Interplay between short-range and critical long-range fluctuations in the out-of-equilibrium behavior of the particle density at quantum transitions

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We address the equilibrium and out-of-equilibrium behavior of the particle density in many-body systems undergoing quantum transitions driven by the chemical potential μ. They originate from a nontrivial interplay between noncritical short-range and critical long-range quantum fluctuations. As a paradigmatic model, we consider the one-dimensional fermionic Kitaev model, for which very accurate numerical studies can be performed, up to $O(10^4)$ chain sites. The search for dynamic scaling behaviors of the particle density is complicated by the fact that its equilibrium (ground-state) behavior is dominated by short-range fluctuations, giving rise to regular background terms and peculiar logarithmic terms from resonances between renormalization-group perturbations associated with the *energy* and *identity* operator families within the conformal field theory. To study these issues, we focus on two dynamic protocols, either instantaneous quenches or quasiadiabatic changes of μ, to the critical value μ_c, unveiling out-of-equilibrium scaling behaviors of the particle density, which arise from the critical modes, within a dynamic finite-size scaling framework.

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I. INTRODUCTION

At continuous quantum transitions in the zero-temperature limit [1,2], the behavior of some interesting observables displays a nontrivial interplay of contributions from (noncritical) short-range and (critical) long-range quantum fluctuations, which can be hard to disentangle. This is the case of the particle density at quantum transitions driven by the chemical potential, whose critical scaling behavior gets hidden by contributions coming from the regular (short-range) term of the free-energy density [3]. This phenomenon may be considered as the quantum counterpart of the analogous interplay between short-range and critical fluctuations at thermal classical transitions, where the energy density at the transition point is dominated by a regular term arising from short-range fluctuations (or mixing with the identity operator), while the critical scaling terms are only subleading [4–9]. Therefore, in equilibrium conditions, the energy density in classical transitions and the particle density at quantum transitions (driven by the chemical potential) generally display nonuniversal leading behaviors at the transition point, while critical scaling behaviors are relatively suppressed. For this reason, they are not considered as optimal observables to probe the universal features at criticality.

However, despite this equilibrium behavior, some numerical analyses of out-of-equilibrium behaviors of three-dimensional N-vector models at classical (thermal) transitions have provided evidence of a peculiar out-of-equilibrium scaling of the energy density [10]. This has been observed along the critical relaxational flow arising from instantaneous quenches of the temperature to the critical point, under a purely relaxational dynamics that leads to the asymptotic large-time thermalization [11,12], starting from equilibrium conditions. The out-of-equilibrium finite-size scaling (FSS) behavior of the energy density (after subtracting its asymptotic value at the critical point) can be expressed in terms of a rescaled time variable $\Theta = t/L^z$ (where t is the time from the quench, z is the dynamic exponent associated with the relaxational dynamics, and L is the system size), analogously to other observables that exhibit scaling at equilibrium. This may be related to the fact that the dynamics of short-range fluctuations is characterized by a significantly shorter timescale than the diverging timescale of critical modes. However, the absence of a scaling behavior at equilibrium, thus at the starting point of the quenching protocol, leaves a noticeable imprinting in the dynamic scaling behavior as a function of Θ, since scaling functions show a peculiar power-law singularity in the $\Theta \to 0$ limit, and in particular a power-law divergence when the specific-heat exponent is negative [10].

In this paper, we investigate whether analogous phenomena emerge in many-body systems at quantum transitions driven by the chemical potential, focusing on the out-of-equilibrium behavior of the particle density arising from instantaneous and slow quenches of the chemical-potential parameter μ to the critical point. Although the classical-to-quantum mapping does not apply, since its applicability is restricted to equilibrium conditions, out-of-equilibrium scaling behaviors similar to those at thermal transitions have also been observed at quantum transitions for standard observables showing asymptotic equilibrium scaling behaviors, such as the correlation function of the order-parameter operator (see, e.g., Refs. [2,13–35]). Again, like classical transitions, the extension to observables dominated by regular terms (mixing with the identity operator) at equilibrium is not straightforward, because of the nontrivial competition between contributions arising from noncritical short-range and critical long-range fluctuations. Moreover, quantum scenarios may substantially differ from those observed at classical transitions under relaxational dynamics, essentially because the quantum dynamics of isolated systems is qualitatively different, being unitary and energy-conserving. Thus, further interesting features may emerge, with respect to those observed along critical relaxational flows at classical transitions, always characterized by an asymptotic large-time thermalization.

To study this issue, we focus on the one-dimensional Kitaev chain [36] as a paradigmatic model for quantum transitions driven by the chemical potential. Its equilibrium particle density at the quantum critical point is dominated by a standard regular term [2,3] and also logarithmic terms arising from peculiar resonances of the renormalization-group (RG) weights of RG operators [3,6], belonging to the *energy* and *identity* conformal families within a conformal-field-theory (CFT) framework (see, e.g., Ref. [37]). We recall that this resonance phenomenon is responsible for the logarithmic divergence of the specific heat at the classical thermal transitions belonging to the two-dimensional (2D) Ising universality class [6]. We consider two relatively simple dynamic protocols: (i) quantum quenches, where μ is instantaneously changed to the critical-point value μ_c, and (ii) quasiadiabatic variations of the time-dependent parameter $\mu(t)$ from $\mu \neq \mu_c$ to μ_c, starting in both cases from equilibrium (ground-state) conditions. We are able to uncover, and characterize, the emergence of out-of-equilibrium scaling behaviors of the particle density arising from the critical modes, within a dynamic FSS framework.

Our numerical study shows that the particle density (after subtracting its large-volume critical-point value) develops a dynamic scaling behavior along the postquench quantum evolution, similar to that observed along critical relaxational flows at classical transitions. The corresponding scaling functions are characterized by a logarithmic divergence in the $\Theta \to 0$ limit, where $\Theta \sim t/\tau$ is the rescaled time with respect to the timescale of the critical modes [$\tau \sim \Delta_c(L)^{-1} \sim L^z$, where $\Delta_c(L)$ is the gap at the critical point and $z = 1$ is the dynamic exponent], somehow reflecting the fact that the subtracted particle density does not have an asymptotic equilibrium scaling behavior. Moreover, the scaling functions show some further logarithmic divergence at finite Θ, related to revival quantum phenomena. We also show that an out-of-equilibrium FSS behavior emerges along protocols entailing slow quasiadiabatic changes of the Hamiltonian parameters, once subtracting the corresponding equilibrium particle density at the instantaneous value of $\mu(t)$.

The paper is organized as follows. In Sec. II we present the fermionic Kitaev wire, providing a paradigmatic model that undergoes a quantum transition driven by the chemical potential. In Sec. III we discuss the equilibrium behavior of the particle density at the quantum transition, which does not show a universal asymptotic scaling. In Sec. IV we outline the dynamic quench protocol and discuss the out-of-equilibrium behavior within a dynamic FSS framework; moreover, we present numerical results showing that the particle density develops an out-of-equilibrium FSS along the postquench quantum evolution, with peculiar singularities of its FSS functions. We also derive the corresponding out-of-equilibrium scaling behavior in the thermodynamic limit. In Sec. V we extend our analysis to quasiadiabatic dynamic protocols entailing slow changes of the chemical potential. Finally, in Sec. VI we summarize and draw our conclusions.

II. THE FERMIONIC KITAEV CHAIN

As a paradigmatic many-body system undergoing a continuous quantum transition driven by the chemical potential, we consider a fermionic Kitaev wire of size L (number of sites), whose quantum unitary dynamics is driven by the Hamiltonian [36]

$$\hat{H} = -J \sum_x (\hat{c}_x^\dagger \hat{c}_{x+1} + \gamma \hat{c}_x^\dagger \hat{c}_{x+1}^\dagger + \text{H.c.}) - \mu \hat{N}, \tag{1}$$

where $\hat{c}_x$ is the fermionic annihilation operator associated with the xth site of the chain ($x = 1, \ldots, L$), while $\hat{N}$ is the particle-number operator:

$$\hat{N} = \sum_x \hat{n}_x, \quad \hat{n}_x = \hat{c}_x^\dagger \hat{c}_x. \tag{2}$$

The Hamiltonian parameter μ denotes the chemical potential, while $\gamma > 0$ controls the relative strength of the terms that do not conserve the fermionic number. In the following, we fix the energy scale by assuming $J = 1$ and also set the Planck and Boltzmann constants $\hbar = k_B = 1$.

