Modified universality hypothesis for the eight-vertex model

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This paper is a comment on recent papers by Kadanoff and Wegner, and Barber and Baxter, concerning the critical exponents of the eight-vertex model. It is shown that the critical exponents of this model as derived by these authors can be obtained from the square Ising lattice critical exponents by a renormalization of the temperature *only*. A new "universality" hypothesis is proposed which is consistent with these results.

Recently a great deal of attention has been given to the two-dimensional eight-vertex model, prompted by Baxter's exact solution[1] for the free energy in zero magnetic field. We shall use the equivalent square-net Ising-model language, as developed by Kadanoff and Wegner[2] and Wu,[3] where the Hamiltonian of the model is given by

$$H = -J\sum_{j,k}(\sigma_{j,k}\sigma_{j+1,k+1} + \sigma_{j+1,k}\sigma_{j,k+1}) - \lambda u, \tag{1}$$

where

$$u = \sum_{j,k}\sigma_{j,k}\sigma_{j+1,k+1}\sigma_{j+1,k}\sigma_{j,k+1}. \tag{2}$$

The original interest centered on Baxter's discovery that the specific-heat critical exponent α is a continuous function of the coupling constant λ, a result which is an apparent violation of universality.[4,5] However, Kadanoff and Wegner showed that this result, at least to first order in λ, is caused by the fact that the Baxter model contains a marginal operator, viz., u, conjugate to λ, which scales like $1/r^d$ and which can consequently give rise to λ-dependent critical exponents. Since that time other exact expressions have been found for the correlation-length exponent[6] ν and the surface-tension exponent μ.[7]

In addition Barber and Baxter[8] have recently given plausible arguments which lead to a conjectured form of the spontaneous-magnetization exponent β. These results, together with the assumption that static scaling holds, lead to the following predictions for the eight-vertex model critical exponents:

$$\begin{aligned} &2-\alpha = \pi/\bar{\mu} = (2-\alpha_0)\nu(\lambda), \quad \mu(\lambda) = \nu(\lambda) = \tfrac{1}{2}\pi/\bar{\mu} \\ &\beta(\lambda) = \tfrac{1}{16}\pi/\bar{\mu} = \beta_0\nu(\lambda), \quad \gamma(\lambda) = \tfrac{7}{8}\pi/\bar{\mu} = \gamma_0\nu(\lambda) \\ &\delta = 15, \ \eta = \tfrac{1}{4}, \end{aligned} \tag{3}$$

where the four-spin coupling constant λ is related to $\bar{\mu}$ by

$$\bar{\mu} = \pi - \cos^{-1}[\tanh(2\lambda)], \tag{4}$$

and where the zero subscripts denote the Ising-model exponents.

A striking consequence of Barber's and Baxter's conjecture is that η and δ, both of which are defined at T_c, are invariant with respect to the perturbation λu and are given by their $\lambda = 0$ (Ising) values. This is in sharp contrast to the other exponents which describe temperature-dependent quantities and which are "renormalized" with respect to their $\lambda = 0$ (Ising) values in a λ-dependent way. In an attempt to rationalize these results we modify the usual universality hypothesis in such a way that the eight-vertex model would satisfy the new form of universality if Eq. (3) is correct. The new postulate is based on the idea of scaling with the correlation length rather than with the reduced temperature and reduces to the current statements of universality in the case of irrelevant operators. We also present arguments as to why the addition of a marginal operator could leave η and δ invariant but produce a change in the correlation-length exponent ν. Finally, we compare our hypothesis with a recent postulate of Suzuki[9] concerning a "new universality of critical exponents." Although the two postulates are essentially the same for the eight-vertex model, Suzuki's hypothesis seems considerably stronger in more general situations.

