Schwinger mechanism of magnon-antimagnon pair production on magnetic field inhomogeneities and the bosonic Klein effect

T. C. Adorno [*]

Department of Physics, School of Mathematics and Physics, Xi'an Jiaotong-Liverpool University, 111 Ren'ai Road, Suzhou Dushu Lake Science and Education Innovation District, Suzhou Industrial Park, Suzhou 215123, People's Republic of China

S. P. Gavrilov [†]

Department of Physics, Tomsk State University, Lenin Avenue 36, 634050, Tomsk, Russia, and Department of General and Experimental Physics, Herzen State Pedagogical University of Russia, Moyka Embankment 48, 191186, St. Petersburg, Russia

D. M. Gitman [‡]

P.N. Lebedev Physical Institute, 53 Leninskiy Prospect, 119991 Moscow, Russia and Institute of Physics, University of São Paulo, Rua do Matão, 1371, CEP 05508-090, São Paulo, Brazil

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Effective field theory of low-energy excitations (magnons) that describe antiferromagnets is mapped into electrodynamics of a charged scalar field interacting with an external magnetic background. In this theory magnons and antimagnons are described by a corresponding scalar field. If the external background is a constant inhomogeneous magnetic field in the quantum version of the model, then there exists vacuum instability which can be analyzed by an analogy with the scalar QED with electric potential steps. Here magnons and antimagnons are treated as charged particles, whereas the magnetic moment plays the role of the electric charge such that magnons and antimagnons differ from each other in the sign of this moment. The vacuum instability is related to the magnon-antimagnon production from the corresponding vacuum by magnetic field inhomogeneities. Characteristics of the vacuum instability can be calculated nonperturbatively using special exact solutions of the Klein-Gordon equation. In particular, we consider examples of the magnetic field that correspond to some regularizations of the Klein step. In the case of smooth-gradient steps, we have derived a universal behavior of the flux density of created magnon-antimagnon pairs. It is noted that there exists an opportunity, for the first time, to observe the Schwinger effect in the case of Bose particle creation. Moreover, it turns out that in the case of the Bose statistics appears a new mechanism for amplifying the effect of pair creation, which we call statistically assisted Schwinger effect.

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I. INTRODUCTION

Magnons, or quantized spin waves, occur in various types of ordered magnets: antiferromagnet, ferromagnet, and ferrimagnet. They present collective magnetic excitations of the electron spin structure in a crystal lattice. The emerging field of magnonics utilizes magnons for information processing; see Ref. [1] for a review. Using magnons as information carriers has various advantages, in particular, the low power-consumption. Although spin systems are originally described as lattice models, similar to Dirac models of nanostructures, one can describe their low-energy dynamics based on a continuum field theory at energy scales much lower than the inverse lattice spacing; see Ref. [2,3] and references therein. Descriptions based on effective field theories (EFT) of spin systems at low energies also allow including external fields in the model. The magnon EFT can incorporate various symmetry-breaking terms. For example, it can be a Zeeman term due to the coupling to an external magnetic field that breaks the symmetry explicitly. It turn out that the magnon EFT that describes antiferromagnets is relativisticlike. Our special interest is the case of an inhomogeneous magnetic field applied to an antiferromagnet in the collinear (homogeneous) ground state. It was recently shown [2] that the magnon EFT with the easy-axis anisotropy can be mapped into electrodynamics of a charged scalar field interacting with an external electromagnetic potential. The mass of this field is determined by the sum of the easy-axis potential and the ratio of magnetization and condensation parameters. Magnetic moment plays here the role of the electric charge, and magnons and antimagnons differ from each other in the sign of the magnetic moment. In the framework of such a consideration, it is important to take into account the vacuum instability (the Schwinger effect) under the magnon-antimagnon production

*Contact author: Tiago.Adorno@xjtlu.edu.cn

†Contact author: gavrilovsergeyp@yahoo.com; gavrilovsp@herzen.spb.ru

‡Contact author: gitman@if.usp.br

on magnetic field inhomogeneities. (an analog of particle-antiparticle creation by constant inhomogeneous electriclike fields). In this article, the leading imaginary part of the one-loop effective action was calculated in the framework of the semiclassical world-line formalism for the case of a linearly varying magnetic field and agrees with the Schwinger's formula for a constant electric field [4]. In the presence of a constant inhomogeneous external magnetic field, one can see that the latter problem is technically reduced to the problem of charged-particle creation from the vacuum by an electric potential step. In this context, it is important to mention recent works addressed the problem of magnon-antimagnon pair creation by a sufficiently high rectangular step (the Klein step) and barrier, formed by magnetic field inhomogeneities. [5–8]. In these cases the world-line formalism does not work and the problem is considered in the framework of the relativistic quantum mechanics. In relativistic quantum mechanics, problems of this type were considered in the relation to the Klein paradox in the pioneer works [9–11] (a detailed historical review can be found in Refs. [12,13]). We note that to avoid a confusion, the Klein paradox should be distinguished from the Klein tunneling through the square barrier. This tunneling without an exponential suppression occurs when a particle is incident on a high barrier, even when it is not high enough to create particles. It is known that attempts to consider overlapping amplitudes as amplitudes of particle transmission and reflection by the Klein step in the same manner as in the relativistic quantum mechanics often leads to contradictions and paradoxes. As it is known processes with changing the number of particles have to be considered in the framework of quantum field theory (QFT). Recently, a consistent nonperturbative treatment of the vacuum instability with respect to charged particle creation was developed in the framework of strong-field quantum electrodynamics (QED) with time-independent external electric potential steps (we call them conditionally x steps) in Refs. [14–16]. In the case of bosons, the latter nonperturbative treatment is based on the existence of special exact solutions of the Klein-Gordon equation with the corresponding x step. This enables the consideration of pair creation by x steps of arbitrary form including, in particular, the Klein step. We hope that the present article will promote consistent application of strong field QED methods in magnonics and will allow avoiding contradictions and nonexistent paradoxes in the interpretation of the obtained theoretical results.

In the present work we use the strong field QED to study the magnon-antimagnon pair production on magnetic field inhomogeneities.[1] The article is organized as follows: In Sec. II the EFT model describing the low-energy dynamics of antiferromagnetic magnons is mapped into scalar electrodynamics with x steps. In Sec. III, we construct a Fock space realization of the EFT model in the framework of strong-field QED with x steps. Initial and final one-particle states are constructed with the help of stationary plane waves satisfying the Klein-Gordon equation. Initial and final vacua are defined and initial and final states of the Fock space are constructed. Mean numbers of magnons and antimagnons created from the vacuum are expressed via overlap amplitudes of the stationary plane waves. Observable physical quantities specifying the vacuum instability are determined. We calculate and analyze the fluxes of energy and magnetic moments of created magnons. In Sec. IV, we present characteristics of the vacuum instability obtained for some magnetic steps that allows exact solving the Klein-Gordon equation. In particular, we consider examples of magnetic steps with very sharp field derivatives $\partial_x U$ that correspond to a regularization of the Klein step. In the case of smooth-gradient steps, we describe a universal behavior of the flux density of created pairs. In the last Sec. V, we summarize the main results of the present work. Some details of the scalar field quantization in the presence of critical potential steps are placed in Appendix A. Examples of some exact solutions with x steps are given in Appendix B.

II. EFT MODEL DESCRIBING LOW-ENERGY DYNAMICS OF MAGNONS

The system under consideration consists of localized spins which live on sites of a cubic-type lattice. These sites are numbered by the index n. The corresponding spin vector operators are denoted by $\hat{\mathbf{s}}^n$. It is assumed that the spins are involved in the antiferromagnetic interaction. Its original $SO(3)$ spin-rotation symmetry is explicitly (but softly) broken due to an external magnetic field $\mathbf{B}$ and an anisotropic interaction C known as single-ion anisotropy. The Hamiltonian describing such a system reads

$$\hat{H}_{\text{spin}} = \sum_n \sum_{i=1}^{d} J\delta^{ab}\hat{s}_a^n \hat{s}_b^{n+\hat{\imath}} - \sum_n \left[\mu B^a(\mathbf{r}_n)\hat{s}_a^n + C^{ab}\hat{s}_a^n\hat{s}_b^n\right]. \tag{1}$$

Here $\hat{s}_a^n$ denote spin operator components on the site n ($[\hat{s}_a^n, \hat{s}_b^n] = i\epsilon_{ab}^c\ \hat{s}_c^n$); $J > 0$ is the antiferromagnetic interaction coupling constant. To describe the nearest-neighbor pairs, the direction $\hat{\imath} = 1, 2, \ldots, d$ with a spatial dimension d is introduced. The sum over $\hat{\imath}$ means summing over nearest neighboring spins and the sum over n means summing over sites n of a cubic-type lattice. $B^a(\mathbf{r}_n)$ are external magnetic field components on the site n, they depend on the coordinates $\mathbf{r}_n = (x_n, y_n, z_n)$ of the site; $\mu > 0$ is the modulus of the magnetic moment projection onto the direction of the magnetic field, which is called magnetic moment in what follows; the single-ion anisotropic interaction is presented by the term $C^{ab}\hat{s}_a^n\hat{s}_b^n$, where the C^{ab} is a constant symmetric rank-two tensor. Here $\hat{\imath}$ is a vector of the length l (l is the lattice spacing) pointing in the i direction. The Kronecker δ δ^{ab} and the Levi-Civita symbol ϵ_{abc} are used for the internal spin indices, $a, b = 1, 2, 3$, and the summation over the repeated indices is implied.

We only consider the simple cubic-type lattice and the G-type antiferromagnet, in which the Néel order appears along all the spatial directions. In the absence of explicit symmetry-breaking terms ($\mu B^a = 0$ and $C^{ab} = 0$) Hamiltonian (1) enjoys $SO(3)$ symmetry. We can study effects of symmetry-breaking terms using the background field (spurion) method if these terms are small enough compared to the symmetric interaction ($\mu B^a,\ C^{ab} \ll J$).

[1]Here we are using the natural system of units $\hbar = 1$.

A continuum field-theoretical description of magnons is given by a $O(3)$ nonlinear sigma model, in which a three-component unit vector $\mathbf{n} = (n^1, n^2, n^3)$ with $n^a n_a = 1$ plays a role as a dynamic degree of freedom. This unit vector expresses the Néel order parameter. Taking the continuum limit of the background field, $B^a(\mathbf{r}_n) \to B^a(\mathbf{r})$, and following the way described in Ref. [2], one can construct a $SO(3)$ gauge invariant effective Lagrangian. The corresponding space-time is parametrized by coordinates $X = (t, \mathbf{r})$, $t = X^0$, $\mathbf{r} = X^j = (x, y, z)$, $j = 1, 2, 3$. The local $SO(3)$ transformation simply acts on the vector field $\mathbf{n}$ as $\mathbf{n} \to g(X)\mathbf{n}$ with $g(X) \in SO(3)$, as in the lattice case. One can identify the background field B^a as the $SO(3)$ gauge field on which the local $SO(3)$ transformation acts as follows:

$$B^a \to g(X)B^a g^{-1}(X) + g(X)\partial_0 g^{-1}(X). \tag{2}$$

The smallness of the terms $\mu B^a/J$ and C^{ab}/J allows us to neglect their higher orders in the future. The only effect of the symmetry-breaking term μB^a is that the constant piece of $B^a(\mathbf{r})$ is used to tune the collinear ground state. In this case, as it will be seen further, the field $B^a(x)$ can be treated as zero component $A_0^a(x)$ of the electromagnetic potential in the theory of the charged scalar field. However, the constant part of $B^a(\mathbf{r})$ is a physical quantity.

In the leading order of the derivative expansion at low-energies, namely, preserving derivatives only up to the second order, the $SO(3)$ gauge invariant effective Lagrangian can be written as

$$\mathcal{L} = \frac{f_t^2}{2}(D_0 n^a)^2 - \frac{f_s^2}{2}(\partial_i n^a)^2 + rC^{ab} n_a n_b, \tag{3}$$

where the covariant derivative D_0 with the $SO(3)$ background gauge field is defined as

$$D_0 n^a = \partial_0 n^a - \epsilon^a_{bc}\, n^b \mu B^c,\ \ \partial_0 = \frac{\partial}{\partial t}, \tag{4}$$

and low-energy parameters f_t, f_s, and r can be determined from the underlying lattice model by the matching condition.

Suppose that our spin system possesses a potential with an easy-axis anisotropy and develops the collinear ground state. We apply an inhomogeneous magnetic field along the spin direction of the ground state. We assume that the magnetic field points to the direction of axis z and depends on the coordinate x, $B^a(x) = B(x)\delta^{a3}$, and the sign of the field is positive, $B(x) > 0$. This field gives the collinear ground state with the Néel vector pointing to the direction of axis z as $\langle \mathbf{n} \rangle = (0, 0, 1)$. Then one can introduce magnon complex scalar fields $\Phi(X)$ and $\Phi^*(X)$ as fluctuations on the top of the ground state, which parametrize the vector $\mathbf{n}$ as

$$\mathbf{n} = \left(\frac{\Phi + \Phi^*}{\sqrt{2}}, \frac{\Phi - \Phi^*}{\sqrt{2}i}, \sqrt{1 - \Phi^*\Phi} \right), \tag{5}$$

where the constraint $n^a n_a = 1$ is explicitly solved. Substituting this parametrization into Eq. (3), one obtains the effective Lagrangian of magnons at the quadratic order of fluctuation fields around the ground state in the following form:

$$\mathcal{L}^{(2)} = f_t^2(D_0\Phi^* D_0\Phi - \Delta^2\Phi^*\Phi) - f_s^2\delta^{ij}\partial_i\Phi^*\partial_j\Phi,$$
$$D_0\Phi = (\partial_0 + iU)\Phi,\ D_0\Phi^* = (\partial_0 - iU)\Phi^*, \tag{6}$$

where the notation $U = \mu B$ and $rC^{ab} = \frac{1}{2}f_t^2\Delta^2\delta^{a3}\delta^{b3}$ are used. We see that the field-theoretical description of antiferromagnetic magnons embedded into an external inhomogeneous magnetic field B can be realized in terms of scalar electrodynamics where a charged complex field $\Phi(X)$ (with μ playing the role of an electric charge) coupled to the zero component of the electromagnetic potential $A_0 = B$. Here the constant $v_s = f_s/f_t$ plays the role of the speed of light and the energy gap Δ plays the role of a mass term. We note that the constant v_s is not related to the speed of the light c and is relatively small, e.g., $\Delta \sim 1$meV and $v_s \sim 60$ m/s for antiferromagnetic MnF_2 [17].

It follows from the effective Lagrangian (6) that the corresponding wave equation for the field $\Phi(X)$ is a modification of the Klein-Gordon equation,

$$\left(D_0^2 - v_s^2\delta^{ij}\partial_i\partial_j + \Delta^2\right)\Phi(X) = 0. \tag{7}$$

Summarizing, we can say that in the example under consideration, the EFT model describing low-energy dynamics of antiferromagnetic magnons can technically be identified with the theory of a charged scalar field interacting with an external constant electric field, which in our terminology is an x step; see Refs. [14–16]. In this theory, the wave equation describing the corresponding charged particles has the form of Eq. (7). A nonperturbative study (with respect to the interaction with the external field) of various quantum effects in such a system, in particular, the study of the vacuum instability, can use the technique [with the necessary modifications due to the specifics of the wave equation (7)] developed earlier by two coauthors (Gavrilov and Gitman) for strong-field QED with x-potential steps and presented in Refs. [14,15]. In the next section, we study the vacuum instability in the system of low-energy magnons following the formulated idea.

III. CONSIDERATION IN THE FRAMEWORK OF STRONG-FIELD QED

A. Solutions of the Klein-Gordon equation with critical x steps

Solutions of the Klein-Gordon equation with critical x steps are known in the form of stationary plane waves. A complete set of such solutions reads

$$\phi_m(X) = \exp(-i\varepsilon t + i\mathbf{p}_\perp \mathbf{r}_\perp)\varphi_m(x),$$
$$\mathbf{r}_\perp = (0, y, z),\ \ m = (\varepsilon, \mathbf{p}_\perp). \tag{8}$$

In fact these are stationary states with well defined the total energy of a particle ε and with definite momenta $\mathbf{p}_\perp$ in the perpendicular to the axis x directions. Substituting Eq. (8) into Eq. (7), we obtain a second-order differential equation for the function $\varphi_m(x)$,

$$\left\{v_s^2\partial_x^2 + [\varepsilon - U(x)]^2 - \pi_\perp^2\right\}\varphi_m(x) = 0,$$
$$\pi_\perp = \sqrt{v_s^2\mathbf{p}_\perp^2 + \Delta^2}, \tag{9}$$

Before considering the case of inhomogeneous magnetic field, where it is necessary to use the recently developed in Refs. [14–16] approach to strong field QED with x steps, it is useful to discuss a simple case of homogeneous magnetic field with $A_0 = B = \text{const} > 0$ such that $U = \mu B < \Delta$. In this case, the field $\Phi(X)$ is a free field satisfying equation (7) with the constant U. A complete set of solutions of this equation reads

$$\phi_m^{(\pm)}(X) = N^{(\pm)} \exp(-i\varepsilon^{(\pm)}t + i\mathbf{pr}),$$
$$\varepsilon^{(\pm)} - U = \pm\sqrt{v_s^2\mathbf{p}^2 + \Delta^2}, \tag{10}$$

where $N^{(\pm)}$ is a normalization factor. A time-independent inner product of solutions of the Klein-Gordon equation is proportional to the matrix elements of a field charge given by the following integral over a spatial volume:

$$\int \{\Phi^*(i\partial_0 - U)\Phi' + \Phi[(i\partial_0 - U)\Phi']^*\}dV. \tag{11}$$

This integral is positive (negative) for any superpositions of $\phi_m^{(+)}$ ($\phi_m^{(-)}$). The collective excitation described by the classical fields $\phi_m^{(+)}$ and $\phi_m^{(-)}$ are fluctuations of the Néel vector on the top of the ground state, that is, semiclassically speaking, these fields describe the fluctuations of the spin vector with clockwise and counterclockwise rotation seen from the north pole. The realization of the scalar field in a Fock space implies that field quanta be boson particles (with the positive frequency $\varepsilon^{(+)}$ and the effective charge μ) and antiparticles (with the negative frequency $\varepsilon^{(-)}$ and the effective charge $-\mu$). In this case, the Hamiltonian of the quantum scalar field is positive defined and the corresponding vacuum is the uncharged Fock state with minimal energy. The normalized plane waves $\phi_m^{(+)}$ and $\phi_m^{(-)}$ describe a single-particle state with the energy $\varepsilon^{(+)} > 0$ and a single-antiparticle state with the energy $-\varepsilon^{(-)} > 0$ as well as with Zeeman energy terms for positive and negative projections of a single-magnon magnetic moment,

$$\pm\varepsilon^{(\pm)} = \sqrt{v_s^2\mathbf{p}^2 + \Delta^2} \pm \mu B. \tag{12}$$

This is a charge conjugation rule. It implies also the following interpretation of momentum quantum number: the quantum number $\mathbf{p}$ is the physical momentum of a particle while the physical momentum of an antiparticle is $-\mathbf{p}$. One sees that $\pm(\varepsilon^{(\pm)} - U)$ are kinetic energies (the energy gap Δ is included in the definition) of a particle and an antiparticle, respectively. In this way, we establish a relation between particle-antiparticle and quantum magnon interpretations. This reminds the electron-hole interpretation of states in semiconductors. In the framework of the effective scalar QED the collinear ground state with the given Néel vector can be considered as a vacuum state. Then a single-particle (antiparticle) is a quant of the quantum magnon field with positive (negative) magnetic moment projection μ ($-\mu$) onto the direction of the magnetic field, respectively. In which follows, keeping in mind this particle-antiparticle interpretation, we'll call these quanta with opposite effective charges magnon and antimagnon, respectively.

In general case with inhomogeneous x-dependent B field of a step form, we chose the potential energy $U(x)$ in the form of a monotonically decreasing function, $\partial_x U(x) < 0$. If the field derivative $\partial_x U(x)$ is not very big, then the terms $\pm[\varepsilon - U(x)]$ can be considered as kinetic energies of particle and antiparticle, respectively. One sees that the kinetic energy of the particle (antiparticle) would grow monotonously along (in inverse direction of) the axis x, that is, the magnon and antimagnon accelerate under the influence of the field derivative $\partial_x U(x)$ in opposite directions and form a spin current.

If the field derivative $\partial_x U(x)$ (playing here the role of the electric field in the scalar electrodynamics) acting on magnons produces big enough work, then the magnon-antimagnon pair production from the corresponding vacuum may take place. We assume that the action of the field derivative contributes significantly to mean numbers of created pairs in the restricted region S_{int} between two planes $x = x_{\text{L}}$ and $x = x_{\text{R}}$ during the sufficiently large (macroscopic) time period T. It is either negligible or switches off in the macroscopic regions $S_{\text{L}} = (x_{\text{FL}}, x_{\text{L}}]$ on the left of the plane $x = x_{\text{L}}$ and in $S_{\text{R}} = [x_{\text{R}}, x_{\text{FR}})$ on the right of the plane $x = x_{\text{R}}$. We also assume that the points x_{FL} and x_{FR} are separated from the origin by macroscopic but finite distances. In this way, the magnetic field B plays the role of an electric potential step, which we call shortly an x step. Its magnitude is

$$\delta U = U_{\text{L}} - U_{\text{R}} > 0,\ U_{\text{L}} = U(-\infty),\ U_{\text{R}} = U(\infty). \tag{13}$$

We distinguish two types of x steps, noncritical $\delta U < 2\Delta$, and critical $\delta U > 2\Delta$. The pair production from the vacuum occurs due to the critical x step.

