Coupled order parameters, symmetry-breaking irrelevant scaling fields, and tetracritical points

Alastair D. Bruce*
Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14850

Amnon Aharony†
Baker Laboratory and Materials Science Center, Cornell University, Ithaca, New York 14850 and Physics Department, Harvard University, Cambridge, Massachusetts 02138

(Received 16 July 1974)

The phase diagrams of systems described by a Hamiltonian containing an anisotropic quadratic term of the form $\frac{1}{2}g\Sigma_{\alpha=1}^{n}c_\alpha\int_{\vec{x}}S_\alpha^2(\vec{x})$, and a cubic anisotropic term $v\Sigma_{\alpha=1}^{n}\int_{\vec{x}}S_\alpha^4(\vec{x})$, are studied using mean-field theory, scaling theory, and expansions in $\epsilon(=4-d)$ and $1/n$. Here, $S_\alpha(\vec{x})$ $(\alpha=1,\ldots,n)$ is a local n-component ordering variable. Systems to which the analysis is applicable include perovskite crystals, stressed along the [100] direction ($n=3$), anisotropic antiferromagnets in a uniform field, uniaxially anisotropic ferromagnets, ferroelectric ferromagnets and crystalline ^{4}He($n=2$). When $g=0$ and $T=T_c$ these systems undergo a phase transition that may be associated (for small n) with the Heisenberg fixed point ($v^*=0$) or (otherwise) with the cubic fixed point ($v^*>0$) of the renormalization group. Although v is an "irrelevant variable" in the former case, it is found to have important effects. For $v<0$, the point $g=0$, $T=T_c$ represents a *bicritical* point in the g-T plane, at which a first-order "spin-flop" line (separating two distinct ordered phases) meets two critical lines. For $v>0$, the "flop" line splits into two critical lines, associated with transitions between each of the ordered phases and a new *intermediate* phase; the point $T=T_c$, $g=0$ is then *tetracritical*. The shape of the boundary of the intermediate phase is given by $T=T_2(g,v)$ with $[T_c-T_2(g,v)]\sim(g/v)^{1/\psi_2}$, where $\psi_2=\phi_g-\phi_v$ (if the tetracritical point is Heisenberg-like) or $\psi_2=\phi_g^C$ (if it is cubic). Here, ϕ_g, ϕ_v, and ϕ_g^C are appropriate crossover exponents associated with the two symmetry-breaking perturbations. The phase diagram of [111]-stressed perovskites is also discussed and the experimental situation briefly reviewed.

I. INTRODUCTION

Many physical systems exhibit critical behavior which depends on the interplay of more than one order parameter (or the different components of a multicomponent order parameter). A well-studied example is that of a uniaxially anisotropic antiferromagnet in a uniform magnetic field,[1-3] which may order antiferromagnetically (for small values of the field) or ferromagnetically (for large values of the field). Other examples are associated with the competition between superfluid and crystal ordering in crystalline ^{4}He,[4] between ferroelectric and ferromagnetic ordering in certain crystals,[5] between two types of magnetic ordering in mixed magnetic crystals, e.g., (Mn, Fe)WO_4 or Fe(Pd, Pt)$_3$,[6,7] and between rotations about the different axes that characterize the displacive phase transitions in perovskite crystals.[8-10]

A Landau-type phenomenological theory (or mean-field theory) has been applied to several of the above examples[2,4,5,7,9,11]; we shall give a systematic development of such a theory in Sec. III. Generally, such an analysis indicates the possibility of two types of phase diagram, as shown schematically in[4] Fig. 1 (for the case of the uniaxially anisotropic antiferromagnet). In one case [Fig. 1(a)], there exist only two distinct ordered phases (antiferromagnetic and "spin-flopped"), separated by a first-order ("spin-flop") line. This line ends at a "spin-flop" point, termed a *bicritical point*,[12-14] where two second-order lines (associated with the transition from the disordered phase into each of the ordered phases) also meet. In some cases, there may be present in the Hamiltonian a term that couples together the degrees of freedom associated with the order parameters of the two ordered phases, in such a way as to favor their *simultaneous* ordering. The existence of such a term may lead (for some values of the parameters) to the appearance of a *third*, *"intermediate"*, ordered phase [Fig. 1 (b)], displaying both types of ordering. In this case, the "flop" line splits into two second-order lines, separating this phase from the two aforementioned ordered phases. The point at which the four second-order lines meet is termed a *tetracritical point*.[4,13] The existence of such an intermediate phase (and thus of a tetracritical point) in crystalline ^{4}He ("supersolid") has been the subject of several recent speculations.[4] It has been seen experimentally in the case of Fe(Pd, Pt)$_3$ and (Mn, Fe)WO_4,[6,7] but not yet, to our knowledge, in other cases.

Although mean-field theory may give a correct description of the behavior of systems far away from their critical points, it is known to fail as

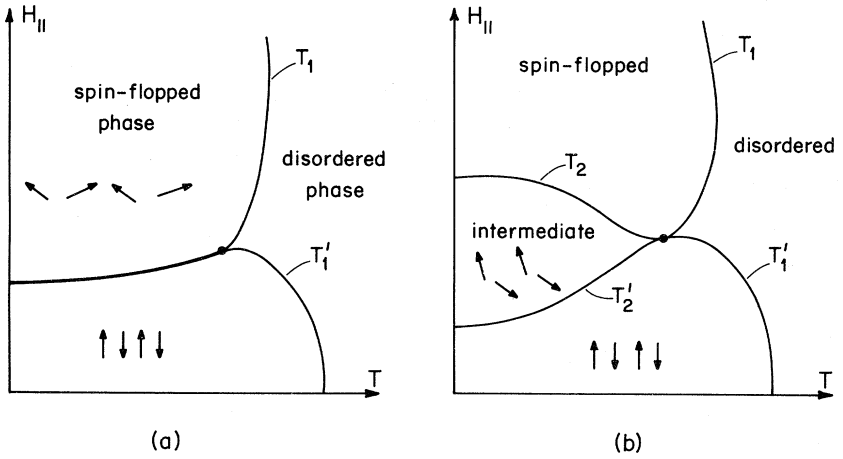

FIG. 1. Schematic phase diagrams of an anisotropic antiferromagnet in a uniform field $H_{\parallel}$. (a) Displaying a bicritical point and a flop line (the bold line in the figure). (b) Displaying a tetracritical point and an intermediate phase.

criticality is approached. Thus, even if mean-field theory predicts, for some system, the existence of an intermediate phase, this phase might exist only far from criticality. In order to obtain a valid description of the critical region one must go beyond mean-field theory, e.g., by a *renormalization group study*[15] or by a general *scaling analysis*.[16] Both methods have recently been applied in analyses of the critical behavior of anisotropic antiferromagnets[13,14] and of stressed perovskite crystals.[10] The essential features of these systems are embodied in a single Hamiltonian which, within a mean-field approximation, can lead either to a bicritical or a tetracritical point. In particular, this Hamiltonian, which we shall describe in more detail in Sec. II, includes a quadratic term (associated with a coupling constant g) reflecting anisotropy in the "exchange" interaction between the n-component (continuous) "spins," and a quartic term (with which we shall associate a coupling constant v) of cubic (or lower[13,14]) symmetry. The mean-field analysis, described in Sec. III, shows that the latter term is instrumental in setting up the competition between the two types of ordering, that leads ultimately to the existence of an intermediate ordered phase, and hence (within the mean-field theory) to a tetracritical point.

A renormalization group analysis shows that, for n larger than a critical value $n_c(d)$ (d is the spatial dimensionality of the system), the isotropic n-vector ("Heisenberg-like") fixed point of the renormalization group is unstable against the "cubic" perturbing term,[17] and the "spin-flop" point can be associated with a *cubic* fixed-point Hamiltonian, characterized by a positive fixed-point value of the cubic parameter v^*. Therefore, for $n > n_c(d)$, the "spin-flop" point may satisfy the conditions for tetracriticality.[4,14] Note that the cubic fixed point will describe the critical behavior only if the parameter v in the Hamiltonian of the system is positive. For negative v, a first-order transition is to be expected.[10]

For $n < n_c(d)$, the *isotropic* fixed point is stable against the cubic (or lower-symmetry) perturbations, and the "spin-flop" point can be identified with this n-component "Heisenberg"-like fixed point.[13] (Again, a first-order transition may occur for values of v smaller than some negative critical value.[10]) The "irrelevance" (in the renormalization group sense[15]) of the v-type perturbation with respect to the Heisenberg fixed point indicates that [for $n < n_c(d)$] *at* the flop point the conditions for tetracriticality are *not* fulfilled, so that (in a sense that will become clearer below) *asymptotically close* to the spin-flop point one may expect a phase diagram with typical *bicritical* geometry. At any finite distance below this point, however, there may appear corrections to the asymptotic scaling form of the free energy.[18] Through such "*corrections to scaling*" the symmetry-breaking v perturbation, though irrelevant, might in principle affect the nature of the ordering, and ensure the persistence of the intermediate ordered phase. We find this to be the case. Indeed, following the mean-field analysis in Sec. III, Sec. IV emphasizes more quantitatively the potentially significant role played by "irrelevant" symmetry-breaking perturbations.

The main part of the paper, however, is concerned with the calculation of the shape of the boundary of the intermediate phase in the vicinity of the spin-flop point. This shape is characterized by an exponent ψ_2, which describes the nonanalytic dependence of the transition temperature upon the symmetry-breaking g perturbation. Since the function $n_c(d)$ cannot be determined accurately at $d = 3$ [a Padé analysis of three terms in the ϵ expansion of $n_c(d)$ gives $n_c(3) \simeq 3.13$, but the errors may be quite large], it is uncertain whether the $n = 3$ case of particular interest should be described by a cubiclike or a Heisenberg-like "flop" point.

We therefore analyze both cases. Following a general scaling analysis in Sec. V, the exponent ψ_2 is calculated to order ϵ $(=4-d)$ for both fixed points in Sec. VI. Aided by a Ward identity, introduced in Sec. VII, we describe the calculation of ψ_2 to order ϵ^2 and to order[19] $1/n$ (for the Heisenberg-like fixed point) in Secs. VIII and IX.

As we shall explain in detail in Sec. II, the Hamiltonian analyzed throughout Secs. II–IX is appropriate for describing the displacive phase transitions of perovskite crystals which are subjected to a stress along the [100] direction.[10] Since the case of [111]-stressed perovskites is also of experimental interest,[8] we give a special analysis of this problem in Sec. X. Finally, in Sec. XI, we discuss the results of our analysis in the light of the current experimental situation.

II. HAMILTONIAN

Our analysis is based on a partition function of the form[15]

$$Z=\int d^nS\, e^{-\bar{\mathcal{H}}}\ , \tag{2.1}$$

where $\vec{S}\equiv(S_1,\ldots,S_n)$ is an n-component continuous-"spin" variable, and where the "reduced" Hamiltonian is

$$\bar{\mathcal{H}}=\bar{\mathcal{H}}_s+\bar{\mathcal{H}}_g+\bar{\mathcal{H}}_v+\bar{\mathcal{H}}_h\ . \tag{2.2}$$

Here,

$$\bar{\mathcal{H}}_s=\frac{1}{2}\int_{\vec{x}}[r_0\vec{S}^2(\vec{x})+(\vec{\nabla}\vec{S})^2]+u\int_{\vec{x}}|\vec{S}|^4\ , \tag{2.3}$$

where $r_0=A(T-T_0)+Bg$ [in the following, we shall choose a temperature scale in which $A=1$; g is defined in (2.4)] and $\vec{S}^2=\sum_{\alpha=1}^{n}S_\alpha^2$. The symmetry-breaking perturbation terms are given by

$$\bar{\mathcal{H}}_g=\tfrac{1}{2}g\sum_{\alpha=1}^{n}c_\alpha\int_{\vec{x}}S_\alpha^2\ ,\qquad \sum_{\alpha=1}^{n}c_\alpha=0\ , \tag{2.4}$$

$$\bar{\mathcal{H}}_v=v\sum_{\alpha=1}^{n}\int_{\vec{x}}S_\alpha^4\ , \tag{2.5}$$

and

$$\bar{\mathcal{H}}_h=-\sum_{\alpha=1}^{n}h_\alpha\int_{\vec{x}}S_\alpha\ . \tag{2.6}$$

The perturbation Hamiltonian $\bar{\mathcal{H}}_g$ represents a "spin" anisotropy.[20] For example, if $g<0$ and

$$\begin{aligned}&c_1=\cdots=c_m=1\ ,\\ &c_{m+1}=\cdots=c_n=-m/(n-m)\ ,\end{aligned} \tag{2.7}$$

then the quadratic ("exchange") interaction between the first m components of $\vec{S}$ is stronger than that among the remaining components, so that the former tend to order at a higher temperature.

The perturbation Hamiltonian $\bar{\mathcal{H}}_v$ represents a "cubic" anisotropy.[17] If $v>0$, mean-field theory shows (see Sec. III) that the "spins" tend to align along the n-dimensional diagonals, e.g., $[1, 1, \ldots, 1]$, whereas if $v<0$ they tend to align along axes, e.g., $[1, 0, \ldots, 0]$.

The last term $\bar{\mathcal{H}}_h$ represents the usual interaction between the variables S_α and the appropriate ordering fields h_α (in units of $1/k_B T$).

To illustrate the relevance of the Hamiltonian (2.2) to the problems mentioned in Sec. I, consider first the case $n=2$, when the Hamiltonian may be written as

$$\begin{aligned}\bar{\mathcal{H}}=\int_{\vec{x}}\{&[\tfrac{1}{2}(r_0+gc_1)S_1^2+\tfrac{1}{2}(r_0+gc_2)S_2^2\\ &+(\vec{\nabla}S_1)^2+(\vec{\nabla}S_2)^2]\\ &+(u+v)(S_1^4+S_2^4)+2uS_1^2S_2^2\}\ .\end{aligned} \tag{2.8}$$

In principle one should also include in (2.8) a term of the form $(S_1^4-S_2^4)$. Within mean-field theory [when the terms $(\nabla S_1)^2$ and $(\nabla S_2)^2$ are ignored], such a term simply amounts to a rescaling of the magnitudes of S_1 and S_2. In the context of the renormalization group, at $n=2$, this term is found to be irrelevant,[14] with a vanishing fixed point value. We thus ignore this term from the outset.

The Hamiltonian (2.8) may be viewed as a sum of two Landau-Ginzburg-like (single-component) free energies, in the order parameters S_1 and S_2, and a coupling term $2u\int_{\vec{x}}S_1^2S_2^2$. Such free energies [without the fluctuation terms $(\vec{\nabla}S_1)^2$ and $(\vec{\nabla}S_2)^2$] have been used to describe the phase diagram of ^{4}He (S_1 and S_2 are then the superfluid and the crystalline order parameters[4]), of ferroelectric ferromagnets (S_1 and S_2 are the electric- and the magnetic-dipole moments[5]) and of (Mn, Fe)WO_4 (S_1 and S_2 are combinations of the Fe and Mn spins[7]). The parameters g and v are functions of the physical variables of the problem, e.g., the pressure in ^{4}He, or the relative concentration of Fe and Mn atoms in (Mn, Fe)WO_4.[7]

For general n, the Hamiltonian (2.2) is directly applicable to the problem of cubic magnets with an anisotropic exchange perturbation (which might, in principle, be induced by an anisotropic stress).