The Hamiltonian (1) can be straightforwardly diagonalized into [38–41]

$$\hat{H} = \sum_k E(k) \left(\hat{a}_k^\dagger \hat{a}_k - \frac{1}{2} \right), \tag{3}$$

where $\hat{a}_k$ are new fermionic annihilation operators, which are obtained through a suitable linear transformation of the original $\hat{c}_x$ operators, and

$$E(k) = 2\sqrt{(\mu/2)^2 + \gamma^2 + \mu \cos k + (1 - \gamma^2) \cos^2 k} \tag{4}$$

is their dispersion relation. In a finite system, the set of k values to be summed over, as well as the allowed states, depend on the boundary conditions.

By means of the Jordan-Wigner transformation (see, e.g., Ref. [1]), the Hamiltonian (1) can also be mapped into a so-called quantum XY chain:

$$\hat{H}_{XY} = -\sum_x \left[\frac{1+\gamma}{2} \hat{\sigma}_x^{(1)} \hat{\sigma}_{x+1}^{(1)} + \frac{1-\gamma}{2} \hat{\sigma}_x^{(2)} \hat{\sigma}_{x+1}^{(2)} + g \hat{\sigma}_x^{(3)} \right], \tag{5}$$

where $\hat{\sigma}_x^{(k)}$ are the spin-1/2 Pauli matrices ($k = 1, 2, 3$). In particular, $\hat{\sigma}_x^{(3)} = 1 - 2\hat{n}_x$, thus $g = -\mu/2$. However, although the bulk behaviors of the two models in the infinite-volume limit (and thus their phase diagram) are analogous, some finite-size features may differ significantly. For example, the nonlocal Jordan-Wigner transformation of the XY chain (5) with periodic and antiperiodic boundary conditions does not map into the fermionic model (1) with analogous boundary conditions. Indeed further considerations apply, leading to a less straightforward correspondence, which also depends on the parity of the particle-number eigenvalue (see, e.g., Refs. [3,39,40]). Therefore, the Kitaev quantum wire and the quantum XY chain cannot be considered as completely equivalent. However, they both undergo a continuous quantum transition, respectively, at $\mu = \mu_c = -2$ or at $g = g_c = 1$, independently of the parameter γ. To simplify the notation, we define the deviation of the relevant parameter μ from its critical value as

$$w \equiv \frac{\mu_c - \mu}{2} = g - g_c \quad (\mu_c = -2,\ g_c = 1). \tag{6}$$

In the following, we consider the fermionic Kitaev chains with antiperiodic boundary conditions [28], i.e., with $\hat{c}_x = -\hat{c}_{L+x}$, which simplify computations of the equilibrium and out-of-equilibrium FSS behaviors, restricting the momenta of the sum in Eq. (3) to

$$k = \left\{\pm\frac{\pi}{L}(2n+1)\right\}, \quad n = 0, 1, \dots, L/2 - 1. \tag{7}$$

When considering antiperiodic boundary conditions, both phases separated by the quantum transition at μ_c are gapped, i.e., the degeneracy of the vacua for $\mu < \mu_c$ in the thermodynamic limit (corresponding to the ordered phase of quantum XY chains) is not realized. The reason for such a substantial difference resides in the fact that the corresponding Hilbert space is restricted with respect to that of the XY chain, so that it is not possible to restore the competition between the two vacua belonging to the symmetric/antisymmetric sectors of the quantum XY chain [2,3,36,39].

III. EQUILIBRIUM BEHAVIOR OF THE PARTICLE DENSITY

A. Quantum criticality

The continuous transition at μ_c belongs to the 2D Ising universality class [1,2], characterized by the length-scale critical exponent $\nu = 1$, related to the RG dimension $y_w = 1/\nu = 1$ of the Hamiltonian parameter w. This implies that, approaching the critical point $w \to 0$ at zero temperature, the lengthscale ξ of the critical quantum fluctuations diverges as $\xi \sim |w|^{-\nu}$. The temperature T represents another relevant RG perturbation at quantum transitions. At the critical point $w = 0$, the lengthscale increases as $\xi \sim T^{-1/z}$ with decreasing T, where z is dynamic exponent $z = 1$ associated with the unitary quantum dynamics within this universality class (it also determines the power law $\Delta \sim \xi^{-z}$ of the vanishing gap with increasing ξ). Moreover, we mention that the RG dimension of the fermionic operators $\hat{c}_x$ and $\hat{c}_x^\dagger$ at the continuous quantum transition is $y_c = 1/2$, and that of the particle density operator $\hat{n}_x$ is [1,2]

$$y_n = d + z - y_w = 2 - y_w = 1. \tag{8}$$

The universal critical exponents enter the asymptotic power laws of the quantum critical behavior as a function of the temperature T and the chemical potential μ. However, the asymptotic critical expansions associated with the 2D Ising universality class are also characterized by the presence of logarithmic terms [6,8,37,42–49]. They arise from a resonance between the *identity* operator of RG dimension 2 and the *energy* operator of RG dimension 1 within the corresponding 2D conformal field theory (CFT) with central charge $c = 1/2$ [37]. In particular, such a resonance mechanism is responsible for the leading logarithmic divergence of the specific heat at the 2D Ising critical point.

Analogous logarithmic terms are found at the quantum critical point of the quantum XY chain, equivalent to the fermionic Kitaev wire in the thermodynamic limit or for open boundary conditions. In particular, they appear in the free-energy density

$$F(w, T, \gamma) = -\frac{T}{L}\ln[\mathrm{Tr}\, e^{-\beta\hat{H}}], \quad \beta = 1/T. \tag{9}$$

In the thermodynamic limit, i.e., when $L/\xi \to \infty$, it can be written as [39]

$$F(w, T, \gamma) = -\int_0^\pi \frac{dk}{2\pi}\,\{E(k) + 2T\ln[1 + e^{-\beta E(k)}]\}, \tag{10}$$

where $E(k)$ is reported in Eq. (4). At the leading order in the critical limit, it behaves as [3]

$$F(w, T, \gamma) \approx F_{\mathrm{reg}}(w, \gamma) + \frac{A}{4\pi}u_w^2 \ln u_w^2 - \frac{2A}{\pi}u_t^2 f(u_w/u_t), \tag{11}$$

where $F_{\mathrm{reg}}(w, \gamma)$ is a regular function at the critical point, u_w and u_t are the scaling fields corresponding to the relevant parameters w and T, which are given by

$$u_w = \frac{w}{A}, \quad u_t = \frac{T}{2A}, \quad A \equiv \sqrt{\gamma^2 + w}, \tag{12}$$

and $f(x)$ is a universal scaling function (apart from a trivial factor and a normalization of the argument) given by

$$f(x) = \int_0^\infty dz\, \ln\left(1 + e^{-\sqrt{x^2+z^2}}\right). \tag{13}$$

Note that the first regular term of the expansion (11) is independent of T, as generally expected at quantum transitions [2,3], and it can be expanded in powers of w:

$$F_{\mathrm{reg}}(w, \gamma) = b_0(\gamma) + b_1(\gamma)\, w + \cdots. \tag{14}$$

B. The particle density in the thermodynamic limit

The thermodynamic-limit behavior of the equilibrium particle density ϱ_e can be derived by differentiating the free-energy density with respect to the chemical potential,

$$\begin{aligned}\varrho_e &\equiv L^{-1}\mathrm{Tr}\,[\rho_G \hat{N}] = -\partial_\mu F = \frac{1}{2}\partial_w F \\ &\approx \varrho_{\mathrm{reg}}(w, \gamma) + \frac{B}{2\pi}\left(u_w \ln u_w^2 + u_w\right) - \frac{B}{\pi}u_t\, g(u_w/u_t),\end{aligned} \tag{15}$$

where ρ_G is the density matrix associated with the Gibbs distribution,

$$\rho_G = \frac{e^{-\beta\hat{H}}}{\mathrm{Tr}[e^{-\beta\hat{H}}]}, \tag{16}$$

and

$$B = A\,\partial_w u_w = 1 - w/(2A^2). \tag{17}$$

In Eq. (15) we only kept the most relevant terms, and $g(x)$ is another scaling function that can be easily derived from $f(x)$, cf. Eq. (13), i.e., $g(x) = \partial_x f(x)$. In the critical limit

$u_w, u_t \to 0$, keeping the ratio $u_w^{z\nu}/u_t = u_w/u_t$ fixed, the particle density is dominated by the contribution of the regular term, which can be expanded as

$$\varrho_{\rm reg}(w,\gamma) = a_0(\gamma) + a_1(\gamma)\, w + \cdots, \tag{18}$$

where a_i are nonuniversal constants depending on γ. In particular, $a_0 = 1/2 - 1/\pi$ for $\gamma = 1$.