Nonperturbative calculations of the vacuum instability effects in the framework of strong-field QED with x steps are possible if there are special complete sets of exact solutions of the corresponding relativistic wave equations (solutions of the Klein-Gordon equations in the case under consideration) orthonormalized on a t-const. hyperplane; see Ref. [14]. In the strong-field QED with x step, which acts during the macroscopic time T, one can construct such sets of solutions, see below. In our work these solutions are chosen in the form of stationary plane waves with given real longitudinal momenta p^{L} and p^{R} in the regions S_{L} and S_{R}. Moreover, it is necessary to construct two types of complete sets of solution in the form of Eq. (8). The first one ${}_\zeta\phi_m(X)$ is constructed with the help of the functions $\varphi_m(x)$ denoted as ${}_\zeta\varphi_m(x)$ and the second one, ${}^\zeta\phi_m(X)$, with the help of functions $\varphi_m(x)$ denoted as ${}^\zeta\varphi_m(x)$. Asymptotically, these functions have the following forms:

$$\begin{aligned}
{}_\zeta\varphi_m(x) &= {}_\zeta\mathcal{N}\exp[ip^{\text{L}}(x - x_{\text{L}})],\ x \in S_{\text{L}},\\
{}^\zeta\varphi_m(x) &= {}^\zeta\mathcal{N}\exp[ip^{\text{R}}(x - x_{\text{R}})],\ x \in S_{\text{R}},\\
p^{\text{L}} &= \frac{\zeta}{v_s}\sqrt{[\pi_0(\text{L})]^2 - \pi_\perp^2},\\
p^{\text{R}} &= \frac{\zeta}{v_s}\sqrt{[\pi_0(\text{R})]^2 - \pi_\perp^2},
\end{aligned} \tag{14}$$

where $\zeta = \pm$, ${}_\zeta\mathcal{N}$ and ${}^\zeta\mathcal{N}$ are normalization constants that shall be determined below. We introduce the quantities $\pi_0(\text{L/R}) = \varepsilon - U_{\text{L/R}}$. Note that $\pi_0(\text{R}) > \pi_0(\text{L})$.

By analogy with how this is done in the time-independent potential scattering due to noncritical steps, it is assumed that there exist time-independent observables in the presence of critical x steps. For example, it seems quite natural that the pair-production rate and the flux of created particles are constant during the macroscopic time T. This means that a leading contribution to the number density of created particle-antiparticle pairs is assumed to be proportional to the large dimensionless parameter $\sqrt{v_s|\partial_x U|}T$ and is independent from switching-on and -off if this parameter satisfies inequality

$$T \gg (v_s|\partial_x U|)^{-1/2} \max\{1, \Delta^2/v_s|\partial_x U|\}. \quad (15)$$

It is clear that the process of pair creation is transient. Nevertheless, the condition of the smallness of backreaction shows there is a window in the parameter range of $\partial_x U$ and T where the constant field approximation is consistent [18].

For any two solutions $\Phi(X)$ and $\Phi'(X)$ of the Klein-Gordon equation, the inner product on the hyperplane $x =$ const has the form of a longitudinal flux:

$$(\Phi, \Phi')_x = i \int [\Phi'(X)\partial_x \Phi^*(X) - \Phi^*(X)\partial_x \Phi'(X)] dt d\mathbf{r}_\perp. \quad (16)$$

Note, that this flux is proportional to the effective charge current. We consider the system under consideration in a large space-time box that has a spatial area $V_\perp = K_y K_z$ and the time dimension T, where all K_y, K_z, and T are macroscopically large. In general the wave packets $\Phi(X)$ and $\Phi'(X)$ can be decomposed into plane waves $\phi_m(X)$ and $\phi'_m(X)$. Along with the introduced plane waves, it is assumed that all the solutions $\Phi(X)$ are periodic under transitions from one box to another. Then the integration in integral (16) over the transverse coordinates runs from $-K_j/2$ to $+K_j/2$, $j = y, z$, and over the time t from $-T/2$ to $+T/2$. We assume that the macroscopic time T is the system surveillance time. Under these suppositions, one can verify, integrating by parts, that the inner product (16) does not depend on x.

Nontrivial solutions ${}_\zeta\phi_m(X)$ and ${}^\zeta\phi_m(X)$ exist only for the quantum numbers m that obey the following relations:

$$[\pi_0(\mathrm{L/R})]^2 > \pi_\perp^2 \Longleftrightarrow \begin{cases} \pi_0(\mathrm{L/R}) > \pi_\perp \\ \pi_0(\mathrm{L/R}) < -\pi_\perp \end{cases}. \quad (17)$$

In the case of critical x steps and for $2\pi_\perp \leqslant \delta U$ there exist five ranges Ω_k, $k = 1, ..., 5$, of quantum numbers m,

$$\begin{aligned} &\pi_0(\mathrm{L}) \geqslant \pi_\perp \text{ if } m \in \Omega_1, \\ &|\pi_0(\mathrm{L})| < \pi_\perp,\ \pi_0(\mathrm{R}) > \pi_\perp \text{ if } m \in \Omega_2, \\ &\pi_0(\mathrm{L}) \leqslant -\pi_\perp,\ \pi_0(\mathrm{R}) \geqslant \pi_\perp \text{ if } m \in \Omega_3, \\ &\pi_0(\mathrm{L}) < -\pi_\perp,\ |\pi_0(\mathrm{R})| < \pi_\perp \text{ if } m \in \Omega_4, \\ &\pi_0(\mathrm{R}) \leqslant -\pi_\perp \text{ if } m \in \Omega_5, \end{aligned} \quad (18)$$

where the solutions ${}_\zeta\phi_m(X)$ and ${}^\zeta\phi_m(X)$ have similar forms and properties for a given $\pi_\perp$. The manifold of all the quantum numbers m is denoted by Ω, so that $\Omega = \Omega_1 \cup \cdots \cup \Omega_5$. In the ranges Ω_2 and Ω_4 we deal with standing waves completed by linear superpositions of solutions ${}_\pm\phi_m(X)$ and ${}^\pm\phi_m(X)$ with corresponding longitudinal fluxes that are equal in magnitude for a given m. In fact, $\phi_m(X)$ for $m \in \Omega_2$ are wave functions that describe an unbounded motion in $x \to \infty$ direction while $\phi_m(X)$ for $m \in \Omega_4$ are wave functions that describe an unbounded motion toward $x = -\infty$. It was demonstrated in the framework of strong-field QED with x steps; see Secs. V and VII and Appendices C1 and C2 in Ref. [14], by using one-particle mean currents and the energy fluxes that, depending on the asymptotic behavior in the regions S_L and S_R, the plane waves ${}_\pm\phi_m(X)$ and ${}^\pm\phi_m(X)$ are identified unambiguously as components of the initial and final wave packets of particles and antiparticles.

The plane waves are subjected to the following orthonormality conditions:

$$\begin{aligned} ({}_\zeta\phi_m, {}_{\zeta'}\phi_{m'})_x &= \zeta\delta_{\zeta,\zeta'}\delta_{m,m'}, \\ ({}^\zeta\phi_m, {}^{\zeta'}\phi_{m'})_x &= \zeta\delta_{\zeta,\zeta'}\delta_{m,m'}. \end{aligned} \quad (19)$$

In fact, integrals (19) represent the flux densities of particles with given m . The normalization factors with respect to the inner product (16) are

$$\begin{aligned} &{}_\zeta\mathcal{N} = {}_\zeta CY,\ {}^\zeta\mathcal{N} = {}^\zeta CY,\ Y = (V_\perp T)^{-1/2}, \\ &{}_\zeta C = |2p^\mathrm{L}|^{-1/2},\ {}^\zeta C = |2p^\mathrm{R}|^{-1/2}. \end{aligned} \quad (20)$$

In the $K_j \to \infty$ and $T \to \infty$ limits, one has to replace $\delta_{m,m'}$ by the factor $\delta(\varepsilon - \varepsilon')\delta(\mathbf{p}_\perp - \mathbf{p}'_\perp)$ in the normalization conditions (19) and then to set $Y = (2\pi)^{-(d-1)/2}$.

Stationary plane waves of type (14) are usually used in potential scattering theory, where they represent one-particle states with corresponding conserved longitudinal currents. It is clear that in the ranges Ω_1 and Ω_2 we have deal with states of a particle whereas in the ranges Ω_4 and Ω_5 the plane waves describe states of an antiparticle. In these ranges particles and antiparticles are subjected to the scattering and the reflection only. Such one-particle consideration does not work in the range Ω_3, where the many-particle quantum field theory consideration is essential. Note that the range Ω_3 is often referred to as the Klein zone. In contrast to the case Ω_1 (and Ω_5), where signs of $\pi_0(\mathrm{L})$ and $\pi_0(\mathrm{R})$ coincide, they are opposite in the Klein zone. This reflects the fact that the interpretation of overlapping between amplitudes ${}_\zeta\phi_m(X)$ and ${}^\zeta\phi_m(X)$ using the quantities $\pi_0(\mathrm{L/R})$ by analogy with potential scattering theory can be erroneous.

Indeed, it is known that attempts to consider the overlapping amplitudes in the range Ω_3 as amplitudes of particle transmission and reflection as its works in the relativistic quantum mechanics led (and often still lead researchers) to contradictions and paradoxes, before the advent of consistent consideration in the framework of QFT. Therefore, we will discuss this issue again below.

It is assumed that each pair of solutions ${}_\zeta\phi_m(X)$ and ${}^\zeta\phi_m(X)$ with given quantum numbers $m \in \Omega_1 \cup \Omega_3 \cup \Omega_5$ is complete in the space of solutions with each given m. Due to Eq. (19) the corresponding mutual decompositions of such solutions have the form:

$$\begin{aligned} {}^\zeta\phi_m(X) &= {}_+\phi_m(X) g({}_+|^\zeta) - {}_-\phi_m(X) g({}_-|^\zeta), \\ {}_\zeta\phi_m(X) &= {}^+\phi_m(X) g({}^+|_\zeta) - {}^-\phi_m(X) g({}^-|_\zeta), \end{aligned} \quad (21)$$

where decomposition coefficients g are given by the relations:

$$\begin{aligned} &({}_\zeta\phi_m, {}^{\zeta'}\phi_{m'})_x = \delta_{m,m'} g({}_\zeta|^{\zeta'}), \\ &\qquad g({}^{\zeta'}|_\zeta) = g({}_\zeta|^{\zeta'})^*, \quad m \in \Omega_1 \cup \Omega_3 \cup \Omega_5. \end{aligned} \quad (22)$$

Substituting Eq. (21) into the orthonormality conditions (19), we derive the following unitary relations for the decomposition coefficients:

$$g(^{\zeta'}|_{+})g(_{+}|^{\zeta}) - g(^{\zeta'}|_{-})g(_{-}|^{\zeta}) = \zeta\delta_{\zeta,\zeta'},$$
$$g(_{\zeta'}|^{+})g(^{+}|_{\zeta}) - g(_{\zeta'}|^{-})g(^{-}|_{\zeta}) = \zeta\delta_{\zeta,\zeta'}. \quad (23)$$

In particular, these relations imply that

$$|g(_{-}|^{+})|^2 = |g(_{+}|^{-})|^2, \ \ |g(_{+}|^{+})|^2 = |g(_{-}|^{-})|^2,$$
$$\frac{g(_{+}|^{-})}{g(_{-}|^{-})} = \frac{g(^{+}|_{-})}{g(^{+}|_{+})}. \quad (24)$$

One can see that all the coefficients g can be expressed via only two of them, e.g., via $g(_{+}|^{+})$ and $g(_{+}|^{-})$. However, even the latter coefficients are not completely independent, they are related as follows:

$$|g(_{+}|^{+})|^2 - |g(_{+}|^{-})|^2 = 1. \quad (25)$$

Nevertheless, in what follows, we will use both coefficients $g(_{+}|^{-})$ and $g(_{+}|^{+})$ in our consideration. This maintains a certain symmetry in important relations.

In canonical quantization of field theory, state vectors in the Fock space are global objects, that is, the definition of vacuum and particle (antiparticle) states has to be realized in the whole space in a given time instant. To this end one has to use a time-independent inner product of solutions of the Klein-Gordon equation. In the case under consideration, this inner product must be adopted for the stationary plane waves (8). We recall that the inner product between two solutions of the Klein-Gordon equation can be defined on t-const. hyperplane as a charge (in the case under consideration, the role of an effective charge plays the magnetic moment). Note that physical states are wave packets that vanish on the remote boundaries, that is why the effective charge of the scalar field is finite. It allows one to integrate by parts in the inner product neglecting boundary terms. The latter property provides the inner product to be time independent. However, considering the stationary plane waves (8) that are generalized states, which do not vanish at the spatial infinity, one should take some additional steps. In the case under consideration, the motion of particles in the x direction is unlimited, therefore the corresponding wave functions cannot be subjected to any periodic boundary conditions in the x direction without changing their physical meaning. For this reason one has to use a special kind of the volume regularization; see Sec. C2 and Appendix B in Ref. [14] and Sec. 2.1 in Ref. [15] for details. Following Refs. [14,15], we mean that the strong field $\partial_x U$ under consideration is located inside the region $S_{\rm int}$ during the time T. Consequently, causally related to the area $S_{\rm int}$ can there be only such parts of the areas $S_{\rm L} = (x_{\rm FL}, x_{\rm L}]$ and $S_{\rm R} = [x_{\rm R}, x_{\rm FR})$, which are located from it at distances not exceeding $v_s T$. The field derivative $\partial_x U$ is either negligible or switches off in these macroscopic regions. We assume that there exist some macroscopic but finite parameters $K^{(\rm L/R)}$ of the volume regularization which are in spatial areas where the contribution of the field $\partial_x U$ is negligible, $|x_{\rm FL}| > K^{(\rm L)} \gg |x_{\rm L}| > 0$ and $x_{\rm FR} > K^{(\rm R)} \gg x_{\rm R} > 0$.

Following Refs. [14,15] we propose the time-independent inner product between two solutions $\Phi(X)$ and $\Phi'(X)$ of the Klein-Gordon equation on t-const. hyperplane as

$$(\Phi, \Phi') = \frac{1}{v_s^2}\int_{V_\perp} d\mathbf{r}_\perp \int_{-K^{(\rm L)}}^{K^{(\rm R)}} \Psi^\dagger(X)\sigma_1\Psi(X)dx,$$
$$\Psi(X) = \begin{pmatrix} i\partial_0 - U(x) \\ 1 \end{pmatrix}\Phi(X), \quad (26)$$

where the integral over the spatial volume $V_\perp$ is completed by an integral over the interval $[K^{(\rm L)}, K^{(\rm R)}]$ in the x direction and σ_1 is a Pauli matrix. The parameters $K^{(\rm L/R)}$ are assumed sufficiently large in final expressions. First, we note that states with different quantum numbers m are independent, therefore decompositions of wave packets Φ into the plane waves (8) in Eq. (26) do not contain interference terms. That is why it is enough to consider Eq. (26) only for a particular case of plane waves $^{\zeta}\phi_m$ and $_{\zeta}\phi_m$ with equal m. Assuming that the areas $S_{\rm L}$ and $S_{\rm R}$ are much wider than the area $S_{\rm int}$,

$$K^{(\rm L)} - |x_{\rm L}|, K^{(\rm R)} - x_{\rm R} \gg x_{\rm R} - x_{\rm L}, \quad (27)$$

and the potential energy $U(x)$ is a continuous function, the principal value of integral (26) is determined by integrals over the areas $x \in [-K^{(\rm L)}, x_{\rm L}]$ and $x \in [x_{\rm R}, K^{(\rm R)}]$, where the field derivative $\partial_x U$ is negligible small. Thus, it is possible to evaluate integrals of the form of Eq. (26) for any form of the external field, using only the asymptotic behavior (14) of functions in the regions $S_{\rm L}$ and $S_{\rm R}$ where particles are free. The form of the field $\partial_x U$ in the area $S_{\rm int}$ affects only coefficients g entering into the mutual decompositions of the solutions given by Eq. (21). One can see that the norms of the plane waves $_{\zeta}\phi_m$ and $_{\zeta}\phi_m$ with respect to the inner product (26) are proportional to the macroscopically large parameters $\tau^{(\rm L)} = K^{(\rm L)}/v^{\rm L}$ and $\tau^{(\rm R)} = K^{(\rm R)}/v^{\rm R}$, where $v^{\rm L} = v_s^2|p^{\rm L}/\pi_0(\rm L)| > 0$ and $v^{\rm R} = v_s^2|p^{\rm R}/\pi_0(\rm R)| > 0$ are absolute values of the longitudinal velocities of particles in the regions $S_{\rm L}$ and $S_{\rm R}$, respectively; see Sec. IIIC.2 and Appendix B in Ref. [14] for details.

It was shown (see Appendix B in Ref. [14]) that the following couples of plane waves are orthogonal with respect to the inner product (26),

$$(_{\zeta}\phi_m, ^{-\zeta}\phi_m) = 0, \ \ m \in \Omega_1 \cup \Omega_5; \ (_{\zeta}\phi_m, ^{\zeta}\phi_m) = 0, \ \ m \in \Omega_3, \quad (28)$$

if the parameters of the volume regularization $\tau^{(\rm L/R)}$ satisfy the condition

$$\tau^{(\rm L)} - \tau^{(\rm R)} = O(1), \quad (29)$$

where $O(1)$ denotes terms that are negligibly small in comparison with the macroscopic quantities $\tau^{(\rm L/R)}$. One can see that $\tau^{(\rm R)}$ and $\tau^{(\rm L)}$ are macroscopic times of motion of particles and antiparticles in the areas $S_{\rm R}$ and $S_{\rm L}$, respectively, and they are equal,

$$\tau^{(\rm L)} = \tau^{(\rm R)} = \tau. \quad (30)$$

It allows one to introduce an unique time of motion τ for all the particles in the system under consideration. This time can be interpreted as a system monitoring time during its evolution. Under condition (29) the norms of the plane waves on the t-const. hyperplane are

$$({}_\zeta\phi_m,\ {}_\zeta\phi_m) = ({}^\zeta\phi_m,\ {}^\zeta\phi_m) = \kappa\mathcal{M}_m,\ \kappa = \begin{cases} 1,\ m\in\Omega_1 \\ -1,\ m\in\Omega_5 \end{cases},$$

$$(\phi_m,\ \phi_m) = \mathcal{M}_m,\ m\in\Omega_2\,;(\phi_m,\ \phi_m) = -\mathcal{M}_m,\ m\in\Omega_4,$$

$$({}_\zeta\phi_m,\ {}_\zeta\phi_m) = -({}^\zeta\phi_m,\ {}^\zeta\phi_m) = \mathcal{M}_m,\ \ m\in\Omega_3,$$

$$\mathcal{M}_m = 2\frac{\tau}{T}\begin{cases} |g({}_+|^+)|^2,\ m\in\Omega_1\cup\Omega_5 \\ 1,\ m\in\Omega_2\cup\Omega_4 \\ |g({}_+|^-)|^2,\ m\in\Omega_3 \end{cases}; \tag{31}$$

see Appendix B in Ref. [14] for details. The ± signs of integrals in Eq. (31) correspond to the signs of the effective charge (magnetic moment) of a particle; see Appendix A) for details.

Thus, there are constructed two linearly independent couples of complete on the t-const. hyperplane states with a given m that are either initial ("in") or final ("out") states,

$$\begin{aligned}
\text{in states}: \phi_{m_1}^{(\text{in},+)} &= {}_+\phi_{m_1},\ \phi_{m_1}^{(\text{in},-)} = {}^-\phi_{m_1};\\
\phi_{m_5}^{(\text{in},+)} &= {}^+\phi_{m_5},\ \phi_{m_5}^{(\text{in},-)} = {}_-\phi_{m_5};\\
\phi_{m_3}^{(\text{in},+)} &= {}_-\phi_{m_3},\ \phi_{m_3}^{(\text{in},-)} = {}^-\phi_{m_3};\\
\text{out states}: \phi_{m_1}^{(\text{out},-)} &= {}_-\phi_{m_1},\ \phi_{m_1}^{(\text{out},+)} = {}^+\phi_{m_1};\\
\phi_{m_5}^{(\text{out},-)} &= {}^-\phi_{m_5},\phi_{m_5}^{(\text{out},+)} = {}_+\phi_{m_5};\\
\phi_{m_3}^{(\text{out},+)} &= {}_+\phi_{m_3},\ \ \phi_{m_3}^{(\text{out},-)} = {}^+\phi_{m_3},\ \ m_k\in\Omega_k.
\end{aligned} \tag{32}$$

Note that standing waves $\phi_m(X)$ for $m\in\Omega_2\cup\Omega_4$ are the same for initial and final sets. In the ranges Ω_1 and Ω_2 we have deal with states of a particle whereas in the ranges Ω_4 and Ω_5 the plane waves describe states of an antiparticle. In these cases, description of the one-particle scattering and reflection in the framework of strong-field QED is the same as in the framework of the potential scattering. In particular, the sign of the flux density, given by Eq. (19), allows one to determine initial and final states, this fact was already used above. In doing so we take into account the charge conjugation, which implies that the physical longitudinal momentum of an antiparticle differs in sign from the quantum numbers $p^{\rm L}$ and $p^{\rm R}$; see Appendix A for more details. We also note that in the range Ω_3 the choice of initial and final states is not so obvious and will be justified below.