The Hamiltonian (2.2) may also be used to describe the uniaxially anisotropic n-component antiferromagnet in a uniform field.[13,14] Strictly, one should again supplement the quartic term in (2.2) with a uniaxial term wS_1^4. Within the context of the renormalization group, however, one finds that the essential features of the possible forms of asymptotic critical behavior are described by the Hamiltonian (2.2).[21]

Our main motivation for writing the Hamiltonian in the form (2.2) is that it is then immediately ap-

plicable to the description of a perovskite crystal, stressed in a [100] direction, in the vicinity of a transition associated with the softening of the zone-corner R_{25} mode of vibration.[22] As we have discussed previously,[10] the effective Hamiltonian (2.2) [with $n=3$, $m=1$ in (2.7)] results when the elastic strain coordinates are integrated out of the partition function for this sytem. The parameter g is then proportional to the applied pressure.

In the context of the applicability of (2.2) to the description of stressed perovskites it should be emphasized that, in addition to the quartic term of cubic symmetry, (2.5), the Hamiltonian (2.2) should also contain a quadratic term of cubic symmetry, $\sum_\alpha \int_{\vec{x}} (\nabla_\alpha S_\alpha)^2$. Such a term reflects the strong anisotropy found in the R_{25} soft-mode dispersion relations, in the perovskite $SrTiO_3$.[23] We do not consider the effects of this term here, since a renormalization-group analysis indicates that it is irrelevant[24]; furthermore, neutron-scattering results show that the dispersion curves are quite isotropic in $LaAlO_3$,[25] which, as we shall see, displays the phase diagram of greatest interest here.

A further advantage of writing the Hamiltonian in the form (2.2) is that the parameters g and v represent *scaling fields*[18] near the isotropic Heisenberg-like fixed point of particular interest. The importance of this will be made clearer in Sec. V. One direct result is that the phase diagrams drawn in Fig. 1 assume a simpler form in the g-T plane (g replacing $H_\parallel$), where the "spin-flop" point lies on the T axis.

The Hamiltonian (2.2) does not apply to perovskites stressed along the [111] direction. We shall give a separate discussion of this problem in Sec. X.

III. MEAN-FIELD THEORY

The basic approximation leading to the phenomenological, or mean-field, theory lies in the assumption that the fluctuations in the order parameters can be ignored. Thus, denoting the (spatially constant) order parameter by $\vec{S}(\vec{x}) \equiv \vec{M}$, Eqs. (2.1)–(2.5) lead to a Helmholtz free-energy density (in zero ordering field) of the form

$$A(T, g, v, \vec{M}) = \tfrac{1}{2} r_0 \vec{M}^2 + u\vec{M}^4 + \tfrac{1}{2} g \sum_\alpha c_\alpha M_\alpha^2 + v \sum_\alpha M_\alpha^4 \quad . \quad (3.1)$$

In order to simplify the analysis, we now assume that [see (2.7)]

$$c_1 = 1, \quad c_2 = \cdots = c_n = -1/(n-1) \quad (3.2)$$

(a uniaxial anisotropy), and define

$$r \equiv r_0 + c_2 g = T - T_0 + [B - 1/(n-1)]g. \quad (3.3)$$

A generalization of this choice is straightforward. The free energy thus becomes

$$A = \tfrac{1}{2} r \vec{M}^2 + \frac{n}{2(n-1)} g M_1^2 + u\vec{M}^4 + v \sum_\alpha M_\alpha^4 \quad . \quad (3.4)$$

The equilibrium state of the system will be described by that vector order parameter $\vec{M}$ which minimizes the free energy A. Before proceeding with the details of finding these minima, we make a few general observations. The direction of the vector $\vec{M}$ is of importance only in the second and last terms in (3.4). Consider first the second term. If $g > 0$, this is minimized when $M_1 = 0$; if $g < 0$, this is at a minimum when $\vec{M}$ is aligned parallel to the 1 axis. Thus this term favors an ordering along the 1 axis or perpendicular to it.

The last term in (3.4) is a little more complicated. However, a direct differentiation, for a fixed length M, shows that $\sum_\alpha M_\alpha^4$ has a minimum when $\vec{M}$ is along a diagonal (e.g., $M_1 = M_2 = \cdots = M_n = M/n^{1/2}$), and a maximum when $\vec{M}$ is along an axis (e.g., $M_1 = M$, $M_2 = \cdots = M_n = 0$). The former will be preferred when $v > 0$, and the latter when $v < 0$.

Thus, ordering along an axis is preferred by both the second and the fourth terms in (3.4) if $v < 0$. If $g < 0$ the ordering will be along the 1 axis, and, if $g > 0$, along one of the other axes. The line $g = 0$ is thus identified as a "spin-flop" line, and the phase diagram assumes the shape presented in Fig. 2(a).[10] This diagram should be appropriate to the case of the perovskite $SrTiO_3$ (stressed along [100]), where model calculations[26] indicate that v is negative, in agreement with the observation that the stress-free crystal distorts to a tetragonal phase.[8] The explicit equations for the second-order lines, from the disordered phase into the [1 0...0] or the [0 1 0...0] phases, follow immediately on substituting $\vec{M} \equiv [M, 0, \ldots, 0]$ (for $g < 0$) or $\vec{M} \equiv [0, M, 0, \ldots, 0]$ (for $g > 0$). With r given by (3.3), we find [Fig. 2(a)]

$$T_1(g) = T_0 - Bg + g/(n-1) \ , \quad g > 0 \quad (3.5)$$

and

$$T_1'(g) = T_0 - Bg - g \ , \quad g < 0 \ . \quad (3.6)$$

Before going on to discuss the case $v > 0$, it is worth mentioning that the system described by the free energy (3.1) becomes unstable when $v < -u$. In this case, one must add to (3.1) terms of sixth order in $\vec{M}$, and the transitions at $T_1(g)$ and $T_1'(g)$ will become first order. The point $v = -u$ is thus *tricritical*. We shall not consider this possibility any further in this paper.[10]

In the case $v > 0$ (appropriate for $LaAlO_3$, which distorts to a trigonal phase,[27] in the absence of an

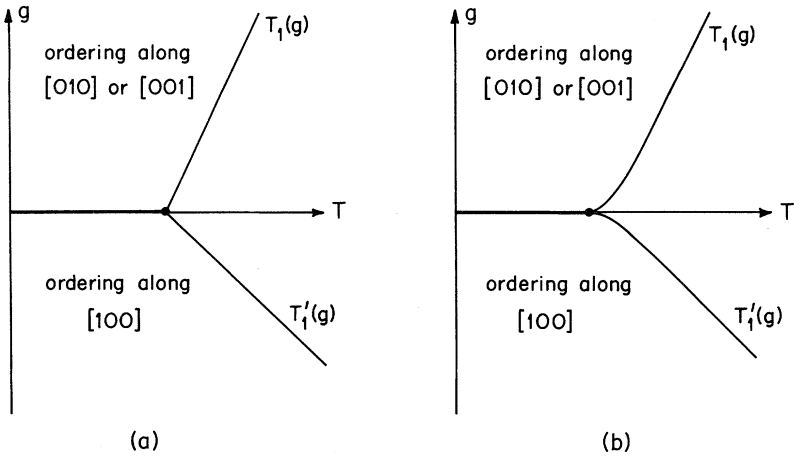

FIG. 2. Phase diagram of the system described by (2.2), with uniaxial anisotropy and $v<0$, displaying a bicritical point, for $n=3$. The bold line represents the spin-flop first-order transition line. (a) Mean-field theory. (b) Scaling and renormalization-group theory.

applied stress), the g term and the v term compete to produce an additional phase with an intermediate ordering (along neither a diagonal nor an axis) and thence a tetracritical point. It is this situation that is of primary interest here.

We now proceed to minimize A with respect to $\vec{M}$, for $v>0$. We first transform to angular coordinates,

$$M_1 = M\cos\theta\ ,\quad M_2 = M\sin\theta\, m_2\ ,\ \ldots,\quad M_n = M\sin\theta\, m_n\ , \tag{3.7}$$

with

$$\sum_{\alpha=2}^{n} m_\alpha^2 = 1\ . \tag{3.8}$$

The free energy (3.4) thus becomes

$$A = \tfrac{1}{2}rM^2 + uM^4 + [n/2(n-1)]gM^2\cos^2\theta + vM^4\left(\cos^4\theta + \sin^4\theta\sum_{\alpha=2}^{n} m_\alpha^4\right)\ . \tag{3.9}$$

The m_α's appear only in the form $\sum_\alpha m_\alpha^4$, in the last term. Since $v>0$, we want to *minimize* this expression. As mentioned above, the minimum occurs when

$$|m_2| = \cdots = |m_n| = 1/(n-1)^{1/2}\ . \tag{3.10}$$

Substituting into (3.9), we thus have

$$A = \tfrac{1}{2}rM^2 + uM^4 + [n/2(n-1)]gM^2\cos^2\theta + vM^4\{\cos^4\theta + [1/(n-1)]\sin^4\theta\}\ . \tag{3.11}$$

Differentiating with respect to θ, and demanding that $\partial A/\partial\theta = 0$, we find three solutions:

$$\text{(I)}\ \cos\theta = 0\ ,$$
$$\text{(II)}\ \cos^2\theta = (1/n)\,(1 - ng/4vM^2)\ , \tag{3.12}$$
$$\text{(III)}\ \sin\theta = 0\ .$$

Clearly, solution II is possible only if

$$-4(n-1)vM^2/n \le g \le 4vM^2/n\ . \tag{3.13}$$

We now differentiate A with respect to M, and find that $\partial A/\partial M = 0$ if $M=0$ (disordered phase) or if M has the following expressions corresponding to the above three solutions:

$$\text{(I)}\ M^2 = -\tfrac{1}{4}r/[u + v/(n-1)]\ ,$$
$$\text{(II)}\ M^2 = -\tfrac{1}{4}[r + g/(n-1)]/(u + v/n)\ , \tag{3.14}$$
$$\text{(III)}\ M^2 = -\tfrac{1}{4}[r + ng/(n-1)]/(u+v)\ .$$

A study of the second derivatives of A with respect to θ and to M now identifies the regions in the g-r (or g-T) plane in which each solution represents the minimum. This leads, after simple algebra, to Fig. 3(a), with $T_1(g)$ and $T_1'(g)$ given by (3.5) and (3.6), while $T_2(g,v)$ and $T_2'(g,v)$ are given [through the inequality (3.13)] as

$$T_2(g,v) = T_0 - Bg - (nu+v)\frac{g}{v},\quad g>0\ . \tag{3.15}$$

and

$$T_2'(g,v) = T_0 - Bg + \frac{1}{n-1}(nu+v)\frac{g}{v}\ ,\quad g<0\ . \tag{3.16}$$

One can explicitly check that the magnetization changes continuously across the phase lines T_2 and T_2'. If one starts at a large value of g, with $T<T_0$, then the ordering is along the $(n-1)$-dimensional diagonal $[0\,1\ldots1]$. As g decreases across $T_2(g,v)$, the vector $\vec{M}$ starts to rotate continuously in the $(x, y, \ldots, y)$ plane (with $x=y$ at $g=0$), until it is fully aligned along the axis $[1\,0\ldots0]$ at $T_2'(g,v)$. For lower (i.e., more negative) values of g, $\vec{M}$ remains in that direction, changing only in magnitude. Phase II is thus an *intermediate* phase, in which

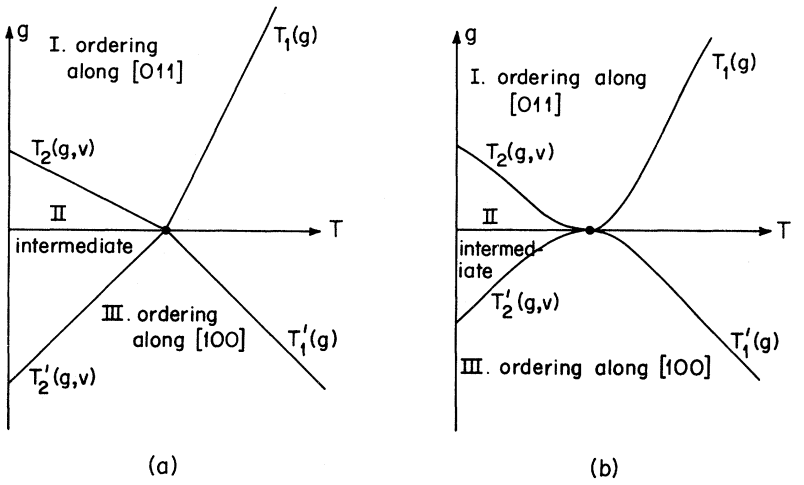

FIG. 3. Phase diagram for $v>0$, displaying a tetracritical point and an intermediate phase, for $n=3$. (a) Mean-field theory. (b) Scaling and renormalization-group theory.

both types of ordering (associated with phases I and III) exist.

The second derivatives of A with respect to θ and to M are directly related to the susceptibility tensor. In particular, $\partial^2 A/\partial\theta^2$ is proportional to the inverse transverse susceptibility in phases I and III, and thus gives a measure of the ease with which the "magnetization" can be rotated. At the phase lines T_2 and T_2', this susceptibility diverges, as is to be expected at a second-order line, with a mean-field exponent $\gamma=1$. The main part of this paper will be devoted to a diagrammatic expansion of this transverse susceptibility, to permit the systematic calculation of corrections to the mean-field results, Eqs. (3.15) and (3.16).

IV. EASY AXES AND "RELEVANCE" OF SYMMETRY-BREAKING IRRELEVANT VARIABLES

We have seen in Sec. III that the "cubic" parameter v in (2.2) plays a particularly important role within mean-field theory, different signs of v leading to different types of ordered phases. For $v>0$ (and $g=0$), the preferred direction for the "magnetization" (the *"easy axis"*) coincides with a cube diagonal, whereas, for $v<0$, the easy axes are the cube edges.

Within the framework of renormalization group theory we expect that the asymptotic critical behavior of a system will reflect the character of the fixed point to which the Hamiltonian "flows" under the renormalization-group transformation.[15] For $n>n_c(d)$ the fixed-point value of v at the stable (cubic) fixed point is nonzero, and in fact positive.[17] Thus the terms of cubic symmetry in the free energy remain nonzero even very close to the critical point and will manifest themselves in the thermodynamic properties, including the nature of the ordered phase. Hence, in this case, it is clear that the notion of "easy axes" is still meaningful, even very close to the critical point.