Actually, the regular background generally provides the leading contribution to the free-energy density, and its derivative with respect to the even relevant Hamiltonian parameter [2], analogous to μ in quantum transitions driven by the chemical potential. In some cases, one obtains an asymptotic scaling behavior in the FSS limit by subtracting its critical-point value, in particular when the dynamic and lengthscale critical exponents are such that $d + z - 2/\nu < 0$ as it occurs in the 2D quantum Ising model [2]. However, in the case of the fermionic Kitaev wire, the particle-density deviation $D_e(w, T, \gamma)$ from its critical-point value $\varrho_c(\gamma)$,

$$D_e(w,T,\gamma) \equiv \varrho_e(w,T,\gamma) - \varrho_c(\gamma), \tag{19}$$

where

$$\varrho_c(\gamma) \equiv \varrho_e(0,0,\gamma) = a_0(\gamma), \tag{20}$$

turns out to be dominated by the logarithmic term arising from the resonance between identity and energy operators, hiding the universal scaling behavior [2,3]

$$\varrho_{\rm scal} \sim T^{y_n/z} g(u_w/u_t) \tag{21}$$

[$y_n = 1$ is the RG dimension of the particle-density operator $\hat{n}_x$; see Eq. (8)], which remains logarithmically suppressed with respect to the leading term.

C. Finite-size behavior of the particle density

The above scaling behaviors can be straightforwardly extended to finite-size systems, within a FSS framework (see, e.g., Refs. [2,3,7,9,37,50–52]). Zero-temperature FSS *Ansätze* can be obtained by introducing the lattice size L, rewriting Eq. (15) in terms of the scaling variable

$$\Phi = u_w(w,\gamma) L^{y_w}, \quad y_w = 1/\nu = 1, \tag{22}$$

and taking the zero-temperature limit $T \sim u_t \to 0$. Therefore, keeping only the most relevant terms, the equilibrium particle density in finite-size systems is expected to behave as

$$\varrho_e(w,\gamma,L) = \varrho_{\rm reg}(w,\gamma) + c_l\, u_w \ln L + c_s L^{-y_n} \mathcal{D}_e(\Phi), \tag{23}$$

where $\varrho_{\rm reg}(w,\gamma)$ is the same regular function appearing in the infinite-volume scaling behavior [2,3], cf. Eq. (15), and $\mathcal{D}_e$ is a universal scaling function (apart from a trivial multiplicative factor and normalization of the argument). Only the last term provides the genuine universal scaling contribution of the critical modes (of course it depends on the boundary conditions), while the other terms are not universal. An analogous behavior is found at classical transitions of 2D Ising systems defined on finite-size square lattices, when considering the finite-size behavior of the energy density (i.e., the derivative of the free-energy density with respect to the temperature) [37].

We now focus on the subtracted particle density

$$D_e(w,\gamma,L) \equiv \varrho_e(w,\gamma,L) - \varrho_c(\gamma), \tag{24}$$

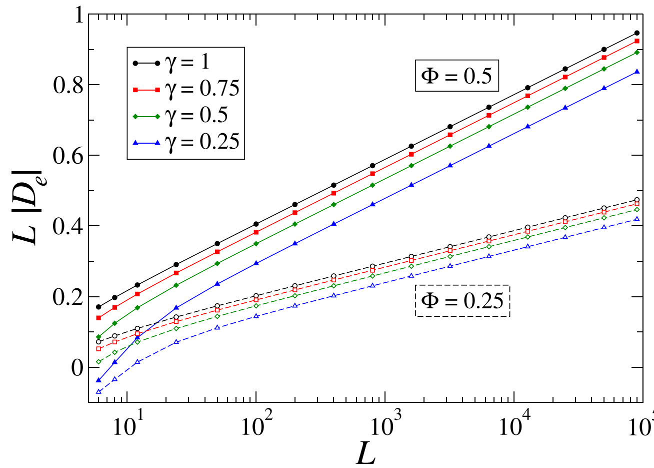

FIG. 1. The absolute value of the equilibrium subtracted particle density D_e as a function of the system size L, keeping the scaling variable Φ fixed. Filled symbols denote numerical data for $\Phi = 0.5$, while empty ones are for $\Phi = 0.25$. The various colors and symbols are for different values of the Hamiltonian parameter γ (see the legend). Lines connecting symbols are drawn to guide the eye. Note the logarithmic scale on the x-axis, so the observed straight-line behavior witnesses the expected large-L behavior $LD_e \sim \Phi \ln L$.

with $\varrho_c(\gamma)$ defined in Eq. (20), whose finite-size behavior can be easily derived using Eqs. (23) and (18). Note that in the FSS limit, i.e., the simultaneous limits $L \to \infty$ and $w \to 0$ keeping $\Phi = u_w L^{y_w}$ fixed, the logarithmic term provides the leading contribution. Therefore, when keeping Φ fixed, the finite-size behavior (23) predicts the large-L behavior

$$L\, D_e(w,\gamma,L) \sim \Phi \ln L. \tag{25}$$

This is clearly confirmed by the numerical results shown in Fig. 1, for various choices of Φ and γ.

On the other hand, a standard asymptotic FSS is expected for the fermionic correlation functions, such as

$$G_P(x_1,x_2) = {\rm Tr}\big[\rho_G\,(\hat{c}_{x_1}^\dagger \hat{c}_{x_2}^\dagger + \hat{c}_{x_2}\hat{c}_{x_1})\big], \tag{26a}$$

$$G_C(x_1,x_2) = {\rm Tr}\big[\rho_G\,(\hat{c}_{x_1}^\dagger \hat{c}_{x_2} + \hat{c}_{x_2}^\dagger\hat{c}_{x_1})\big]. \tag{26b}$$

Indeed, assuming translational invariance, thus $G_\#(x) \equiv G_\#(x_1, x_1 + x)$, and in the zero-temperature limit, their equilibrium FSS is given by [2]

$$G_\#(x,w,L) \approx L^{-2y_c} \mathcal{G}_\#(X,\Phi), \quad X \equiv x/L \neq 0 \tag{27}$$

($y_c = 1/2$ is the RG dimension of the operator $\hat{c}_x$).

In passing, it is worth mentioning that one may also consider higher derivatives of the free-energy density with respect to the chemical potential. However, their connection with the correlation functions of the particle-density operator $\hat{n}_x$ is not straightforward. In fact, due to the nontrivial derivative of the Gibbs exponential operator [53]

$$\frac{d}{d\mu} e^{-\beta\hat{H}} = -\beta \int_0^1 e^{-z\beta\hat{H}} \hat{N} e^{-(1-z)\beta\hat{H}} dz, \tag{28}$$

the derivative $\partial\varrho/\partial\mu$ is not related to the connected expectation value $\langle \hat{N}^2 \rangle_c$. Of course, the relation (15) can be easily derived using the fact that

$$\mathrm{Tr}\left[\frac{d}{d\mu}e^{-\beta\hat{H}}\right] = -\beta\,\mathrm{Tr}[e^{-\beta\hat{H}}\hat{N}]. \tag{29}$$

In conclusion, the above scaling analyses show that the asymptotic behavior of the particle density at the critical point is characterized by competing contributions from different sources: the regular background term of the free-energy density, a peculiar logarithmic resonance term, and the critical scaling term controlled by the critical exponents. Even the subtracted particle density (24) does not show an asymptotic scaling behavior, which turns out to be hidden by a logarithmic contribution and the regular term arising from mixings with the identity operator. In the following we investigate whether, and how, out-of-equilibrium conditions arising from quantum quenches can disentangle the various contributions, recovering a well-defined universal out-of-equilibrium FSS behavior.