B. Quantization in terms of particles

In the absence of an explicit time evolution of physical quantities presented by the stationary plane waves (8), an interpretation of these plane waves as states of initial and final particle or antiparticle in the Klein zone Ω_3 requires consideration the process of breaking vacuum stability, taking into account the effects of switching on and off. It is clear that the effect of pair creation is transient and therefore must be limited in time. However, it can be assumed that moderate intensity of pair creation one can neglect backreaction and that the external field remains unchanged for some macroscopic period of time T. Then physically it make sense to believe that the field of the x step, $\partial_x B(x)$, should be considered as a part of a time-dependent inhomogeneous field $E_{\rm pristine}$ directed along the x direction, which was switched on at the time instant t_1 sufficiently fast before the time instant $t_{\rm in}$, by this time it had time to spread to the whole area $S_{\rm int}$ and disappear in the macroscopic regions $S_{\rm L}$ and $S_{\rm R}$. Thus, in the region $S_{\rm int}$ a time-independent field configuration $E_{\rm pristine} = \partial_x B(x)$ is formed, whereas in the two regions $S_{\rm L}$ and $S_{\rm R}$ the field $\partial_x B(x)$ is zero. However, there are two different uniform fields $B(x_{\rm L}) \neq 0$ in $S_{\rm L}$ and $B(x_{\rm R}) \neq 0$ in $S_{\rm R}$; $\mu[B(x_{\rm L}) - B(x_{\rm R})] = \delta U$. Such a field configuration remains unchanged for quite a long time T. Then at time instant t_2 the field $E_{\rm pristine}$ is switched off sufficiently fast just after the time instant $t_{\rm out} = t_{\rm in} + T$. The main effect of the pair creation occurs in the region $S_{\rm int}$ during the time period T. However, the switching on and off of an external field $E_{\rm pristine}$ may also lead to a pair creation. We believe that its contribution is much less then the one in the region $S_{\rm int}$. The created magnons and antimagnons enter in the regions $S_{\rm L}$ and $S_{\rm R}$, respectively, already as free particles and remain there separately after switching off the field $E_{\rm pristine}$. Thus, by observing over a long period of time T fluxes of created particles crossing the boundaries of the field at the planes $x = x_{\rm L}$ and $x = x_{\rm R}$, respectively, it is possible to determine the parameters of these fluxes without waiting for switching off the field $E_{\rm pristine}$. This can be done using stationary plane wave solutions of the Klein-Gordon equation in the framework of approach presented in Ref. [14]. (We note that some relevant details of scalar field quantization in the presence of critical potential steps is given in Appendix A).

Now you can set a quantitative criterion that allows one to evaluate the accuracy of the approximation in which the effects of switching on and off are neglected. Let $N^{\rm true}$ be the total number of pairs created from the vacuum due to field $E_{\rm pristine}$ from the time it is turned on t_1 to the time it is turned off t_2, and $N^{\rm cr}$ be the total number of pairs created from the vacuum by the field $\partial_x B(x)$ from a moment $t_{\rm in} > t_1$ to a moment $t_{\rm out} = t_{\rm in} + T$.

According to a widely accepted formulation of QED with a strong background there exist initial $|0, \text{true in}\rangle$ and final $|0, \text{true out}\rangle$ vacua of free particles related by an unitary transformation under the condition that $N^{\rm true}$ is finite; see, e.g., Refs. [19,20]. In the Heisenberg representation these vacua are null vectors for particle number operators $\hat{N}(t_1)$ and $\hat{N}(t_2)$,

$$\hat{N}(t_1)|0, \text{true in}\rangle = 0,\ \hat{N}(t_2)|0, \text{true out}\rangle = 0, \tag{33}$$

where $\hat{N}(t_1)$ is free particle number operator at the time t_1 and $\hat{N}(t_2)$ is free particle number operator at the time t_2. The difference of the true final vacuum from the initial one can be specified by the total number of pairs created from the vacuum,

$$N^{\rm true} = \sum_m N_m^{\rm true} = \langle 0, \text{true in}|\hat{N}(t_2)|0, \text{true in}\rangle, \tag{34}$$

and $N_m^{\rm true}$ are differential mean numbers of created from the vacuum pairs with given quantum numbers m. If there exists a complete set of solutions of the Klein-Gordon equation with the background under consideration, then one can find all the numbers $N_m^{\rm true}$. From general considerations, it is possible to establish some properties of these numbers without knowing their form explicitly. It seems quite natural that the pair-production rate and the flux of created particles are constant during the macroscopic time T. It means that a leading contribution to the number density N^{true} is assumed to be proportional to the large dimensionless parameter $\sqrt{v_s|\partial_x U|}T$ and is independent from switching-on and -off if this parameter satisfies inequality (15). In this case, the numbers N^{true} can be approximated by N^{cr},

$$N^{\mathrm{true}} = N^{\mathrm{cr}}\{1 + O([\sqrt{v_s|\partial_x U|}T]^{-1})\},$$
$$N^{\mathrm{cr}} = \sum_{m\in\Omega_3} N_m^{\mathrm{cr}}, \tag{35}$$

where the terms $N_m^{\mathrm{cr}} = N_m^{\mathrm{true}}$ if $m \in \Omega_3$ appear due to the time-independent part of the field $E_{\mathrm{pristine}} = \partial_x B(x)$ and do not depend on the oscillations related to fast switching-on and -off of the field E_{pristine}. This possibility does exist due to the fact that both fast switching-on and -off produce particle-antiparticle pairs with quantum numbers in a tiny range of the kinetic energy, such that one can neglect the corresponding contributions to total characteristics of vacuum instability that are determined by sums over all the kinetic energies; see Ref. [15] for more details.

The numbers N_m^{cr} can be found by using the decomposition coefficients g, given by Eq. (22). Here one interprets stationary plane wave solutions of the Klein-Gordon equation as states of initial and final particle or antiparticle in the Klein zone Ω_3.

Neglecting effects of fast switching-on and -off, one can use instead of the true vacua $|0,\mathrm{true\ in}\rangle$ and $|0,\mathrm{true\ out}\rangle$ some states $|0,\mathrm{in}\rangle$ and $|0,\mathrm{out}\rangle$, respectively. Namely, we choose these states as ones with minimum kinetic energies (kinetic energies of the states $|0,\mathrm{in}\rangle$ and $|0,\mathrm{out}\rangle$ are the same) and such that the leading contribution to the quantity N^{true} is determined by the number N^{cr}. In the Heisenberg representation these vacua are null vectors of the particle number operators $\hat{N}(\mathrm{in})$ and $\hat{N}(\mathrm{out})$,

$$\hat{N}(\mathrm{in})|0,\mathrm{in}\rangle = 0,\ \ \hat{N}(\mathrm{out})|0,\mathrm{out}\rangle = 0. \tag{36}$$

The difference between these vectors is determined by the total number of created pairs,

$$N^{\mathrm{cr}} = \langle 0,\mathrm{in}|\hat{N}(\mathrm{out})|0,\mathrm{in}\rangle. \tag{37}$$

In what follows, the number operators $\hat{N}(\mathrm{in})$ and $\hat{N}(\mathrm{out})$ will be expressed via corresponding annihilation and creation operators and the states $|0,\mathrm{in}\rangle$ and $|0,\mathrm{out}\rangle$ are called the initial and the final vacua, respectively. Accordingly, magnon and antimagnon excitations over these vacua are called initial and final particles, respectively. Mean fluxes of the effective charge and the energies of the vacuum states $|0,\mathrm{in}\rangle$ and $|0,\mathrm{out}\rangle$ are quite distinct. That allows us to define unambiguously these vacua and to construct initial and final states in a Fock space, using the plane waves (8); see Appendix A for details.

One can decompose the Heisenberg operators of the scalar fields $\hat{\Phi}(X)$ and $\hat{\Phi}(X)^\dagger$ into solutions of either the initial or final complete sets (32). It is useful to represent $\hat{\Phi}(X)$ (and $\hat{\Phi}(X)^\dagger$, respectively) as sums of five operators, $\hat{\Phi}(X) = \sum_{k=1}^{5}\hat{\Phi}_k(X)$, where the operators $\hat{\Phi}_k(X)$ are defined in the ranges Ω_k; see Eqs. (A13) and (A14) in Appendix A for details. The operators $\hat{\Phi}_k(X)$, $k = 1, 2$, are decomposed via the creation and annihilation operators a and $a^\dagger$ of the magnons, while the operators $\hat{\Phi}_k(X)$, $k = 4, 5$, are decomposed via the creation and annihilation operators b and $b^\dagger$ of antimagnons. In the range Ω_3 the operators $\hat{\Phi}_3(X)$ and $\hat{\Phi}_3^\dagger(X)$ are decomposed via creation and annihilation operators of both magnons and antimagnons,

$$\hat{\Phi}_3(X) = \sum_{m\in\Omega_3}\mathcal{M}_m^{-1/2}[\,{}_{-}a_m(\mathrm{in})\,{}_{-}\phi_m(X) + {}^{-}b_m^\dagger(\mathrm{in})\,{}^{-}\phi_m(X)]$$
$$= \sum_{m\in\Omega_3}\mathcal{M}_m^{-1/2}[\,{}_{+}a_m(\mathrm{out})\,{}_{+}\phi_m(X) + {}^{+}b_m^\dagger(\mathrm{out})\,{}^{+}\phi_m(X)]. \tag{38}$$

Here $\mathcal{M}_m$ are normalization factors given by Eq. (31). All the operators labeled by the argument "in" are interpreted as the operators of the initial particles, whereas all the operators labeled by the argument "out" are interpreted as the operators of the final particles. This identification can be confirmed as follows. The equal-time commutation relations given by Eq. (A3) in Appendix A yield the standard commutation rules for the creation and annihilation in- or out-operators introduced. The vacuum vectors $|0,\mathrm{in}\rangle$ and $|0,\mathrm{out}\rangle$ are null vectors for all annihilation operators a and b given by Eq. (A14) in Appendix A. In particular, in the range Ω_3 nonzero commutators are

$$[\,{}_{\mp}a_m(\mathrm{in/out}),\ {}_{\mp}a_{m'}^\dagger(\mathrm{in/out})]$$
$$= [\,{}^{\mp}b_m(\mathrm{in/out}),\ {}^{\mp}b_m^\dagger(\mathrm{in/out})] = \delta_{m,m'}, \tag{39}$$

and one-particle states of initial (final) magnon and antimagnon have the form

$${}_{\mp}a_{m'}^\dagger(\mathrm{in/out})|0,\mathrm{in/out}\rangle,\ \ {}^{\mp}b_m^\dagger(\mathrm{in/out})|0,\mathrm{in/out}\rangle. \tag{40}$$

Using the both alternative decompositions (38) for $\hat{\Phi}_3(X)$ and the orthonormality conditions (19), one can find the following linear canonical transformation between the introduced in- and out-creation and annihilation operators

$${}_{+}a_m(\mathrm{out}) = g({}^{-}|_{+})^{-1}g({}^{-}|_{-})\,{}_{-}a_m(\mathrm{in}) - g({}^{-}|_{+})^{-1}\,{}^{-}b_m^\dagger(\mathrm{in}),$$
$${}^{+}b_m^\dagger(\mathrm{out}) = -g({}_{-}|^{+})^{-1}\,{}_{-}a_m(\mathrm{in}) + g({}_{-}|^{+})^{-1}g({}_{-}|^{-})\,{}^{-}b_m^\dagger(\mathrm{in}). \tag{41}$$

One can verify that the transformation is unitary; see Sec. VII D in Ref. [14]. The inverse transformation reads

$${}_{-}a_m(\mathrm{in}) = g({}^{+}|_{-})^{-1}g({}^{+}|_{+})\,{}_{+}a_m(\mathrm{out}) + g({}^{+}|_{-})^{-1}\,{}^{+}b_m^\dagger(\mathrm{out}),$$
$${}^{-}b_m^\dagger(\mathrm{in}) = g({}_{+}|^{-})^{-1}\,{}_{+}a_m(\mathrm{out}) + g({}_{+}|^{-})^{-1}g({}_{+}|^{+})\,{}^{+}b_m^\dagger(\mathrm{out}). \tag{42}$$

The $\pm$ sign of integrals in Eq. (31) corresponds to the sign of the effective charge (magnetic moment) of a particle, which justifies the above interpretation of states as magnons and antimagnons in the ranges Ω_k, $k = 1, 2, 4, 5$. In the ranges Ω_1 and Ω_5 the flux densities of particles through the surfaces $x = x_{\mathrm{L}}$ and $x = x_{\mathrm{R}}$ given by Eq. (19) allow one to define initial and final states of particles. In particular, it can be seen that the sign of the flux densities of magnons with a given m is equal to ζ in the range Ω_1, while the sign of the flux densities of antimagnons with a given m is equal to $-\zeta$ in the range Ω_5. The corresponding reasons in the framework of QFT are presented in Appendix A. One can see that the partial vacua in the Fock subspaces with a given m are stable in Ω_k, $k = 1, 2, 4, 5$. In the ranges Ω_1 and Ω_5 we meet a realization of rules of the potential scattering theory in the framework of QFT and see that relative probabilities of reflection and transmission (under the condition that the vacuum remain the vacuum),

$$|R_m|^2 = |g({}_+|^+)^{-1}g({}_-|^+)|^2,\ \ |T_m|^2 = |g({}_+|^+)|^{-2}, \quad (43)$$

coincide with mean currents of reflected particles $J_R = |R_m|^2$ and transmitted particles $J_T = |T_m|^2$. The correct result $J_R + J_T = 1$ follows from the unitary relation (25).

In contrast to the ranges Ω_1 and Ω_5, one can see from Eq. (31) that in the range Ω_3 the magnetic moment of the particle states ${}_\zeta\phi_m$ is positive while the magnetic moment of the particle states ${}^\zeta\phi_m$ is negative. This allows us to use the magnon/antimagnon classification for the creation and annihilation operators in Eq. (38). Considering diagonalized forms of the kinetic energy and magnetic moment operators given by Eq. (A21) in Appendix A, we see that such identification is also confirmed in the framework of QFT. In particular, we see that the spectrum of the kinetic energy operator is positively defined and one-particle states ${}_\zeta\phi_m$ represent magnons with the kinetic energy ${}_\zeta E_m > 0$ and the magnetic moment μ, whereas one-particle states ${}^\zeta\phi_m$ are antimagnons with the kinetic energy $-{}^\zeta E_m > 0$.

Since transformations (41) and (42) entangle annihilation and creation operators, the vacua $|0, \text{in}\rangle$ and $|0, \text{out}\rangle$ are essentially different. The total vacuum-to-vacuum transition amplitude c_v is formed due to the vacuum instability in the range Ω_3. Differential mean numbers $N_m^a(\text{out})$ and $N_m^b(\text{out})$, $m \in \Omega_3$, of the magnons and antimagnons, respectively, created from the vacuum are equal, $N_m^b(\text{out}) = N_m^a(\text{out}) = N_m^{\text{cr}}$, and have the forms

$$\begin{aligned} N_m^a(\text{out}) &= \langle 0, \text{in}|\, {}_+a_m^\dagger(\text{out})\, {}_+a_m(\text{out})|0, \text{in}\rangle \\ &= |g({}_+|^-)|^{-2}, \\ N_m^b(\text{out}) &= \langle 0, \text{in}|\, {}^+b_m^\dagger(\text{out})\, {}^+b_m(\text{out})|0, \text{in}\rangle \\ &= |g({}_-|^+)|^{-2}, \end{aligned} \quad (44)$$

where the coefficients g are given by Eq. (22).

To distinguish initial and final states in the range Ω_3, one needs to consider one-particle mean values of the operators of the fluxes, of the energy and the effective charge (that is, the magnetic moment current) through the surfaces $x = x_L$ and $x = x_R$, given by Eqs. (A11) and (A12) in Appendix A. In the beginning we note that in the range Ω_3 the spatial distribution of physical states, presented by wave packets of plane waves, is the same as in the ranges Ω_2 and Ω_4. Therefore, it can be shown that particles (magnons) can be situated only in the region S_R, whereas antiparticles (antimagnons) can be situated only in the region S_L. The field $\partial_x U(x)$ does not allow particles to penetrate through the region S_{int}, and turns them in the opposite direction. For the plane waves such a behavior can be easily seen in the case of weak external fields (but still strong enough, $\delta U > 2\pi_\perp$, to provide the existence of the Ω_3-range) using a semiclassical approximation. If N_m^{cr} tends to zero, then $|g({}_+|^-)|^2 \to \infty$ and, at the same time, $|g({}_+|^+)|^2 \to \infty$ in accordance to relation (25). Relations (21) imply that for an arbitrary $m \in \Omega_3$ the magnon densities $|\ {}_\zeta\phi_m(X)|^2$ are concentrated in the region S_R, whereas the antimagnon densities $|\ {}^\zeta\phi_m|^2$ are concentrated in the region S_L.

In the general case when the quantities N_m^{cr} are not small, it is natural to expect a similar behavior, namely: the region S_L is not available for magnons, and the region S_R is not available for antimagnons. However, when the quantities N_m^{cr} are not small, the latter property may hold only for the corresponding wave packets, but not for the separate plane waves. That means that these plane waves may be different from zero in the whole space. Namely, this fact leads often to a misinterpretation, since the behavior of these plane waves looks like the one in the ranges Ω_1 and Ω_5, where they represent one-particle densities both in the region S_L and S_R. However, this similarity is misleading. Indeed, within our context it is assumed that the magnons and antimagnons in one of corresponding asymptotic regions may occupy quasistationary states, i.e., they should be described by wave packets that pertain their form a sufficiently long time in these regions. Note, that in the ranges Ω_1 and Ω_5, the sign of the longitudinal momentum $p^{L/R}$ is related to the sign of the mean energy flux in the region $S_{L/R}$. In the range Ω_3 the magnon states ${}_\zeta\phi_m$ are states with a definite quantum number p^L, whereas the antimagnon states ${}^\zeta\phi_m$ are states with a definite quantum number p^R. This fact together with relation (21) implies, for example, that a partial wave ${}_+\phi_m$ of a magnon, in the region where this particle can really be observed, i.e., in the region S_R, is always a superposition of two waves ${}^+\phi_m$ and ${}^-\phi_m$ with opposite signs of the quantum number p^R. Thus, the sign of the mean energy flux in the region S_R cannot be related to the sign of an asymptotic momentum in this region. Similarly, one can see, for example, that the partial wave ${}^+\phi_m$ of an antimagnon, in the region S_L, is always a superposition of two waves with quantum number p^L of opposite signs and, therefore, the sign of the mean energy flux cannot be related to the sign of an asymptotic momentum in the region where this particle can really be observed. However, as it will be demonstrated, these are states with well-defined asymptotic energy flux and, therefore, with a corresponding well-defined asymptotic field momentum. Namely, these properties of the constituent plane waves are responsible for the fact that stable magnon wave packets can exist only in the region S_R, whereas stable antimagnon wave packets can exist only in the region S_L; see Appendix D in Ref. [14] for details.

Taking into account such a space separation of the magnons and antimagnons one can use the one-particle mean values of fluxes, of the kinetic energy, and the magnetic moment, given by Eq. (A23) in Appendix A, to differ initial and final states in the range Ω_3. So if the flux of the magnetic moment and the kinetic energy in the region S_R coincides with the acceleration direction of a magnon in the region S_{int}, then the state under consideration is a final state of the magnon, since such a particle can only move away from the region S_{int} ($x \to \infty$). And viceversa, if these fluxes are opposite to the acceleration direction of a magnon in the region S_{int}, then the state under consideration is an initial state of the magnon, since such a particle can only move to the region S_{int}. In the case of antimagnons, the direction of the flux of the kinetic energy coincides with the direction of the flux density, but is opposite to the direction of the flux of the magnetic moment. The antimagnons do exist in the region S_L only. Therefore, if the direction of the flux of the kinetic energy in the region S_L coincides with the acceleration direction of the antimagnon in the region $S_{\rm int}$, then the state under consideration is the final state of an antimagnon. And viceversa, if the direction of the flux of the kinetic energy in the region $S_{\rm L}$ is opposite to the acceleration direction of an antimagnon in the region $S_{\rm int}$, then the state under consideration is an initial state of an antimagnon. Namely, in such a manner initial and final states in Eq. (32) are defined.

C. Observable physical quantities specifying the vacuum instability

Above, using representations (44) only differential mean number of magnon-antimagnon pairs created from the vacuum were calculated. However, one can obtain additional characteristics of the vacuum instability. This section is devoted to their study.