When $n<n_c(d)$, however, the asymptotic critical behavior should be dominated by the Heisenberg fixed-point Hamiltonian, for which the fixed-point value of the coupling constant v is equal to zero. The fact that near this fixed point v is an *irrelevant variable* leads us to expect those terms in the Hamiltonian which involve v to have a decreasing weight as the critical point is approached. *Asymptotically* close to the critical point, these "cubic" terms are essentially zero, and full rotational invariance is achieved. As we shall see below, this effect can be related to an increase in the transverse fluctuations of the order parameter. As the critical point is approached, these fluctuations become very large, corresponding to a divergence of the transverse susceptibility. It is the purpose of this section to make these ideas more quantitative, and to argue that the notion of an *easy axis* is still meaningful at any *finite* distance below criticality.

In the interests of clarity we shall restrict our discussion to the problem of an $n=2$ component cubic system [$g=0$ in Eq. (2.2)] in zero field. We shall also consider only the general functional form of the ϵ-expansion corrections to mean-field theory. Explicit expressions are given in the calculations described in Secs. VI–IX. We thus consider the Hamiltonian

$$\overline{\mathcal{H}}=\int_{\vec{x}}\{\tfrac{1}{2}[r_0\vec{S}^2+(\vec{\nabla}\vec{S})^2]+u\vec{S}^4+v(S_1^4+S_2^4)\}\ . \qquad (4.1)$$

Below criticality, the "magnetization" vector $\vec{M}=\langle\vec{S}\rangle$ may *a priori* lie in a general direction in the S_1-S_2 plane. We denote the angle between $\vec{M}$ and the 1 axis by θ (to be determined later). Following Wallace,[28] we can now rotate the coordinate axes of our "spin" space by an angle θ, so that the new 1 axis coincides with the magnetization. We thus define new "spin" coordinates Q_1 and Q_2, by

$$S_1 = Q_1 \cos\theta - Q_2 \sin\theta \ , \qquad S_2 = Q_1 \sin\theta + Q_2 \cos\theta \ . \tag{4.2}$$

We now have

$$\langle Q_1 \rangle = M \ , \quad \langle Q_2 \rangle = 0 \ . \tag{4.3}$$

As usual in the analysis of ordered phases,[29] we shift the variable Q_1,

$$Q_1 \to Q_1 + M \ , \tag{4.4}$$

so that

$$\langle \vec{Q} \rangle = 0 \ . \tag{4.5}$$

It is also convenient to introduce exact susceptibilities by[29]

$$\int_{\vec{x}} \langle Q_\alpha(0)\, Q_\beta(\vec{x}) \rangle = \gamma_\alpha^{-1}\, \delta_{\alpha\beta} \ . \tag{4.6}$$

We have assumed here that the true susceptibility tensor is diagonal in the Q_1-Q_2 representation, so that

$$\int_{\vec{x}} \langle Q_1(0)\, Q_2(\vec{x}) \rangle = 0 \ . \tag{4.7}$$

We shall later check the consistency of this assumption.

We now write our Hamiltonian as

$$\bar{\mathcal{H}} = \bar{\mathcal{H}}_0 + \bar{\mathcal{H}}_1 \ , \tag{4.8}$$

with the free part

$$\bar{\mathcal{H}}_0 = \frac{1}{2} \int_{\vec{x}} [r_1 Q_1^2 + r_2 Q_2^2 + (\vec{\nabla}\vec{Q})^2] \tag{4.9}$$

and a perturbation

$$\begin{aligned} \bar{\mathcal{H}}_1 = \int_{\vec{x}} \{ & [r_0 + 4uM^2 + 4vM^2 a(\theta)] M Q_1 - 4vM^3 b(\theta) Q_2 + \tfrac{1}{2}[r_0 - r_1 + 12uM^2 + 12vM^2 a(\theta)] Q_1^2 + \tfrac{1}{2}[r_0 - r_2 + 4uM^2 \\ & + 24vM^2 c(\theta)] Q_2^2 - 12vM^2 b(\theta) Q_1 Q_2 + 4uMQ_1 \vec{Q}^2 + 4vMa(\theta) Q_1^3 + 4vMb(\theta) Q_2 (Q_2^2 - 3Q_1^2) + 24vMc(\theta) Q_1 Q_2^2 + u\vec{Q}^4 \\ & + va(\theta)(Q_1^4 + Q_2^4) + 4vb(\theta) Q_1 Q_2 (Q_2^2 - Q_1^2) + 12vc(\theta) Q_1^2 Q_2^2 \} \ , \end{aligned} \tag{4.10a}$$

with

$$a(\theta) = \sin^4\theta + \cos^4\theta \ , \qquad b(\theta) = \sin\theta\cos\theta\,(\cos^2\theta - \sin^2\theta) \ , \qquad c(\theta) = \sin^2\theta\cos^2\theta \ . \tag{4.10b}$$

The values of M and of θ may now be obtained by a perturbation expansion (in $\bar{\mathcal{H}}_1$) of Eqs. (4.5) and (4.6). Consider first the condition $\langle Q_2 \rangle = 0$. Contributions to this equation will come only from terms in the diagrammatic expansion which involve one vertex (or an odd number of vertices) with an odd power of Q_2. Inspection of (4.10a) shows that all such terms must then involve the angular factor $b(\theta)$, so that the diagrammatic expansion leads to an equation of the form

$$0 = 4vMb(\theta)\,[M^2 + O(1)] \ . \tag{4.11}$$

Recalling that, within the context of an ϵ expansion, M^2 is to be regarded as being of order[29] $1/\epsilon$ [this follows from the equation of state, $\langle Q_1 \rangle = 0$; see Eq. (4.13) below] we may reasonably assume that the only solution of (4.11) (for nonzero v, M) is given by

$$b(\theta) = \sin\theta\cos\theta\,(\cos^2\theta - \sin^2\theta) = 0 \tag{4.12}$$

to all orders in perturbation theory. We therefore identify two types of easy axes, namely, a coordinate axis ($\sin\theta = 0$ or $\cos\theta = 0$) or a diagonal ($\sin^2\theta = \cos^2\theta = \frac{1}{2}$).

Turning now to the consistency check of (4.7), we see that the only terms which might contribute to the nondiagonal element of the susceptibility tensor, also involve the angular factor $b(\theta)$, so that (4.7) is, indeed, satisfied.

We can now proceed to calculate the magnetization M and the susceptibilities r_1 and r_2, using the remaining Eqs. (4.5) and (4.6). These yield

$$M[r_0 + 4uM^2 + 4vM^2 a(\theta) + O(u,v)] = 0 \ , \tag{4.13}$$

$$r_1 = r_0 + 12uM^2 + 12vM^2 a(\theta) + O(u,v) \ , \tag{4.14}$$

and

$$r_2 = r_0 + 4uM^2 + 24vM^2 c(\theta) + O(u,v) \ . \tag{4.15}$$

Thus, for $M \neq 0$,

$$r_2 = 4vM^2 [6c(\theta) - a(\theta) + O(u,v)] \ . \tag{4.16}$$

This yields

$$\begin{aligned} r_2 &= 4vM^2 [1 + O(u,v)] \quad \text{if } \cos^2\theta = \sin^2\theta = \tfrac{1}{2} \ , \\ &= -4vM^2 [1 + O(u,v)] \quad \text{if } \sin\theta\cos\theta = 0 \ . \end{aligned} \tag{4.17}$$

Since the correction terms in the brackets are much smaller than unity, the sign of r_2 is determined by the sign of v. For thermodynamic stability, we must have $r_2 \geq 0$. We thus conclude that the ordering is along [1 1] ($\cos^2\theta = \sin^2\theta = \frac{1}{2}$) if $v > 0$

and along [0 1] or [1 0] ($\cos\theta = 0$ or $\sin\theta = 0$) if $v < 0$.

Although, in Eq. (4.17), we have explicitly retained only the leading (mean-field) term, we shall see below [see, for example, Eq. (8.6)], and, indeed, Wallace has already shown within the context of the $1/n$ expansion,[28] that the only effect of the critical fluctuations [which manifest themselves in the diagrammatic contributions to (4.17)] is to modify the power of the prefactor M^2. The "irrelevance" of v at the Heisenberg fixed point merely increases the rate at which r_2 vanishes as the transition temperature is approached ($M \to 0$). For any finite M (i.e., at any finite distance from criticality), $r_2 > 0$, and the notion of easy axes is still meaningful.

V. SCALING ANALYSIS

The mean-field analysis presented in Sec. III ignores the effects of critical fluctuations. In this section we shall develop general scaling arguments[16] that indicate in what way such fluctuations may affect the phase diagrams which we study. In the interests of clarity we shall again consider the special case of uniaxial anisotropy, Eq. (3.2). Without loss of generality we shall also assume that $g > 0$.

As discussed in Sec. IV, we expect the identification of the various phases to remain as in Figs. 2(a) and 3(a). We assume that an intermediate phase exists, with two second-order boundaries, and we study the generalizations of the mean-field results [Figs. 2(a) and 3(a)] allowed by scaling theory. The final results of this analysis are exhibited in Figs. 2(b) and 3(b).

We shall first consider the case $n < n_c(d)$, when the Heisenberg-like fixed point is stable. The "irrelevance" of v ensures that, for $g = 0$, all the "spin" components order (simultaneously) via an n-component Heisenberg-like transition at a critical temperature T_c (T_c is a function of v, as it is of all other irrelevant variables). The temperature T_c is thus to be identified with[30] $T_2(0, v)$ and $T_1(0)$ of Fig. 3. Note that we have emphasized the v dependence of T_2, and not that of T_1, only because, as we have seen from mean-field theory, the *existence* of a second phase transition depends crucially on the cubic perturbation. Hereafter we shall use a scaling temperature field[31]

$$t = (T - T_c)/T_c \quad . \tag{5.1}$$

For $T > T_1(g)$ we now make the usual *extended scaling Ansatz* for the free energy,[16,18] at zero ordering field,

$$F(t, g, v) \approx t^{2-\alpha}\, \mathfrak{F}(g t^{-\phi_g},\ v t^{-\phi_v}) \quad . \tag{5.2}$$

Similar relations, with different scaling functions, may be written for $T < T_1(g)$ and/or $g < 0$. The exponent α is the Heisenberg specific-heat exponent, while ϕ_g and ϕ_v give a measure of the stability of the Heisenberg-like fixed point against the perturbations (2.4) and (2.5). The "relevance" of the g perturbation is expressed in the positivity of the "crossover" exponent ϕ_g,[20b,20c] while, for $n < n_c(d)$, ϕ_v is negative.[17] In general, we might include in (5.2) other irrelevant variables. However, since we expect that only symmetry-breaking fields, like v, are important, we ignore these other variables.

For $g > 0$ $T \geq T_1(g)$, we expect F to be singular at $T = T_1(g)$, when the last $n - 1$ "spin"-components order. Thus, $\mathfrak{F}(x, y)$ has a line of singularities in the XY plane. This line may be described by an equation

$$\bar{x} = X_1(\bar{y}) \quad , \tag{5.3}$$

where $\bar{x} = g t_1^{-\phi_g}$, $\bar{y} = |v|\, t_1^{-\phi_v}$, and $t_1 = [T_1(g) - T_c]/T_c$. For small values of $\bar{y}$ we expect

$$X_1(\bar{y}) \simeq a_1 \bar{y}^{\theta_1} + b_1 \quad , \tag{5.4}$$

where a_1 and b_1 are constants, and where θ_1 is some exponent characteristic of the Heisenberg fixed point. The fact that mean-field theory gives a phase transition at $T_1(g)$, even in the absence of a cubic perturbation ($\bar{y} = 0$), indicates that b_1 will in general be positive ($g > 0$) and that $\theta_1 > 0$. Thus, for sufficiently small t_1 (i.e., sufficiently close to the "spin-flop" point, when $\bar{y}$ is very small; note that $\phi_v < 0$) we expect $g t_1^{-\phi_g} \approx b_1$, or

$$t_1 = [T_1(g) - T_c]/T_c \sim g^{1/\psi_1} \quad , \tag{5.5}$$

where the *shift exponent* ψ_1 (within our framework of extended scaling) is simply given as the spin-anisotropy crossover exponent,[32]

$$\psi_1 = \phi_g \quad . \tag{5.6}$$

Since $\phi_g > 1$,[20] one concludes[13] that the critical lines $T_1(g)$ and $T_1'(g)$ come in *tangentially* to the T axis [Figs. 2(b) and 3(b)].

Note that in general the shift in the transition temperature, Eq. (5.5), also contains an analytic piece, i.e., a term *linear* in g.[31] *Only* in the *asymptotic* scaling region ($g \to 0$) can this term be ignored. Outside this region, the dependence of T_1 on g may be *effectively linear*.

Our discussion so far is applicable for v positive or negative. We must now examine the second phase transition, expected only when $v > 0$. Thus, in the remaining part of this section, and, indeed, in Secs. VI–IX, we shall consider only $v > 0$.

For $T_2(g, v) < T < T_1(g)$ we repeat the above analysis to yield an equation analogous to (5.3), namely,

$$g t_2^{-\phi_g} = X_2(v t_2^{-\phi_v}) \quad , \tag{5.7}$$

with $t_2 = [T_c - T_2(g, v)]/T_c$. Again, we expect

$$X_2(\bar{y}) \simeq a_2 \bar{y}^{\theta_2} + b_2 \tag{5.8}$$

for small $\bar{y}$. In this case, however, mean-field theory shows that for $v \leq 0$ there is no second phase transition, since there is nothing to destabilize the Hamiltonian for $T < T_1(g)$. Consequently, we expect to find no solution for t_2 of (5.7) when $v = 0$ (except the trivial solution $g = t_2 = 0$). This implies that $\theta_2 > 0$, $b_2 \leq 0$ (we assume $g > 0$). The mean-field result (3.15) would imply that

$$\lim_{g \to 0} T_2(g, v) = T_c \ , \tag{5.9}$$

and that b_2 is thus equal to zero. However, scaling arguments alone cannot rule out the possibility of a negative b_2 (presumably of order ϵ), corresponding to a violation of (5.9). Since the calculations described below afford no evidence for such a situation, we shall pursue this scaling analysis on the assumption that $b_2 = 0$. Then, for sufficiently small t_2, $g t_2^{-\phi_g} \simeq a_2 (v t_2^{-\phi_v})^{\theta_2}$, or

$$t_2 \sim (g/v^{\theta_2})^{1/\psi_2} \ , \tag{5.10}$$

with

$$\psi_2 = \phi_g - \theta_2 \phi_v \ . \tag{5.11}$$

Since $\theta_2 > 0$ and $\phi_v < 0$, we conclude that $\psi_2 > \phi_g$. Therefore, as the line $T_2(g, v)$ comes in to the tetracritical point it approaches the T axis *faster* than the line $T_1(g)$ [Fig. 3(b)]. In the *asymptotic* scaling region one might therefore say that the two lines $T_2(g, v)$ and $T_2'(g, v)$ coincide to give a single "flop" line. However, they do, in fact, represent two distinct second-order lines all the way into the tetracritical point. We shall see below that explicit expansions in ϵ and $1/n$ are in accord with (5.10) and (5.11), with $\theta_2 = 1$.