IV. DYNAMICS AFTER A QUENCH

The dynamics of quantum many-body systems is often studied by considering protocols based on instantaneous quantum quenches of the model parameters (see, e.g., Refs. [2,23,26,35,54–62]), or protocols entailing slow changes of such parameters like those associated with the so-called Kibble-Zurek (KZ) problem (see, e.g., Refs. [2,19,20,24,32–34,63–65]).

In this section, we focus on the quantum evolution arising from a so-called soft quench around the critical point, for which the variation of the parameters associated with the quench is sufficiently small to maintain the system close to criticality. We address the out-of-equilibrium scaling behavior along the postquench critical quantum evolution, paying particular attention to the behavior of the particle density.

We also report numerical results for the out-of-equilibrium evolution after the quench, up to very large $O(10^4)$ lattice sizes, obtained by a straightforward diagonalization of the Hamiltonian (1) [see also Eq. (3)]. With antiperiodic boundary conditions, this can be done by decoupling $\hat{H}$ into a sum of $L/2$ independent terms, each of them acting in the four-dimensional Hilbert subspace generated by the k and $-k$ modes, for a given value of n in Eq. (7), and then exploiting the conservation of fermion parity to further reduce it to two dimensions. Then, the unitary time evolution can be easily computed numerically by a direct integration of the Schrödinger's equation on each of such subspaces.

A. Soft quench protocol

We perform an instantaneous quench of the chemical-potential parameter, from $\mu \neq \mu_c$ to μ_c, or correspondingly from $w \equiv (\mu_c - \mu)/2 \neq 0$ to $w_c = 0$, in such a way as to study the critical out-of-equilibrium quantum evolution. In practice, we consider the following protocol: (i) At $t = 0$ the system is prepared in the ground state $|\Psi_{\rm GS}(w)\rangle$ of the Hamiltonian $\hat{H}(w, \gamma)$, cf. Eq. (1), for a given value of $w \neq 0$. (ii) At $t > 0$, the system evolves unitarily driven by the critical Hamiltonian $\hat{H}(\mu_c, \gamma)$, i.e.,

$$i\frac{d}{dt}|\Psi(t)\rangle = \hat{H}(w_c, \gamma)\,|\Psi(t)\rangle, \quad |\Psi(0)\rangle \equiv |\Psi_{\rm GS}(w)\rangle. \tag{30}$$

We remark that we only consider soft quenches starting from initial conditions close to the critical point (i.e., for small values of $|w|$), so that the system stays always within the critical regime during the postquench quantum evolution.

The arising out-of-equilibrium dynamics can be monitored using the particle density and the fermionic correlation functions, analogous to the equilibrium definitions in Eqs. (15) and (26), respectively, replacing the Gibbs density matrix with the time-dependent density matrix of the evolving (pure) state

$$\rho_\Psi(t) = |\Psi(t)\rangle\langle\Psi(t)|. \tag{31}$$

B. Out-of-equilibrium finite-size scaling

As shown in Refs. [2,26,33], the postquench quantum evolution of standard observables characterized by an asymptotic equilibrium FSS, such as the fermionic correlations of the fermionic Kitaev wires and the longitudinal spin correlations of the quantum XY chain, develops an out-of-equilibrium FSS behavior at quantum transitions. This is essentially obtained by adding a further dependence on the time scaling variable $\Theta \sim t/\tau$ to the equilibrium FSS behaviors, where τ is the timescale of the critical modes, which is expected to be related to the gap $\Delta_c(L)$ (energy difference of the lowest eigenmodes) at the critical point, i.e., $\tau \sim 1/\Delta_c(L) \sim L^z$. Actually, since the gap corresponds to the lowest excitation energy (for $k = \pm\pi/L$) and thus its finite-size dependence at the critical point is given by

$$\Delta_c(L,\gamma) = E\left(k = \frac{\pi}{L}, \mu = \mu_c, \gamma\right) = \frac{2\pi\gamma}{L} + O(L^{-3}), \tag{32}$$

where $E(k, \mu, \gamma)$ is the function defined in Eq. (4), we define

$$\Theta = \frac{\gamma\, t}{L} \sim \Delta_c(L,\gamma)\,t, \tag{33}$$

including a γ-dependent normalization.

Then, along the quench protocol outlined in Sec. IV A, the fixed-time two-point functions $G_P(t, x, w, L)$ and $G_C(t, x, w, L)$ are expected to develop the out-of-equilibrium FSS [2,26],

$$G_\#(t, x, w, L) \approx L^{-2y_c}\,\mathcal{G}_\#(X, \Phi, \Theta), \quad y_c = 1/2, \tag{34}$$

asymptotically in the out-of-equilibrium FSS limit (i.e., the large-L and large-t limits, keeping the scaling variables $X \equiv x/L$, $\Phi \equiv u_w L^{y_w}$, and Θ fixed). Note that the dependence on Φ is essentially related to the initial condition, while no scaling variable is associated with the postquench value of $w = w_f$, because it is always set to zero (otherwise the additional scaling variable $\Phi_f = u_{w_f} L^{y_w}$ should have been added). The out-of-equilibrium FSS functions $\mathcal{G}_\#$ are universal, therefore they must be independent of γ, apart from a multiplicative factor and possible nonuniversal normalizations of the scaling variables Φ and Θ. Actually, their definitions (22) and (33) already contain the correct γ dependence to avoid further γ-dependent normalizations. The asymptotic out-of-equilibrium FSS is expected to be approached with power-law scaling corrections, like at equilibrium. Figure 2

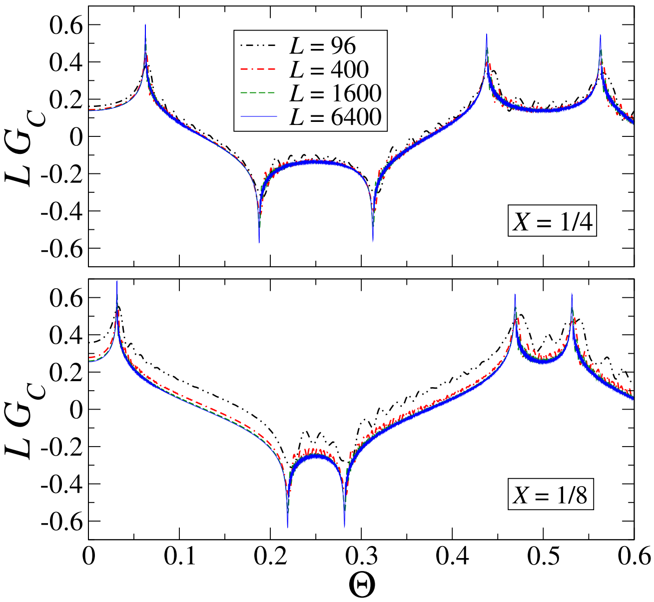

FIG. 2. Out-of-equilibrium FSS of the fermionic correlation G_C as a function of the rescaled time Θ, for fixed $X = x/L = 1/4$ (upper panel) and $X = 1/8$ (lower panel), after a soft quench from the initial condition $\Phi \equiv u_w L = -0.5$ to the critical point $\Phi = 0$, for $\gamma = 1$ [cf. Eq. (34)]. The various curves correspond to different system sizes L, as indicated in the legend. The approach to the asymptotic out-of-equilibrium FSS is consistent with a simple $1/L$ power-law behavior, as expected [2].

shows some numerical results for G_C, clearly confirming the out-of-equilibrium FSS in Eq. (34). We also note the presence of spikes at finite values of the rescaled time Θ, which will be discussed in detail later.

The above out-of-equilibrium FSS behaviors have been obtained by a natural extension of their equilibrium scaling behaviors, simply adding a time dependence through the time scaling variable Θ. However, it is not straightforward to extend this simple picture to the out-of-equilibrium time dependence of quantities whose equilibrium behavior is affected by regular and singular contributions involving the identity operator, such as those appearing in the particle density behavior at the critical point. As discussed in Sec. III, the scaling term of the equilibrium particle density ϱ_e, and also of its subtracted definition D_e, is generally hidden by the analytical background and the logarithmic resonance term; cf. Eq. (23).