The probability of the transition from the vacuum $|0,\text{in}\rangle$ to the vacuum $|0,\text{out}\rangle$,

$$P_v = |c_v|^2,\ \ c_v = \langle 0,\text{out}\,|0,\text{in}\rangle, \tag{45}$$

is related to the mean numbers $N_m^{\rm cr}$ as

$$\ln P_v = \sum_{m\in\Omega_3} \ln p_m,\ \ p_m = \left(1+N_m^{\rm cr}\right)^{-1}; \tag{46}$$

see Appendix A in Ref. [14] for details. However, this probability can be represented via the imaginary part of a one-loop effective action S by the seminal Schwinger formula [4],

$$P_{\rm v} = \exp\left(-2{\rm Im}S\right). \tag{47}$$

A relation of this representation with the one that follows from the locally constant field approximation for the Schwinger's effective action was found in Ref. [21]. The probabilities of the magnon reflection and the magnon-antimagnon pair creation can be expressed via the mean numbers $N_m^{\rm cr}$ as follows:

$$\begin{aligned} P(+|+)_{m,m'} &= |\langle 0,\text{out}|\ {}_{+}a_m(\text{out})\ {}_{-}a_m^{\dagger}(\text{in})|0,\text{in}\rangle|^2 \\ &= \delta_{m,m'}(1+N_m^{\rm cr})^{-1}P_v, \\ P(+-|0)_{m,m'} &= |\langle 0,\text{out}|\ {}_{+}a_m(\text{out})\ {}^{+}b_m(\text{out})|0,\text{in}\rangle|^2 \\ &= \delta_{m,m'}N_m^{\rm cr}\left(1+N_m^{\rm cr}\right)^{-1}P_v. \end{aligned} \tag{48}$$

The probabilities of the antimagnon reflection and the magnon-antimagnon pair annihilation coincide with the quantities $P(+|+)$ and $P(+-|0)$, respectively. In the case of bosons in a given state m any number of pairs can be created from the vacuum and from the one particle state. By this reason probabilities (48) are not representative if the mean numbers $N_m^{\rm cr}$ are large. In the partial state with a given m the probability of the creation of any pairs with given m is $1-p_m$ where p_m is the probability that the partial vacuum state remains a vacuum, given by Eq. (46). If all the mean numbers $N_m^{\rm cr}$ are sufficiently small, $N_m^{\rm cr} \ll 1$, then the simple relations $p_m \approx 1-N_m^{\rm cr}$ and $1-P_v \approx N^{\rm cr} \ll 1$ hold true in the leading approximation. In this case $P(+|+)_{m,m} \approx 1$ and $P(+-|0)_{m,m} \approx N_m^{\rm cr}$. Therefore, information about the quantity P_v allows one to estimate the total number $N^{\rm cr}$. It is in this case that the Schwinger's effective action approach [4] to calculating P_v turns out to be useful. We note that this approach is a base of a number of approximation methods; see, e.g., Ref. [22] for a review. In this relation, it should be noted that the probability P_v by itself is not very useful in the case of strong fields when $P_v \ll 1$.

Taking into account Eq. (44), the total number of pairs created from the vacuum reads

$$N^{\rm cr} = \sum_{m\in\Omega_3} N_m^{\rm cr} = \sum_{m\in\Omega_3} |g({}_+|^-)|^{-2}. \tag{49}$$

Magnons and antimagnons created with quantum numbers m leaving the area $S_{\rm int}$ enter the areas $S_{\rm L}$ and $S_{\rm R}$, respectively. At the same time, the magnons continue to move in the x direction with a constant velocity $v^{\rm R}$. The motion of the magnons forms the flux density

$$\langle j_x\rangle_m = N_m^{\rm cr}(TV_\perp)^{-1}, \tag{50}$$

in the area $S_{\rm R}$, while the antimagnon motion in the opposite direction with the constant velocities $-v^{\rm L}$ forms the flux density $-(j_x)_m$ in the area $S_{\rm L}$. Here it is taken into account that differential mean numbers of created magnons and antimagnons with a given m are equal. The total flux densities of the magnons and the antimagnons are

$$\langle j_x\rangle = \sum_{m\in\Omega_3}\langle j_x\rangle_m = N^{\rm cr}(TV_\perp)^{-1} \tag{51}$$

and $-\langle j_x\rangle$, respectively. The effective charge (the magnetic moment) current density of both created magnons and antimagnons is $J_x^{\rm cr} = \mu\langle j_x\rangle$. This corresponds to the spin current $\langle j_x\rangle$. It is conserved in the x direction.

During the time T, the created magnons carry the magnetic moment $\mu\langle j_x\rangle_m T$ over the unit area $V_\perp$ of the surface $x=x_{\rm R}$. This magnetic moment is evenly distributed over the cylindrical volume of the length $v^{\rm R}T$. Thus, the magnetic moment density of the magnons created with a given m is $\mu j_m^0({\rm R})$, where $j_m^0({\rm R}) = \langle j_x\rangle_m/v^{\rm R}$ is the number density of the magnons. During the time T, the created antimagnons carry the magnetic moment $\mu\langle j_x\rangle_m T$ over the unit area $V_\perp$ of the surface $x=x_{\rm L}$. Taking into account that this magnetic moment is evenly distributed over the cylindrical volume of the length $v^{\rm L}T$, we can see that the magnetic moment density of the antimagnons created with a given m is $-\mu j_m^0({\rm L})$, where $j_m^0({\rm L}) = \langle j_x\rangle_m/v^{\rm L}$ is the number density of the magnons. The total magnetic moment density of the created particles reads

$$\rho^{\rm cr}(x) = \mu\begin{cases} -\sum_{m\in\Omega_3} j_m^0({\rm L}), & x\in S_{\rm L} \\ \sum_{m\in\Omega_3} j_m^0({\rm R}), & x\in S_{\rm R}\end{cases}. \tag{52}$$

Due to a relation between the velocities $v^{\rm L}$ and $v^{\rm R}$, the total number densities of the created magnons and antimagnons are the same,

$$\sum_{m\in\Omega_3} j_m^0({\rm L}) = \sum_{m\in\Omega_3} j_m^0({\rm R}). \tag{53}$$

We also note that the created magnons and antimagnons are spatially separated and carry magnetic moments that tend to smooth out the inhomogeneity of the external magnetic field.

In the same manner, one can derive some representation for the nonzero components of EMT of the created particles:

$$T_{\rm cr}^{00}(x) = \begin{cases} \sum_{m\in\Omega_3} j_m^0({\rm L})|\pi_0({\rm L})|, & x\in S_{\rm L} \\ \sum_{m\in\Omega_3} j_m^0({\rm R})\pi_0({\rm R}), & x\in S_{\rm R}\end{cases},$$

$$T_{\rm cr}^{11}(x)=\begin{cases}\sum_{m\in\Omega_3}\langle j_x\rangle_m|p^{\rm L}|, x\in S_{\rm L}\\ \sum_{m\in\Omega_3}\langle j_x\rangle_m|p^{\rm R}|, x\in S_{\rm R}\end{cases},$$

$$T_{\rm cr}^{kk}(x)=\begin{cases}\sum_{m\in\Omega_3}\langle j_x\rangle_m(p_k)^2/|p^{\rm L}|, x\in S_{\rm L}\\ \sum_{m\in\Omega_3}\langle j_x\rangle_m(p_k)^2/|p^{\rm R}|, x\in S_{\rm R}\end{cases},\ k=2,3,$$

$$T_{\rm cr}^{10}(x)=\begin{cases}-\frac{1}{v_s}\sum_{m\in\Omega_3}\langle j_x\rangle_m|\pi_0({\rm L})|, x\in S_{\rm L}\\ \frac{1}{v_s}\sum_{m\in\Omega_3}\langle j_x\rangle_m\pi_0({\rm R}), x\in S_{\rm R}\end{cases}. \tag{54}$$

Here $T_{\rm cr}^{00}(x)$ and $T_{\rm cr}^{kk}(x)$, $k=1,2,3$, are energy density and components of the pressure of the particles created in the areas $S_{\rm L}$ and $S_{\rm R}$, respectively, whereas $T_{\rm cr}^{10}(x)v_s$, for $x\in S_{\rm L}$ or $x\in S_{\rm R}$, is the energy flux density of the created particles through the surfaces $x=x_{\rm L}$ or $x=x_{\rm R}$, respectively. In a strong field, or in a field with the sufficiently large potential step δU, the energy density and the pressure along the direction of the axis x are near equal.

Let us consider effects of the backreaction on the external field due to the vacuum instability, to establish the so-called consistency conditions. We assume that the volume $V=V_\perp(x_{\rm R}-x_{\rm L})$ contains the area $S_{\rm int}=(x_{\rm L},x_{\rm R})$. The total energy of the created particles in the volume V is given by the corresponding volume integral of the energy density $T_{\rm cr}^{00}(t,x)$. The corresponding energy conservation low reads

$$\frac{\partial}{\partial t}\int_{V_\perp}d\mathbf{r}_\perp\int_{x_{\rm L}}^{x_{\rm R}}T_{\rm cr}^{00}(t,x)dx=-\oint_\Sigma v_sT_{\rm cr}^{k0}(x)df_k, \tag{55}$$

where Σ is a surface surrounding the volume V and df_k, $k=1,2,3$, are the components of the surface element $d\mathbf{f}$. Taking into account that $T_{\rm cr}^{00}(t,x)$ does not depend on the transversal coordinates, and $T_{\rm cr}^{k0}(x)=0$ for $k\neq1$, we find using Eq. (54) that the rate of the energy density change of the created particles in the region $S_{\rm int}$ per unit of the spatial area $V_\perp$ is

$$\frac{\partial}{\partial t}\int_{x_{\rm L}}^{x_{\rm R}}T_{\rm cr}^{00}(t,x)dx=v_s\big[T_{\rm cr}^{10}(x)\big|_{x\in S_{\rm L}}-T_{\rm cr}^{10}(x)\big|_{x\in S_{\rm R}}\big]$$
$$=-\delta U\,j_x. \tag{56}$$

It characterizes the loss of the energy that the created particles carry away from the region $S_{\rm int}$. At the same time, the constant rate (56) determines the power of the constant effective field $E_{\rm pristine}=\partial_xU(x)$ spent on the pair creation. Integrating this rate over the time duration of the field from $t_{\rm in}$ to $t_{\rm out}$, and using the notation

$$\Delta T_{\rm cr}^{00}(x)=-\int_{t_{\rm in}}^{t_{\rm out}}\frac{\partial}{\partial t}T_{\rm cr}^{00}(t,x)dt,$$

we find the total energy density of created pairs per unit of the area $V_\perp$ as

$$\int_{x_{\rm L}}^{x_{\rm R}}\Delta T_{\rm cr}^{00}(x)dx=\delta U\frac{N^{\rm cr}}{V_\perp}. \tag{57}$$

In strong-field QED it is usually assumed that just from the beginning there exists a classical effective field having a given energy. The system of particles interacting with this field is closed, that is, the total energy of the system is conserved.[2] It is clear that due to pair creation from the vacuum, the constant effective field $E_{\rm pristine}=\partial_xU(x)$ is losing its energy and should depleted with time. Thus, the applicability of the constant field approximation, which is used in the formulation of strong field QED with x step, is limited by the smallness of the backreaction. The relation (57) allows one to find conditions that provide this smallness, we call these relations the consistency conditions. These conditions can be obtained from the requirement that the energy density given by Eq. (57) is essentially smaller than the energy density of the constant effective field per unit of the area $V_\perp$.

Note that the presence of the matter in the initial state increases the mean number of created bosons. It is an obvious consequence of the Bose-Einstein statistics. In the case of fermions, the presence of the matter at the initial state prevents the pair creation. Assuming that $N_m^{(+)}({\rm in})$ and $N_m^{(-)}({\rm in})$ are the mean numbers of particles and antiparticles with quantum numbers m at the initial time instant, one obtains that the differential mean numbers of final particles and antiparticles are

$$N_m^{(\zeta)}=\big(1+N_m^{\rm cr}\big)N_m^{(\zeta)}({\rm in})+N_m^{\rm cr}\big[1+N_m^{(-\zeta)}({\rm in})\big], \tag{58}$$

respectively. The differential mean numbers of particles and antiparticles created by the external field are given by an increment $\Delta N_m^{(\zeta)}=N_m^{(\zeta)}-N_m^{(\zeta)}({\rm in})$. One can see that the increments of the numbers of particles and antiparticles are equal,

$$\Delta N_m^{(+)}=\Delta N_m^{(-)}=\Delta N_m,$$
$$\Delta N_m=N_m^{\rm cr}[1+N_m^{(+)}({\rm in})+N_m^{(-)}({\rm in})]. \tag{59}$$

In contrast to the previously used methods for studying the production of bosonic pairs by external fields, our approach allows us to consider the case of special inhomogeneous external fields supporting the spatial separation of particles and antiparticles (in the case under consideration, these are

[2]One can, however, imagine an alternative situation when these effective charges are getting out of the regions $S_{\rm L}$ and $S_{\rm R}$ with the help of the work done by an external storage battery. For example, dealing with graphene devices, it is natural to assume that the constant electric strength on the graphene plane is due to the applied fixed voltage, i.e., we are dealing with an open system of fermions interacting with a classical electromagnetic field. In that case there would be no backreaction problem. Note that the evolution of the mean electromagnetic field in the graphene, taking into account the backreaction of the matter field to the applied time-dependent external field, was considered in Ref. [26].

magnons and antimagnons) in the Klein zone. In such a way, one can see that the equal increments of mean numbers of particles in the area S_R and antiparticles in the area S_L do not depend on the symmetry between the mean numbers of particles and antiparticles in the initial state. For example, assuming the absence of the initial antiparticles, $N_m^{(-)}(\mathrm{in}) = 0$, with the number of initial particles being not zero in the Klein zone, $N_m^{(+)}(\mathrm{in}) \neq 0$, one can see that the number of created antiparticles is growing in comparison with the one created from the vacuum, $\Delta N_m = N_m^{\mathrm{cr}}[1 + N_m^{(+)}(\mathrm{in})]$. Therefore, the flux of created antiparticles in the area S_L is growing proportionally to the flux of coming particles from the area S_R. Such a behavior can be called statistically assisted Schwinger effect.

That is why operating with the concept of probability turns out to be unfruitful in the case when the mean number N_m^{cr} is not relatively small. In our considerations the presence of particles in the initial state implies that these are ingoing particles and the mean numbers $N_m^{(\zeta)}(\mathrm{in})$ are proportional to densities of ingoing fluxes,

$$\left\langle j_x^{(\zeta)}(\mathrm{in})\right\rangle_m = N_m^{(\zeta)}(\mathrm{in})(TV_\perp)^{-1}. \tag{60}$$

Densities of outgoing fluxes are

$$\left\langle j_x^{(\zeta)}\right\rangle_m = N_m^{(\zeta)}(TV_\perp)^{-1}. \tag{61}$$

Both ingoing and outgoing magnons are situated in the area S_R while both ingoing and outgoing antimagnons are situated in the area S_L. For example, assuming the absence of initial antiparticles, $N_m^{(-)}(\mathrm{in}) = 0$, the presence of particles in the initial state, $N_m^{(+)}(\mathrm{in}) \neq 0$, leads to the fact that the density of the outgoing particle flux turns out to be more than the density of incoming particle flux,

$$\langle j_x^{(+)}\rangle_m|/\langle j_x^{(+)}(\mathrm{in})\rangle_m = 1 + N_m^{\mathrm{cr}}(1 + 1/N_m^{(+)}(\mathrm{in})). \tag{62}$$

Thus, the flux proportional to N_m^{cr} of particles born from the vacuum is added to the total flux of reflected particles. A similar picture is observed for antiparticle fluxes in the case when $N_m^{(+)}(\mathrm{in}) = 0$ while $N_m^{(-)}(\mathrm{in}) \neq 0$. In the areas of Ω_3 adjoining the borders of the ranges Ω_2 and Ω_4, the pair creation is absent, $N_m^{\mathrm{cr}} \to 0$, and the only the total reflection takes place. However, in general, in the Klein zone, fluxes due to the total reflection cannot be separated from the fluxes due to the pair creation, that is one more reason not to use probabilities of the reflection.

IV. EXAMPLES OF EXACT SOLUTIONS WITH *X* STEPS

In this section, we present a collection of external magnetic fields that allow calculating magnon pair production characteristics based on exact solutions of Eq. (9). For the sake of convenience, we discuss examples separately and list pertinent results only. Further details are placed in Appendix B.

A. Differential quantities

1. L-constant step

The L-constant magnetic step is a model of magnetic field inhomogeneity that grows linearly with x within S_{int} and is constant outside of it, $B(x)|_{x\leqslant x_L} \neq B(x)|_{x\geqslant x_R}$. We call this field "$L$-constant" magnetic step due to its analogy with the "L-constant electric field," which is a type of electric field that creates electron-positron pairs from the vacuum if it is strong enough; see Ref. [31] for a discussion. The field has the following form:

$$B(x) = \begin{cases} B'L/2, & x \in S_L = (-\infty, -L/2], \\ -B'x, & x \in S_{\mathrm{int}} = (-L/2, L/2), \\ -B'L/2 & x \in S_R = [L/2, +\infty), \end{cases} \tag{63}$$

where $B' > 0$, $L > 0$, and we set $x_L = -L/2 = -x_R$ for simplicity.

Beyond the intermediate interval potential energies are constants, $U_L = +\mu B'L/2$ and $U_R = -\mu B'L/2$, and exact solutions to Eq. (9) are plane waves, classified according to Eqs. (14). As for the intermediate interval, S_{int}, we perform a change of variable

$$\xi(x) = \frac{\varepsilon + \mu B'x}{\sqrt{v_s\mu B'}}, \tag{64}$$

to rewrite Eq. (9) as

$$\left(\frac{d^2}{d\xi^2} + \xi^2 - \lambda\right)\varphi_m(\xi) = 0, \quad \lambda = \frac{\pi_\perp^2}{v_s\mu B'}. \tag{65}$$

This is Weber's parabolic cylinder differential equation [32], whose independent sets of solutions are $D_\nu[(1-i)\xi]$, $D_{-\nu-1}[(1+i)\xi]$, or $D_\nu[-(1-i)\xi]$, and $D_{-\nu-1}[-(1+i)\xi]$, where $\nu = -i\lambda/2$.

With the aid of the exact solutions (B1) and the coefficient (B5) discussed in Appendix B, the differential mean numbers of magnon-antimagnon pairs created from the vacuum by the external field (44) has the form

$$N_m^{\mathrm{cr}} = \frac{8e^{-\pi\lambda/2}}{\sqrt{\xi_1^2-\lambda}\sqrt{\xi_2^2-\lambda}}|f_1^{(-)}(\xi_2)f_2^{(-)}(\xi_1) - f_2^{(-)}(\xi_2)f_1^{(-)}(\xi_1)|^{-2}. \tag{66}$$

Here, $\xi_1 = \xi(x_L)$, $\xi_2 = \xi(x_R)$, and

$$f_1^{(\pm)}(\xi) = \left(1 \pm \frac{i}{\sqrt{\xi^2-\lambda}}\frac{d}{d\xi}\right)D_{-\nu-1}[\pm(1+i)\xi],$$
$$f_2^{(\pm)}(\xi) = \left(1 \pm \frac{i}{\sqrt{\xi^2-\lambda}}\frac{d}{d\xi}\right)D_\nu[\pm(1-i)\xi]. \tag{67}$$

Optimal conditions for the magnon pair production occur when step (63) is high enough and stretches over a wide region of the space, characterized by the inequalities

$$\sqrt{\frac{|\mu B'|}{v_s}}L \gg \max\left\{1, \frac{\Delta^2}{v_s\mu B'}\right\}. \tag{68}$$

If these conditions are met and $\sqrt{\lambda}$ is fixed, in the sense that $\sqrt{\lambda} < K_\perp$, where $K_\perp$ is a reasonably large number obeying the conditions $\sqrt{|\mu B'|/v_s}L/2 \gg K_\perp^2 \gg \max\{1, \Delta^2/v_s\mu B'\}$, then

$|\xi_1|$ and ξ_2 are large

$$\xi_2 \geqslant \sqrt{\frac{|\mu B'|}{v_s}}\frac{L}{2}, \quad -\sqrt{\frac{|\mu B'|}{v_s}}\frac{L}{2} \leqslant \xi_1 \leqslant -K,$$
$$K_\perp^2 < K \ll \sqrt{\frac{|\mu B'|}{v_s}}\frac{L}{2}, \tag{69}$$

which means that we can use asymptotic representations of Weber parabolic cylinder functions (WPCFs) given by Eqs. (1)–(3) in Sec. 8.4 of Ref. [32], to show that the mean numbers acquire the form

$$N_m^{\mathrm{cr}} = \exp(-\pi\lambda)\left[1 + O(|\xi_1|^{-3}) + O(\xi_2^{-3})\right]. \tag{70}$$

In the limit where the inhomogeneity of the field spreads over the entire x axis, i.e., when $L \to \infty$ (thus $|\xi_1| \to \infty$, $\xi_2 \to \infty$), we obtain

$$N_m^{\mathrm{cr}} \to N_m^{\mathrm{uni}} = \exp(-\pi\lambda). \tag{71}$$

This is a well-known expression that was originally obtained in the context of electron-positron pair creation from the vacuum by a constant uniform electric field [34]. Its maximum value $\max N_m^{\mathrm{uni}} = N_m^{\mathrm{uni}}|_{\mathbf{p}_\perp=\mathbf{0}}$ becomes pronounced if the derivative B' is of the order of the critical value $B'_c = \Delta^2/v_s\mu$, which plays the role of the Schwinger's critical field [4] in the case under consideration.