We being by summarizing in what seems a particularly simple and suggestive way the above results for the eight-vertex model and to raise certain questions which are suggested by them. We first summarize the static-scaling assumption for the Baxter model for the magnetization $\langle M\rangle$ (we do not use the free energy in order to avoid the special case of logarithmic divergencies), and the spin-spin correlation function $\langle\sigma_0\sigma_r\rangle$:

$$\langle M\rangle = \epsilon^{\beta(\lambda)}m^*(h/\epsilon^{\Delta(\lambda)})$$

and

$$\langle \sigma_0 \sigma_r \rangle = \epsilon^{2\beta(\lambda)} g^*(h/\epsilon^{\Delta(\lambda)}, \epsilon^{\nu(\lambda)} r), \tag{5}$$

where h is the magnetic field, $\Delta(\lambda) = \delta\beta(\lambda)$, and $\epsilon = [T - T_c(\lambda)]/T_c(\lambda)$. The form of the scaling functions m^* and g^* are, in general, also dependent on λ; this can affect the amplitudes of their singular parts. We now note that, introducing the zero-field inverse correlation length $\kappa = \epsilon^{\nu(\lambda)}$ and using Eq. (4), Eq. (5) can be rewritten

$$\langle M \rangle = m(\kappa, h) = \kappa^{\beta_0} m^*(h/\kappa^{\Delta_0}),$$

$$\langle \sigma_0 \sigma_r \rangle = g(r, \kappa, h) = \kappa^{2\beta_0} g^*(h/\kappa^{\Delta_0}, \kappa r). \tag{6}$$

Stating the scaling hypothesis in terms of the inverse correlation length rather than ϵ, one sees that these equations can be interpreted as a "modified smoothness" or "modified universality" hypothesis, as they bear some similarity to the original "smoothness" hypothesis of Kadanoff and Griffiths[5] (which does not hold in its original form for the eight-vertex model) in that the exponents which enter into the equations are those for the unperturbed (Ising) problem but differ from it in two aspects:

(a) There is a temperature-dependent renormalization of the correlation length κ such that there is a continuous dependence on the coupling constant λ via

$$\kappa \sim \epsilon^{\nu(\lambda)}. \tag{7}$$

(b) The form of the scaling functions may depend on λ. The original form of smoothness would give, for example,

$$m^*(h/\kappa^{\Delta_0}) = a m_0^*(bh/c\kappa^{\Delta_0}), \tag{8}$$

where $a, b,$ and c depend on λ, and where m_0^* is the Ising scaling function. However, this would not seem likely to hold for the eight-vertex model, although we know of no results which preclude it at the moment.

The interesting aspect of these equations is that they display vividly what is implied by the recent conjecture of Barber and Baxter, namely that the critical properties of the eight-vertex model differ from the original pair of uncoupled Ising models, to which it reduces as $\lambda \to 0$, only by a temperature-dependent renormalization of the correlation length, but that exponents which are independent of temperature, such as η and δ, are left invariant. Such a "renormalization" was in fact shown by Kadanoff and Wegner in first-order perturbation theory, where they showed that for small λ

$$\kappa \sim (\epsilon^*)^{\nu_0},$$

$$\langle M \rangle_{\text{spont.}} \sim (\epsilon^*)^{\beta_0}, \tag{9}$$

etc., with $\epsilon^* = \epsilon^{1 - 4\lambda/\pi}$. However, this now seems to be true for all λ, with $\epsilon^* = \epsilon^{\nu(\lambda)}$.

If this conjecture is true it raises several interesting questions:

(a) Why does the four-spin coupling term leave η and δ invariant?

(b) Is there a physical explanation for the temperature renormalization ϵ^*, as one has, for example, in Fisher renormalization where there is $\epsilon^* = \epsilon^{1/(1-\alpha)}$?

(c) Is the four-spin coupling term a marginal operator for all λ, i.e., if one could construct a rigorously exact renormalization transformation for the eight-vertex model would this operator have an eigenvalue 1? This remains a fundamental problem to resolve before one can really say that one understands why the eight-vertex model has a continuous variation of the critical exponents with the coupling constant λ.