The question is whether the modes related to the identity operator share the same timescale $\tau \sim L^z$ of the critical modes, or if their timescale τ_I is significantly shorter, as may be suggested by the fact they are expected to arise from short-range quantum fluctuations. If the ratio τ_I/τ vanishes in the large-L limit, then the out-of-equilibrium particle density may develop a scaling behavior characterized by the same power law of the scaling term of the equilibrium particle density; cf. Eq. (21). An analogous phenomenon occurs along the postquench critical relaxational flow at classical thermal transitions [10], where the energy density shows an out-of-equilibrium FSS behavior, even though it does not scale at equilibrium, like the equilibrium particle density at quantum transitions driven by the chemical potential.

To investigate the out-of-equilibrium behavior of the particle density under the postquench critical quantum evolution, we consider the subtracted particle density

$$D(t, w, \gamma, L) \equiv \frac{1}{L}\mathrm{Tr}\,[\rho_\Psi(t)\,\hat{N}] - \varrho_c(\gamma), \tag{35}$$

where ϱ_c is the γ-dependent value of the particle density at the critical point in the thermodynamic limit; cf. Eq. (20). If the timescales of the different contributions differ substantially, then the postquench quantum evolution may disentangle the scaling contribution from the terms arising from the mixing with the identity operator. As we shall see, the postquench out-of-equilibrium behavior of the particle density turns out to develop the nontrivial asymptotic out-of-equilibrium FSS

$$D(t, w, \gamma, L) \approx L^{-y_n}\mathcal{D}(\Phi, \Theta), \tag{36}$$

which is analogous to a standard scaling behavior, like the case of the fermionic correlations, cf. Eq. (34).

However, like the energy density at classical transitions with negative specific-heat exponent [10], the out-of-equilibrium FSS function $\mathcal{D}$ develops a nontrivial singularity for $\Theta \to 0$, essentially related to the fact that at $\Theta = 0$, and therefore at equilibrium, the subtracted particle density does not show a universal asymptotic FSS, behaving as $L^{-1}\ln L$. Indeed, by matching the out-of-equilibrium FSS of the subtracted particle density, put forward in Eq. (36), with the leading logarithmic term of the equilibrium behavior [cf. Eq. (25)], one would predict a logarithmic divergence of the scaling function $\mathcal{D}(\Phi, \Theta)$ when $\Theta \to 0$ keeping Φ fixed.

C. Numerical results

The typical out-of-equilibrium behavior of the subtracted particle density $D(t, w, \gamma, L)$ after a soft quench is reported in Fig. 3 for initial conditions corresponding to a fixed scaling variable $\Phi = -0.5$ [cf. Eq. (22)]. Analogous results are obtained for other values of Φ, both negative and positive (not shown). The numerical data for different sizes L (upper panel) nicely support the scaling *Ansatz* (36) at generic values of Θ. We have also performed simulations for various values of γ (lower panel), showing that the scaling function $\mathcal{D}(\Phi, \Theta)$ is universal (i.e., independent of γ). Aside from this expected behavior, two special occurrences which correspond to specific values of Θ have to be carefully addressed.

First of all, at small values of Θ, our numerics clearly evidences a logarithmic divergence with L of the product $L\,D(t, w, \gamma, L)$, which is somehow reconstructing the logarithmic equilibrium behavior arising from the resonance between the energy and identity operators, as already mentioned at the end of Sec. IV B. This is shown in Fig. 4, displaying a zoom of the curves in the upper panel of Fig. 3, for $\Theta \to 0$. We also note that the out-of-equilibrium FSS sets in for $\Theta \gtrsim \tau_s/L$, where the timescale τ_s turns out to be independent of L (see also the inset, which displays the same data as a function of the bare time t). Thus, τ_s is the timescale after which the logarithmic singularity in Θ, for $\Theta \to 0$, starts emerging. Such time should be identified, or at least strictly connected, with the timescale τ_I of the modes related to the identity operator, which is needed to equilibrate short-range fluctuations, i.e., $\tau_s \sim \tau_I \sim O(L^0)$ with increasing L.

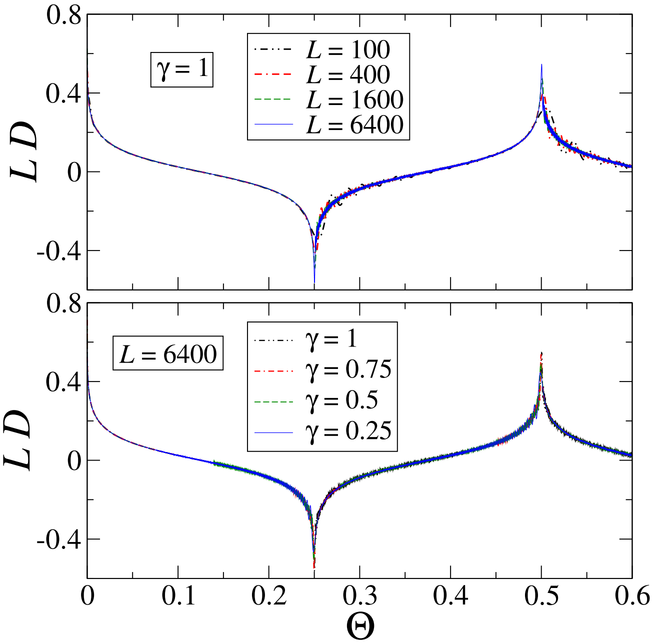

FIG. 3. Scaling of the subtracted particle density D with time, after a quench from $\Phi = -0.5$ to the critical point [cf. Eq. (36)]. Upper panel: curves are for different values of L, while $\gamma = 1$ is kept fixed. Lower panel: curves are for different values of γ, while $L = 6400$ is kept fixed: a collapse of the various curves with γ supports universality of the scaling function $\mathcal{D}(\Phi, \Theta)$.

Second, it is worth stressing that the curves reported in Fig. 3 for the subtracted particle density develop peculiar spikes at finite values of Θ. For example, the spikes occurring around $\Theta = 0.25$ are highlighted in Fig. 5, showing strong

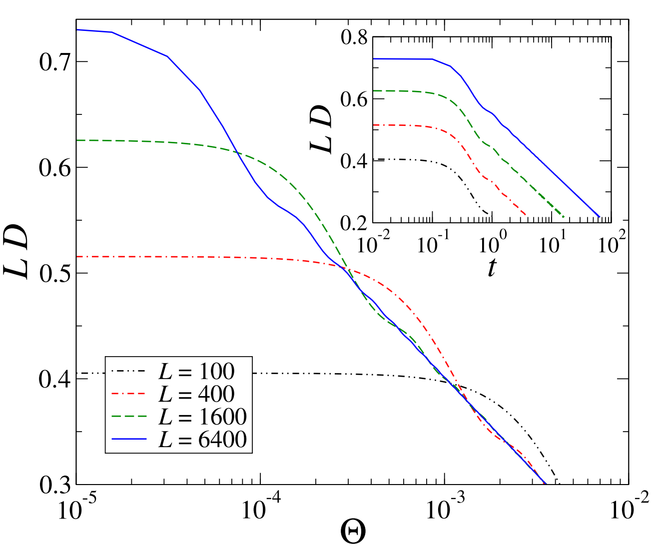

FIG. 4. A zoom of Fig. 3 upper panel for small values of the rescaled time ($\Theta \leqslant 10^{-2}$). The x-axis is in log scale, such that a straight-line behavior denotes a logarithmic divergence of the scaling function $LD \approx \mathcal{D}(\Phi, \Theta)$ for $\Theta \to 0$, as $\mathcal{D} \sim \log \Theta$. The inset shows the same data, but with the real time t (and not the rescaled time Θ) on the x-axis. Note that curves start deviating from the equilibrium value at a time $\tau_s \sim O(L^0)$, which is independent of L.