Another configuration of the external field worth of discussion is when its spacial inhomogeneity varies "abruptly" along the x direction. We call this configuration "sharply varying" or "steep" field configuration. A steep L-constant magnetic step is characterized by the set of inequalities,

$$\delta U = |\mu B'|L = \mathrm{const} > 2\Delta, \quad \delta U L/v_s \ll 1, \tag{72}$$

which, in turn, implies in the conditions

$$\max\left(|\pi_0(\mathrm{L})|\frac{L}{v_s}, \pi_0(\mathrm{R})\frac{L}{v_s}\right) \ll 1, \tag{73}$$

as quantum numbers are bounded in the Klein zone Ω_3. As a result, coefficients involving asymptotic momenta are also small since $|p^{\mathrm{L/R}}| < |\pi_0(\mathrm{L/R})|$. In this case, the argument of the WPCFs are sufficiently small in Ω_3 and we may use the power-series expansion to show that

$$N_m^{\mathrm{cr}} \approx \frac{4|p^{\mathrm{L}}||p^{\mathrm{R}}|}{||p^{\mathrm{L}}| - |p^{\mathrm{R}}| + i\sigma|^2},$$
$$\sigma = \left[|p^{\mathrm{L}}||p^{\mathrm{R}}| + (i + \lambda)\frac{|\mu B'|}{v_s}\right]L. \tag{74}$$

Notice that the limit $L \to 0$ is admissible provided the difference $||p^{\mathrm{L}}| - |p^{\mathrm{R}}||$ is larger compared $|\sigma|$. In particular, if $||p^{\mathrm{L}}| - |p^{\mathrm{R}}|| \gg |\sigma|$, then the mean number (74) admits form

$$N_m^{\mathrm{cr}} \approx \frac{4k}{(1-k)^2}, \tag{75}$$

where $k = |p^{\mathrm{R}}|/|p^{\mathrm{L}}|$. It is in agreement with results obtained at $p_\perp = 0$ in Refs. [9,10,12,13] for the case of the Klein step formed by an electric field. Additionally, Eq. (75) can also be reproducible by other magnetic steps, as shall be seen below.

2. Sauter-like step

The Sauter-like magnetic step[3]—a more realistic, smoothed version of the L-constant magnetic step—is another example of magnetic field inhomogeneity for which exact solutions of Eq. (9) are known. The Sauter (or Sauter-like) electric field is a popular example of an electric field that may violate the vacuum stability; see Ref. [14] for an extensive discussion of the phenomenon in QED with x steps.

In the present case, the Sauter-like magnetic step has the form:

$$B(x) = -B'L_{\mathrm{S}}\tanh(x/L_{\mathrm{S}}), \quad B' > 0, \quad L_{\mathrm{S}} > 0. \tag{76}$$

At remote regions $x \to \mp\infty$, the magnetic field is constant $B(\mp\infty) = \pm B'L_{\mathrm{S}}$, which means that $U_{\mathrm{L}} = \mu B'L_{\mathrm{S}} = -U_{\mathrm{R}}$. Therefore, the magnitude of the potential step (13) in this case is $\delta U = U_{\mathrm{L}} - U_{\mathrm{R}} = 2\mu B'L_{\mathrm{S}}$.

Performing the change of variable

$$\chi(x) = \tfrac{1}{2}[1 + \tanh(x/L_{\mathrm{S}})], \tag{77}$$

and seeking for solutions in the form

$$\varphi_m(x) = \chi^{-iL_{\mathrm{S}}|p^{\mathrm{L}}|/2}(1-\chi)^{iL_{\mathrm{S}}|p^{\mathrm{R}}|/2} f(\chi), \tag{78}$$

allows us to express Eq. (9) in the same form as the differential equation for the Gauss hypergeometric function [32],

$$\chi(1-\chi)f'' + [c - (a+b+1)\chi]f' - abf = 0, \tag{79}$$

provided

$$a = \frac{1}{2}\left[iL_{\mathrm{S}}(|p^{\mathrm{R}}| - |p^{\mathrm{L}}|) + 1 + i\sqrt{\left(\frac{L_{\mathrm{S}}\delta U}{v_s}\right)^2 - 1}\right],$$
$$b = \frac{1}{2}\left[iL_{\mathrm{S}}(|p^{\mathrm{R}}| - |p^{\mathrm{L}}|) + 1 - i\sqrt{\left(\frac{L_{\mathrm{S}}\delta U}{v_s}\right)^2 - 1}\right],$$
$$c = 1 - iL_{\mathrm{S}}|p^{\mathrm{L}}|. \tag{80}$$

Using the exact solutions (B8) and (B9) and the coefficient (B11) discussed in Appendix B, we find that

$$|g(_+|^-)|^{-2} = \frac{\sinh(\pi L_{\mathrm{S}}|p^{\mathrm{R}}|)\sinh(\pi L_{\mathrm{S}}|p^{\mathrm{L}}|)}{\sinh^2[\pi L_{\mathrm{S}}(|p^{\mathrm{R}}| - |p^{\mathrm{L}}|)/2] + \cosh^2\left(\frac{\pi}{2}\sqrt{(L_{\mathrm{S}}\delta U/v_s)^2 - 1}\right)}. \tag{81}$$

[3]We name the field (76) "Sauter-like," in reference to F. Sauter [11], who first solved relativistic wave equations for a charged particle with a potential step of the form $-\alpha E\tanh(x/\alpha)$.

Result (81) holds in the ranges Ω_1, Ω_5, and Ω_3. Taking into account relation (25), one finds $|g({}_+|{}^+)|^2$. Using these coefficients in Eq. (43), one can calculate the relative probabilities of the reflection, $|R_m|^2 = 1 - |T_m|^2$, and the transmission, $|T_m|^2 = [1 + |g({}_+|{}^-)|^2]^{-1}$ in the ranges Ω_1 and Ω_5. In the range Ω_3 according to relation (44) the coefficient $|g({}_+|{}^-)|^{-2}$ gives the differential mean numbers of the magnon-antimagnon pairs created from the vacuum, N_m^{cr}. In particular, for any $\pi_\perp \neq 0$, one of the following limits holds true:

$$|g({}_+|{}^-)|^{-2} \sim |L_{\text{S}} p^{\text{R}}| \to 0, \quad |g({}_+|{}^-)|^{-2} \sim |L_{\text{S}} p^{\text{L}}| \to 0. \tag{82}$$

This means that in the range Ω_3, the mean numbers tend to zero, $N_m^{\text{cr}} \to 0$, while in the ranges Ω_1 and Ω_5 the relative probability of the transmission reads $|T_m|^2 \to 0$ if m tends to the boundary with either the range Ω_2 ($|p^{\text{L}}| \to 0$) or the range Ω_4 ($|p^{\text{R}}| \to 0$).

In cases where the magnetic step is high enough and stretches over a wide region of the space, such that $L_{\text{S}}\delta U/v_s \gg 1$, the mean number of created pairs can be approximated as

$$N_m^{\text{cr}} \approx N_m^{\text{as}} = e^{-\pi\tau}, \quad \tau = L_{\text{S}}(2|\mu B'|L_{\text{S}}/v_s - |p^{\text{R}}| - |p^{\text{L}}|). \tag{83}$$

When $L_{\text{S}}\delta U/v_s \to \infty$, one obtains $N_m^{\text{cr}} \to N_m^{\text{uni}}$ where N_m^{uni} is given by Eq. (71).

Sharp-gradient configuration

$$\delta U = |\mu B'|L_{\text{S}} = \text{const} > 2\Delta, \quad \delta U L_{\text{S}}/v_s \ll 1, \tag{84}$$

corresponds to a very sharp field derivative $\partial_x U$, highly concentrated near the origin $x = 0$, described by a very "steep" potential step. This configuration has a special interest because it corresponds to a regularization of the Klein step (originally an electric step potential)

$$U(x) = \begin{cases} U_{\text{L}} \text{ if } x < 0 \\ U_{\text{R}} \text{ if } x > 0 \end{cases}, \tag{85}$$

where U_{R} and U_{L} are constants, and may be useful in a discussion of the Klein paradox. In the ranges Ω_1 and Ω_5 the energy $|\varepsilon|$ is not restricted from the above, that is why in what follows we consider only the subranges, where $\max\{L_{\text{S}}|p^{\text{L}}|, L_{\text{S}}|p^{\text{R}}|\} \ll 1$. Note that in these ranges $||p^{\text{L}}| - |p^{\text{R}}|| > \delta U$, then the parameter $k = |p^{\text{R}}|/|p^{\text{L}}|$ does not achieve the unit value, $k \neq 1$. Then one has

$$|g({}_+|{}^-)|^{-2} \approx \frac{4k}{(1-k)^2} \tag{86}$$

and obtains the transmission coefficient as

$$|T_m|^2 \approx \frac{4k}{(1+k)^2}, \tag{87}$$

that is in agreement with results of the nonrelativistic consideration obtained in any textbook for one-dimensional quantum motion. In the range Ω_3 for any given $\pi_\perp$ the absolute values of $|p^{\text{R}}|$ and $|p^{\text{L}}|$ are restricted from above,

$$0 \leqslant ||p^{\text{L}}| - |p^{\text{R}}|| \leqslant \sqrt{\delta U(\delta U - 2\pi_\perp)}. \tag{88}$$

As it follows from Eq. (81), in the range Ω_3 the differential mean numbers of created magnon-antimagnon pairs read

$$N_m^{\text{cr}} = |g({}_+|{}^-)|^{-2} \approx \frac{4|p^{\text{L}}||p^{\text{R}}|}{\left(\frac{\delta U^2 L_{\text{S}}}{2v_s^2}\right)^2 + (|p^{\text{L}}| - |p^{\text{R}}|)^2}. \tag{89}$$

They have a maximum at $k = 1$ that can be quite large,

$$\max N_m^{\text{cr}} = \frac{4}{(L_{\text{S}}\delta U/v_s)^2}\left[1 - \left(\frac{2\pi_\perp}{\delta U}\right)^2\right]. \tag{90}$$

The limit $L_{\text{S}} \to 0$ in Eq. (89) is possible only when the difference $|p^{\text{L}}| - |p^{\text{R}}|$ is not very small, namely when

$$\left(\frac{\delta U^2 L_{\text{S}}}{2v_s^2}\right)^2 \ll (|p^{\text{L}}| - |p^{\text{R}}|)^2. \tag{91}$$

Only under the latter condition one can neglect an L_{S} -depending term in Eq. (89) to obtain the form given by Eq. (75). Thus, we have an another example of the regularization of the Klein step.

3. Exponential step

We present here an example of the magnetic field inhomogeneity whose analytical form is a piecewise, continuous exponential functions of x. In this case we have a possibility to consider various asymmetric peak configurations. The electric-analog of this field in QED was considered in Ref. [33]. The magnetic step has the form

$$B(x) = B'\begin{cases} k_1^{-1}(1 - e^{k_1 x}), & x \in \text{I} = (-\infty, 0] \\ k_2^{-1}(e^{-k_2 x} - 1), & x \in \text{II} = (0, +\infty) \end{cases}, \tag{92}$$

where k_j, $j = 1, 2$, are positive constants that characterizes how steep or smooth the field decays from $x = -\infty$ to $x = +\infty$. Similar to the preceding examples, the exponential magnetic step (92) reaches constant values at remote regions which means that $U_{\text{L}} = \mu B' k_1^{-1}$ and $U_{\text{R}} = -\mu B' k_2^{-1}$. As a result, the magnitude of the potential step is $\delta U = U_{\text{L}} - U_{\text{R}} = \mu B'(k_1^{-1} + k_2^{-1})$.

To solve Eq. (9) with this field, we perform the change of variables

$$\eta_1 = ih_1 e^{k_1 x}, \quad h_1 = \frac{2\mu B'}{k_1^2 v_s}, \quad x \in \text{I},$$
$$\eta_2 = ih_2 e^{-k_2 x}, \quad h_2 = \frac{2\mu B'}{k_1^2 v_s}, \quad x \in \text{II}, \tag{93}$$

and represent the scalar functions in the form

$$\varphi_m(x) = e^{-\eta_j/2}\eta_j^{\nu_j} R_j(\eta_j),$$
$$\nu_1 = \frac{i|p^{\text{L}}|}{k_1}, \quad \nu_2 = \frac{i|p^{\text{R}}|}{k_2}, \tag{94}$$

to learn that the functions $R_j(\eta_j)$ obey the confluent hypergeometric equations

$$\eta_j R_j'' + (c_j - \eta_j)R_j' - a_j R_j = 0, \tag{95}$$

provided

$$c_j = 2\nu_j + 1,$$
$$a_1 = \nu_1 + \frac{1}{2} + \frac{i\pi_0(\mathrm{L})}{k_1 v_s},$$
$$a_2 = \nu_2 + \frac{1}{2} - \frac{i\pi_0(\mathrm{R})}{k_2 v_s}. \quad (96)$$

Fundamental pairs of solutions to Eq. (95) with special asymptotic properties at remote regions are proportional to confluent hypergeometric functions $\Phi(a_j, c_j; \eta_j)$ and $\eta_j^{1-c_j} e^{\eta_j}\Phi(1-a_j, 2-c_j; -\eta_j)$.

Using exact solutions and their connection via g coefficients discussed in Appendix B, we find

$$|g(_+|^-)|^{-2} = \frac{4|p^{\mathrm{L}}||p^{\mathrm{R}}|}{\exp\left[-\pi\left(k_1^{-1}|p^{\mathrm{L}}| - k_2^{-1}|p^{\mathrm{R}}|\right)\right]} \times \left|\left(k_1 h_1 y_1^2 \frac{d}{d\eta_1} y_2^1 + k_2 h_2 y_2^1 \frac{d}{d\eta_2} y_1^2\right)\Big|_{x=0}\right|^{-2}. \quad (97)$$

Taking into account relation (25) one finds $|g(_+|^+)|^2$. In the ranges Ω_1 and Ω_5 use of these coefficients in Eq. (43) gives the relative probabilities of the reflection, $|R_m|^2$, and the transmission, $|T_m|^2$. In the range Ω_3 according to relation (44) the coefficient (97) gives the differential mean numbers of pairs created from the vacuum, N_m^{cr}. In particular, if either $|p^{\mathrm{R}}|$ or $|p^{\mathrm{L}}|$ tends to zero for any $\pi_\perp \neq 0$, then one of the following limits holds true:

$$|g(_-|^+)|^{-2} \sim |p^{\mathrm{R}}| \to 0, \quad |g(^+|_-)|^{-2} \sim |p^{\mathrm{L}}| \to 0. \quad (98)$$

If the step is high enough and stretches over a wide region of the space, so that the conditions are satisfied,

$$\min(h_1, h_2) \gg \max\left\{1, \frac{\Delta^2}{v_s \mu B'}\right\}, \quad \frac{k_1}{k_2} = \text{fixed}, \quad (99)$$

then it is possible to show based on the results of Ref. [33] that the mean numbers of created pairs, given by Eq. (97), admit simpler forms,

$$N_m^{\mathrm{cr}} = |g(_-|^+)|^{-2} \approx \begin{cases} \exp\left\{-\frac{2\pi}{k_1}[|\pi_0(\mathrm{L})| - |p^{\mathrm{L}}|]\right\}, & 0 \leqslant \varepsilon < U_{\mathrm{L}} - \pi_\perp, \\ \exp\left\{-\frac{2\pi}{k_2}[\pi_0(\mathrm{R}) - |p^{\mathrm{R}}|]\right\}, & U_{\mathrm{R}} + \pi_\perp \leqslant \varepsilon < 0. \end{cases} \quad (100)$$

When $h_1, h_2 \to \infty$, one obtains $N_m^{\mathrm{cr}} \to N_m^{\mathrm{uni}}$.

One can consider an essentially asymmetric configuration with $k_1 \gg k_2$.

By choosing large parameters $k_1, k_2 \to \infty$ with a fixed ratio k_1/k_2, one can obtain sharp gradient fields that exist only in a small area in a vicinity of the origin $x = 0$. We assume that the corresponding asymptotic potential energies U_{R} and U_{L} define finite magnitudes of the potential steps ΔU_1 and ΔU_2 for increasing and decreasing parts,

$$U_{\mathrm{L}} = \Delta U_1, \quad U_{\mathrm{R}} = -\Delta U_2, \quad (101)$$

respectively, and satisfy the following inequalities:

$$\Delta U_1/k_1 \ll 1, \quad \Delta U_2/k_2 \ll 1. \quad (102)$$

This case corresponds to a very sharp peak with a given step magnitude $\delta U = \Delta U_1 + \Delta U_2$. In the ranges Ω_1 and Ω_5 the energy $|\varepsilon|$ is not restricted from the above, that is why in what follows we consider only the subranges, where

$$\max(|\pi_0(\mathrm{L})|/k_1, |\pi_0(\mathrm{R})|/k_2) \ll 1. \quad (103)$$

In these ranges taking into account that $||p^{\mathrm{L}}| - |p^{\mathrm{R}}|| > \delta U$ we obtain that the $|g(_+|^-)|^{-2}$ and the transmission coefficient can be presented by the functions (86) and (87) of the parameter $k = |p^{\mathrm{R}}|/|p^{\mathrm{L}}|$.

In the range Ω_3 the difference $||p^{\mathrm{L}}| - |p^{\mathrm{R}}||$ are restricted from above by Eq. (88) and can tend to zero. That is why the differential mean number of pairs created is

$$N_m^{\mathrm{cr}} \approx \frac{4|p^{\mathrm{L}}||p^{\mathrm{R}}|}{(|p^{\mathrm{L}}| - |p^{\mathrm{R}}|)^2 + b^2},$$
$$b = \frac{2\Delta U_1}{k_1}\left[\frac{\Delta U_1}{4} + |\pi_0(\mathrm{L})|\right] + \frac{2\Delta U_2}{k_2}\left[\frac{\Delta U_2}{4} + \pi_0(\mathrm{R})\right]. \quad (104)$$

It has a maximum at $k = 1$ that can be large, $N_m^{\mathrm{cr}} = 4|p^{\mathrm{L}}||p^{\mathrm{R}}|/b^2$. Under the condition $b^2 \ll (|p^{\mathrm{L}}| - |p^{\mathrm{R}}|)^2$ one can verify that the mean number N_m^{cr} is given by Eq. (75). Thus, we have additional example of the regularization of the Klein step.

4. Inverse-square step

As a last example, we present below a model of the magnetic field inhomogeneity that is also a piecewise and a continuous function of x. In QED, we call the electric field corresponding to the potential step inverse-square electric field, whose solutions to relativistic wave equations were found by us recently in Ref. [35]. The magnetic step has the form:

$$B(x) = B'\begin{cases} \varrho_1[1 - (1 - x/\varrho_1)^{-1}], & x \in \mathrm{I} = (-\infty, 0] \\ \varrho_2[(1 + x/\varrho_2)^{-1} - 1], & x \in \mathrm{II} = (0, +\infty) \end{cases}, \quad (105)$$

where ϱ_j, $j = 1, 2$, are positive constants that characterize how the field grows/decays along the x axis. The magnitude of the potential step for this field is $\delta U = U_{\mathrm{L}} - U_{\mathrm{R}} = \mu B'(\varrho_1 + \varrho_2)$.