If $n > n_c(d)$, the exponent ϕ_v is positive, and the asymptotic critical behavior (in the absence of the g perturbation) is characteristic of the cubic fixed point.[17] Nevertheless, for a sufficiently small cubic perturbation, the preceding analysis should still describe the shape of the phase boundary in some region close to the "spin-flop" point (although ψ_2 will be less than ϕ_g, since $\phi_v > 0$): Only very close to this point will the relevance of v alter the character of the critical behavior. This observation motivates the calculation of ψ_2 to order $1/n$, described in Sec. IX, where we do indeed find the results (5.10) and (5.11) (for sufficiently small v) even though the cubic perturbation is certainly relevant in this (large-n) limit.

Ultimately, however, for $n > n_c(d)$, the relevance of v will be felt. To describe the resulting phase boundary we need only repeat the above analysis, writing a scaling *Ansatz* for the free energy in terms of the scaling fields appropriate to the *cubic* fixed point.

The situation for $T \gtrsim T_1(g)$ is essentially the same as that described [for $n < n_c(d)$] by (5.5) and (5.6), except that ϕ_g in Eq. (5.6) is replaced by ϕ_g^C,[33] the exponent that characterizes the instability of the *cubic* fixed point against the anisotropic g perturbation. However, in contrast to the case $n < n_c(d)$, one does not have to invoke symmetry-breaking corrections to scaling to permit the second phase transition, since (unlike the Heisenberg fixed-point Hamiltonian which displays rotational invariance) the cubic fixed-point Hamiltonian has a nonzero cubic "symmetry-breaking" term. By a trivial extension of the above analysis one then finds that the exponent ψ_2 in this case is the same as ψ_1, and is simply equal to the crossover exponent ϕ_g^C. Again, the ϵ-expansion results in Sec. VI are in accord with this conclusion.

VI. CALCULATION OF ψ_2 TO ORDER ϵ

We are now ready to go beyond mean-field theory, and obtain systematic corrections to the results of Sec. III, in particular those for $T_2(g, v)$ and $T_2'(g, v)$, Eqs. (3.15) and (3.16). In order to include both in a single analysis, we use the general form of the coefficients c_α, given in Eq. (2.7), and consider only $g > 0$. Thus, at zero external ordering field, our Hamiltonian is

$$\mathcal{H} = \int_{\vec{x}} \left\{ \frac{1}{2} \left[(r_0 + g) \sum_{\alpha=1}^{m} S_\alpha^2 + \left(r_0 - \frac{m}{n-m} g \right) \sum_{\alpha=m+1}^{n} S_\alpha^2 + (\vec{\nabla} \vec{S})^2 \right] + u |\vec{S}|^4 + v \sum_{\alpha=1}^{n} S_\alpha^4 \right\} \ . \tag{6.1}$$

Since $g > 0$, the last $(n - m)$ "spin" components tend to order first, at the line $T_1(g)$ in Fig. 3(b). A renormalization-group analysis, using recursion relations similar to those studied by Fisher and Pfeuty,[20b] immediately leads to the conclusion that this ordering occurs through an $(n - m)$-component Heisenberg-like transition.[34] Close to the tetracritical point, one thus observes a crossover from n-component to $(n - m)$-component critical behavior. [Similarly, the transition at $T_1'(g)$ is m-component Heisenberg-like.]

Phase I in Fig. 3(b) now has an ordering, or "magnetization," along the $[0, \ldots, 0, 1, \ldots, 1]$ direction, where the first m components are zero and the last $(n - m)$ components are all equal. As we have seen in Sec. IV, symmetry-breaking corrections to scaling [associated with the cubic term v in (6.1)], about the $(n - m)$-Heisenberg-like fixed

point, ensure the persistence of this "easy axis" for any finite positive $[T_1(g) - T]$. We thus assume that in phase I,

$$\langle S^\alpha \rangle = \begin{cases} 0 & , \quad \alpha = 1, \ldots, m \\ (n-m)^{-1/2} M & , \quad \alpha = m+1, \ldots, n \end{cases} \quad . \tag{6.2}$$

As the temperature $T_2(g, v)$ is approached from above, we expect the correlations between the first m "spin" components to grow larger, until they become of an infinite range at $T_2(g, v)$, accompanied by the divergence of the appropriate susceptibility. It is through this divergence that we shall identify the temperature $T_2(g, v)$. In order to study the nature of this second transition, one must construct the recursion relations appropriate to the Hamiltonian (6.1) in the ordered phase I. Since only the susceptibility related to the first m"spin" components diverges at this transtion, while the other susceptibilities remain finite (and, of course, positive), and since the symmetry in the m-component subspace is unchanged by the ordering of the other components, one expects the transition at $T_2(g, v)$ to be described by an m-component Heisenberg-like fixed point.[34] Indeed, this is borne out by a study of the recursion relations.[35]

The calculation of $T_2(g, v)$ is simplified by a rotation of the coordinate system.[36] Thus we write

$$\vec{S} = \sum_{\alpha=1}^{n-1} Q_\alpha \vec{e}_\alpha + (Q_n + M) \vec{e}_n \quad , \tag{6.3}$$

where the vectors $\vec{e}_\alpha$ $(\alpha = 1, \ldots, n)$ form a complete orthonormal set, the first m of which coincide with the first m of the original coordinate axes, while the nth is given by

$$\vec{e}_n = (n-m)^{-1/2} [0, 0, \ldots, 0, 1, 1, \ldots, 1] \quad , \tag{6.4}$$

and thus is parallel to the "magnetization" vector (the first m components are zero). Under the transformation (6.3), Eq. (6.2) is replaced by

$$\langle Q_\alpha \rangle = 0 \quad , \tag{6.5}$$

while the only term in the Hamiltonian (6.1) that transforms nontrivially is the last one, which becomes

$$\sum_{\alpha=1}^{n} S_\alpha^4 = \sum_{\alpha=1}^{m} Q_\alpha^4 + \frac{1}{n-m} \Bigg((Q_n + M)^4 + 6(Q_n + M)^2 \sum_{\alpha=m+1}^{n-1} Q_\alpha^2 + 4(n-m)^{1/2} (Q_n + M) \sum_{\alpha,\beta,\gamma=m+1}^{n-1} a_{\alpha\beta\gamma} Q_\alpha Q_\beta Q_\gamma + (n-m) \times \sum_{\alpha,\beta,\gamma,\delta=m+1}^{n-1} b_{\alpha\beta\gamma\delta} Q_\alpha Q_\beta Q_\gamma Q_\delta \Bigg) \quad , \tag{6.6}$$

where

$$a_{\alpha\beta\gamma} = \sum_{i=m+1}^{n} e_\alpha^i e_\beta^i e_\gamma^i \tag{6.7}$$

(e_α^i is the ith component of $\vec{e}_\alpha$), and

$$b_{\alpha\beta\gamma\delta} = \sum_{i=m+1}^{n} e_\alpha^i e_\beta^i e_\gamma^i e_\delta^i \quad . \tag{6.8}$$

As we shall see, the particular values of $a_{\alpha\beta\gamma}$ and $b_{\alpha\beta\gamma\delta}$ (which reflect the details of the coordinate transformation) will not enter into the final result.

As usual in diagrammatic expansions,[29] it is convenient to use propagators which involve the true susceptibilities. Symmetry considerations show that in this case we may have three distinct susceptibilities. We thus define

$$\int_{\vec{x}} \langle Q_\alpha(0)\, Q_\beta(\vec{x}) \rangle = \begin{cases} r_1^{-1} \delta_{\alpha\beta} & , \quad \alpha, \beta = 1, \ldots, m \\ r_2^{-1} \delta_{\alpha\beta} & , \quad \alpha, \beta = m+1, \ldots, n-1 \\ r_3^{-1} & , \quad \alpha = \beta = n \quad . \end{cases} \tag{6.9}$$

We can now write the Hamiltonian as

$$\bar{\mathcal{H}} = \bar{\mathcal{H}}_0 + \bar{\mathcal{H}}_1 \quad , \tag{6.10}$$

with

$$\bar{\mathcal{H}}_0 = \frac{1}{2} \int_{\vec{x}} \left(r_1 \sum_{\alpha=1}^{m} Q_\alpha^2 + r_2 \sum_{\alpha=m+1}^{n-1} Q_\alpha^2 + r_3 Q_n^2 + (\vec{\nabla}\vec{Q})^2 \right) \tag{6.11}$$

and

$$\bar{\mathcal{H}}_1 = \int_{\vec{x}} \Bigg\{ \left[r_0 - \frac{m}{n-m} g + 4\left(u + \frac{v}{n-m}\right) M^2 \right] M Q_n + \tfrac{1}{2} (r_0 + g - r_1 + 4uM^2) \sum_{\alpha=1}^{m} Q_\alpha^2 + \tfrac{1}{2} \left[r_0 - \frac{m}{n-m} g - r_2 + 4\left(u + \frac{3v}{n-m}\right) M^2 \right] \times \sum_{\alpha=m+1}^{n-1} Q_\alpha^2 + \tfrac{1}{2} \left[r_0 - \frac{m}{n-m} g - r_3 + 12\left(u + \frac{v}{n-m}\right) M^2 \right] Q_n^2 + 4M \left[\left(u + \frac{v}{n-m}\right) Q_n^3 + u Q_n \sum_{\alpha=1}^{m} Q_\alpha^2 + \left(u + \frac{3v}{n-m}\right) Q_1 \times \sum_{\alpha=m+1}^{n-1} Q_\alpha^2 + v(n-m)^{-1/2} \sum_{\alpha,\beta,\gamma=m+1}^{n-1} a_{\alpha\beta\gamma} Q_\alpha Q_\beta Q_\gamma \right] + u |\vec{Q}|^4 + v \left(\sum_{\alpha=1}^{m} Q_\alpha^4 + \frac{1}{n-m} Q_n^4 + \frac{6}{n-m} Q_n^2 \sum_{\alpha=m+1}^{n-1} Q_\alpha^2 \right.$$

$$+4(n-m)^{-1/2} Q_n \sum_{\alpha,\beta,\gamma=m+1}^{n-1} a_{\alpha\beta\gamma} Q_\alpha Q_\beta Q_\gamma + \sum_{\alpha,\beta,\gamma,\delta=m+1}^{n-1} b_{\alpha\beta\gamma\delta} Q_\alpha Q_\beta Q_\gamma Q_\delta \Big)\Big\} \ . \tag{6.12}$$

We must now consider the four self-consistency conditions associated with Eqs. (6.5) and (6.9). Introducing the propagator

$$G(r,\vec{q}) = (r+q^2)^{-1} \ , \tag{6.13}$$

and expanding in the perturbation $\bar{\mathcal{H}}_1$, Eq. (6.5) (equivalent to the equation of state in zero field) reduces to

$$r_0 - \frac{m}{n-m} g + 4\left(u+\frac{v}{n-m}\right)M^2 + 4mu\int_{\vec{q}} G(r_1,\vec{q}) + 4(n-m-1)\left(u+\frac{3v}{n-m}\right)\int_{\vec{q}} G(r_2,\vec{q})$$

$$+12\left(u+\frac{v}{n-m}\right)\int_{\vec{q}} G(r_3,\vec{q}) + O(u^2,uv,v^2) = 0 \ , \tag{6.14}$$

where we have used the standard notation[15]

$$\int_{\vec{q}} \equiv (2\pi)^{-d}\int d^d q \ \text{ over } |\vec{q}| < 1 \ .$$

Similarly, Eq. (6.9) leads to the equations

$$r_1 = r_0 + g + 4uM^2 + 4[(m+2)u+3v]\int_{\vec{q}} G(r_1,\vec{q}) + 4(n-m-1)u\int_{\vec{q}} G(r_2,\vec{q}) + 4u\int_{\vec{q}} G(r_3,\vec{q}) - 64u^2M^2$$

$$\times \int_{\vec{q}} G(r_1,\vec{q})\,G(r_3,\vec{q}) + O(u^2,uv,v^2) \ , \tag{6.15}$$

$$r_2 = r_0 - [m/(n-m)]g + 4[u+3v/(n-m)]M^2 + O(u,v) \tag{6.16}$$

and

$$r_3 = r_0 - [m/(n-m)]g + 12[u+v/(n-m)]M^2 + O(u,v) \ . \tag{6.17}$$

Subtracting (6.14) from (6.16) and (6.17) we find

$$r_2 = [8v/(n-m)]M^2 + O(v) \tag{6.18}$$

and

$$r_3 = 8[u+v/(n-m)]M^2 + O(u,v) \ . \tag{6.19}$$

Thus, if $v=0$ the "transverse" susceptibility r_2 is identically zero, as expected for an isotropic Heisenberg-like system in zero field.[29] Similarly, $r_1 = 0$ if both $v=0$ and $g=0$. However, in general $r_1 \neq 0$ for nonzero g and v. It is our aim now to find conditions under which $r_1 = 0$, corresponding to the onset of long-range order in the first m spin components. Thus, the solution to the equation

$$r_1 = 0 \tag{6.20}$$

defines the equation of the transition line $T = T_2(g,v)$. Substituting (6.20) in (6.15), and subtracting it from (6.14), we now find

$$\frac{n}{n-m} g = \frac{4v}{n-m}\bar{M}^2 + \frac{12v}{n-m}(n-m-1)I_1(\bar{r}_2) + 4\left(2u+\frac{3v}{n-m}\right)I_1(\bar{r}_3) - \frac{64u^2\bar{M}^2 I_1(\bar{r}_3)}{\bar{r}_3} + O(uv,v^2) \ , \tag{6.21}$$

where $\bar{r}_2$, $\bar{r}_3$, and $\bar{M}$ are the values of r_2, r_3, and M on the transition line, when $r_1 = 0$, and

$$I_1(r) \equiv \int_{\vec{q}} [G(r,\vec{q}) - G(0,\vec{q})]$$

$$\simeq \tfrac{1}{2} K_d\, r \ln r + O(\epsilon) \ , \tag{6.22}$$

where

$$K_d^{-1} = 2^{d-1}\pi^{d/2}\,\Gamma(\tfrac{1}{2}d) \ . \tag{6.23}$$

Substituting the results (6.18), (6.19), and (6.22), Eq. (6.21) may be written

$$ng = 4v\bar{M}^2\left\{1 + 4(5u+3v)K_d \ln\left[8\left(u+\frac{v}{n-m}\right)\bar{M}^2\right] + 12vK_d\left(1-\frac{1}{n-m}\right)\ln\left(\frac{v}{(n-m)u+v}\right) + O(u^2, uv, v^2, u\epsilon, v\epsilon)\right\} . \tag{6.24}$$