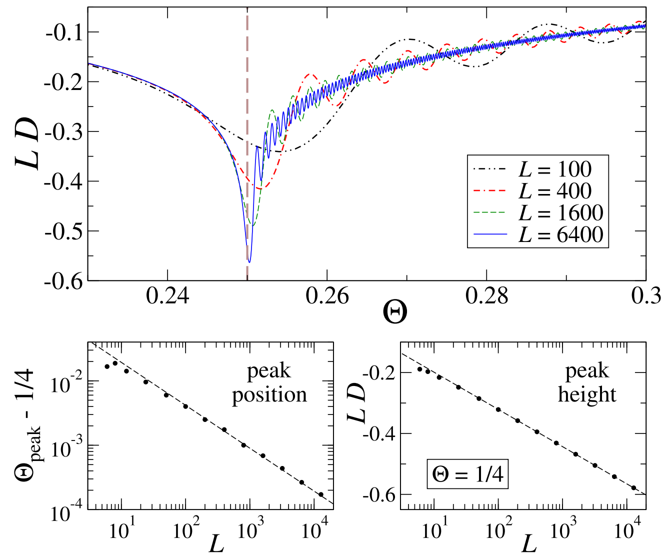

FIG. 5. A zoom of Fig. 3 upper panel, around the first spike that develops for $\Theta \to \Theta_1 = 1/4$ (vertical dashed line). Bottom left panel: the location of the peak with L; the asymptotic value $1/4$ is approached with $O(L^{-2/3})$ corrections [73] (represented by the dashed straight line to guide the eye; note the log-log scale of the plot). Bottom right panel: the behavior of D with L at Θ_1; the straight line is a logarithmic fit $LD = a + b \log L$ for $L > 10^3$, with a and b fitting parameters (note the log scale on the x-axis).

evidence of the presence of another logarithmic divergence of LD, evaluated at $\Theta = 0.25$, with L (bottom right panel). Analogous results are found for $\Theta = 0.5$ and, in general, for multiples of $\Theta = 0.25$ (not shown). These singularities can be related to revival phenomena (see, e.g., Refs. [62,66–73]), due to the finite size of the system. Indeed, they appear at times

$$\Theta_k \equiv \frac{\gamma t_k}{L}, \quad t_k = \frac{kL}{2v_m}, \quad v_m = 2\gamma \tag{37}$$

for $k = 1, 2, \ldots$, where v_m is the maximum velocity of the quasiparticle modes at the critical point [57,74,75]. In fact, our numerics shows that the first emerging spikes are asymptotically (i.e., for $L \to \infty$) located at $\Theta_1 = 1/4$ and $\Theta_2 = 1/2$ with a great accuracy (see Fig. 3). Such values of Θ correspond to the rescaled time for the quasiparticle modes to run across half lattice size (or multiple of it), thus confirming their interpretation in terms of revival phenomena. Finite-size corrections to the Θ-location of such peaks are power-law suppressed with L, as reported in the bottom-left panel of Fig. 5 and already found in Ref. [73] for similar revival phenomena.

It is also worth pointing out that analogous spikes can be observed in the postquench behavior of the fermionic correlations; see Fig. 2. Again, they appear to be associated with logarithmic divergences in L (which have been previously overlooked [28]), as reported in Fig. 6 for the peak in the correlation function G_C with $X = 1/4$ emerging at $\Theta = 0.0625$ (the upper panel is a magnification of the data in Fig. 6 around $\Theta = 0.0625$, while the bottom-right panel displays the behavior of $L\,G_C$, evaluated exactly at $\Theta = 0.0625$, with L). Their location differs from those of the particle density, essentially because they are nonlocal observables characterized by a

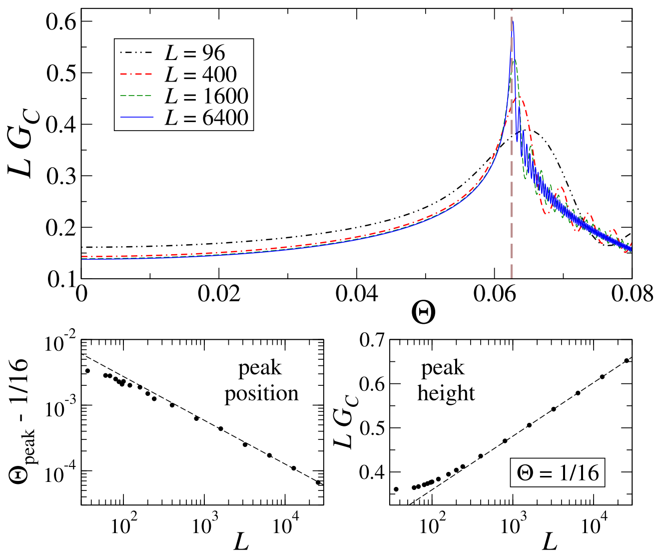

FIG. 6. A zoom of Fig. 2 upper panel, for $\Theta \in [0, 0.08]$, which highlights the spike that develops for $\Theta \to \Theta_1 \equiv 0.0625$ (vertical brown dashed line). Also note that, for $\Theta \to 0$, curves approach the expected equilibrium scaling behavior in Eq. (27). Bottom-left panel: the location of the peak with L; analogously as for the subtracted particle density, finite-size deviations from the asymptotic value $1/16$ scale as $L^{-2/3}$ (dashed line), see also Ref. [73], where analogous revival phenomena are reported. Bottom-right panel: the behavior of G_C with L, evaluated at Θ_1; the straight line is a logarithmic fit $LG_C = a + b \log L$ of the numerical data for $L > 10^3$, with a and b fitting parameters.

scaling distance X. Actually it corresponds to the rescaled time to run across half rescaled distance X, and the complementary value $1 - X$, at the maximum speed of the quasiparticle modes, i.e., the first logarithmic spikes are (asymptotically) located at

$$\Theta_1 = \frac{\gamma X}{2v_m}, \quad \Theta_2 = \frac{\gamma(1-X)}{2v_m}, \tag{38}$$

respectively; see Fig. 2. This may be interpreted as an emerging singularity analogous to the particle density when the quasiparticle modes starting at relative distance X meet. Analogously as before, at finite size, the position of the peaks approaches the asymptotic value as a power-law with L (bottom right panel of Fig. 6).

The above results fully confirm the out-of-equilibrium FSS behavior after a soft quench, put forward in Sec. IV B, and in particular that of the particle density given in Eq. (36). Since the particle density should be quite accessible experimentally and numerically, its out-of-equilibrium behavior under quantum quenches may provide a further effective probe of the universal features at quantum transitions, unlike its equilibrium behavior that is essentially dominated by nonuniversal short-range fluctuations.

D. Out-of-equilibrium scaling in the thermodynamic limit

The previous results show that the subtracted particle density develops a peculiar out-of-equilibrium FSS, according to Eq. (36), characterized by various singularities of the corresponding scaling function, in particular in the $\Theta \to 0$ limit keeping Φ fixed. We now focus on the out-of-equilibrium scaling behavior of the subtracted particle density in the thermodynamic limit.

To derive a scaling *Ansatz* from the FSS behavior (36), we note that the thermodynamic limit corresponds to the limit $L/\xi \to \infty$, where $\xi \sim w^{-\nu}$ is the correlation length of the system. This is achieved by taking the $\Phi \to \infty$ limit keeping the product $\Phi^{z\nu}\Theta = \Phi\Theta = \gamma\, u_w\, t$ fixed. Actually, for simplicity, we consider the scaling variable

$$\theta = w\,t, \tag{39}$$

so that $\theta \approx \Phi\Theta$ apart from irrelevant subleading terms in the scaling limit [taking into account that $u_w = w/\gamma + O(w^2)$]. Then, the thermodynamic limit of the out-equilibrium FSS Eq. (36) can be straightforwardly obtained by taking the limit $\Phi \to \infty$ keeping θ fixed, so that

$$D_\infty(t, w, \gamma) = D(t, w, \gamma, L \to \infty) \approx \frac{w}{\gamma}\,\mathcal{D}_\infty(\theta). \tag{40}$$

Note that we keep the γ dependence in the prefactor of the right-hand side of Eq. (40), which turns out to be useful to check the universality of the scaling behavior with respect to the Hamiltonian parameter γ.

The above out-of-equilibrium scaling behavior can be checked using the known analytical results for the postquench time dependence of the transverse magnetization of the quantum XY chain in the thermodynamic limit, already reported in Refs. [54,55], where the postquench time dependence of the transverse magnetization is expressed in terms of an integral over the momenta (as a function of the coupling $g_i > g_c$ of the initial ground state before the quench, the postquench coupling g, and the parameter γ). In the thermodynamic limit, the boundary conditions become irrelevant, therefore these computations apply also to the particle density of the Kitaev wire in the corresponding infinite-size limit.