The only insight that so far has been given is contained in the work of Kadanoff and Wegner, who find in first-order perturbation theory that η and δ are invariant to $O(\lambda^2)$. This result follows from an application of the operator algebra and the observation that the relevant coefficient a_σ in the differential equation $\partial X_\sigma / \partial\lambda = -2\pi a_\sigma$ in this algebra, where $\eta = 2X_\sigma$, is identically zero. Here one can see that the result is intimately linked to the fact that the marginal operator $u_r = E_r^{(1)} E_r^{(2)}$ is compounded from a second member of the operator algebra of the Ising model, namely the energy density $E_r^{(i)}$ and *not*, say, from the operator whose fluctuations are the largest in the Ising model, namely the order parameter. Hence, if η and δ are invariant for the eight-vertex model it is presumably due to the type of perturbation added to the Ising model and to the subtleties of the operator algebra. Other perturbations, for example, might change η and δ, as is possibly the case in the fcc Ising model with pure three-spin interactions, where Griffiths and Wood[10] estimate $\delta \simeq 24$ from low-temperature expansions.

It is tempting to generalize the conjecture embodied in (6) for the eight-vertex model in the form of a modified universality hypothesis. Namely, when two Hamiltonians which are otherwise equivalent in the original universality sense (i.e., they both have the same dimensionality, the same symmetry of the order parameter, and the same potential range parameter σ) differ by a perturbation λu, "universality" holds in the sense of Eq. (6). This has as a consequence the conjectured results of Suzuki, namely that the reduced critical exponents $\hat{\gamma} = \gamma/\nu$, $\hat{\beta} = \beta/\nu'$, $\hat{\phi} = (2-\alpha)/\nu$, and η and δ are independent of the details of the Hamiltonian (i.e., these reduced exponents are independent of λ). A plausible argument for the hypothesis can be given which depends upon an interpretation of the difference in the physical significance of the two correlation-function exponents η and ν. The exponent η is not an exponent which is associated with some field-scaling variable (such as temperature or magnetic field) but is rather a measure of the behavior of the order parameter *at* the critical point. One can interpret it either as a measure of the spatial decay of the order-parameter correlation function at the critical point or in a related way as characterizing the absolute magnitude of the fluctuations of the order parameter at the critical point.[11] Hence if the operator involved in the perturbation λu has fluctuations smaller than the order parameter itself (as with the eight-vertex model), one would expect that this perturbation could not change the magnitude of the fluctuations at the critical point. Hence such a perturbation should leave η (and through the scaling law, δ) invariant. On the other hand, the exponent ν is a measure of the correlation range as one *approaches* the critical point *in units* of the *temperature variable*. Thus it is *not* a measure of the absolute fluctuations at the critical point but is rather a measure of the rate of growth of the fluctuations in units of temperature. Therefore, since, as was first pointed out by Suzuki,[9] there would seem to be nothing special about the temperature variable, one could argue that the exponent ν might depend upon the details of the perturbation, i.e., $\nu = \nu(\lambda)$. However, if one "scales" with this correlation length κ^{-1}, rather than with ϵ, as in Eq. (6), all details of the perturbation disappear. They arise only when one characterizes the critical behavior in terms of ϵ.

The distinction between η and other exponents such as ν noted above has also been made by Wegner[12] in the context of the renormalization group theory. He has shown that η is associated with a "redundant operator" which can be transformed away from the renormalization group equations, in contrast to exponents associated with "scaling operators." A further discussion of the distinction between these two classes of operators is given in Wegner's paper.

Our conjecture about scaling with the correlation length is essentially the same as has been proposed by Suzuki. However, Suzuki does not attempt to justify his assumption that η and δ are invariant, other than to assert that they do not involve the temperature variable. Furthermore, he seems to assume that the usual form of universality might be incorrect even if the operator added is not marginal as is implied by his discussion of the λ transition in He.[4] If so, his postulate would be considerably stronger than ours.

Convincing answers to all of these issues must of course await more detailed theoretical studies.

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