon_o)\kappa_o^2$ from (4) and (5). Thus, the second bracket in dispersion equation (2) should vanish. It brings us to the equation

$$\frac{\kappa_e}{\varepsilon_e} + \frac{\kappa_i}{\varepsilon_i} = 0, \quad (\text{D1})$$

which is similar to the well-known SPP dispersion equation. The solution is

$$n_z^2 = \frac{\varepsilon_i \varepsilon_o (\varepsilon_i - \varepsilon_e)}{\varepsilon_i^2 - \varepsilon_o \varepsilon_e} = \frac{\varepsilon_o \chi (1 + \eta + \chi)}{\chi^2 - 1 - \eta} \quad (\text{D2})$$

and exists in two cases: $\varepsilon_i < 0$ and $\varepsilon_o, \varepsilon_e > 0$ and $\varepsilon_i > 0$ and $\varepsilon_o, \varepsilon_e < 0$ [14,17,18]. Also, we need to require n_z to be real. If $\varepsilon_o > 0$ then the additional condition is $\chi > \sqrt{1 + \eta}$ in our notation. This corresponds to domains I, IV, and V in Fig. 2(a). In the case $\varepsilon_o < 0$ the condition $0 < \chi < \sqrt{1 + \eta}$ should be satisfied. The corresponding domains are I and II [Fig. 3(a)]. It can be seen from (A2) and (A3) that only the ‘‘extraordinary’’ wave in an anisotropic medium and TM wave in an isotropic medium remain.

It may seem from the right side of Eq. (D2) that the solution is also possible for $\varepsilon_e < 0$ and $\varepsilon_o > 0$. This is incorrect. Any deviation from the optic axis, no matter how small, leads to the mixing of an ordinary wave. The wave vector q of the surface polariton cannot change significantly. The attenuation decrement κ_o of an ordinary wave [Eq. (4)] is imaginary for such q . It means that the ordinary wave component of a polariton is not localized at the interface. Thus, this solution does not correspond to a well-defined surface polariton. Such a type of solution is called a virtual surface polariton in [17,18].

If the direction of propagation is perpendicular to the optic axis, then $n_z = 0$. The only way to satisfy Eq. (2) is to make the first bracket equal to zero. This yields a dispersion equation similar to (D1)

$$\frac{\kappa_o}{\varepsilon_o} + \frac{\kappa_i}{\varepsilon_i} = 0,$$

and its solution is

$$n_y^2 = \frac{\varepsilon_o \varepsilon_i}{\varepsilon_o + \varepsilon_i}.$$

It is the well-known solution for SPP wave vector at the boundary of two isotropic media with dielectric permittivities ε_o and ε_i . To the usual SPP existence conditions $\varepsilon_o + \varepsilon_i < 0$ and $\varepsilon_o \varepsilon_i < 0$, we need to add $n_y^2 > \varepsilon_e$ for the same reason as in the previous case. In our notation they can be expressed as follows:

$$\varepsilon_i < 0 : \begin{cases} \chi > 1 & \text{if } \eta \leq 0, \\ 1 < \chi < 1 + \eta^{-1} & \text{if } \eta > 0, \end{cases}$$

$$\varepsilon_i > 0 : \begin{cases} \chi < 1 & \text{if } \eta \geq -1, \\ 1 + \eta^{-1} < \chi < 1 & \text{if } \eta < -1. \end{cases}$$

The corresponding domains for the first case are III and IV in Fig. 2(a) and I and II in Fig. 5(a). For the second case we have II and III in Fig. 3(a) and I in Fig. 6(a).

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