By performing the change of variables

$$z_1(x) = 2i|p^{\mathrm{L}}|\varrho_1(1 - x/\varrho_1), \quad x \in \mathrm{I},$$
$$z_2(x) = 2i|p^{\mathrm{R}}|\varrho_2(1 + x/\varrho_2), \quad x \in \mathrm{II}, \quad (106)$$

we may write the second-order differential equation (9) as

$$\left(\frac{d^2}{dz_j^2} - \frac{1}{4} + \frac{\kappa_j}{z_j} + \frac{1/4 - \mu_j^2}{z_j^2}\right)\varphi_m(x) = 0, \quad (107)$$

whose solutions are proportional to the Whittaker functions $W_{\kappa_j,\mu_j}(z_j)$ and $W_{-\kappa_j,\mu_j}(e^{-i\pi}z_j)$, provided the parameters κ_j

and μ_j have the form

$$\kappa_1 = -i\frac{\mu B'\varrho_1^2}{v_s}\frac{\pi_0(\mathrm{L})/v_s}{|p^{\mathrm{L}}|}, \quad \kappa_2 = i\frac{\mu B'\varrho_2^2}{v_s}\frac{\pi_0(\mathrm{R})/v_s}{|p^{\mathrm{R}}|},$$
$$\mu_1 = -\sqrt{\frac{1}{4}-\left(\frac{\mu B'\varrho_1^2}{v_s}\right)^2}, \quad \mu_2 = \sqrt{\frac{1}{4}-\left(\frac{\mu B'\varrho_2^2}{v_s}\right)^2}. \tag{108}$$

Using the exact solutions (B18), (B21) and coefficient (B24) that connects different solutions as is discussed in Appendix B, we obtain

$$N_m^{\mathrm{cr}} = |p^{\mathrm{L}}||p^{\mathrm{R}}|\left|\left[\left|p^{\mathrm{L}}|w_1^2(z_2)\frac{d}{dz_1}w_2^1(z_1) + w_2^1(z_1)|p^{\mathrm{R}}|\frac{d}{dz_2}w_1^2(z_2)\right]\right|_{x=0}\right|^{-2}. \tag{109}$$

Finally, considering cases where the step is sufficiently high and stretches over a wide region of the space, specified by the conditions

$$\min\left(U_{\mathrm{L}}\varrho_1, |U_{\mathrm{R}}|\varrho_2\right) \gg \max\left\{1, \frac{\Delta^2}{v_s\mu B'}\right\}, \quad \frac{\varrho_1}{\varrho_2} = \text{fixed}, \tag{110}$$

it is possible to demonstrate based on our previous results [35] that the mean numbers (109) admit the asymptotic forms

$$N_m^{\mathrm{cr}} \approx \begin{cases}\exp(2\pi\omega_1^+), & 0 \leqslant \varepsilon < U_{\mathrm{L}} - \pi_\perp,\\ \exp(2\pi\omega_2^-), & U_{\mathrm{R}} + \pi_\perp \leqslant \varepsilon < 0,\end{cases} \tag{111}$$

where

$$\omega_1^{\pm} = \pm i(\kappa_1 \pm \mu_1) = \frac{U_{\mathrm{L}}\varrho_1}{v_s}\left[\sqrt{1-\left(\frac{2U_{\mathrm{L}}\varrho_1}{v_s}\right)^{-2}} \pm \frac{\pi_0(\mathrm{L})/v_s}{|p^{\mathrm{L}}|}\right],$$
$$\omega_2^{\pm} = \mp i(\kappa_2 \pm \mu_2) = \frac{|U_{\mathrm{R}}|\varrho_2}{v_s}\left[\sqrt{1-\left(\frac{2|U_{\mathrm{R}}|\varrho_2}{v_s}\right)^{-2}} \pm \frac{\pi_0(\mathrm{R})/v_s}{|p^{\mathrm{R}}|}\right]. \tag{112}$$

When $U_{\mathrm{L}}\varrho_1, |U_{\mathrm{R}}|\varrho_2 \to \infty$, we have $N_m^{\mathrm{cr}} \to N_m^{\mathrm{uni}}$ where N_m^{uni}. One can see that all of the above presented examples in the model of smooth-gradient step can be seen as regularizations of linearly growing magnetic field.

Another field configuration worthy of discussion is the case where the step is still sufficiently high but the field inhomogeneity is concentrated in a narrow region of the x axis, characterized by the inequality

$$\max\left(\frac{U_{\mathrm{L}}\varrho_1}{v_s}, \frac{|U_{\mathrm{R}}|\varrho_2}{v_s}\right) \ll 1, \quad \frac{\varrho_1}{\varrho_2} \text{ fixed}. \tag{113}$$

In the range Ω_3, the most significant contribution to the mean numbers (35) comes from values for $|\varepsilon|$ that are sufficiently away from the borders with Ω_2 and Ω_4, in the sense that the supplementary inequalities

$$\max\left(\frac{|\pi_0(\mathrm{L})|\varrho_1}{v_s}, \frac{\pi_0(\mathrm{R})\varrho_2}{v_s}\right) \ll 1, \quad \frac{\varrho_1}{\varrho_2} \text{ fixed} \tag{114}$$

are also satisfied. In this case, the argument of the Whittaker functions (B18) are also small, which means that one can use the connection formulas given by Eqs. (119) in Ref. [35] and expand the Whittaker functions $M_{\kappa_j,\mu_j}(z_j)$ around the origin, i.e., $M_{\kappa_j,\mu_j}(z_j) = z^{\mu_j+1/2}[1 - z\kappa_j/(2\mu_j + 1) + O(z^2)]$, to show that the differential mean numbers of magnon-antimagnon pairs created from the vacuum is approximately given by the equations

$$N_n^{\mathrm{cr}} \approx \frac{4|p^{\mathrm{L}}||p^{\mathrm{R}}|}{(|p^{\mathrm{L}}| - |p^{\mathrm{R}}|)^2 + d^2},$$
$$d = \frac{\pi_0(\mathrm{L})}{U_{\mathrm{L}}}|p^{\mathrm{L}}|^2\varrho_1 + \frac{\pi_0(\mathrm{R})}{U_{\mathrm{R}}}|p^{\mathrm{R}}|^2\varrho_2. \tag{115}$$

Similar to the previous examples, the limit in which the field is infinitely steep, given by $\varrho_1 \to 0$ and $\varrho_2 \to 0$, is admissible as long as the difference $||p^{\mathrm{L}}| - |p^{\mathrm{R}}||$ is larger compared to $|d|$. If, in particular, $||p^{\mathrm{L}}| - |p^{\mathrm{R}}|| \gg |d|$, then we find the same expression obtained for the Klein step, given by Eq. (75). In addition to the previous examples, the compatibility between Eq. (115) and (75) shows that the field (105) is also a regularization of the Klein step.

B. Integral quantities

According to the discussion in preceding section, the total number of the magnon-antimagnon pairs produced from the vacuum by magnetic steps is a sum of the differential numbers over the quantum numbers within the Klein zone Ω_3, given by Eq. (49). The created magnons continue to move in the x direction and form the flux density j_x, given by Eq. (51), in the area S_{R}, while the antimagnon motion in the opposite direction form the flux density $-j_x$ in the area S_{L}. These flux densities can be represented in an integral form as

$$j_x = \frac{1}{(2\pi)^3}\int_{U_{\mathrm{R}}+\pi_\perp}^{U_{\mathrm{L}}-\pi_\perp} d\varepsilon \int d\mathbf{p}_\perp N_m^{\mathrm{cr}}. \tag{116}$$

In general, analytical integration in Eq. (116) is impossible. Nevertheless, due to the analogy between QED with x steps and the present study, we may use universal expressions in a strong electric field from Ref. [21] to conclude that if the magnetic step is sufficiently high but evolves gradually along the x axis (smooth-gradient step), one can approximate Eq. (116) as

$$j_x \approx \tilde{j}_x = \frac{1}{(2\pi)^3}\int_{x_{\mathrm{L}}}^{x_{\mathrm{R}}} dx\, U'(x) \int d\mathbf{p}_\perp N_n^{\mathrm{uni}}(x),$$
$$N_n^{\mathrm{uni}}(x) = \exp\left[-\pi\frac{\pi_\perp^2}{v_s U'(x)}\right], \tag{117}$$

where the prime over the potential denotes differentiation with respect to x, $U'(x) = \mu dB(x)/dx$. The quantity $N_n^{\mathrm{uni}}(x)$ has a universal form which can be used to calculate any total characteristic of the pair creation effect. Integrating the latter expression over $d\mathbf{p}_\perp$ one obtain the final form,

$$j_x \approx \tilde{j}_x = \frac{1}{(2\pi)^3}\int_{x_{\mathrm{L}}}^{x_{\mathrm{R}}} dx U'(x)^2 \exp\left[-\pi\frac{\Delta^2}{v_s U'(x)}\right]. \tag{118}$$

For the examples discussed in Secs. IV A 1–IV A 4, the total density of the magnon pair production has the following form:

$$\tilde{j}_x = r^{\rm cr}\frac{\delta U}{|\mu B'|}\begin{cases}1, & \text{for } L\text{-constant step}\\ \tilde{\delta}/2, & \text{for Sauter-like step}\\ G\left(2, \pi\frac{\Delta^2}{v_s|\mu B'|}\right), & \text{for exponential step}\\ \frac{1}{2}G\left(\frac{3}{2}, \pi\frac{\Delta^2}{v_s|\mu B'|}\right), & \text{for inverse-square step}\end{cases}, \quad (119)$$

where $\delta U = U_{\rm L} - U_{\rm R}$ is the magnitude of the step in all the examples,

$$r^{\rm cr} = \frac{|\mu B'|^2}{(2\pi)^3}\exp\left(-\pi\frac{\Delta^2}{v_s|\mu B'|}\right), \quad (120)$$

and

$$\tilde{\delta} = \int_0^\infty dt t^{-1/2}(1+t)^{-5/2}\exp\left(-\frac{\pi t\Delta^2}{v_s|\mu B'|}\right) = \sqrt{\pi}\Psi\left(\frac{1}{2}, -2; \pi\frac{\Delta^2}{v_s|\mu B'|}\right),$$
$$G(\alpha, z) = \int_1^\infty \frac{ds}{s^{\alpha+1}}e^{-x(s-1)} = e^x x^\alpha \Gamma(-\alpha, x). \quad (121)$$

Here $\Psi(\alpha, \beta; z)$ and $\Gamma(\beta, z)$ denote a confluent hypergeometric function and an incomplete Γ function, respectively.

From the general expression for the vacuum-to-vacuum transition probability, given by Eq. (46), the universal form of the vacuum-to-vacuum transition probability for the present case reads

$$P_{\rm v} \approx \exp\left\{-\frac{V_\perp T}{(2\pi)^3}\sum_{l=1}^\infty (-1)^{l-1}\int_{x_{\rm L}}^{x_{\rm R}} dx\frac{U'(x)^2}{l^2} \times \exp\left[-\pi\frac{l\Delta^2}{v_s U'(x)}\right]\right\}. \quad (122)$$

Representation (122) coincides with the vacuum-to-vacuum transition probabilities obtained from the imaginary part of an effective action in the locally constant field approximation [23,24]. In this approximation, the effective action S is expanded about the constant field case, in terms of derivatives of the background field strength $F_{\mu\nu}$,

$$S = S^{(0)}[F_{\mu\nu}] + S^{(2)}[F_{\mu\nu}, \partial_\mu F_{\nu\rho}] + ..., \quad (123)$$

where $S^{(0)}$ involves no derivatives of the background field strength $F_{\mu\nu}$ (that is, $S^{(0)}$ is a locally constant field approximation for S that has a form of the Heisenberg-Euler action), while the first correction $S^{(2)}$ involves two derivatives of the field strength, and so on; see Ref. [25] for a review. Using representation (47), one can see that in the locally constant field approximation the probability $P_{\rm v}$ is given by Eq. (47) where the action S is replaced by $S^{(0)}$.

V. CONCLUSION

In this work we present a Fock space realization of the effective field model describing low-energy dynamics of the antiferromagnetic magnons. Mapping the model to the theory of a charged scalar field interacting with an external constant electric field we apply recently developed approach of the strong-field QED with step potentials to study the Schwinger effect of the magnon-antimagnon pair production on magnetic field inhomogeneities. Initial and final one-particle states are constructed from stationary plane waves satisfied the Klein-Gordon equation. Initial and final vacua are defined and initial and final states of the Fock space are constructed. Mean numbers of magnons and antimagnons created from the vacuum are expressed via overlap amplitudes of the stationary plane waves. Observable physical quantities specifying the vacuum instability are determined. The fluxes of energy and magnetic moments of created magnons are analyzed. Characteristics of the vacuum instability obtained for the number of magnetic steps that allows exact solving the Klein-Gordon equation are presented. In the case of a smooth-gradient step, universal behavior of the flux density of created pairs is described and the relation to the imaginary part of a one-loop effective action S in a locally constant field approximation is established. The results are quite general and are not limited to the simple cubic-type lattice and the G-type antiferromagnet. The presented study demonstrates a consistent application of strong field QED methods in magnonics avoiding contradictions and nonexistent paradoxes in the interpretation of the obtained theoretical results. As a result, one can now apply many of the results known from strong-field QED in magnonics.

Condensed matter systems provide a possibility for experimental verification of quantum vacuum effects stimulated by strong fields, in particular, a laboratory observation the Schwinger effect of the violation of the vacuum instability due to particle-antiparticle pair production, previously considered possible only in supercritical fields in astrophysical situations. This is due to the fact that in many cases low energy dynamics of quasiparticle excitations in condensed matter systems can be described by the Dirac model in which quasiparticles are virtually massless. That is why external field intensities needed for breaking the vacuum stability are relatively small and can be observed in the laboratory conditions. Usually the literature discusses the possibility to observe effects related to the electron-hole creation from vacuum in the case of Dirac and Weyl semimetals, such as the graphene and topological insulators, where the low energy excitations are fermions; see, e.g., Refs. [22,27,28] for the review. Since, the low-energy magnons are bosons with small effective mass, then for the first time it becomes possible to observe the Schwinger effect in the case of the Bose statistics, in particular, the bosonic Klein effect in laboratory conditions. As was already mentioned above, in the case of the Bose statistics appears a new mechanism for amplifying the effect of the pair creation, which we call statistically assisted Schwinger effect. A possible way to detect the Schwinger mechanism of antiferromagnetic magnons in experiments is discussed in Ref. [2]. For example, one can experimentally confirm this mechanism by detecting the spin (the magnetic moment) current using an inverse spin Hall effect.

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APPENDIX A: SOME DETAILS OF SCALAR FIELD QUANTIZATION IN THE PRESENCE OF CRITICAL POTENTIAL STEPS

The Heisenberg operator of the Klein-Gordon field $\hat{\Phi}(X)$ is assigned to the scalar field $\Phi(X)$. It is convenient to consider the canonical pair of the field operator $\hat{\Phi}(X)$ and its canonical momentum $\hat{\Pi}(X)$ as a column $\hat{\Psi}(X)$,

$$\hat{\Psi}(X) = \begin{pmatrix} i\hat{\Pi}^{\dagger}(X) \\ \hat{\Phi}(X) \end{pmatrix}. \tag{A1}$$

The latter satisfies both the Klein-Gordon equation (7) given in the Hamiltonian form,

$$[i\partial_0 - U(x)]\hat{\Psi}(X) = H^{\text{kin}}\hat{\Psi}(X),$$
$$H^{\text{kin}} = \begin{pmatrix} 0 & -v_s^2\delta^{ij}\partial_i\partial_j + \Delta^2 \\ 1 & 0 \end{pmatrix}, \tag{A2}$$

and the equal time canonical commutation relations

$$[\hat{\Psi}(X), \hat{\Psi}(X')]_-|_{t=t'} = 0,$$
$$[\hat{\Psi}(X), \hat{\Psi}^{\dagger}(X')]_-|_{t=t'} = \delta(\mathbf{r}-\mathbf{r}')\sigma_1, \tag{A3}$$

where H^{kin} is the one-particle kinetic energy operator; see, e.g., Refs. [29,30]. It follows from Eq. (A2) that

$$i\hat{\Pi}^{\dagger}(X) = [i\partial_0 - U(x)]\hat{\Phi}(X), \tag{A4}$$

The Hamiltonian $\widehat{\mathbb{H}}$ of the quantized scalar field has the form

$$\widehat{\mathbb{H}} = \widehat{\mathbb{H}}^{\text{kin}} - \frac{1}{v_s^2}\int \hat{\Psi}^{\dagger}(X)\sigma_1 U(x)\hat{\Psi}(X)d\mathbf{r},$$
$$\widehat{\mathbb{H}}^{\text{kin}} = \int \hat{T}^{00}d\mathbf{r} - \mathbb{H}_0,$$
$$\hat{T}^{00} = \frac{1}{v_s^2}\hat{\Psi}^{\dagger}(X)\sigma_1[i\partial_0 - U(x)]\hat{\Psi}(X), \tag{A5}$$

where $\widehat{\mathbb{H}}^{\text{kin}}$ is a kinetic energy operator, which, just from the beginning, we write in a renormalized form with a constant (in general, infinite) term $\mathbb{H}_0$ that corresponds to the kinetic energy of vacuum fluctuations. In the case under consideration, in the similar manner as discussing the inner product (26), it is possible to evaluate integrals (A5) for arbitrary field $U(x)$, using only the asymptotic behavior (14) of functions in the regions S_{L} and S_{R} where particles are free. Decomposing the field $\hat{\Psi}(X)$ over the complete set (8), and dividing integral (A5) in three integrals within the regions S_{L}, S_{int}, and S_{R}, we reduce calculating the quantity (A5) to calculating its one-particle matrix elements in the regions S_{L} and S_{R}; see Sec. IVB.1 and Appendix B in Ref. [14] for details. One can see that the matrix elements of the Hamiltonian $\widehat{\mathbb{H}}$ depends on the total energy of a particle ε. However, probabilities of the magnon scattering, reflection, and the magnon-antimagnon pair production depend on the kinetic energy terms $\pi_0(\text{L/R}) = \varepsilon - U_{\text{L/R}}$, which is a combination of ε and $U_{\text{L/R}}$. This is due to the fact that not the field $U(x)$ itself, acting on the magnons, does the work, but its derivative $\partial_x U(x)$. That is why in the field-theoretical description of magnons embedded into an external inhomogeneous magnetic field namely eigenvalues of the operator $\widehat{\mathbb{H}}^{\text{kin}}$ define vacuum states and other state vectors in the Fock space.[4]

The formal expression of the effective charge (magnetic moment) operator $\hat{Q}$ is

$$\hat{Q} = \int \hat{\rho}d\mathbf{r},\ \ \hat{\rho} = \frac{\mu}{2v_s^2}[\hat{\Psi}^{\dagger}(X)\sigma_1, \hat{\Psi}(X)]_+, \tag{A6}$$

where $[A, B]_+ = AB + BA$ stands for the anticommutator. The eigenvalues of the operator $\widehat{\mathbb{H}}^{\text{kin}}$, together with one-particle mean values of the effective charge, allow one to distinguish particles and antiparticles.

Before proceeding to the definition of the operators of interest (operators of fluxes), which usually are not considered in QFT, it is useful to note that even in the framework of the corresponding classical field theory an observable $\mathcal{F}$ can be realized as an inner product of type (26) of localizable wave packets $\Phi(X)$ and $\hat{F}\Phi'(X)$,

$$\mathcal{F}(\Phi, \Phi') = (\Phi, \hat{\mathcal{F}}\Phi'), \tag{A7}$$

where $\hat{\mathcal{F}}$ is a differential operator, whereas $\Phi(X)$ and $\Phi'(X)$ are solutions of the Klein-Gordon equation. Assuming that an observable $\mathcal{F}(\Phi, \Phi')$ is time-independent during the time T one can represent it in the form of an average over the period T,

$$\langle\mathcal{F}\rangle = \frac{1}{T}\int_{-T/2}^{+T/2}\mathcal{F}(\phi, \phi')dt. \tag{A8}$$

In general, the wave packets $\Phi(X)$ and $\Phi'(X)$ can be decomposed into plane waves $\phi_m(X)$ and $\phi'_m(X)$,

$$\Phi(X) = \sum_m \alpha_m\phi_m(X),\ \ \Phi'(X) = \sum_m \alpha'_m\phi'_m(X), \tag{A9}$$

where $\phi_m(X)$ and $\phi'_m(X)$ are superpositions of the solutions ${}_\zeta\phi_m(X)$ and ${}^\zeta\phi_m(X)$. Taking into account the orthogonality relation (19), one finds that the corresponding decomposition of $\langle\mathcal{F}\rangle$ does not contain interference terms,

$$\langle\mathcal{F}\rangle = \sum_m \mathcal{F}(\alpha_m\phi_m, \alpha'_m\phi'_m). \tag{A10}$$

A physical quantity useful for the further analysis is the average over the period T of the effective charge current through the surface $x = \text{const}$. Since the regions S_{L} and S_{R} supposed to be macroscopic and the particles that come there are free, then such a semiclassical statement of the problem seems to be justified. Moreover, this is how the problem statement is formulated in the theory of the potential scattering. The corresponding QFT operator of the effective charge (magnetic moment) current is proportional to the inner product on this

[4]In QED, where $U_{\text{L/R}}$ are potentials of an electromagnetic field, the operator $\widehat{\mathbb{H}}^{\text{kin}}$ is gauge invariant, which implies that it is an observable physical quantity in contrast with the Hamiltonian $\widehat{\mathbb{H}}$. In the case under consideration the constant values $U_{\text{L/R}}$ are physical quantities and are used to tune the collinear ground state within the regions S_{L} and S_{R}.

surface given by Eq. (16),

$$\widehat{\mathbb{J}} = \frac{1}{T}\int \hat{J}_x dt d\mathbf{r}_\perp,$$
$$\hat{J}_x = \frac{\mu i}{2}\{[\hat{\Phi}(X), \partial_x\hat{\Phi}^\dagger(X)]_+ - [\hat{\Phi}^\dagger(X), \partial_x\hat{\Phi}(X)]_+\}. \quad (A11)$$

Here $\hat{J}_x$ is the longitudinal component of the operator of the effective current density $\hat{\mathbf{J}}$. Note that by virtue of Eq. (7) the latter operator and the operator of the effective charge density $\hat{\rho}$ satisfy the continuity equation $\nabla\hat{\mathbf{J}} + \partial_t\hat{\rho} = 0$.

In what follows, we consider the energy flux of the scalar field through the surface $x = \text{const}$. Its QFT operator has the form

$$\widehat{\mathbb{F}}(x) = \frac{1}{T}\int v_s\hat{T}^{10} dt d\mathbf{r}_\perp,$$
$$\hat{T}^{10}(x) = -[\partial_x\hat{\Phi}^\dagger(X)]\hat{\Pi}^\dagger(X) - \hat{\Pi}(X)\partial_x\hat{\Phi}(X), \quad (A12)$$

where $\hat{T}^{10}$ is the component of the operator of the energy momentum tensor; see Refs. [14,15] for more details.