Applying the analytical expressions of Refs. [54,55] to the soft quench protocol outlined in Sec. IV A, and taking the out-of-equilibrium scaling limit $w = g_i - g_c \to 0$ and $t \to \infty$ keeping θ fixed, we obtain the curves shown in Fig. 7. They clearly show the scaling behavior (40) (upper panel) and universality with respect to variations of γ (lower panel). Moreover, the scaling curve displays again a logarithmic divergence for $\theta \to 0$, analogously to what has been observed in Fig. 4, in the FSS framework.

V. QUASIADIABATIC KIBBLE-ZUREK PROTOCOL

We now discuss the out-of-equilibrium behavior of the particle density arising from slow changes of the chemical potential approaching the quantum transition, such as the dynamic protocols related to the so-called KZ problem, related to the defect production when crossing continuous transitions from disordered to ordered phases (see, e.g., Refs. [2,18–20,24,63–65]).

Quasiadiabatic evolutions at quantum transitions can be obtained by slowly varying the Hamiltonian parameter μ, according to the linear time dependence

$$\mu(t) = \mu_c - 2w(t), \qquad w(t) = -t/t_s, \tag{41}$$

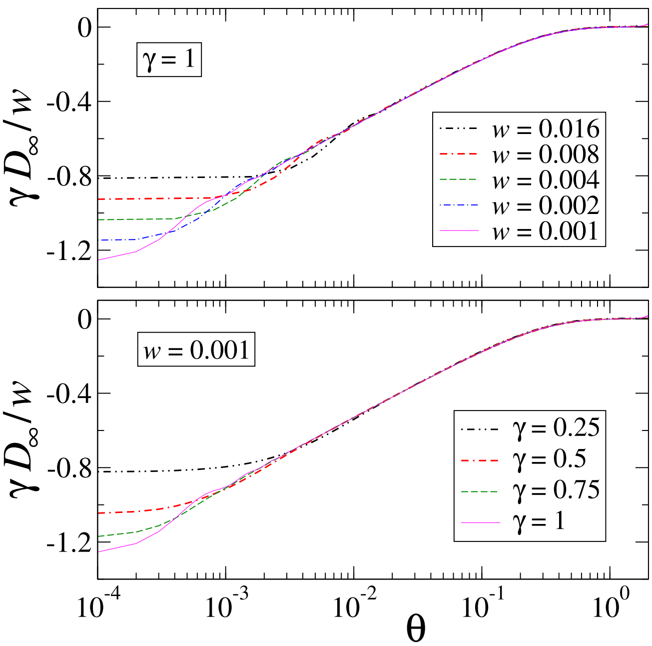

FIG. 7. Scaling of the subtracted particle density D_∞ with time, in the thermodynamic limit [cf. Eq. (40)]. Upper panel: curves are for different values of w, keeping $\gamma = 1$ fixed. Lower panel: curves are for different values of γ, keeping $w = 10^{-3}$ fixed.

with a large timescale t_s. More precisely, we consider the following quasiadiabatic protocol: (i) The quantum evolution of finite fermionic wires of size L starts at a time $t_i < 0$ from the ground state $|\Psi_{\rm GS}(w_i)\rangle$ associated with the initial value $w_i = -t_i/t_s > 0$. (ii) Then the system evolves unitarily according to the Schrödinger equation

$$i\frac{d}{dt}|\Psi(t)\rangle = \hat{H}[w(t),\gamma]\,|\Psi(t)\rangle, \quad |\Psi(0)\rangle = |\Psi_{\rm GS}(w_i)\rangle, \tag{42}$$

where w varies linearly as in Eq. (41) up to the final value $w_f = 0$, corresponding to $t_f = 0$, thus $w(t) \geqslant 0$ along the whole protocol. If we assume w_i fixed with increasing t_s, then $t_i \to -\infty$ in the large-t_s limit.

At a quantum transition, the development of an out-of-equilibrium dynamics is inevitable (in the thermodynamic limit $L \to \infty$, before taking the critical limit) even for very slow changes of the parameter w, because large-scale modes are unable to equilibrate the long-distance critical correlations emerging at the transition point. As a consequence, when starting from equilibrium states at the initial value w_i, the system cannot pass through equilibrium states associated with the values of $w(t)$ at the transition point, thus departing from an adiabatic dynamics before arriving at $w = 0$. Such a departure develops peculiar out-of-equilibrium scaling phenomena in the limit of large timescale t_s of the time variation of $w(t)$. In particular, during the quantum evolution of finite systems under KZ protocols, an out-of-equilibrium FSS emerges in the large-L and large-t_s limits, keeping the appropriate scaling variables fixed (see, e.g., Refs. [2,32,34,35]).

The results of this kind of dynamics can be monitored by looking at some observables and correlations at fixed time, such as the fermionic correlations G_P and G_C [cf. Eqs. (26)], replacing ρ_G with $\rho(t) = |\Psi(t)\rangle\langle\Psi(t)|$. Their out-of-equilibrium FSS along the KZ protocol can be written in terms the scaling variables [2]

$$S_1 \propto w(t)L^{y_w}, \qquad S_2 \propto t\,\Delta_c(L) \sim t/L^z, \tag{43}$$

where Δ_c is the critical gap, or more convenient combinations such as

$$\Upsilon \propto -\frac{S_2}{S_1} \propto \frac{t_s}{L^\zeta}, \qquad \Omega \propto \frac{S_2}{\Upsilon^\kappa} \propto \frac{t}{t_s^\kappa}, \tag{44}$$

where

$$\zeta = y_w + z = 2, \qquad \kappa = \frac{z}{y_w + z} = \frac{1}{2}. \tag{45}$$

Actually, analogously to the definitions of the scaling variables Φ and Θ associated with the quench protocol [cf. Eqs. (22) and (33), respectively], we may allow for a γ-dependent normalization of the scaling variables to simplify the universality checks with respect to variations of γ, without including further factors in the dependence of the scaling variables. Taking into account the fact that $u_w \sim w/\gamma$ [cf. Eq. (12)] and that the critical gap is asymptotically proportional to γ [cf. Eq. (32)], we may refine the definitions of S_1 and S_2, so that $S_1 = w(t)L/\gamma$ and $S_2 = \gamma t/L$, obtaining

$$\Upsilon = \frac{\gamma^2 t_s}{L^2}, \qquad \Omega = \frac{t}{\sqrt{t_s}}. \tag{46}$$

Assuming that the initial value w_i remains fixed in the large-L and large-t_s limit, keeping Υ and Ω fixed, the out-of-equilibrium KZ FSS of the fermionic correlations is given by [2]

$$G_\#(t,x,t_s,w_i,\gamma,L) \approx L^{-2y_c}\widehat{\mathcal{G}}_\#(X,\Upsilon,\Omega), \tag{47}$$

where $y_c = 1/2$ and $X = x/L$. Thus, the scaling behavior turns out to be independent of the initial value w_i. The scaling functions $\widehat{\mathcal{G}}_\#$ are expected to be universal and, in particular, independent of γ, apart from a possible multiplicative factor.

The out-of-equilibrium FSS of particle density following a KZ protocol requires a more careful analysis, although its equilibrium behavior at the transition is understood. We may again consider the working hypothesis that short-range fluctuations, responsible for the leading equilibrium contributions to the particle density, get equilibrated faster than the critical modes, so that the slow dynamics at the KZ protocol, in the large t_s limit keeping Υ and Θ fixed, can be considered as effectively adiabatic for them. Therefore, the emerging out-of-equilibrium behavior in the KZ FSS limit should only be related to the critical modes. We again focus on the deviation of the particle density from its equilibrium value [cf. Eq. (35)],

$$\widehat{D}(t,t_s,w_i,\gamma,L) = \frac{1}{L}{\rm Tr}\,[\rho_\Psi(t)\hat{N}] - \varrho_e[w(t),\gamma,L], \tag{48}$$

where $\varrho_e = \langle\Psi_{\rm GS}[w(t)]|\hat{n}_x|\Psi_{\rm GS}[w(t)]\rangle$ is the ground-state particle density at the instantaneous value $w(t)$ and size L of the system, whose size dependence at the transition was reported in Eq. (23). Then, analogously to Eq. (47), the above scenario naturally leads to the conjecture

$$\widehat{D}(t,t_s,w_i,\gamma,L) \approx L^{-y_n}\widehat{\mathcal{D}}(\Upsilon,\Omega). \tag{49}$$