The scaling behavior of expressions such as (6.24) becomes apparent only when leading corrections to scaling are eliminated by appropriate choices of the coupling constants u and v.[37] When the "flop" point is associated with the isotropic Heisenberg fixed point [$n < n_c(d)$], we require[29,38]

$$u = u_{0c}^H = \frac{\epsilon}{4K_d(n+8)}\left(1 + \frac{9n+42}{(n+8)^2}\epsilon + O(\epsilon^2)\right) \tag{6.25}$$

and

$$v \ll u_{0c}^H . \tag{6.26}$$

Substituting (6.25) into (6.24), we then find

$$ng \sim \bar{M}^{\rho} + O(v^2) , \tag{6.27}$$

with

$$\rho = 2(1 + [5/(n+8)]\epsilon + O(\epsilon^2)) . \tag{6.28}$$

For $n > n_c(d)$, when the flop point is to be associated with the cubic fixed point, the appropriate choices of the coupling constants are[17]

$$u = u_{0c}^C = \frac{\epsilon}{12K_d n} + O(\epsilon^2) ,$$
$$v = v_{0c}^C = \frac{\epsilon(n-4)}{36K_d n} + O(\epsilon^2) . \tag{6.29}$$

Substituting these in (6.24) we again find $ng \sim \bar{M}^{\rho^C}$, with

$$\rho^C = 2(1 + [(n+1)/3n]\epsilon + O(\epsilon^2)) . \tag{6.30}$$

To find the difference $[T_c - T_2(g,v)]$, we need the values of $r_0 (= T - T_0 + Bg)$ at these two temperatures. At $g = 0$, Eq. (6.27) shows that $\bar{M} = 0$, and Eqs. (6.18) and (6.19) then give $\bar{r}_2 = \bar{r}_3 = 0$. Substituting in Eq. (6.15) (with $r_1 = 0$), and subtracting the result from (6.15), gives (henceforth we set[31] $B = 0$)[39]

$$T_c - T_2(g,v) = \bar{r}_0(g=0) - \bar{r}_0(g) = (nu+v)\frac{g}{v}\left\{1 - 8nuK_d\frac{u+v}{nu+v}\ln\left[\frac{2ng}{v}\left(u+\frac{v}{n-m}\right)\right]\right.$$
$$\left. - 8nuK_d\frac{(n-m-1)v}{(n-m)(nu+v)}\ln\left(\frac{v}{(n-m)u+v}\right) + O(u^2, uv, v^2)\right\} . \tag{6.31}$$

This can be readily exponentiated at each of the fixed points considered, leading to a result of the form

$$T_c - T_2(g,v) \sim (g/v)^{1/\psi_2} . \tag{6.32}$$

At the Heisenberg-like fixed point, we find

$$\psi_2 = 1 + [2/(n+8)]\epsilon + O(\epsilon^2) . \tag{6.33}$$

Noting the known expansion of[20b,20c,37] ϕ_g and of ϕ_v,[17,20c,38]

$$\phi_g = 1 + \frac{n}{2(n+8)}\epsilon\left(1 + \frac{n^2+24n+68}{2(n+8)^2}\epsilon + O(\epsilon^2)\right) \tag{6.34}$$

and

$$\phi_v = \frac{n-4}{2(n+8)}\epsilon + \frac{n^3+16n^2+4n+240}{2(n+8)^3}\epsilon^2 + O(\epsilon^3) , \tag{6.35}$$

we readily identify

$$\psi_2 = \phi_g - \phi_v + O(\epsilon^2) , \tag{6.36}$$

and we confirm (to order ϵ) the scaling prediction (5.11) with

$$\theta_2 = 1 + O(\epsilon^2) . \tag{6.37}$$

In Secs. VIII–IX we pursue the calculation of ψ_2 to order ϵ^2 and to order $1/n$, and find that in both cases θ_2 remains exactly equal to unity.

At the cubic fixed point, (6.31) leads to (6.32) with

$$\psi_2^C = 1 + \tfrac{1}{6}\epsilon + O(\epsilon^2) = \phi_g^C , \tag{6.38}$$

with ϕ_g^C as given elsewhere.[33] Again, this is in agreement with the scaling prediction of Sec. V.

It is interesting to note that at both fixed points the relation

$$\psi_2 = \rho\beta \tag{6.39}$$

holds, where β is the exponent describing in temperature dependence of the "magnetization" below the tetracritical point at zero ordering field.[29,36] In Sec. VIII we shall return to discuss and exploit this result.

VII. WARD IDENTITY

In this section we shall develop some formalism that will facilitate the extension of the calculations described in Sec. VI, to order ϵ^2 and $1/n$. Since the higher-order analysis is, naturally, more com-

plex than the order-ϵ calculations, we shall henceforth consider only the case of a uniaxial anisotropy [$g<0$, $m=1$ in Eq. (2.7)] that favors the ordering of a single component, followed, at the lower temperature $T_2'(g,v)$, by the ordering of the remaining $n-1$ components. The discussion up to Eq. (7.8) will, however, be quite general.

We shall label the component which orders first "longitudinal," and the other components "transverse." As in Sec. VI, we are interested in a diagrammatic evaluation of the *transverse susceptibility* r_T. As noted by Wallace,[28] this evaluation, for a system described by a Hamiltonian such as (2.2), is considerably simplified by the use of a *Ward identity*. We shall now outline the appropriate extensions of Wallace's result, which was derived for a system with only cubic anisotropy [described by (2.2) with $g=0$].

We consider the effect of applying, to the Hamiltonian (2.2), an infinitesimal rotation of the "spin" variables,

$$S_\alpha(\vec{x}) \to S'_\alpha(\vec{x}) + \sum_\beta D_{\alpha\beta} S'_\beta(\vec{x}) \quad , \tag{7.1}$$

where $D_{\alpha\beta}$ is an antisymmetric matrix, with infinitesimal elements. Under this transformation the form of the $O(n)$ invariant piece, $\bar{\mathcal{H}}_s$, of (2.2) is unchanged,

$$\bar{\mathcal{H}}_s[\{\vec{S}\}] \to \bar{\mathcal{H}}_s[\{\vec{S}'\}] \quad , \tag{7.2}$$

while the field term (2.6) transforms as

$$\bar{\mathcal{H}}_h[\{\vec{S}\}] \to \bar{\mathcal{H}}_h[\{\vec{S}'\}] + \int_{\vec{x}} \sum_{\alpha,\beta} D_{\alpha\beta} h_\beta S'_\alpha \tag{7.3}$$

The two additional symmetry-breaking perturbations (2.4) and (2.5) are transformed thus,

$$\bar{\mathcal{H}}_g[\{\vec{S}\}] \to \bar{\mathcal{H}}_g[\{\vec{S}'\}] + g \int_{\vec{x}} \sum_{\alpha,\beta} D_{\alpha\beta} c_\alpha S'_\alpha S'_\beta \tag{7.4}$$

and

$$\bar{\mathcal{H}}_v[\{\vec{S}\}] \to \bar{\mathcal{H}}_v[\{\vec{S}'\}] + 4v \int_{\vec{x}} \sum_{\alpha,\beta} D_{\alpha\beta} (S'_\alpha)^3 S'_\beta \quad . \tag{7.5}$$

Gathering together the results (7.2)–(7.5) we find that the transformation of (2.2) under (7.1) is given by

$$\bar{\mathcal{H}}[\{\vec{S}\}] \to \bar{\mathcal{H}}[\{\vec{S}'\}] + \bar{\mathcal{H}}_A[\{\vec{S}'\}] \quad , \tag{7.6}$$

where

$$\bar{\mathcal{H}}_A[\{\vec{S}'\}] \equiv \int_{\vec{x}} \sum_{\alpha,\beta} D_{\alpha\beta} [h_\beta S'_\alpha + 4v(S'_\alpha)^3 S'_\beta + g c_\alpha S'_\alpha S'_\beta] \, . \tag{7.7}$$

Since the partition function (2.1) is invariant under the transformation (7.1), the expectation value of $\bar{\mathcal{H}}_A$ in the ensemble defined by $\bar{\mathcal{H}}$, must vanish:

$$\int_{\vec{x}} \sum_{\alpha,\beta} D_{\alpha\beta} \{h_\beta \langle S'_\alpha(\vec{x})\rangle + 4v \langle [S'_\alpha(\vec{x})]^3 S'_\beta(\vec{x})\rangle + g c_\alpha \langle S'_\alpha(\vec{x}) S'_\beta(\vec{x})\rangle\} = 0 \quad . \tag{7.8}$$

Dropping the (now redundant) primes on the spin indices we now differentiate with respect to a specific transverse component (denoted by the subscript T) of the magnetic field, and set all such components equal to zero, obtaining the result

$$r_T = \frac{h}{M} + \frac{r_T}{M D_{TL}} \int_{\vec{x}} \sum_{\alpha,\beta} D_{\alpha\beta} \times [4v \langle S^3_\alpha(\vec{x}) S_\beta(\vec{x}) S_T(0)\rangle + g c_\alpha \langle S_\alpha(\vec{x}) S_\beta(\vec{x}) S_T(0)\rangle] \quad , \tag{7.9}$$

where the magnetization $\vec{M}$ and the field $\vec{h}$ coincide with the "longitudinal" axis of anisotropy, which we denote by the subscript L.

Introducing the usual shift of the longitudinal component of the spin,

$$S_L(\vec{x}) \to S_L(\vec{x}) + M \quad , \tag{7.10}$$

symmetry arguments lead easily to the final result

$$r_T = \frac{h}{M} - \frac{ng}{n-1} - 4vM^2 + 4r_T \int_{\vec{x}} \Big(v \Big[\langle S^3_T(\vec{x}) S_T(0)\rangle + \frac{1}{M} \langle S_L(\vec{x}) S^3_T(\vec{x}) S_T(0)\rangle - 3M \langle S_L(\vec{x}) S_T(\vec{x}) S_T(0)\rangle - 3 \langle S^2_L(\vec{x}) S_T(\vec{x}) S_T(0)\rangle - \frac{1}{M} \langle S^3_L(\vec{x}) S_T(\vec{x}) S_T(0)\rangle \Big] - \frac{ng}{(n-1)M} \langle S_L(\vec{x}) S_T(\vec{x}) S_T(0)\rangle \Big) \quad . \tag{7.11}$$

VIII. CALCULATION OF ψ_2 TO ORDER ϵ^2

With the aid of the Ward identity developed in Sec. VII we can now extend the analysis of Sec. VI to the calculation of the exponent ψ_2 to second order in ϵ. In the interest of simplicity we limit ourselves to the uniaxially anisotropic system considered in Sec. VII. In addition, we shall consider only the case $n < n_d(d)$, when the tetracritical point may be identified with the n-component Heisenberg fixed point: There is no reason to doubt the simple scaling prediction that, around the cubic fixed point, $\psi_2 = \phi^C_g$. (We did, in fact, check this explicitly in Sec. VI.) Thus, in the following, the coupling constant u is chosen to have its characteristic Heisenberg value [Eq. (6.25)], of order ϵ, while $v \ll u$, so that only terms linear in v need be retained in this analysis.

The basic problem to which we address ourselves is the evaluation of the right-hand side of Eq. (7.11), at $h=0$. In particular, we look for a solution of the equation $r_T=0$, which will identify the transition temperature $T_2'(g,v)$. The calculation is somewhat lengthy, and we shall emphasize only the essential features before quoting the results.

The calculation proceeds in the usual manner: The spin variables in the Hamiltonian (2.2) (with the appropriate c_α's) are shifted, according to Eq. (7.10), and the Hamiltonian split up into a "free" part and a perturbation, as described in Sec. VI. Expressions for each of the expectation values in (7.11), in turn, are then derived to the appropriate order in ϵ (remembering that M is to be regarded as being of order[29] $\epsilon^{-1/2}$) by usual diagrammatic perturbation theory. The resulting expressions involve integrals, over reciprocal space, of products of longitudinal and transverse propagators [Eq. (6.13) with $r=r_L$, r_T]. We remark that the topology of each contributing diagram is such as to produce a factor r_T^{-1} (coming from a transverse propagator at zero momentum) that cancels the r_T prefactor multiplying the integral in (7.11). We note that, since we are concerned with the solution of Eq. (7.11) when $r_T=0$, we need not consider those terms in the diagrammatic expansion whose temperature dependence enters only through r_T: Such terms could not contribute to our final result, since we shall ultimately subtract from (7.11) the corresponding equation at the tetracritical point ($g=M=0$, $T=T_c$).

We must emphasize also that we are not concerned with the *manner* in which r_T arrproaches zero at the second phase transition. Were this our aim, we should have to extract those terms on the right-hand side of Eq. (7.11) that go to zero with r_T nonanalytically. Such terms would, presumably, combine with the r_T term on the left-hand side of this equation, to lead ultimately to the result

$$r_T \sim [T - T_2'(g,v)]^{\mathring{\gamma}} \quad , \tag{8.1}$$

where $\mathring{\gamma}$ is the $(n-1)$ Heisenberg susceptibility exponent. However, since we choose the coupling constant u so as to elimate the leading corrections to scaling about the *Heisenberg* fixed point, it would be difficult (within an ϵ expansion) to reconstruct the power law (8.1) unambiguously, because of corrections to scaling around the $(n-1)$-Heisenberg fixed point. To do so would amount to a determination of the n Heisenberg $\to (n-1)$ Heisenberg susceptibility crossover function.[20d]

Finally, we remark that certain terms in the diagrammatic expansion of (7.11) are formally divergent in the limit as $r_T \to 0$. However, as noted by Brézin *et al.*,[29] such a "divergence" also exists among the order-ϵ terms in the diagrammatic expansion of the longitudinal susceptibility. We may write the latter expansion, formally, as

$$r_L = r_L^{(0)} + \epsilon r_L^{(1)} \quad , \tag{8.2}$$

where we have already shown that the leading (zeroth-order) term is [cf. Eq. (6.19) with $n-m=1$]

$$r_L^{(0)} = 8(u+v)M^2 \quad . \tag{8.3}$$

We then find that, within a consistent ϵ expansion, the diverging $\epsilon r_L^{(1)}$ terms in (8.2) [whose contributions to r_L must be included in longitudinal propagators entering diagrams that contribute to the integral in (7.11) at order *zero* in ϵ] precisely cancel the diverging *first*-order terms in the integral in (7.11). The remaining (fully convergent for $r_T \to 0$) contributions to (7.11) may be evaluated with the aid of the integrals tabulated in Ref. 29. Gathering together terms we then find that Eq. (7.11) gives (at $h=0$)

$$r_T = 0 = -\frac{ng}{n-1}\left(1 + (4uK_d + 16u^2K_d^2)\ln\bar{r}_L^{(0)} + [\tfrac{1}{4}(n+10)u^2K_d^2 - \epsilon uK_d]\ln^2\bar{r}_L^{(0)}\right)$$
$$- 4v\bar{M}^2\{1 + (24uK_d - 288u^2K_d^2)\ln\bar{r}_L^{(0)} + [(480+24n)u^2K_d^2 - 6\epsilon uK_d]\ln^2\bar{r}_L^{(0)}\} \quad , \tag{8.4}$$

where, following the notation of Sec. VI, we denote by $\bar{r}_L^{(0)}$ the value of $r_L^{(0)}$ at $T=T_2'(g,v)$, where $M\equiv\bar{M}$. Thus, in the vicinity of the Heisenberg fixed point, where $v \ll u$, $u=u_{0c}^H$,

$$\bar{r}_L^{(0)} = 8u_{0c}^H\bar{M}^2 \quad . \tag{8.5}$$

Using the value of u_{0c}^H given in Eq. (6.25) we find that (8.4) may be written as[40]

$$r_T = 0 = -A_v v\bar{M}^{\sigma_v} - A_g[ng/(n-1)]\bar{M}^{\sigma_g} \quad , \tag{8.6}$$

where

$$A_v \simeq 4\,[1 + [6\epsilon/(n+8)]\ln[2\epsilon/(n+8)K_d]] \quad ,$$
$$A_g \simeq 1 + [\epsilon/(n+8)]\ln[2\epsilon/(n+8)K_d] \quad , \tag{8.7}$$

and

$$\sigma_g = \frac{2\epsilon}{n+8}\left(1 + \frac{10(n+5)}{(n+8)^2}\epsilon\right) + O(\epsilon^3) \quad .$$
$$\sigma_v = 2 + \frac{12\epsilon}{n+8}\left(1 + \frac{6(n+3)}{(n+8)^2}\epsilon\right) + O(\epsilon^3) \quad . \tag{8.8}$$