One can decompose the operator $\hat{\Phi}(X)$ into solutions of the either initial or final complete sets (32) to construct in- and out-states in an adequate Fock space:

$$\hat{\Phi}(X) = \sum_m \mathcal{M}_m^{-1/2}\big[A_m(\text{in})\phi_m^{(\text{in},+)}(X) + B_m^\dagger(\text{in})\phi_m^{(\text{in},-)}(X)\big]$$
$$= \sum_m \mathcal{M}_m^{-1/2}\big[A_m(\text{out})\phi_m^{(\text{out},+)}(X) + B_m^\dagger(\text{out})\,\phi_m^{(\text{out},-)}(X)\big]. \quad (A13)$$

The operator-valued coefficients can be determined with the help of the inner product (26),

$$A_m(\text{in}) = \frac{\mathcal{M}_m^{1/2}\big(\phi_m^{(\text{in},+)}, \hat{\Phi}\big)}{\big(\phi_m^{(\text{in},+)}, \phi_m^{(\text{in},+)}\big)},$$
$$B_m^\dagger(\text{in}) = \frac{\mathcal{M}_m^{1/2}\big(\phi_m^{(\text{in},-)}, \hat{\Phi}\big)}{\big(\phi_m^{(\text{in},-)}, \phi_m^{(\text{in},-)}\big)},$$
$$A_m(\text{out}) = \frac{\mathcal{M}_m^{1/2}\big(\phi_m^{(\text{out},+)}, \hat{\Phi}\big)}{\big(\phi_m^{(\text{out},+)}, \phi_m^{(\text{out},+)}\big)},$$
$$B_m^\dagger(\text{out}) = \frac{\mathcal{M}_m^{1/2}\big(\phi_m^{(\text{out},-)}, \hat{\Phi}\big)}{\big(\phi_m^{(\text{out},-)}, \phi_m^{(\text{out},-)}\big)},$$

where M_m are normalization factors given by Eq. (31). These operators define annihilation and creation operators of initial or final particles in each the range Ω_k ($m_k \in \Omega_k$) as follows:

$$A_{m_1}(\text{in}) = {}_+a_{m_1}(\text{in}),\ B_{m_1}^\dagger(\text{in}) = {}^-a_{m_1}(\text{in});$$
$$A_{m_1}(\text{out}) = {}^+a_{m_1}(\text{out}),\ B_{m_1}^\dagger(\text{out}) = {}_-a_{m_1}(\text{out});$$
$$A_{m_2}(\text{in}) = A_{m_2}(\text{out}) = a_{m_2},\ B_{m_2}^\dagger(\text{in}) = B_{m_2}^\dagger(\text{out}) = 0\,;$$
$$A_{m_3}(\text{in}) = {}_-a_{m_3}(\text{in}),\ B_{m_3}^\dagger(\text{in}) = {}^-b_{m_3}^\dagger(\text{in});$$
$$A_{m_3}(\text{out}) = {}_+a_{m_3}(\text{out}),\ B_{m_3}^\dagger(\text{out}) = {}^+b_{m_3}^\dagger(\text{out});$$
$$A_{m_4}(\text{in}) = A_{m_4}(\text{out}) = 0,\ B_{m_4}^\dagger(\text{in}) = B_{m_4}^\dagger(\text{out}) = b_m^\dagger\,;$$
$$A_{m_5}(\text{in}) = {}^+b_{m_5}^\dagger(\text{in}),\ B_{m_5}^\dagger(\text{in}) = {}_-b_{m_5}^\dagger(\text{in});$$
$$A_{m_5}(\text{out}) = {}_+b_{m_5}^\dagger(\text{out}),\ B_{m_5}^\dagger(\text{out}) = {}^-b_{m_5}^\dagger(\text{out}). \quad (A14)$$

Here a and b are annihilation and $a^\dagger$ and $b^\dagger$ are creation operators, the operators a and $a^\dagger$ describe magnons and the operators b and $b^\dagger$ describe antimagnons. The vacuum vectors $|0, \text{in}\rangle$ and $|0, \text{out}\rangle$ are null vectors for all the annihilation operators a and b. Operators labeled by the argument "in" are interpreted as initial particle operators, whereas operators labeled by the argument "out" are interpreted as final particle operators. Indeed, the commutation relations (A3) yield the standard commutation rules for the introduced creation and annihilation in- or out-operators. One-particle states of initial (final) magnon and antimagnon are

$$a_m^\dagger(\text{in/out})|0, \text{in/out}\rangle,\ b_m^\dagger(\text{in/out})|0, \text{in/out}\rangle, \quad (A15)$$

where $a_m^\dagger(\text{in/out})$ and $b_m^\dagger(\text{in/out})$ are given by Eq. (A14) for each the range Ω_k . The unitary transformation (21) implies a canonical transformations between the in and out-operators.

Interpretation of the magnons and the antimagnon states which follows from in Eqs. (A14) is consistent with a spectrum analysis of the kinetic energy operator $\widehat{\mathbb{H}}^{\text{kin}}$ and the effective charge operator $\hat{Q}$. Inserting decompositions (A13) in Eqs. (A5) and (A6), we obtain diagonal representations for these operators in terms of the introduced creation and annihilation operators. The operators $\widehat{\mathbb{H}}^{\text{kin}}$ and $\hat{Q}$ can be represented as sums of five contributions, each one in the range Ω_k,

$$\widehat{\mathbb{H}}^{\text{kin}} = \sum_{k=1}^{5}\sum_{m\in\Omega_k}\widehat{\mathbb{H}}_m,\ \hat{Q} = \sum_{k=1}^{5}\sum_{m\in\Omega_k}\hat{Q}_m. \quad (A16)$$

Note that a stationary state

$$\Psi_m(X) = \begin{pmatrix}[\varepsilon_m - U(x)]\phi_m(X)\\ \phi_m(X)\end{pmatrix}, \quad (A17)$$

where ϕ_m is one of the solutions from Eq. (32), satisfies the following eigenvalue problem:

$$H^{\text{kin}}\Psi_m(X) = [\varepsilon_m - U(x)]\Psi_m(X). \quad (A18)$$

This implies that the kinetic energy term of the one-particle state reads

$$E_m = (\,\phi_m,\ \phi_m)^{-1}\int_{V_\perp} d\mathbf{r}_\perp\int_{-K^{(\text{L})}}^{K^{(\text{R})}}\Psi_m^\dagger(X)\sigma_1 \times\ [\varepsilon_m - U(x)]\ \Psi_m(X)dx, \quad (A19)$$

where the quantities $(\,\phi_m,\ \phi_m)$ are positive for $m \in \Omega_1 \cup \Omega_2$ and are negative for $m \in \Omega_4 \cup \Omega_5$. In the range Ω_3 we have that $(\,{}_\zeta\phi_m,\ {}_\zeta\phi_m) > 0$, while $(\,{}^\zeta\phi_m,\ {}^\zeta\phi_m) < 0$. Note that the kinetic energy terms ${}_\zeta E_m$ corresponding to the states ${}_\zeta\phi_m$ and the terms ${}^\zeta E_m$ corresponding to the states ${}^\zeta\phi_m$ are different in the general case. The principal value of integral (A19) is determined by integrals over the areas $x \in [-K^{(\text{L})}, x_\text{L}]$ and $x \in [x_\text{R}, K^{(\text{R})}]$, where the field derivative $\partial_x U$ is negligible small. Thus, it is possible to evaluate integrals (A19) for any form of the external field, using only the asymptotic behavior (14) of functions in the regions S_L and S_R where particles are free; see Sec. IV and Appendix B in Ref. [14] for details. Note that $\varepsilon_m - U(x) = \pi_0(\text{L/R})$ in the regions S_L/S_R, respectively.

It can be easily seen from Eq. (A16) that in the range $\Omega_1 \cup \Omega_2$ a one-particle state is the state of a magnon with the kinetic energy $E_m > 0$ and the magnetic moment μ whereas in the range $\Omega_4 \cup \Omega_5$ a one-particle state is the state of an antimagnon with the kinetic energy $-E_m > 0$ and the magnetic moment $-\mu$.

Inserting decompositions (A13) in operators (A11) and (A12), we obtain a renormalized (with respect to the corresponding vacua) in- and out-operators of the effective charge (magnetic moment) current and energy flux flowing through the surfaces $x = x_\mathrm{L}$ and $x = x_\mathrm{R}$, respectively,

$$\begin{aligned}
\widehat{\mathbb{J}}(\mathrm{in}) &= \widehat{\mathbb{J}} - \langle 0, \mathrm{in}|\widehat{\mathbb{J}}|0, \mathrm{in}\rangle, \\
\widehat{\mathbb{J}}(\mathrm{out}) &= \widehat{\mathbb{J}} - \langle 0, \mathrm{out}|\widehat{\mathbb{J}}|0, \mathrm{out}\rangle, \\
\widehat{\mathbb{F}}(x|\mathrm{in}) &= \widehat{\mathbb{F}}(x) - \langle 0, \mathrm{in}|\widehat{\mathbb{F}}(x)|0, \mathrm{in}\rangle, \\
\widehat{\mathbb{F}}(x|\mathrm{out}) &= \widehat{\mathbb{F}}(x) - \langle 0, \mathrm{out}|\widehat{\mathbb{F}}(x)|0, \mathrm{out}\rangle.
\end{aligned} \tag{A20}$$

The one-particle mean values of the fluxes, the kinetic energy, and the effective charge through the surfaces $x = x_\mathrm{L}$ and $x = x_\mathrm{R}$, given by Eqs. (A20), are proportional to the inner product on these surfaces given by Eq. (19), that is, these fluxes are proportional to the flux densities of the particles with given m. Of course, in the range $\Omega_2 \cup \Omega_4$ the flux densities of particles, given by standing waves, are zero.

With account taken of the charge conjugation, it can be seen that the sign of the flux densities of magnons with given m is equal to ζ in the range Ω_1, whereas the sign of the flux densities of antimagnons with given m is equal to $-\zeta$ in the range Ω_5 . In the ranges Ω_1 and Ω_5 an initial state may be localized both in the regions S_L and S_R. This follows from the way of choosing initial conditions. Taking into account directions of motion of magnons and antimagnons in the regions S_L and S_R, we define initial and final states as it is done in Eqs. (A14) and (32).

In the Klein zone Ω_3 the identification of states demands a special consideration. To this end, we represent explicitly the operators $\widehat{\mathbb{H}}_m$ and $\hat{Q}_m$ as follows:

$$\begin{aligned}
\widehat{\mathbb{H}}_m &= {}_+E_m \; {}_+a_m^\dagger(\mathrm{out})\, {}_+a_m(\mathrm{out}) \\
&\quad - {}^+E_m \; {}^+b_m^\dagger(\mathrm{out})\, {}^+b_m(\mathrm{out}) \\
&= {}_-E_m \; {}_-a_m^\dagger(\mathrm{in})\, {}_-a_m(\mathrm{in}) \\
&\quad - {}^-E_m \; {}_-b_m^\dagger(\mathrm{in})\, {}_-b_m(\mathrm{in}); \\
\hat{Q}_m &= \mu[{}_+a_m^\dagger(\mathrm{out})\, {}_+a_m(\mathrm{out}) \\
&\quad - {}^+b_{m_3}^\dagger(\mathrm{out})\, {}^+b_{m_3}(\mathrm{out})] \\
&= \mu\, [{}_-a_{m_3}^\dagger(\mathrm{in})\, {}_-a_{m_3}(\mathrm{in}) \\
&\quad - {}_-b_{m_3}^\dagger(\mathrm{in})\, {}_-b_{m_3}(\mathrm{in})], \quad m \in \Omega_3.
\end{aligned} \tag{A21}$$

Here ${}_\zeta E_m$ and ${}^\zeta E_m$ are principal values of integral (A19) for ${}_\zeta\Psi_m$ and ${}^\zeta\Psi_m$, respectively. They read

$$\begin{aligned}
{}_\zeta E_m &= \pi_0(\mathrm{R}) + \frac{\delta U}{2}|g({}_+|{}^-)|^{-2}, \\
{}^\zeta E_m &= \pi_0(\mathrm{L}) - \frac{\delta U}{2}|g({}_+|{}^-)|^{-2}.
\end{aligned} \tag{A22}$$

Thus, we see that in the range Ω_3 the one-particle state ${}_\zeta\Psi_m$ is the magnon state with the kinetic energy ${}_\zeta E_m > 0$ and the magnetic moment μ whereas the one-particle state ${}^\zeta\Psi_m$ is the antimagnon state with the kinetic energy $-\,{}^\zeta E_m > 0$ and the magnetic moment $-\mu$.

Let us find one-particle mean values of fluxes of the kinetic energy and the effective charge through the surfaces $x = x_\mathrm{L}$ and $x = x_\mathrm{R}$, given by Eqs. (A20), in the range Ω_3. With the help of Eq. (19) we obtain:

$$\begin{aligned}
J_m^a(\mathrm{in}) &= \langle 0, \mathrm{in}|\, {}_-a_m(\mathrm{in})\widehat{\mathbb{J}}(\mathrm{in})\, {}_-a_m^\dagger(\mathrm{in})|0, \mathrm{in}\rangle = -\mu(\mathcal{M}_m T)^{-1}, \\
J_m^a(\mathrm{out}) &= \langle 0, \mathrm{out}|\, {}_+a_m(\mathrm{out})\widehat{\mathbb{J}}(\mathrm{out})\, {}_+a_m^\dagger(\mathrm{out})\, |0, \mathrm{out}\rangle = \mu(\mathcal{M}_m T)^{-1}, \\
J_m^b(\mathrm{in}) &= \langle 0, \mathrm{in}|\, {}^-b_m(\mathrm{in})\widehat{\mathbb{J}}(\mathrm{in})\, {}^-b_m^\dagger(\mathrm{in})|0, \mathrm{in}\rangle = -\mu(\mathcal{M}_m T)^{-1}, \\
J_m^b(\mathrm{out}) &= \langle 0, \mathrm{out}|\, {}^+b_m^\dagger(\mathrm{out})\widehat{\mathbb{J}}(\mathrm{out})\, {}^+b_m^\dagger(\mathrm{out})|0, \mathrm{out}\rangle = \mu(\mathcal{M}_m T)^{-1}; \\
F_m^a(\mathrm{in}) &= \langle 0, \mathrm{in}|\, {}_-a_m(\mathrm{in})\hat{\mathbb{F}}(x_\mathrm{R}, \mathrm{out})\, {}_-a_m^\dagger(\mathrm{in})|0, \mathrm{in}\rangle = -(\mathcal{M}_m T)^{-1}\pi_0(\mathrm{R}), \\
F_m^a(\mathrm{out}) &= \langle 0, \mathrm{out}|\, {}_+a_m(\mathrm{out})\hat{\mathbb{F}}(x_\mathrm{R}, \mathrm{out})\, {}_+a_m(\mathrm{out})|0, \mathrm{out}\rangle = (\mathcal{M}_m T)^{-1}\pi_0(\mathrm{R}), \\
F_m^b(\mathrm{in}) &= \langle 0, \mathrm{in}|\, {}^-b_m(\mathrm{in})\hat{\mathbb{F}}(x_\mathrm{L}, \mathrm{in})\, {}^-b_m^\dagger(\mathrm{in})|0, \mathrm{in}\rangle = (\mathcal{M}_n T)^{-1}|\pi_0(\mathrm{L})|, \\
F_m^b(\mathrm{out}) &= \langle 0, \mathrm{out}|\, {}^+b_m^\dagger(\mathrm{out})\hat{\mathbb{F}}(x_\mathrm{L}, \mathrm{out})\, {}^+b_m^\dagger(\mathrm{out})|0, \mathrm{out}\rangle = -(\mathcal{M}_n T)^{-1}|\pi_0(\mathrm{L})|.
\end{aligned} \tag{A23}$$

Taking into account the space separation of magnons and antimagnons in the region S_R and S_L, one can use the mean values (A23) to distinguish initial and final states in the range Ω_3.

The QFT operators of the effective charge (magnetic moment) current density $\hat{J}_x$ and energy flux density $v_s\hat{T}^{10}$ flowing through the surfaces $x = x_\mathrm{L}$ and $x = x_\mathrm{R}$, that were introduced above in the framework of the approximation under consideration, are time-independent. It is clear that these operators are defined up to certain C-numbers, which may affect explicit forms of the corresponding vacuum means. This circumstance allows one to relate matrix elements of these operators with matrix elements of the corresponding exact strong-field QED operators (which are time-dependent in the presence of the time-dependent field E_pristine). This can be done based on the following physical considerations: Let us consider a relation of the time-independent quantity J_x^cr, obtained in the framework of approximation under consideration, to the matrix elements of the time-dependent longitudinal component $\hat{J}_x^\mathrm{true}(t)$ of an exact current density operator of the strong-field

QED. According to the general theory the difference δJ_x^{true} of the true final vacuum from the initial one is due to the contribution of the current density of the created particles and antiparticles,

$$\delta J_x^{\text{true}} = \langle 0, \text{true in}|\left[\hat{J}_x^{\text{true}}(t_2) - \hat{J}_x^{\text{true}}(t_1)\right]|0, \text{true in}\rangle, \quad \text{(A24)}$$

where $t_1 < t_{\text{in}}$ and $t_2 > t_{\text{out}}$ are the time instants of switching on and off of the field E_{pristine}, respectively. Assuming that effects of fast switching-on and -off are small, we can use approximation (35) for the total number of created pairs. In this case we obtain

$$\delta J_x^{\text{true}} = J_x^{\text{cr}}\{1 + O([\sqrt{v_s|\partial_x U|}T]^{-1})\}. \quad \text{(A25)}$$

We see that the quantity J_x^{cr} can be represented as the following mean value with respect to the $|0, \text{in}\rangle$ vacuum:

$$J_x^{\text{cr}} \approx \langle 0, \text{in}|[\hat{J}_x^{\text{true}}(t_2) - \hat{J}_x^{\text{true}}(t_1)]|0, \text{in}\rangle. \quad \text{(A26)}$$

With the help of this result we may find a relation of the operators $\hat{J}_x^{\text{true}}(t_2)$ and $\hat{J}_x^{\text{true}}(t_1)$ with the time-independent current density operator $\hat{J}_x$ given by Eq. (A11). In particular, we see that the difference $\hat{J}_x^{\text{true}}(t_2) - \hat{J}_x^{\text{true}}(t_1)$ can be approximated by a C-number,

$$\hat{J}_x^{\text{true}}(t_2) - \hat{J}_x^{\text{true}}(t_1) \approx J_x^{\text{cr}}. \quad \text{(A27)}$$

Then, for example, the current density operator $\hat{J}_x^{\text{true}}(t_1)$ can be approximated by the time-independent current density operator $\hat{J}_x$, $\hat{J}_x^{\text{true}}(t_1) \approx \hat{J}_x$. In this case, we have $\hat{J}_x^{\text{true}}(t_2) \approx \hat{J}_x + J_x^{\text{cr}}$, therefore the normal form of both $\hat{J}_x^{\text{true}}(t_1)$ and $\hat{J}_x^{\text{true}}(t_2)$ with respect to the in- and out-operators of creation and annihilation are the same. Thus, calculating one-particle mean values of the effective charge current in Eq. (A23) one can use the operator $\hat{J}_x$. In a similar way, one can relate the time-independent operator $\hat{T}^{10}(x)$ to time-dependent components of the exact operator EMT, for example, $\hat{T}^{10}_{\text{true}}(t_1, x) \approx \hat{T}^{10}(x)$ and $\hat{T}^{10}_{\text{true}}(t_2, x) \approx \hat{T}^{10}(x) + T^{10}_{\text{cr}}(x)$. In addition, calculating one-particle mean values of fluxes of the kinetic energy in Eq. (A23) one can use the operator $\hat{T}^{10}(x)$.

APPENDIX B: EXAMPLES OF EXACT SOLUTIONS WITH *X* STEPS

In this Appendix, we provide additional information on the computation of differential quantities that are relevant to the investigation of magnon pair production stimulated by the external fields mentioned in Sec. IV. We present results only and refer the reader to Refs. [14,31,33,35] for more comprehensive discussions.

