Reply to "Comment on 'Towards exact solutions for the superconducting T_c induced by electron-phonon interaction' "

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(Received 12 June 2024; accepted 10 July 2024; published 22 July 2024)

In a series of papers, we have proposed a nonperturbative field-theoretic approach to deal with strong electron-phonon and strong Coulomb interactions. The key ingredient of such an approach is to determine the full fermion-boson vertex corrections by solving a number of self-consistent Ward-Takahashi identities. Palle argued that our Ward-Takahashi identities failed to include some important additional terms and thus are incorrect. We agree that our Ward-Takahashi identities have ignored some potentially important contributions and here give some remarks on the role played by the additional terms.

DOI: 10.1103/PhysRevB.110.026502

Strong fermion-boson interactions cannot be treated by weak-coupling perturbation theory. While the complete set of Dyson-Schwinger (DS) equations are exact and in principle contain a major part of interaction-induced effects, they are not closed and thus intractable. In a series of papers [1–5], we proposed a nonperturbative approach to determine fermion-boson vertex corrections by solving a number of self-consistent Ward-Takahashi identities (WTIs) and also applied this approach to study strong electron-phonon and Coulomb interactions in a few condensed matter systems.

In the preceding comment, Palle [6] questioned the validity of our approach and argued that the WTIs derived in our papers [1–5] are not satisfied by the results obtained via perturbative calculations. Palle [6] further pointed out that the mistake arises from the ignorance of the $z - z'$ dependence of the current vertex functions in our manipulation of point-slitting technique. After considering the $z - z'$ dependence, Palle [6] showed that the modified WTIs should contain some additional terms that cannot be expressed purely in terms of the full fermion propagator $G(p)$, which signals the invalidity of our approach.

We have actually analyzed the potential contribution of the $z - z'$ dependence, but eventually decided not to include it for the following reason. The two points z and z' come from one point z, and the limit $z \to z'$ must be taken at the end. If an electron propagates between z and z', it must carry a momentum $\mathbf{k}$ that is conjugate with $\mathbf{z} - \mathbf{z}'$. As $z - z'$ approaches to zero, the absolute value $|\mathbf{k}|$ should become extremely large so as to obey the Heisenberg uncertainty principle. However, $|\mathbf{k}|$ should not take large values because the low-energy properties of a quantum many-fermion system are dominantly governed by the electrons excited around the Fermi surface. It thus appeared to us that the inclusion of an internal momentum $\mathbf{k}$ between z and z' would lead to an inconsistency.

On the other hand, however, we agree with Palle that the current vertex function $\Gamma_t(q, p)$ derived from our WTIs at one-loop level are different from the one-loop result of $\Gamma_t(q, p)$ calculated by carrying out perturbative expansion. Therefore, our previous analysis of the $z - z'$ dependence needs to be reexamined more carefully. The WTIs presented in Refs. [1–5] are incorrect and should be properly modified to accommodate the $z - z'$ dependence. Palle [6] argued that the WTIs should contain a number of additional terms given by

$$\Delta_m(q, p) = \int_k (\xi_{\mathbf{p}+\mathbf{q}+\mathbf{k}} - \xi_{\mathbf{p}+\mathbf{k}} - \xi_{\mathbf{p}+\mathbf{q}} + \xi_{\mathbf{p}})\tilde{\Gamma}_m(q, p, k),$$

where $\tilde{\Gamma}_m(q, p, k)$ arises from a straightforward Fourier transformation of $\tilde{\Gamma}_m(z_3 - z', z - z_4, z - z')$. Although it is not entirely clear to us how to define the internal interval of momentum $\mathbf{k}$ in a self-consistent way, we agree that such additional terms should be incorporated in the WTIs.

The contribution of additional terms might be small under suitable conditions. Below are some remarks.

(1) For a one-dimensional interacting electron system, the electrons have a linear dispersion $\xi_{\mathbf{p}} = v_{\mathrm{F}}(|\mathbf{p}| - p_{\mathrm{F}})$, where p_{F} is Fermi momentum. In this case, the additional terms

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$\Delta_m(q, p)$ vanish. Then the current vertex function can be expressed purely in terms of $G(p)$, which reproduces the result previously obtained by Dzyaloshinskii and Larkin [7]. For two- and three-dimensional systems, the additional terms do not necessarily vanish.

(2) The interfacial superconductivity of one-unit-cell $FeSe/SrTiO_3$ system studied in Ref. [1] is induced by the interaction between electrons and optical phonons. Such an interaction is strongly peaked at $\mathbf{q} = 0$. In the limit $\mathbf{q} \to 0$, the additional terms

$$\Delta_m(q, p) = \int_k \frac{\mathbf{k} \cdot \mathbf{q}}{m_e} \tilde{\Gamma}_m(q, p, k) \tag{1}$$

vanish, as long as $\tilde{\Gamma}_m(q, p, k)$ is analytic in $\mathbf{q}$. Moreover, the Fermi-surface approximation, i.e., $\xi_{\mathbf{p}} = \xi_{\mathbf{p}_F} = 0$, was employed in the numerical computations of T_c in Ref. [1]. Under such an approximation, both $\xi_{\mathbf{p}+\mathbf{q}} - \xi_{\mathbf{p}}$ and $\Delta_m(p, q)$ vanish. Thus, the numerical results of the transition temperature T_c reported in Ref. [1] are not expected to be changed by the additional terms.

(3) For fermion-boson interacting systems that are not dominated by zero-$\mathbf{q}$ forward scattering, additional terms are small only under the approximation $\xi_{\mathbf{p}} = \xi_{\mathbf{p}_F} = 0$. If such an approximation is not justified, the contribution of additional terms cannot be ignored.

(4) When electrons are coupled to two sorts of bosons, the DS equations become much more complicated than the case of a single fermion-boson coupling owing to the presence of four different interaction vertex functions, as shown by Eq. (27) in Ref. [3] and Eq. (40) in Ref. [5]. The identities satisfied by the interaction and current vertex functions given by Eqs. (40) and (41) in Ref. [3] and Eqs. (50) and (52) in Ref. [5] are still valid and can be used to simplify the DS equation of $G(p)$. One can see from Eq. (42) in Ref. [3] and Eq. (53) in Ref. [5] that the simplified DS equation of $G(p)$ contains, apart from $G(p)$ and three free propagators, merely one single current vertex function. Even though such a current vertex function cannot be determined rigorously, one could compute it approximately by means of series expansion if the system has a small coupling parameter.

In summary, we admit that the DS equations of the full fermion propagator $G(p)$ derived in Refs. [1–5] are not self-closed once the $z - z'$ dependence of current vertex functions $\tilde{\Gamma}_m(z_3 - z', z - z_4, z - z')$ is taken into account. The additional terms appearing in modified WTIs are small under certain approximations, but generically cannot be neglected. The structure of the additional terms remains unknown and needs to be carefully investigated. The modified WTIs impose exact constraints on current vertex functions. Such constraints do not suffice to make the DS equation of $G(p)$ self-closed, but provide useful guidance for the exploration of suitable approximate form of current vertex functions.

We thank G. Palle for insightful comments.

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