It is interesting to note that an *extended scaling Ansatz* for the Helmholtz free energy shows that *if* one can express the condition $r_T = 0$ in the form (8.6), i.e.,

$$A_g[ng/(n-1)]\bar{M}^{\sigma_g} + A_v v\bar{M}^{\sigma_v} = 0, \tag{8.9}$$

then σ_g and σ_v are given by[41]

$$\sigma_g = (\gamma - \phi_g)/\beta \ , \qquad \sigma_v = (\gamma - \phi_v)/\beta \ , \tag{8.10}$$

and ρ [cf. (6.27) and (6.39)] is given by

$$\rho = \sigma_v - \sigma_g = (\phi_g - \phi_v)/\beta \ . \tag{8.11}$$

Here, γ and β are the usual Heisenberg exponents. Indeed, one finds that the known ϵ expansion of[37] γ and β,[29] and ϕ_g, ϕ_v [Eqs. (6.34) and (6.35)] are in accord with (8.8) and (8.10). The calculation of ψ_2 to order $1/n$, to which we turn in Sec. IX, will clarify the origins of the general form (8.9), and we defer further discussion of this point until then.

To complete our calculation of the transition temperature shift we need only note that an explicit examination of the equation of state (or a simple scaling argument) shows that the leading behavior of the magnetization is simply given by

$$M \sim (T_c - T)^{\beta} \ . \tag{8.12}$$

It is this result, together with (8.9), that leads to the identification (6.39). Substituting (8.12) into (8.6) we find the result

$$T_c - T_2'(g, v) \sim (g/v)^{1/\psi_2} \ , \tag{8.13}$$

where we may now make the identification

$$\psi_2 = \phi_g - \phi_v + O(\epsilon^3) \ . \tag{8.14}$$

Thus we have confirmed, in order ϵ^2, the scaling prediction (5.11), with $\theta_2 = 1$.

IX. CALCULATION OF ψ_2 TO ORDER $1/n$

Our investigation of the shape of the phase boundary near the tetracritical point is completed with a calculation of ψ_2 to order $1/n$,[19] for the same system as we have treated in Secs. VII and VIII. Some features of this calculation are instructive, so we shall give rather more detail than in the corresponding ϵ^2 calculation.

As we remarked in Sec. V, the cubic perturbation is certainly relevant with respect to the Heisenberg fixed point in the large n limit [$n \gg n_c(d)$!]. Nevertheless, we may still expect to recover the scaling result (5.11) if we treat v as a perturbation, small in comparison with u, and do not work so close to T_c that the crossover to cubic behavior, due to the relevance of v, is felt. Thus we regard u and v as being of order $1/n$,[28] with $v \ll u$, and note that consistency of the expansion demands that we treat M as being of order $n^{1/2}$.[42]

Formally, the procedure is identical to that discussed in Sec. VIII: Only the graph counting is different. We note, following Ref. 42, that there is a zeroth-order self-energy renormalization of the longitudinal propagator, through the stream of bubbles formed by transverse propagators. The renormalized longitudinal propagator is then given by[42]

$$[G_L(q)]^{-1} = r_L + q^2 + 8uM^2 \times\left(\frac{1}{1+4unI_2(q, r_T)} - \frac{1}{1+4unI_2(0, r_T)}\right) \ , \tag{9.1}$$

where

$$I_2(q, r_T) = \int_{\vec{p}} G_T(p)\, G_T(|\vec{q}+\vec{p}|) \ , \tag{9.2}$$

with the transverse propagator still having the form (6.13),

$$[G_T(q)]^{-1} = r_T + q^2 \ . \tag{9.3}$$

Turning to the term-by-term evaluation of the right-hand side of Eq. (7.11) (at $h = 0$) we note first that those expectation values associated with a factor $1/M$ vanish in the ensemble of the "free" Hamiltonian. Their leading contribution comes through diagrams involving a three-point vertex, proportional to uM. These contributions are thus of order vu (the v coming from the prefactor), with no compensating factor of n from sums over transverse propagators. They are thus of order $1/n^2$, and may be neglected.

The first term in the integral (7.11) does give a nonzero contribution of order unity, but it involves no temperature-dependent term other than r_T. As discussed in Sec. VIII, it may thus be ignored. The third term in the integral in (7.11) gives contributions of order unity from diagrams involving one three-point vertex and an arbitrary number of four-point vertices; the result is

$$-3M\int_{\vec{x}} \langle S_L(\vec{x})\, S_T(\vec{x}) S_T(0)\rangle = (24uM^2/r_T)\int_{\vec{q}} G_L(q)\, G_T(q) \times[1 + 4unI_2(q, r_T)]^{-1} + O(1/n) \ . \tag{9.4}$$

The remaining term with a v prefactor gives

$$-3\int_{\vec{x}} \langle S_L^3(\vec{x})\, S_T(\vec{x})\, S_T(0)\rangle = -(3/r_T)\int_{\vec{q}} G_L(q) + O(1/n) \ . \tag{9.5}$$

Finally, the last term in (7.11) is given directly by the result (9.4).

On subtracting from (7.11) the corresponding equation at the tetracritical point ($T = T_c$, $g = M = r_T = r_L = 0$) we have [setting $r_T = 0$, $M = \bar{M}$, $r_L = \bar{r}_L$ at $T = T_2'(g, v)$]

$$r_T = 0 = -4v\left[\bar{M}^2\left(1 - 24u\int_{\vec{q}} \frac{\bar{G}_L(q)\bar{G}_T(q)}{1+4umI_2(q,0)}\right) + 3\int_{\vec{q}} [G_L(q) - q^{-2}]\right] - \frac{ng}{n-1}\left(1 - 8u\int_{\vec{q}} \frac{\bar{G}_L(q)\bar{G}_T(q)}{1+4umI_2(q,0)}\right) , \tag{9.6}$$

where the bars on the propagators denote that they are to be evaluated at $T = T_2'(g, v)$. Note that the "cubic" contributions to this equation are in agreement with the result of Wallace.[28]

The integral (9.2) may be evaluated using Feynman parameters to give, for $r_T = 0$,

$$I_2(q, 0) = \tfrac{1}{2} K_d S_d^{-1} q^{-\epsilon} , \tag{9.7}$$

where[19]

$$S_d^{-1} = [\pi(1 - \tfrac{1}{2}\epsilon) B(1 - \tfrac{1}{2}\epsilon,\ 1 - \tfrac{1}{2}\epsilon)]/\sin(\tfrac{1}{2}\pi\epsilon) , \tag{9.8}$$

with B the Beta function. Using this result, Eq. (9.6) gives

$$0 \approx -4v\left[\bar{M}^2\left(1 - \frac{12S_d}{nK_d} J_1(\bar{r}_L, \bar{M})\right) + 3J_2(\bar{r}_L, \bar{M})\right] - \frac{ng}{n-1}\left(1 - \frac{4S_d}{nK_d} J_1(\bar{r}_L, \bar{M})\right) , \tag{9.9}$$

where

$$J_1(\bar{r}_L, \bar{M}) = \int_{\vec{q}} \frac{1}{q^{2-\epsilon}[\bar{r}_L + q^2 + (4\bar{M}^2 S_d/nK_d) q^\epsilon]} ,$$
$$J_2(\bar{r}_L, \bar{M}) = \int_{\vec{q}} \left(\frac{1}{\bar{r}_L + q^2 + (4\bar{M}^2 S_d/nK_d) q^\epsilon} - \frac{1}{q^2}\right) . \tag{9.10}$$

Our last requirement is an expression for $\bar{r}_L$ (to zeroth order in $1/n$). A direct perturbation expansion, supplemented by the zeroth-order result from (9.6), gives

$$\bar{r}_L = -ng/(n-1) + 12v\bar{M}^2 . \tag{9.11}$$

Since $\bar{r}_L = O(g, v)$, we find from (9.10) (ignoring terms that will ultimately make contributions of order v^2 or gv)

$$J_1(\bar{r}_L, \bar{M}) \simeq -\frac{K_d}{2(1 - \frac{1}{2}\epsilon)} \ln\left(\frac{4\bar{M}^2 S_d}{nK_d}\right) ,$$
$$J_2(\bar{r}_L, \bar{M}) \simeq \frac{2\bar{M}^2 S_d}{n(1 - \frac{1}{2}\epsilon)} \ln\left(\frac{4\bar{M}^2 S_d}{nK_d}\right) . \tag{9.12}$$

Finally, substituting (9.12) into (9.9), we have

$$r_T = 0 \simeq -A_v v\bar{M}^{\sigma_v} - A_g \frac{ng}{(n-1)} \bar{M}^{\sigma_g} ,$$

with

$$A_v \simeq 4\left[1 + \frac{12S_d}{n(1 - \frac{1}{2}\epsilon)} \ln\left(\frac{4S_d}{nK_d}\right)\right] ,$$
$$A_g \simeq 1 + \frac{2S_d}{n(1 - \frac{1}{2}\epsilon)} \ln\left(\frac{4S_d}{nK_d}\right) , \tag{9.13}$$

and

$$\sigma_g = 4S_d/n(1 - \tfrac{1}{2}\epsilon) .$$
$$\sigma_v = 2 + 24S_d/n(1 - \tfrac{1}{2}\epsilon) . \tag{9.14}$$

We note that the ϵ expansions of the amplitudes (9.13) are in accord with the $1/n$ expansions of the results (8.7). Noting the $1/n$ expansions of ϕ_g [43] and ϕ_v, [41]

$$\phi_g = 2/(2 - \epsilon) - 8S_d/n(1 - \tfrac{1}{2}\epsilon) ,$$
$$\phi_v = \epsilon/(2 - \epsilon) - 4(\epsilon + 3) S_d/n(1 - \tfrac{1}{2}\epsilon) \tag{9.15}$$

and the corresponding results for β[42] and γ, [19] we find that (9.14) is compatible with the nonrigorous scaling "prediction" (8.10) [cf. remarks preceding (8.9), and below]. Using again the result (8.11) (and thus assuming implicitly that we are not so close to the tetracritical point that the relevance of v will be felt) we obtain the result (8.13) with

$$\psi_2 = 1 + \frac{4(1+\epsilon)}{1 - \frac{1}{2}\epsilon} \frac{S_d}{n} = \phi_g - \phi_v + O\left(\frac{1}{n^2}\right) \tag{9.16}$$

confirming, to order $1/n$, the scaling prediction (5.11), with $\theta_2 = 1$.

We now return to discuss the general form, Eq. (8.9), of the equation $r_T = 0$. It is clear from the Ward identity (7.11) that such a form is plausible *provided* the terms in the integral in this equation give only logarithms of $\bar{M}$, and *not* logarithms of g or v. It is not obvious, *a priori*, that this condition is fulfilled, and the ϵ-expansion analysis leaves some cause for doubt, since it does not reveal the fact that, in the absence of the symmetry breaking g and v perturbations, r_L is actually zero on the coexistence curve ($h = 0$).[29] As noted by Brézin and Wallace,[42] the $1/n$ expansion is much more satisfactory in this respect, since the vanishing of r_L for $h = g = v = 0$ is given even in the spherical model limit [in contrast to the mean-field limit, (8.5)]. Indeed, Eq. (9.11) shows explicitly that $\bar{r}_L$ goes to zero with g and v. In the light of this observation one might expect to pick up [in the evaluation of the right-hand side of (7.11)] logarithms

of g and v that would invalidate the form (8.9). However, one sees from the $1/n$ analysis that the propagator renormalization (9.1) ensures that (provided one ignores terms of order v^2 or gv) one obtains only logarithms of $\overline{M}$ (and *not* of $\overline{r}_L$, and thence g and v) in this expansion, thus guaranteeing the form (8.9).

Finally we remark that a similar $1/n$ expansion about the *cubic* fixed point is more complicated, since v should then be regarded as being of order unity and cannot be treated as a perturbation.[44] A preliminary analysis shows that one would have to keep terms of order $1/n$ in order to find the leading term in (6.32), which behaves as $(g/v)^{1/\psi_2}$, with

$$\psi_2 = \phi_g^C = 1/(1-\alpha_I) + O(1/n) \tag{9.17}$$

(α_I is the Ising specific-heat exponent).[3]

X. PHASE DIAGRAMS OF PEROVSKITE CRYSTALS STRESSED ALONG [111]

As noted in Sec. II, the form of anisotropic "exchange" perturbation introduced into the Hamiltonian (2.2) in Eq. (2.4) does not allow study of the phase diagram of [111]-stressed perovskites. In view of the recent experimental investigations of $SrTiO_3$ subjected to such a stress,[8] we devote this section to a discussion of this problem.

As shown in Ref. 10, the generalization of the effective Hamiltonian (2.2) necessary to describe the effects of a general stress [with components T_i $(i=1,\ldots,6)$ in the Voigt notation] is, for $n=3$,

$$\overline{\mathcal{H}} = \int_{\vec{x}} \left(\tfrac{1}{2}[(T-T_0)\vec{S}^2 + (\vec{\nabla}\vec{S})^2] + u\vec{S}^4 + v\sum_{\alpha=1}^{3} S_\alpha^4 - \sum_{\alpha=1}^{3} T_\alpha [(L_1-L_2)S_\alpha^2 + L_2\vec{S}^2] - L_3(S_1S_2T_6 + S_2S_3T_4 + S_3S_1T_5) \right) . \tag{10.1}$$

For a stress along [100], only T_1 is nonzero, and (10.1) reduces to (2.2).