For the L-constant magnetic step (63), solutions of Eq. (9) to all intervals can be expressed in the form

$$^{-}\varphi_m(x) = Y\begin{cases} {}_{+}C\exp[i|p^{\text{L}}|(x - x_{\text{L}})]g({}_{+}|^{-}) - {}_{-}C\exp[-i|p^{\text{L}}|(x - x_{\text{L}})]g({}_{-}|^{-}), & x \in S_{\text{L}},\\ {}^{-}C\{b_1 D_\nu[-(1-i)\xi] + b_2 D_{-\nu-1}[-(1+i)\xi]\}, & x \in S_{\text{int}},\\ {}^{-}C\exp[-i|p^{\text{R}}|(x - x_{\text{R}})], & x \in S_{\text{R}},\end{cases} \quad \text{(B1)}$$

$$^{+}\varphi_m(x) = Y\begin{cases} {}_{+}Cg({}_{+}|^{+})\exp[i|p^{\text{L}}|(x - x_{\text{L}})] - {}_{-}Cg({}_{-}|^{+})\exp[-i|p^{\text{L}}|(x - x_{\text{L}})], & x \in S_{\text{L}},\\ {}^{+}C\{b'_1 D_\nu[(1-i)\xi] + b'_2 D_{-\nu-1}[(1+i)\xi]\}, & x \in S_{\text{int}},\\ {}^{+}C\exp[i|p^{\text{R}}|(x - x_{\text{R}})], & x \in S_{\text{R}},\end{cases} \quad \text{(B2)}$$

where b_j, b'_j, $g({}^{\zeta}|_{\zeta'})$, $g({}_{\zeta}|^{\zeta'})$ are constants, which can be obtained via the continuity conditions:

$$\begin{aligned} {}^{+}_{-}\varphi_m(x_{\text{L/R}} - 0) &= {}^{+}_{-}\varphi_m(x_{\text{L/R}} + 0),\\ \frac{d}{dx}\,{}^{+}_{-}\varphi_m(x_{\text{L/R}} - 0) &= \frac{d}{dx}\,{}^{+}_{-}\varphi_m(x_{\text{L/R}} + 0). \end{aligned} \quad \text{(B3)}$$

By demanding continuity of the above functions and their derivatives at $x = x_{\text{R}}$ we find

$$\begin{aligned} b_j &= (-1)^{j+1}\exp\left[\frac{i\pi}{2}\left(\nu + \frac{1}{2}\right)\right]\sqrt{\frac{\xi_2^2 - \lambda}{2}}f_j^{(-)}(\xi_2),\\ b'_j &= (-1)^{j+1}\exp\left[\frac{i\pi}{2}\left(\nu + \frac{1}{2}\right)\right]\sqrt{\frac{\xi_2^2 - \lambda}{2}}f_j^{(+)}(\xi_2), \end{aligned} \quad \text{(B4)}$$

where $\lambda = \pi_\perp^2/v_s\mu B'$ and $f_j^{(\pm)}(\xi)$ were defined before; see Eqs. (67). Now, by imposing the continuity of the functions and derivatives at $x = x_{\text{L}}$ we obtain the coefficients $g({}_{+}|^{+})$ and $g({}_{+}|^{-})$:

$$g({}_{+}|^{-}) = e^{\frac{i\pi}{2}(\nu+\frac{1}{2})}\sqrt{\frac{\sqrt{\xi_1^2 - \lambda}\sqrt{\xi_2^2 - \lambda}}{8}}[f_1^{(-)}(\xi_2)f_2^{(-)}(\xi_1) - f_2^{(-)}(\xi_2)f_1^{(-)}(\xi_1)], \quad \text{(B5)}$$

$$g\left({}_{+}|^{+}\right) = e^{\frac{i\pi}{2}\left(\nu+\frac{1}{2}\right)}\sqrt{\frac{\sqrt{\xi_1^2-\lambda}\sqrt{\xi_2^2-\lambda}}{8}}\left[f_1^{(+)}(\xi_2)\tilde{f}_2^{(-)}(\xi_1) - f_2^{(+)}(\xi_2)\tilde{f}_1^{(-)}(\xi_1)\right], \tag{B6}$$

where

$$\tilde{f}_1^{(\pm)}(\xi) = \left(1 \pm \frac{i}{\sqrt{\xi^2-\lambda}}\frac{d}{d\xi}\right)D_{-\nu-1}[\mp(1+i)\xi],$$
$$\tilde{f}_2^{(\pm)}(\xi) = \left(1 \pm \frac{i}{\sqrt{\xi^2-\lambda}}\frac{d}{d\xi}\right)D_{\nu}[\mp(1-i)\xi]. \tag{B7}$$

For the Sauter-like magnetic step (76), solutions of Eq. (9) and (79) with real asymptotic momenta (14) in remote regions $x \to \mp\infty$ read

$$_{\zeta}\varphi_m(x) = {}_{\zeta}\mathcal{N}\exp(i\zeta|p^{\mathrm{L}}|x)[1+\exp(2x/L_{\mathrm{S}})]^{-iL_{\mathrm{S}}(\zeta|p^{\mathrm{L}}|+|p^{\mathrm{R}}|)/2}\,{}_{\zeta}u_m(x),$$
$$^{\zeta}\varphi_m(x) = {}^{\zeta}\mathcal{N}\exp(i\zeta|p^{\mathrm{R}}|x)[1+\exp(-2x/L_{\mathrm{S}})]^{iL_{\mathrm{S}}(|p^{\mathrm{L}}|+\zeta|p^{\mathrm{R}}|)/2}\,{}^{\zeta}u_m(x), \tag{B8}$$

where

$$_{-}u_m(x) = F(a,b;c;\chi),$$
$$_{+}u_m(x) = F(a+1-c,b+1-c;2-c;\chi),$$
$$^{-}u_m(x) = F(a,b;a+b+1-c;1-\chi),$$
$$^{+}u_m(x) = F(c-a,c-b;c+1-a-b;1-\chi), \tag{B9}$$

where $\chi(x)$ is the change of variable defined in Eq. (77) and a, b, c are given by Eqs. (80). Using the above solutions and Kummer's connection formulas [32]

$$(1-\chi)^{c-a-b}\,{}^{+}u_m(x) = \frac{\Gamma(c+1-a-b)\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)}\,{}_{-}u_m(x) + \frac{\Gamma(c+1-a-b)\Gamma(c-1)}{\Gamma(c-a)\Gamma(c-b)}\chi^{1-c}\,{}_{+}u_m(x),$$
$$^{-}u_m(x) = \frac{\Gamma(a+b+1-c)\Gamma(1-c)}{\Gamma(a+1-c)\Gamma(b+1-c)}\,{}_{-}u_m(x) + \frac{\Gamma(a+b+1-c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)}\chi^{1-c}\,{}_{+}u_m(x), \tag{B10}$$

we conclude that

$$g\left({}_{+}|^{+}\right) = \frac{^{+}\mathcal{N}}{_{+}\mathcal{N}}\frac{\Gamma(c+1-a-b)\Gamma(c-1)}{\Gamma(c-a)\Gamma(c-b)},$$
$$g\left({}_{+}|^{-}\right) = \frac{^{-}\mathcal{N}}{_{+}\mathcal{N}}\frac{\Gamma(a+b+1-c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)}. \tag{B11}$$

For the exponential step (92), general solutions of Eq. (9) with such a field can be presented as a linear combination of two functions, $y_1^j(\eta_j)$ and $y_2^j(\eta_j)$,

$$y_1^j(\eta_j) = e^{-\eta_j/2}\eta_j^{\nu_j}\Phi(a_j,c_j;\eta_j)$$
$$= e^{\eta_j/2}\eta_j^{\nu_j}\Phi(c_j-a_j,c_j;-\eta_j),$$
$$y_2^j(\eta_j) = e^{\eta_j/2}\eta_j^{-\nu_j}\Phi(1-a_j,2-c_j;-\eta_j)$$
$$= e^{-\eta_j/2}\eta_j^{-\nu_j}\Phi(a_j-c_j+1,2-c_j;\eta_j), \tag{B12}$$

where $\eta_j(x)$ are defined in Eqs. (93) and the parameters a_j, c_j by Eqs. (96). In particular, solutions with special asymptotic properties in remote regions $x \to \mp\infty$ are classified as

$$_{+}\varphi_m(x) = {}_{+}\mathcal{N}e^{-i\pi\nu_1/2}y_1^1(\eta_1),$$
$$_{-}\varphi_m(x) = {}_{-}\mathcal{N}e^{i\pi\nu_1/2}y_2^1(\eta_1),$$
$$^{+}\varphi_m(x) = {}^{+}\mathcal{N}e^{i\pi\nu_2/2}y_2^2(\eta_2),$$
$$^{-}\varphi_m(x) = {}^{-}\mathcal{N}e^{-i\pi\nu_2/2}y_1^2(\eta_2). \tag{B13}$$

With the aid of this classification, we may express solutions to all intervals in the form

$$ {}^{+}\varphi_m(x) = \begin{cases} {}_{+}\varphi_m(x)g({}_{+}|^{+}) - {}_{-}\varphi_m(x)g({}_{-}|^{+}), & x \in \mathrm{I}, \\ {}^{+}\mathcal{N}e^{i\pi\nu_2/2}y_2^2(\eta_2), & x \in \mathrm{II}, \end{cases} \tag{B14} $$

$$ {}^{-}\varphi_m(x) = \begin{cases} {}_{+}\varphi_m(x)g({}_{+}|^{-}) - {}_{-}\varphi_m(x)g({}_{-}|^{-}), & x \in \mathrm{I}, \\ {}^{-}\mathcal{N}e^{-i\pi\nu_2/2}y_1^2(\eta_2), & x \in \mathrm{II}, \end{cases} \tag{B15} $$

where $\mathrm{I} = -\infty < x \leqslant 0$, and $\mathrm{II} = 0 < x < +\infty$. To calculate the decomposition coefficients $g({}_{\zeta}|^{\zeta'})$, $g({}^{\zeta}|_{\zeta'})$, we impose continuity of the functions and their derivatives at $x = 0$:

$$ \begin{aligned} {}^{\zeta}\varphi_m(x-0) &= {}^{\zeta}\varphi_m(x+0), \\ \frac{d}{dx}\,{}^{\zeta}\varphi_m(x-0) &= \frac{d}{dx}\,{}^{\zeta}\varphi_m(x+0). \end{aligned} \tag{B16} $$

In particular, the coefficients $g({}_{+}|^{-})$ and $g({}_{+}|^{+})$ have the form

$$ \begin{aligned} g({}_{+}|^{+}) &= -\frac{\exp\left[\frac{i\pi}{2}(\nu_1+\nu_2)\right]}{2\sqrt{|p^{\mathrm{L}}||p^{\mathrm{R}}|}}\left(k_1h_1y_2^2\frac{d}{d\eta_1}y_2^1 + k_2h_2y_2^1\frac{d}{d\eta_2}y_2^2\right)\bigg|_{x=0}, \\ g({}_{+}|^{-}) &= -\frac{\exp\left[\frac{i\pi}{2}(\nu_1-\nu_2)\right]}{2\sqrt{|p^{\mathrm{L}}||p^{\mathrm{R}}|}}\left(k_1h_1y_1^2\frac{d}{d\eta_1}y_2^1 + k_2h_2y_2^1\frac{d}{d\eta_2}y_1^2\right)\bigg|_{x=0}. \end{aligned} \tag{B17} $$

Last, exact solutions to Eq. (9) with inverse-square magnetic steps (105) can be represented as a linear combination of Whittaker functions $W_{\kappa_j,\mu_j}(z_j)$, $W_{-\kappa_j,\mu_j}(e^{-i\pi}z_j)$,

$$ \begin{aligned} w_1^j(z_j) &= e^{-i\pi\kappa_j/2}W_{\kappa_j,\mu_j}(z_j), \\ w_2^j(z_j) &= e^{-i\pi\kappa_j/2}W_{-\kappa_j,\mu_j}(e^{-i\pi}z_j), \end{aligned} \tag{B18} $$

where the variables $z_j(x)$ are given by Eqs. (106) and the parameters κ_j, μ_j by Eqs. (108). Their Wronskian determinant $\mathbb{W}$ reads [36]

$$ \mathbb{W} = w_1^j(z_j)\frac{d}{dz_j}w_2^j(z_j) - w_2^j(z_j)\frac{d}{dz_j}w_1^j(z_j) = 1. \tag{B19} $$

Sometimes, it is convenient to represent the set of solutions (B18) in terms of confluent hypergeometric functions

$$ \begin{aligned} w_1^j(z_j) &= \exp\left[-\frac{i\pi}{2}\left(\kappa_j - \mu_j - \frac{1}{2}\right)\right]e^{-z_j/2}|z_j|^{c_j/2}\Psi(\tilde{a}_j, \tilde{c}_j; z_j), \\ w_2^j(z_j) &= \exp\left[-\frac{i\pi}{2}\left(\kappa_j + \mu_j + \frac{1}{2}\right)\right]e^{z_j/2}|z_j|^{c_j/2}\Psi(\tilde{c}_j - \tilde{a}_j, \tilde{c}_j; e^{-i\pi}z_j), \end{aligned} \tag{B20} $$

in which $\tilde{a}_j = \mu_j - \kappa_j + 1/2$, $\tilde{c}_j = 1 + 2\mu_j$.

Based on asymptotic properties of the Whittaker functions with large argument [36], solutions with real asymptotic momenta (14) in remote regions $x \to \mp\infty$ are classified as follows:

$$ \begin{aligned} {}_{+}\varphi_m(x) &= {}_{+}\mathcal{N}w_1^1(z_1), \quad {}_{-}\varphi_m(x) = {}_{-}\mathcal{N}w_2^1(z_1), \\ {}^{+}\varphi_m(x) &= {}^{+}\mathcal{N}w_2^2(z_2), \quad {}^{-}\varphi_m(x) = {}^{-}\mathcal{N}w_1^2(z_2). \end{aligned} \tag{B21} $$

Thanks to the above classification, we may represent solutions valid at all x in two equivalent forms:

$$ {}^{+}\varphi_m(x) = \begin{cases} {}_{+}\varphi_m(x)g({}_{+}|^{+}) - {}_{-}\varphi_m(x)g({}_{-}|^{+}), & x \in \mathrm{I}, \\ {}^{+}\mathcal{N}w_2^2(z_2), & x \in \mathrm{II}, \end{cases} \tag{B22} $$

$$ {}^{-}\varphi_m(x) = \begin{cases} {}_{+}\varphi_m(x)g({}_{+}|^{-}) - {}_{-}\varphi_m(x)g({}_{-}|^{-}), & x \in \mathrm{I}, \\ {}^{-}\mathcal{N}w_1^2(z_2), & x \in \mathrm{II}. \end{cases} \tag{B23} $$

Demanding continuity of the solutions and their derivatives at $x = 0$ (B16) we discover that $g({}_{+}|^{+})$ and $g({}_{+}|^{-})$ admit the forms

$$ \begin{aligned} g({}_{+}|^{+}) &= \frac{1}{\sqrt{|p^{\mathrm{L}}||p^{\mathrm{R}}|}}\left[|p^{\mathrm{L}}|w_2^2(z_2)\frac{d}{dz_1}w_2^1(z_1) + |p^{\mathrm{R}}|w_2^1(z_1)\frac{d}{dz_2}w_2^2(z_2)\right]\bigg|_{x=0}, \\ g({}_{+}|^{-}) &= \frac{1}{\sqrt{|p^{\mathrm{L}}||p^{\mathrm{R}}|}}\left[|p^{\mathrm{L}}|w_1^2(z_2)\frac{d}{dz_1}w_2^1(z_1) + w_2^1(z_1)|p^{\mathrm{R}}|\frac{d}{dz_2}w_1^2(z_2)\right]\bigg|_{x=0}. \end{aligned} \tag{B24} $$

[1] H. Y. Yuan, Y. Cao, A. Kamra, R. A. Duine, and P. Yan, Quantum magnonics: When magnon spintronics meets quantum information science, Phys. Rep. **965**, 1 (2022).
[2] M. Hongo, T. Fujimori, T. Misumi, M. Nitta, and N. Sakai, Effective field theory of magnons: Chiral magnets and the Schwinger mechanism, Phys. Rev. B **104**, 134403 (2021).
[3] F. Kämpfer, M. Moser, and U.-J. Wiese, Systematic low-energy effective theory for magnons and charge carriers in an antiferromagnet, Nucl. Phys. B **729**, 317 (2005).
[4] J. Schwinger, On gauge invariance and vacuum polarization, Phys. Rev. **82**, 664 (1951).
[5] J. S. Harms, H. Y. Yuan, and R. A. Duine, Enhanced magnon spin current using the bosonic Klein paradox, Phys. Rev. Appl. **18**, 064026 (2022).
[6] J. S. Harms, H. Y. Yuan, and R. A. Duine, Antimagnonics, arXiv:2210.16698.
[7] C. Zhou, L. Wang, and D.-X. Yao, Klein tunneling of Weyl magnons, arXiv:1909.04109.
[8] S. Yuan, C. Sui, Z. Fan, J. Berakdar, D. Xue, and C. Jia, Magnonic Klein and acausal tunneling enabled by breaking the anti parity-time symmetry in antiferromagnets, Commun. Phys. **6**, 95 (2023).
[9] O. Klein, Die Reflexion von Elektronen einem Potentialsprung nach der relativistischen Dynamik von Dirac, Z. Phys. **53**, 157 (1929); Elelrtrodynamik und Wellenmechanik vom Standpunkt des Korrespondenzprinzips, **41**, 407 (1927).
[10] F. Sauter, Über das Verhalten eines Elektrons im homogenen elektrischen Feld nach der relativistischen Theorie Diracs, Z. Phys. **69**, 742 (1931).
[11] F. Sauter, Zum "Klenschen Paradoxon," Z. Phys. **73**, 547 (1932).
[12] N. Dombey and A. Calogeracos, Seventy years of the Klein paradox, Phys. Rep. **315**, 41 (1999); A. Calogeracos and N. Dombey, History and physics of the Klein paradox, Contemp. Phys. **40**, 313 (1999).
[13] A. Hansen and F. Ravndal, Klein's paradox and its resolution, Phys. Scr. **23**, 1036 (1981).
[14] S. P. Gavrilov and D. M. Gitman, Quantization of charged fields in the presence of critical potential steps, Phys. Rev. D **93**, 045002 (2016).
[15] S. P. Gavrilov and D. M. Gitman, Regularization, renormalization and consistency conditions in QED with x-electric potential steps, Eur. Phys. J. C **80**, 820 (2020).
[16] A. I. Breev, S. P. Gavrilov, and D. M. Gitman, Calculations of vacuum mean values of spinor field current and energy-momentum tensor in a constant electric background, Eur. Phys. J. C **83**, 108 (2023).
[17] S. Bayrakci, T. Keller, K. Habicht, and B. Keimer, Spin-wave lifetimes throughout the Brillouin Zone, Science **312**, 1926 (2006).
[18] S. P. Gavrilov and D. M. Gitman, Consistency restrictions on maximal electric field strength in QFT, Phys. Rev. Lett. **101**, 130403 (2008).
[19] E. S. Fradkin, D. M. Gitman, and S. M. Shvartsman, *Quantum Electrodynamics with Unstable Vacuum* (Springer-Verlag, Berlin, 1991).
[20] S. P. Gavrilov, D. M. Gitman, and J. L. Tomazelli, Density matrix of a quantum field in a particle-creating background, Nucl. Phys. B **795**, 645 (2008).
[21] S. P. Gavrilov, D. M. Gitman, and A. A. Shishmarev, Pair production from the vacuum by a weakly inhomogeneous space-dependent electric potential, Phys. Rev. D **99**, 116014 (2019).
[22] A. Fedotov, A. Ilderton, F. Karbstein, B. King, D. Seipt, H. Taya, and G. Torgrimsson, Advances in QED with intense background fields, Phys. Rep. **1010**, 1 (2023).
[23] H. Gies and F. Karbstein, An addendum to the Heisenberg-Euler effective action beyond one loop, J. High Energy Phys. 03 (2017) 108.
[24] F. Karbstein, Heisenberg-Euler effective action in slowly varying electric field inhomogeneities of Lorentzian shape, Phys. Rev. D **95**, 076015 (2017).
[25] G. V. Dunne, Heisenberg-Euler effective lagrangians: Basics and extensions, in *From Fields to Strings: Circumnavigating Theoretical Physics*, edited by M. Shifman, A. Vainshtein, and J. Wheater, I. Kogan Memorial Volume (World Scientific, Singapore, 2005).
[26] S. P. Gavrilov, D. M. Gitman, and N. Yokomizo, Dirac fermions in strong electric field and quantum transport in graphene, Phys. Rev. D **86**, 125022 (2012).
[27] O. Vafek and A. Vishwanath, Dirac fermions in solids: From high-Tc cuprates and graphene to topological insulators and Weyl semimetals, Annu. Rev. Condens. Matter Phys. **5**, 83 (2014).
[28] F. Gelis and N. Tanji, Schwinger mechanism revisited, Prog. Part. Nucl. Phys. **87**, 1 (2016).
[29] S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper & Row, New York, NY, 1961).
[30] D. M. Gitman and I. V. Tyutin, *Quantization of Fields with Constraints* (Springer-Verlag, Berlin, 1990).
[31] S. P. Gavrilov and D. M. Gitman, Scattering and pair creation by a constant electric field between two capacitor plates, Phys. Rev. D **93**, 045033 (2016).
[32] *Higher Transcendental Functions*, edited by A. Erdelyi *et al.*, (McGraw-Hill, New York, NY, 1953), Vols. 1 and 2.
[33] S. P. Gavrilov, D. M. Gitman, and A. A. Shishmarev, Particle scattering and vacuum instability by exponential steps, Phys. Rev. D **96**, 096020 (2017).
[34] A. I. Nikishov, Barrier scattering in field theory: Removal of Klein paradox, Nucl. Phys. B **21**, 346 (1970).
[35] T. C. Adorno, S. P. Gavrilov, and D. M. Gitman, Vacuum instability in a constant inhomogeneous electric field: A new example of exact nonperturbative calculations, Eur. Phys. J. C **80**, 88 (2020).
[36] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, *NIST Handbook of Mathematical Functions* (Cambridge University Press, New York, NY, 2010).