Two point correlation function for general fields and temperatures in the critical region*

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A detailed calculation, to order ϵ^2 ($\epsilon = 4 - d$), of the two-point-correlation function of an Ising-like system in the whole critical region is presented. The scaling function is shown to be a cut-off-independent function of two variables which is universal in the context of a sharp cut off. Explicit asymptotic expansions in the large and small momentum (relative to the inverse correlation length) are given. Particular attention is paid to the corrections from the Ornstein-Zernike theory. These corrections are two orders of magnitude larger below T_c than above. Numerical comparison with series-expansion results agree surprisingly well. A powerful technique of evaluating diagrams using the Fourier transform of the propagator is also presented.

I. INTRODUCTION

It is well known that as one approaches the critical point of most systems, the scattering intensity shows a dramatic increase in certain directions (near a reciprocal-lattice or superlattice vector for systems with a periodic structure and near the forward direction for liquids). Within the first Born approximation, it can be shown that the quasielastic scattering intensity is proportional to the Fourier-transformed order-order correlation function.[1-4]

The study of critical scattering thus reduces to the study of the function

$$G(\vec{q}, T, m) \propto \sum_{\vec{R}} e^{i\vec{q}\cdot\vec{R}} \times \langle [\vec{s}(\vec{0}) - \langle \vec{s}(\vec{0})\rangle] \cdot [\vec{s}(\vec{R}) - \langle \vec{s}(\vec{R})\rangle] \rangle . \tag{1.1}$$

Here $\vec{R}$ labels the sites of a d-dimensional lattice, $\vec{s}$ denotes an n-component vector or "spin" field, and $m = \langle \vec{s} \rangle$ is the order parameter of the system. For a magnetic system, a pure fluid or binary mixture, the order parameter will be, respectively, the magnetization, density, or composition deviation from critical. With improving scattering techniques (using x rays, neutrons, and light) one may expect that highly precise measurements of the scattering intensity in the whole critical region of such systems will soon be possible. Such measurements are of fundamental theoretical importance as a test of scaling[2-5] and operator-product-expansion[6,7] hypotheses. In particular, scattering experiments are the only way to determine the exponent η directly.[2]

There have been several theoretical studies of the correlation function in the critical region, the aim of which is to find the form of the deviations from the classical or Ornstein-Zernike theories,[1] which predict that the scattering intensity at fixed temperature has a pure Lorentzian line shape. For the two-dimensional Ising model it has been shown[2,8] that these theories fail badly. This failure has also been demonstrated for higher-dimensional systems by extrapolation of high-temperature series results.[9-11]

Recently there have been several investigations using renormalization-group techniques. Aharony and Fisher,[12] using the Wilson-Fisher ϵ and $1/n$ perturbation expansions[13] have discussed extensively the situation for systems with an order parameter with an arbitrary number of components in zero field above T_c and both large and small momentum transfer (measured in units of inverse correlation length). Brezin and coworkers[14,15] using the Callen-Symanzik equation have obtained results for arbitrary fields, temperatures, and n, but are restricted to the large-momentum regime.

If $t = (T - T_c)/T_c$ is the reduced temperature and $qa \ll 1$ (a is the lattice spacing), the correlation scaling hypothesis asserts that the Fourier-transformed order-order correlation function may be written asymptotically as[2-5]

$$G(\vec{q}, t, m) \approx \chi(t, z) D(x, z) \tag{1.2}$$

with $z = tm^{-1/\beta}$ and $x = q\xi$; the correlation length ξ varies as[3]

$$\xi(t, z) = \tilde{f} f(z) |t|^{-\nu} \qquad [f(\infty) = 1] . \tag{1.3}$$

Moreover, we choose to normalize the scaling function $D(x, z)$ by

$$D(0, z) = 1, \ \frac{d}{dx^2} D(x, z)\bigg|_{x=0} = -1 , \tag{1.4}$$

so that $\chi(t, z)$ is the reduced susceptibility (or compressibility or composition fluctuation, etc.), varying as[3]

$$\chi^{-1}(t,z)=CN(z)|t|^{\gamma}\quad [N(\infty)=1]\,, \tag{1.5}$$

and ξ is identified as the second-moment correlation length.[4]

The amplitudes C and $\tilde{f}$ are then nonuniversal numbers depending on the details of the system. On the other hand, the exponents γ and ν, the functions $f(z)$ and $N(z)$, and the scaling function $D(x,z)$ are expected to be universal depending only on a few general features of the system such as the dimensionality d and the number of components of the order parameter.[1,2,16]

In this paper, we use the ϵ expansion to discuss the situation for general momentum, temperature, and field, but are restricted to a single-component order parameter. Our results are, therefore, applicable to Ising-like magnetic systems, pure fluids, binary mixtures, etc. We show explicitly the universality of $D(x,z)$, $f(z)$, and $N(z)$, in the context of a sharp cut off in the direct space, up to to ϵ^2.

To analyze properly experimental results, particularly those subject to resolution corrections,[17] it is necessary to have an interpolation formula for $D(x,z)$ over the whole range of x. We have paid particular attention to the amplitudes of the deviations from the simple Ornstein-Zernike form, $\Sigma_4(z)$ and $\Sigma_6(z)$, for small x.[18,19] These deviations are quoted as a function of z as well as in the parametric representation, more directly accessible to experiments. Numerical comparisons with series-expansion predictions by Fisher and Tarko agree surprisingly well. The agreement is equally good for the $q\xi \gg 1$ region.