When the stress is along [111], we have $T_\alpha \equiv -\frac{1}{3}p$, and thus

$$\overline{\mathcal{H}} = \int_{\vec{x}} \left(\tfrac{1}{2}\{[r_0 + \tfrac{2}{3}(L_1+2L_2)p]\vec{S}^2 + (\vec{\nabla}\vec{S})^2\} + u\vec{S}^4 + v\sum_{\alpha} S_\alpha^4 + \tfrac{1}{3}L_3 p(S_1S_2 + S_2S_3 + S_3S_1) \right) . \tag{10.2}$$

As we shall see shortly, it is convenient to rotate the soft-mode coordinates, so that one component is along [111]. To be specific, we choose

$$\vec{S} = \sum_{i=1}^{3} Q_i \vec{e}_i , \quad \vec{e}_1 = (1/\sqrt{3})[1\,1\,1] , \quad \vec{e}_2 = (1/\sqrt{2})[1\,-1\,0] , \quad \vec{e}_3 = (1/\sqrt{6})[1\,1\,-2] . \tag{10.3}$$

Equation (10.2) can now be written

$$\overline{\mathcal{H}} = \int_{\vec{x}} \{ \tfrac{1}{2}[r_1 Q_1^2 + r_2(Q_2^2+Q_3^2) + (\vec{\nabla}\vec{Q})^2] + u\vec{Q}^4 + v[\tfrac{1}{3}Q_1^4 + 2Q_1^2(Q_2^2+Q_3^2) + 2\sqrt{2}\,Q_1Q_3(Q_2^2 - \tfrac{1}{3}Q_3^2) + \tfrac{1}{2}(Q_2^2+Q_3^2)^2]\} , \tag{10.4}$$

with

$$r_1 = r_0 + \tfrac{2}{3}L_3 p, \quad r_2 = r_0 - \tfrac{1}{3}L_3 p , \tag{10.5}$$

where $r_0 = T - T_0 + \frac{2}{3}(L_1+2L_2)p$.

Thus (in this representation) the quadratic terms in $\vec{Q}$ have a form similar to that discussed above, with a uniaxial "spin" anisotropy. However, the "cubic" term, proportional to v, now has a more complicated form, involving terms which are odd in Q_1 and in Q_3. These terms are crucial, as they lead to a variety of new effects.

As in the previous case, we begin with a mean-field analysis, studying the Helmholtz free energy density,

$$A(T,p,v,\vec{M}) = \tfrac{1}{2}r_1 M_1^2 + \tfrac{1}{2}r_2(M_2^2+M_3^2) + u\vec{M}^4 + v[\tfrac{1}{3}M_1^4 + 2M_1^2(M_2^2+M_3^2) + 2\sqrt{2}\,M_1M_3(M_2^2 - \tfrac{1}{3}M_3^2) + \tfrac{1}{2}(M_2^2+M_3^2)^2] . \tag{10.6}$$

Differentiating with respect to the components of $\vec{M}$, we find that necessary conditions for equilibrium are

$$\frac{\partial A}{\partial M_1} = r_1 M_1 + 4u\vec{M}^2 M_1 + v[\tfrac{4}{3}M_1^3 + 4M_1(M_2^2+M_3^2) + 2\sqrt{2}\,M_3(M_2^2 - \tfrac{1}{3}M_3^2)] = 0 , \tag{10.7}$$

$$\frac{\partial A}{\partial M_2} = \{r_2 + 4u\vec{M}^2 + v[4M_1^2 + 4\sqrt{2}\,M_1M_3 + 2(M_2^2+M_3^2)]\}M_2 = 0 , \tag{10.8}$$

and

$$\frac{\partial A}{\partial M_3} = r_3 M_3 + 4u\vec{M}^2 M_3 + v[4M_1^2M_3 + 2\sqrt{2}\,M_1(M_2^2 - M_3^2) + 2(M_2^2+M_3^2)M_3] = 0 . \tag{10.9}$$

One immediate solution (apart from the trivial disordered phase result $\vec{M}=0$) is

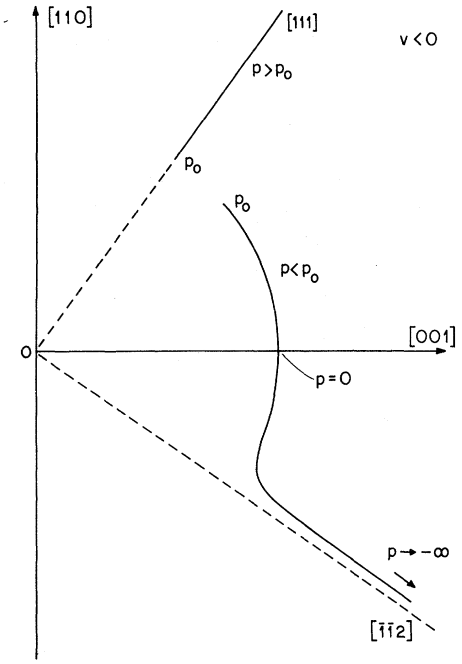

FIG. 4. Schematic $(1\bar{1}0)$ section of soft-mode-coordinate ("spin") space, for $v<0$ ($SrTiO_3$). The curve is the locus of calculated stable points as a function of a stress p applied parallel to the [111] axis at a temperature below the zero-stress critical temperature. The critical stress p_0 separates "trigonal" and "pseudotetragonal" phases. (The $p>0$ region follows Slonczewski, Ref. 9.)

$$M_1^2 = -\tfrac{1}{4}r_1/(u+\tfrac{1}{3}v)\ , \quad M_2 = M_3 = 0\ . \qquad (10.10)$$

In the original coordinates, this corresponds to an ordering along [111]. Analysis shows that a second-order transition, from the disordered phase into this phase, occurs at a temperature $T_1(p)$ (for $p>0$) at which r_1, Eq. (10.5), is zero.

As noted by Slonczewski,[9] this ordered phase becomes unstable for pressures

$$p < p_L = -6vr_0[(9u+7v)L_3 + 4v(L_1+2L_2)]^{-1}\ . \qquad (10.11)$$

[We assume that the term in the square brackets on the right-hand side is negative, as it is for $SrTiO_3$ (Ref. 26).]

One might also look for solutions of (10.7)–(10.9) with $M_1=0$ [ordering in the (111) plane]. A direct calculation shows that none of these solutions is stable against rotations out of the (111) plane. Thus, *all stable solutions with* $p<p_L$ *must have ordering both along* [111] *and perpendicular to* [111]. This type of ordering is similar to the one found in the "intermediate" phase in Sec. III. However, there are two crucial differences. One is that there exists no ordered phase with the ordering *purely* in the (111) plane, i.e., there is no phase equivalent to phase III of Sec. III. The second difference lies in the order of the transition between the [111] phase and "intermediate" phase.

In order to study in more detail the nature of the "intermediate" phase, we must resort to a numerical solution of Eqs. (10.7)–(10.9). We first note that when $M_1=0$ there is a symmetry between the coordinates M_2 and M_3: All states in the (111) plane have the same energy [only the last term in the square brackets in Eq. (10.6) remains]. Slonczewski[9] assumed this symmetry to hold generally, and used it to solve, numerically, equations which are equivalent to (10.7)–(10.9), putting $M_2=0$ and searching for solutions for M_1 and M_3. Since it is difficult to solve Eqs. (10.7)–(10.9) generally, we follow Slonczewski, and look only for equilibrium states with $M_2=0$ (checking explicitly that these states are, indeed, stable against the addition of an M_2 component).

Choosing parameters that are qualitatively appropriate to $SrTiO_3$ we find the equilibrium states, for various pressures, shown schematically in Fig. 4. The upper part of the figure (positive pressures) is in accord with Slonczewski's results: As the pressure is lowered, at a critical pressure $p_0>p_L$, a first-order phase transition occurs, from the [111]-ordered "*trigonal*" phase into a "*pseudotetragonal*" phase, in which the vector $\vec{M}$ is rotated by a finite angle. For lower pressures, the vector $\vec{M}$ rotates continuously towards the [001] axis, along which it aligns at zero pressure. Continuing the same analysis for negative stresses we find a continuing rotation, with the vector $\vec{M}$ tending *asymptotically* (as $p\to-\infty$) to align along the $[\bar{1}\bar{1}2]$ direction.

An analysis carried out for $p<0$ and $T>T_0$ ($r_0>0$) shows that the system undergoes a second-order transition at a temperature $T_1'(p)$, at which $r_2=0$ [Eq. (10.5)]. At this temperature, both M_1 and M_3 are zero. Both start growing immediately below $T_1'(p)$, but close to $T_1'(p)$ one has $|M_1/M_3| \ll 1$. This ratio tends to zero as criticality is approached from below. The phase diagram, schematically drawn in Fig. 5(a), thus contains a *bicritical point*, at which two second-order lines and one first-order line meet. However, the first-order line is no longer a "flop" line (in the sense in which we have used this term), since the order parameter of the lower phase cannot be associated with a distinct (temperature- and pressure-independent) direction.

For completeness of the mean-field analysis, we have performed a similar calculation for the case $v>0$, appropriate for $LaAlO_3$. The results for the order parameter, at $T<T_0$, for various stresses, are shown in Fig. 6: The phase [111] remains stable down to zero stress. At this point, the or-

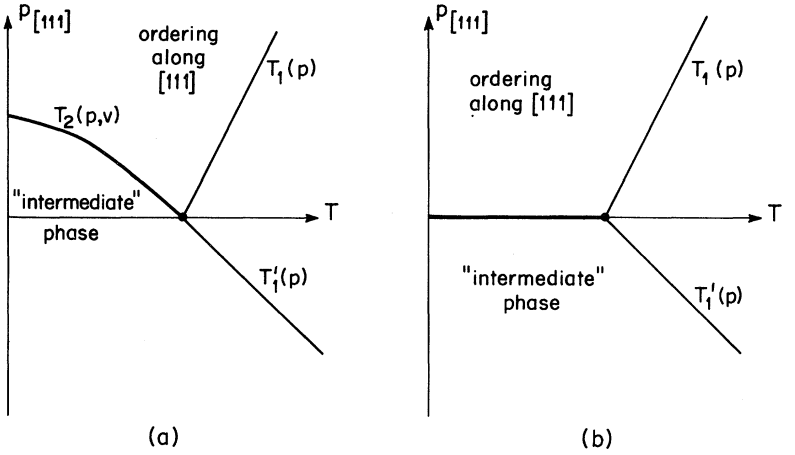

FIG. 5. Schematic mean-field phase diagrams for a perovskite crystal stressed along the [111] axis. (a) $v<0$ ($SrTiO_3$); (b) $v>0$ ($LaAlO_3$).

der parameter "flops" from the [111] direction to the $[\bar{1}\bar{1}1]$ direction (or to the equivalent symmetry-related phases); it then rotates continuously towards the $[\bar{1}\bar{1}2]$ direction. Thus, the picture is qualitatively similar to that described above, except for the fact that the first-order line now coincides with the T axis [Fig. 5(b)].

We can now proceed beyond mean-field theory, with the aid of renormalization-group arguments. Consider first the situation for temperatures above the critical lines $T_1(p)$ and $T_1'(p)$ (Fig. 5). For $SrTiO_3$, $L_3<0$.[26] Thus, for $p>0$ we have $r_2>r_1$ [Eq. (10.5)], and therefore Q_1 tends to order at a higher temperature than Q_2 or Q_3. We can choose the temperature so that r_1 varies very slowly under the renormalization-group transformation,[20b] whereas $r_2\to\infty$, eliminating from the recursion relations all the contributions from terms in $\overline{\mathcal{H}}$ involving Q_2 or Q_3. We are then left with effective recursion relations for r_1 and $(u+\frac{1}{3}v)$, which are the same as those of the Ising model. Thus, the line $T_1(p)$ represents an Ising-like second-order transition into a [111]-ordered phase.

The situation for $p<0$ is rather more complicated: now $r_2<r_1$, and the ordering (at least near $d=4$) tends to be XY-like, with Q_2 and Q_3 as order parameters. However, the operator $Q_1Q_3(Q_2^2-\frac{1}{3}Q_3^2)$ in (10.4) may not be irrelevant (at $d=3$) and might affect the critical behavior. This problem deserves a separate study.

To identify the nature of the bicritical point itself, it is sufficient to study the Hamiltonian at zero stress. Then, (10.2) coincides with the usual Hamiltonian of cubic symmetry, yielding a Heisenberg-like or a cubic fixed point. For $n<n_c(d)$ we thus conclude that the bicritical point here is Heisenberg-like. We can now proceed using scaling arguments (as in Sec. V), or direct diagrammatic expansion, to calculate the transition temperatures $T_1(p)$ and $T_1'(p)$. The result is still of the form (for sufficiently small p)

$$T_1(p)-T_1(0)\propto p^{1/\phi_g}\ , \tag{10.12}$$

with ϕ_g being the crossover exponent associated with spin anisotropy.

The situation in the ordered phases is now more complicated: $T_2(p,v)$ now represents a first-order

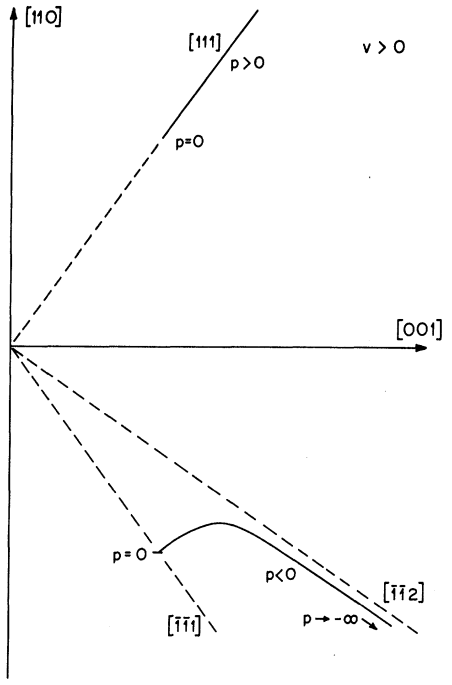

FIG. 6. Schematic $(1\bar{1}0)$ section of soft-mode-coordinate space, for $v>0$ ($LaAlO_3$). The curve is the locus of calculated stable points as a function of a stress p applied parallel to the [111] axis at a temperature below the zero-stress critical temperature.

transition, so that the techniques of Secs. VI–IX cannot be used (the "transverse" susceptibility does not diverge at the transition!). However, having identified the bicritical point as Heisenberg-like, we can still try to use scaling arguments about this point. From the mean-field analysis we know that the line $T_2(p, v)$ tends towards the T axis as $v \to 0$. Thus, we still expect this line to be given by a relation between the scaling fields p/t^{ϕ_g} and v/t^{ϕ_v}. Following the lines of the arguments of Sec. V we thus speculate

$$T_2(p, v) - T_2(0, v) \propto (p/v)^{1/\psi_2} , \qquad (10.13)$$

with ψ_2 equal to $\phi_g - \phi_v$. Thus, the phase diagram in this case may have the form shown in Fig. 7.