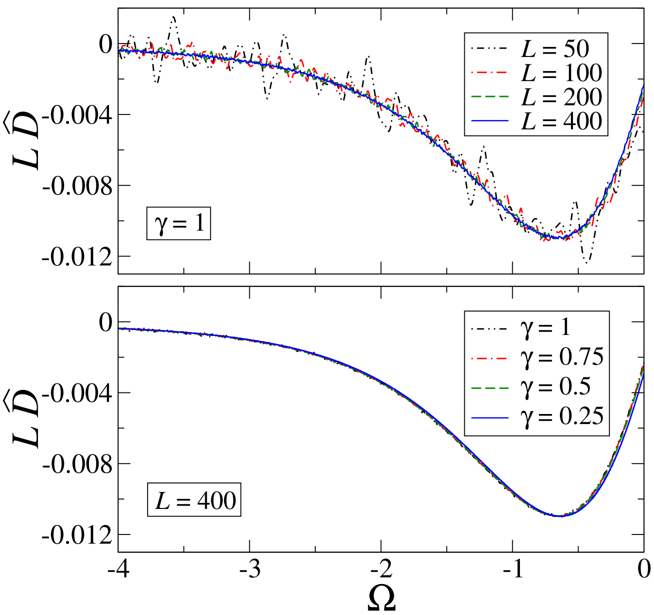

FIG. 8. Deviation of the particle density from its equilibrium value $\widehat{D}$, along a KZ protocol from $w_i = 0.5$ to $w_f = w_c = 0$. Data are shown against the rescaled time Ω, in a window close to the ending point $\Omega = 0$. We fix $\Upsilon = 0.1$ (analogous results are obtained for other values of Υ). Upper panel: curves are for different values of L, while $\gamma = 1$ is kept fixed. They appear to approach an asymptotic curve, supporting Eq. (49). Lower panel: curves are for different values of γ while $L = 400$ is kept fixed, which are hardly distinguishable. Their agreement provides strong evidence of universality of the scaling function $\widehat{\mathcal{D}}(\Upsilon, \Omega)$.

As shown in Fig. 8, this conjecture is fully supported by our numerical results for different sizes L (upper panel). Besides that, they also show that the asymptotic out-of-equilibrium scaling behavior is universal, i.e., independent of the value of γ (bottom panel). In this respect we do not even apparently need a multiplicative factor, while the γ dependence in the definitions of the scaling variables Υ and Ω [cf. Eq. (46)] already takes correctly into account their nonuniversal normalizations.

VI. CONCLUSIONS

We have addressed the equilibrium (ground-state) and out-of-equilibrium behaviors of the particle density in many-body systems undergoing quantum transitions driven by the chemical potential, which arise from a nontrivial interplay between noncritical short-range and critical long-range quantum fluctuations. To study this issue, we consider the fermionic Kitaev chain [36] as a paradigmatic model for quantum transitions driven by the chemical potential, for which very accurate numerical calculations, and thus checks of the scaling *Ansatz*, can be performed up to $O(10^4)$ sites.

The equilibrium behavior of the particle density (or equivalently the transverse magnetization in the quantum *XY* chain) is related to the derivative of the free-energy density with respect to the chemical-potential parameter μ. Its behavior at continuous quantum transitions driven by μ is generally dominated by contributions arising from short-range fluctuations, which may be interpreted as mixings with the identity operators within CFT frameworks [37]. In particular, within the paradigmatic fermionic Kitaev wires, they appear as a regular background term in the free-energy density (actually this generally occurs at continuous quantum transitions [2,3]), and also logarithmic terms arising from peculiar resonances [6] between operators of the *energy* and *identity* CFT families [3,37]. These contributions hide the genuine scaling behavior arising from the long-range critical modes, making the particle density nonoptimal to study the universal critical properties at continuous quantum transitions. The absence of an asymptotic universal equilibrium scaling makes the out-of-equilibrium behavior unclear. To study this issue, we focus on two simple dynamic protocols: (i) instantaneous soft quenches, where the chemical-potential is instantaneously changed from $\mu \neq \mu_c$ within the critical regime to its critical value μ_c, and (ii) quasiadiabatic protocols, where μ gets slowly and linearly changed from μ to μ_c.

After subtracting its infinite-volume value at the critical point, the subtracted particle density D [cf. Eq. (35)] shows an out-of-equilibrium FSS along the quantum evolution after a soft quench of μ. The resulting asymptotic out-of-equilibrium FSS, $D(t, w, \gamma, L) \approx L^{-y_n}\mathcal{D}(\Phi, \Theta)$, is controlled by the RG dimension y_n of the particle-density operator $\hat{n}_x$ and is defined in the large-L limit keeping the scaling variables $\Phi \sim (\mu - \mu_c)L^{1/\nu}$ (associated with the chemical-potential parameter) and $\Theta \sim t/L^z$ (associated with the time interval from the quench) fixed. This dynamic FSS appears analogous to that of other observables possessing an asymptotic equilibrium FSS [2,26], such as that of the fermionic correlations [cf. Eq. (34)]. However, unlike them, the scaling functions are now characterized by a logarithmic divergence in the $\Theta \to 0$ limit, which is somehow related to the logarithmic singularity at equilibrium [cf. Eq. (25)], which must be somehow reconstructed in the $\Theta \to 0$ limit. Analogous results are obtained in the thermodynamic limit, in terms of the time scaling variable $\theta \sim \Phi\Theta \sim (\mu - \mu_c)t$ remaining after the thermodynamic infinite-size limit.

Within the out-of-equilibrium FSS framework, we also spotlight logarithmic divergences at finite values of Θ (both for the subtracted particle density and for the fermionic correlations), which can be related to revival phenomena in finite-size systems, already observed in various contexts (see, e.g., Refs. [62,66–73]).

Then we consider quasiadiabatic KZ-like protocols, where the chemical potential is slowly changed at the quantum transition, with a large timescale (see Sec. V). Along these protocols, the critical large-scale modes are generally unable to equilibrate the long-distance critical correlations emerging at the transition point, giving rise to peculiar out-of-equilibrium KZ FSS behaviors [2]. However, the case of the particle density is again particular, because its equilibrium behavior is dominated by the short-range contributions. Under the assumption that the timescale of the changes of the short-range modes, and therefore of the changes of their contributions, is much smaller than that driving the critical modes, the slow dynamics is expected to disentangle the effects of the out-of-equilibrium critical modes from those associated with the short-range modes. This scenario leads us to conjecture that the quasiadiabatic KZ dynamics in the out-of-equilibrium FSS limit is effectively adiabatic with respect to the short-range modes, and thus the out-of-equilibrium behavior is only associated with the critical modes that give rise to a well-defined out-of-equilibrium FSS. This is indeed observed when looking at the difference between the out-of-equilibrium particle density and its equilibrium value at the instantaneous value of the chemical-potential parameter [cf. Eq. (48)], which shows an out-of-equilibrium KZ FSS analogous to the observables defined from the fermionic correlations, whose equilibrium FSS is not affected by short-range contributions.

We remark that analogous out-of-equilibrium scaling behaviors, at both instantaneous quenches and quasiadiabatic protocols, are expected at any quantum transition, when considering the behavior of observables related to the derivative of the free-energy density with respect to the Hamiltonian parameter that drives the quantum transition preserving the symmetry (such as the transverse magnetization at the quantum transitions of d-dimensional quantum Ising systems, or the square angular momentum at the quantum transitions of d-dimensional quantum rotor models [1,2]).

We finally mention that recent theoretical proposals [76,77], as well as experimental attempts to realize fermionic Kitaev wires, by means of quantum dots [78], integrated circuits [79], or even quantum computers [80–83], have been put forward with the purpose of manipulating Majorana zero modes. Since, in this context, particle density measurements should be quite accessible, we believe there could be the possibility to study its out-of-equilibrium behavior, thus providing a further effective probe of the universal features at quantum transitions (unlike its equilibrium behavior, which is essentially dominated by the nonuniversal short-range fluctuations). More generally, the main features of the dynamic scaling behaviors of the particle density for the fermionic Kitaev wire are expected to extend to generic quantum transitions driven by the chemical potential when the particle-number operator is not conserved. Further theoretical as well as experimental studies could help in validating this scenario.

ACKNOWLEDGMENT

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