We remark, however, that the measured phase diagram[8] (for $p > 0$) definitely seems to be more consistent with the mean-field prediction [Fig. 5(a)] than with the form suggested in Fig. 7. This may be due to a shift in the effective $p = 0$ line due to systematic residual strains,[10] or to reasons similar to those discussed following Eq. (5.6).[45]

Finally it is interesting to note that none of the above complications occur for $n = 2$. In this case the transformed Hamiltonian, corresponding to Eq. (10.4), simply reduces to

$$\overline{\mathcal{H}} = \int_{\vec{x}} \{\tfrac{1}{2}[r_1 Q_1^2 + r_2 Q_2^2 + (\vec{\nabla}\vec{Q})^2] + u\vec{Q}^4 + v(\tfrac{1}{2}Q_1^4 + 3Q_1^2 Q_2^2 + \tfrac{1}{2}Q_2^2)\} = \int_{\vec{x}} \{\tfrac{1}{2}[r_1 Q_1^2 + r_2 Q_2^2 + (\vec{\nabla}\vec{Q})^2] + (u + \tfrac{3}{2}v)Q^4 - v(Q_1^4 + Q_2^4)\} . \qquad (10.14)$$

Thus, the quartic terms have the same forms as those in Eq. (2.2), except for the change in the sign of the "cubic" term. We thus expect a bicritical point for $v > 0$, and a tetracritical point for $v < 0$.

XI. SUMMARY AND DISCUSSION

The essential results of Secs. V–IX are succinctly summarized in Fig. 2(b) [$v < 0$ in (2.2)] and Fig. 3(b) [$v > 0$ in (2.2)]. We shall first review the latter situation. For $n > n_c(d)$, the cubic fixed point of the Hamiltonian (2.2) (with $g = 0$) is stable; the asymptotic critical behavior on the line $g = 0$ is then expected to be "cubic," and the phase diagram displays a tetracritical point with $\psi_2 = \psi_1 = \phi_g^c$.

For $n < n_c(d)$, the stable fixed point is the Heisenberg one, and the cubic perturbation (2.5) is irrelevant with respect to this fixed point, entering only as a correction to the leading scaling behavior. However, we have demonstrated the importance of such corrections both as regards the directions of (indeed the *existence* of) easy axes, and the geometry of the phase diagram. The rather tedious perturbation theory described in Secs. VIII and IX establishes that the exponent θ_2, introduced in the scaling analysis of Sec. V, is equal to unity to within correction terms that are at most of order ϵ^3 and $1/n^2$. Thus, while ψ_1 is given simply by ϕ_g, the exponent ψ_2 is equal to $\phi_g - \phi_v$, to within the same order of correction terms. It seems plausible (cf. the discussion at the end of Sec. IX) that these results are true to all orders. Even for $n < n_c(d)$, therefore, we expect a phase diagram of the general form of Fig. 3(b), in which, however, the second-order lines approaching the tetracritical point from temperatures less than T_c, merge together more rapidly than their counterparts, $T_1(g)$ and $T_1'(g)$. In the *asymptotic scaling region* (by the definition of this term) the lines $T_2(g, v)$ and $T_2'(g, v)$ actually merge into a single "flop" line, making the *tetracritical* point *apparently bicritical*. However, although it is not possible to estimate ϕ_v reliably within the ϵ-expansion,[41] since $n_c(3)$ is close to 3, it is clear that ϕ_v is quite small for $n = 2$ or 3. Thus, the difference between ψ_2 and ψ_1 is itself small and one may have to be very close to the tetracritical point before this asymptotic scaling region is reached.

Even within the asymptotic region, however, the true *tetracritical* nature of the spin-flop point should be revealed in a *divergence* [as the line $T = T_2(g, v)$ is approached from above, at constant temperature] of the susceptibility related to the degree of freedom in which ordering occurs at $T_2(g, v)$. *No* such divergence is expected when a

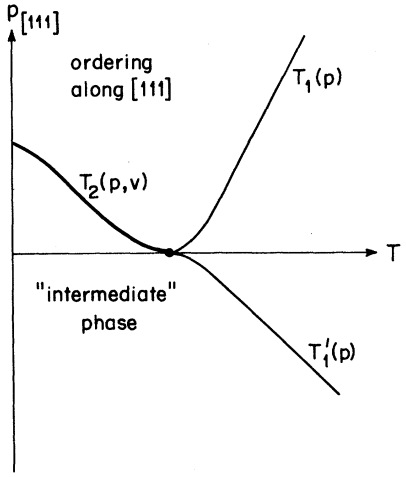

FIG. 7. Conjectured phase diagram for a perovskite lattice stressed along the [111] axis ($v < 0$, $SrTiO_3$).

truly first-order "flop" line (terminating in a bicritical point) is approached.[46]

While the relative stability of the Heisenberg and cubic fixed points is, of course, unaffected by the sign of the parameter v in the system Hamiltonian, the above results are nevertheless applicable only to the case $v > 0$. If v is negative and $n > n_c(d)$ the $(g = 0)$ system cannot evolve (under the renormalization-group transformation) to the only stable fixed point (the cubic fixed point), since this has $v^* > 0$, while the renormalization-group transformation preserves the sign of v (in the absence of any other symmetry-breaking irrelevant variables). Thus, the transition at the "flop" point $T = T_c$, $g = 0$ is likely to be first order.[10] It is not clear what effect this will have on the phase diagram in the vicinity of the flop point (although the qualitative three-phase structure of Fig. 2 should be preserved).

For $n < n_c(d)$ and v negative (but not too negative![47]) the system *can* flow to the stable (Heisenberg) fixed point. As in the case $v > 0$, there will be cubic "corrections to scaling" but, for $v < 0$, these will play no essential role (since they will not favor tetracritical behavior) and the appropriate phase diagram is that of Fig. 2(b).[10]

At present the experimental situation offers little with which to compare the results of our analysis. As mentioned in Sec. I, only the experiments on mixed magnetic crystals[6] have, as yet, exhibited the intermediate phase. Unfortunately, the nature of these experiments (which can study only a relatively small number of different mixtures) precludes the determination of detailed structure of the phase diagram in the vicinity of the tetracritical point. In any case, the direct applicability of our theory to such translationally nonivariant systems also needs further study.[7]

As we have emphasized throughout this work, the paradigm cases of systems exhibiting phase transitions described by the Hamiltonian (2.2) are to be found among the perovskite crystals, stressed along [100]. Indeed, study of the phase diagram of [100]-stressed $LaAlO_3$ (which has[27] $v > 0$) may offer the most favorable circumstances for testing the predictions made here, although the experiments may be made harder by the relatively high transition temperature of the unstressed crystal.[48] We hope that this paper will stimulate such experiments which, in principle at least, offer direct means of determining the exponent ϕ_g and even (if one is particularly optimistic!) the exponent ϕ_v.

ACKNOWLEDGMENTS

The authors have benefited considerably from discussions with Professor Michael E. Fisher and David R. Nelson. Informative conversations with Professor Kenneth G. Wilson, Professor K. Alex Müller, Dr. J. Michael Kosterlitz, and Professor James T. Bartis have also been appreciated. The support of the British Science Research Council (ADB), of the National Science Foundation, in part through the Materials Science Center at Cornell University and of the Fulbright-Hays Committee, through a scholarship (AA), is gratefully acknowledged.

---

*Permanent address: Physics Department, University of Edinburgh, James Clerk Maxwell Building, Mayfield Road, Edinburgh, United Kingdom.

†Address after Jan. 1, 1975: Institute of Pure and Applied Physical Sciences, University of California, San Diego, La Jolla, California 92037.

[1]L. Néel, Ann. Phys. (Paris) 18, 5 (1932); C. R. Acad. Sci. 203, 304 (1936).

[2]C. J. Gorter and T. van Peski-Tinbergen, Physica 22, 273 (1956).

[3]M. E. Fisher, Rep. Prog. Phys. 30, 615 (1967).

[4]K. S. Liu and M. E. Fisher, J. Low Temp. Phys. 10, 655 (1973).

[5]G. A. Smolenski, Fiz. Tverd. Tela 4, 1095 (1962) [Sov. Phys. -Solid State 4, 807 (1962)].

[6]H. Weitzel, Z. Kristallogr. 131, 289 (1970); H. A. Obermayer, H. Dachs and H. Schröcke, Solid State Commun. 12, 779 (1973); Ch. Wissel, Phys. Status Solidi B 51, 669 (1972).

[7]F. J. Wegner, Solid State Commun. 12, 785 (1973). In this paper, Wegner describes the $(Mn, Fe)WO_4$ system by a mean-field free energy of a form similar to the one we use. It is not yet clear if our generalized Hamiltonian, which includes fluctuations in the "spin" coordinates, fully applied to these systems, as we do not take special care of their translational noninvariance.

[8]K. A. Müller, W. Berlinger, and J. C. Slonczewski, Phys. Rev. Lett. 25, 734 (1970); K. A. Müller (private communication).

[9]J. C. Slonczewski, Phys. Rev. B 2, 4646 (1970). The first Landau-type description of the phase transitions in perovskite-type crystals including interactions with elastic strain was given by H. Thomas and K. A. Müller, [Phys. Rev. Lett. 21, 1256 (1968)].

[10]A. Aharony and A. D. Bruce, Phys. Rev. Lett. 33, 427 (1974); A. Aharony, paper presented at the Conference on Critical Phenomena in Multicomponent Systems, Athens, Ga., April 1974 (unpublished).

[11]Y. Imry, D. J. Scalapino, and L. Gunther, Phys. Rev. B 10, 2900 (1974).

[12]R. B. Griffiths, Phys. Rev. Lett. 24, 715 (1970); Phys. Rev. B 7, 545 (1973).

[13]M. E. Fisher and D. R. Nelson, Phys. Rev. Lett. 32, 1350 (1974). The term "tetracritical point" has also been used in another context by J. C. Bonner and J. F. Nagle [J. Chem. Phys. 54, 279 (1971)].

[14]D. R. Nelson, J. M. Kosterlitz and M. E. Fisher, Phys. Rev. Lett. 33, 813 (1974).

[15]K. G. Wilson and J. Kogut, Phys. Rept. C 12, 77 (1974).

[16]M. E. Fisher and D. Jasnow, in *Theory of Correlation in the Critical Region*, edited by C. Domb and M. S.

Green (Academic, New York, to be published).
[17]A. Aharony, Phys. Rev. B 8, 4270 (1973).
[18]F. J. Wegner, Phys. Rev. B 5, 4529 (1972).
[19]See, e.g., S. K. Ma, Phys. Rev. A 7, 2172 (1973); Rev. Mod. Phys. 45, 589 (1973).
[20](a) E. K. Riedel and F. J. Wegner, Z. Physik 225, 195 (1969); (b) M. E. Fisher and P. Pfeuty, Phys. Rev. B 6, 1889 (1973); (c) F. J. Wegner, Phys. Rev. B 6, 1891 (1973); (d) P. Pfeuty, M. E. Fisher, and D. Jasnow, AIP Conf. Proc. 10, 817 (1973); and Phys. Rev. B 10, 2088 (1974).
[21]The biconical fixed point, found by Nelson *et al.* (Ref. 14), is unstable against a cubic perturbation [D. R. Nelson (unpublished)], and the system ultimately crosses over to a cubiclike critical behavior. For $n=2$ the biconical and cubic fixed points coincide.
[22]R. A. Cowley and A. D. Bruce, J. Phys. C 6, L191 (1973).
[23]W. G. Stirling, J. Phys. C 5, 2711 (1972).
[24]A. D. Bruce, J. Phys. C 7, 2089 (1974).
[25]J. K. Kjems, G. Shirane, K. A. Müller, and H. J. Scheel, Phys. Rev. B 8, 1119 (1973).
[26]A. D. Bruce and R. A. Cowley, J. Phys. C 6, 2422 (1973).
[27]J. C. Slonczewski and H. Thomas, Phys. Rev. B 1, 3599 (1970). The trigonal ordering of $LaAlO_3$ was experimentally observed by D. Derighetti *et al.*, Acta Crystallogr. 18, 557 (1965).
[28]D. J. Wallace, J. Phys. C 6, 1390 (1973). Note that there is an error in Eq. (27) of this paper.
[29]E. Brézin, D. J. Wallace, and K. G. Wilson, Phys. Rev. B 7, 232 (1973).
[30]See, however, the discussion following Eq. (5.9).
[31]In general, t could be a linear combination of $(T-T_c)/T_c$ and of g. However, in the *asymptotic* critical region this analytic term, linear in g, may be ignored. See also the discussion following Eq. (5.6).
[32]This is confirmed by direct calculation [A. D. Bruce, Phys. Lett. 48A, 317 (1974)].
[33]A. Aharony, Phys. Lett. A 49, 221 (1974).
[34]Since in the cases of interest $n \leq 3$, we can safely assume that $(n-m)$ and m are smaller than $n_c(d)$, so that these transitions are Heisenberg-like and not cubic like.
[35]J. M. Kosterlitz (private communication).
[36]A. Aharony, Phys. Rev. B 10, 3006 (1974).
[37]K. G. Wilson, Phys. Rev. Lett. 28, 540 (1972).
[38]We give explicit expressions to a higher order in ϵ than necessary in this section, for reference in following sections.
[39]A similar result for $T_2'(g,v)$ is obtained by replacing m by $(n-m)$ and g by $-mg/(n-m)$.
[40]The first term in Eq. (8.6) has been previously derived, in the context of the $1/n$ expansion, by Wallace, Ref. 28 (our σ_v corresponds to his ψ).
[41]The scaling relation between σ_v and ϕ_v has been noted by I. J. Ketley and D. J. Wallace [J. Phys. A 6, 1667 (1973)]. In their notation $\alpha_2-\alpha_1=-\phi_v/\gamma$. Our Padé approximant value $n_c(3)=3.13$ is based on their expansion of $\alpha_2-\alpha_1$ to order ϵ^3.
[42]E. Brézin and D. J. Wallace, Phys. Rev. B 7, 1967 (1973).
[43]S. Hikami and R. Abe, Prog. Theor. Phys. 52, 369 (1974). See also R. Oppermann, Phys. Lett. A 47, 383 (1974).
[44]A. Aharony, Phys. Rev. Lett. 31, 1494 (1973).
[45]This seems to be the case for the line $T_1(g)$ in the anisotropic antiferromagnet MnF_2: Y. Shapiro, S. Foner, and A. Misetich, Phys. Rev. Lett. 23, 98 (1969).
[46]Note, however, that in practice this distinction may be blurred by the fact that *all* susceptibilities are large in the vicinity of the "spin-flop" point.
[47]For v less than some critical value we again expect Heisenberg-like transitions to become first order. See also Ref. 10.
[48]K. A. Müller and W. Berlinger, Phys. Rev. Lett. 26, 13 (1971).