Finally, from a technical point of view, we have developed a powerful method of evaluating the various diagrams in the problem by working mainly in coordinate space, which greatly facilitates the calculations. This technique should prove useful in more complicated situations. Some important details of this method are explained in the Appendix.

In outline, the paper goes as follows. In Sec. II the model is defined and the perturbation expansion to $O(\epsilon^2)$ derived. In Sec. III we discuss the normalizations and calculate the subsidiary scaling functions of the problem both as functions of the variable z and of θ, the angular variable of the parametric representation.[20] Section IV outlines the derivation of the asymptotic expansions of the correlation function and contains the main results of the paper in Eqs. (4.1), (4.4), and (4.6).[21] The results are discussed in Sec. V, and some further possible extensions are mentioned. The coordinate-space method of calculating Feynman diagrams is discussed in the Appendix, and the evaluation of the individual graphs is outlined.

II. MODEL AND GRAPHICAL EXPANSION OF THE CORRELATION FUNCTION

For simplicity we shall use magnetic language in the following, although the results are valid for any system with a one-component order parameter and short-range interactions. Following Wilson[13] we consider a continuous spin system in an external ordering field. Making the usual shift on the spin in order to avoid the inclusion of tadpole insertions and, provided all self-energy diagrams are subtracted at zero external momentum, we may work with the effective Hamiltonian

$$\bar{\mathcal{H}}=\frac{1}{2}\int (q^2+r)\sigma_{\vec{q}}\sigma_{-\vec{q}}+4u\tilde{m}\int_{\vec{q}_1}\int_{\vec{q}_2}\sigma_{\vec{q}_1}\sigma_{\vec{q}_2}\sigma_{-\vec{q}_1-\vec{q}_2}+u\int_{\vec{q}_1}\int_{\vec{q}_2}\int_{\vec{q}_3}\sigma_{\vec{q}_1}\sigma_{\vec{q}_2}\sigma_{\vec{q}_3}\sigma_{-\vec{q}_1-\vec{q}_2-\vec{q}_3}\,, \tag{2.1}$$

where $\int_{\vec{q}}$ means $(2\pi)^{-d}\int d^d q$, r is the exact inverse susceptibility and m is the magnetization. Note that a perturbation expansion in u will automatically produce a scaling form for $G(q,t,m)$ when u is fixed at its critical value u_c. A different choice of u will produce corrections not of a scaling form but such terms will vanish rapidly near the critical point. In this paper we fix $u=u_c$ and ignore the problem of corrections to scaling. Such a perturbation expansion implicitly assumes the existence of a renormalization-group fixed point and the possibility of making a valid expansion about it.

We consider all graphs up to order ϵ^2. Since $\tilde{m}=O(\epsilon^{-1/2})$[22] and $u_c=O(\epsilon)$, all the diagrams in Fig. 1 must be evaluated. The graph B has already been evaluated by Aharony and Fisher.[12] Below T_c or in a field one needs to evaluate all the other ones. As discussed in some detail in the Appendix, the evaluation of the graphical integrals has been greatly eased by working mostly in real space. The nonuniversal parts may be calculated either by using a sharp cut off Λ in momentum space or a sharp cut off L in real space. The latter gives rise to oscillations in $G_0(\vec{p})$ for $pL>1$, but this contributes only to certain nonuniversal constants which either cancel or are absorbed into normalizations of physical quanti-

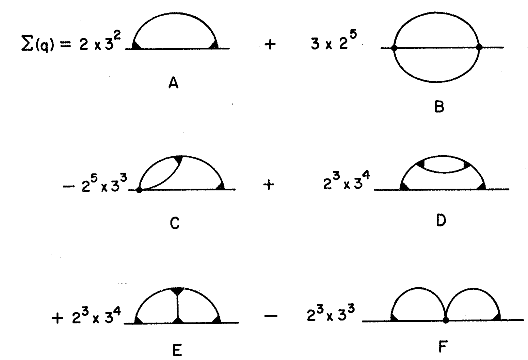

FIG. 1. Graphs of the perturbation expansion of the self-energy to order ϵ^2. The three-point vertex denoted by a triangle is $4u_c \tilde{m}$ and the four-point vertex denoted by a circle is u_c. Solid lines represent bare propagators G_0.

ties. In particular, using the cut off L, the critical value u_c of the coupling constant is different at $O(\epsilon^2)$ from the conventional value[13] (calculated with Λ). We find

$$u_c = \frac{\epsilon}{n+8}\, 2^{d-3}\pi^{d/2}\Gamma(d/2)L^{-\epsilon} \times \left[1 + \epsilon\left(\frac{3(3n+14)}{(n+8)^2} + \ln 2 - c\right) + O(\epsilon^2)\right], \quad (2.2)$$

where n is the number components of the order parameter and c is Euler's constant.

The graphical expansion of the correlation function to $O(\epsilon^2)$ now follows in a straightforward manner. Dyson's equation reads

$$G^{-1}(q,\tilde{m},r) = q^2 + r - [\Sigma(q) - \Sigma(0)]\,, \quad (2.3)$$

$\Sigma(q)$ being depicted in Fig. 1.

Let

$$u_c = \epsilon u_0[1 + \epsilon(\bar{u}_0 - \ln L)]\,, \quad (2.4)$$

where $u_0 = \frac{2}{9}\pi^2$ and $\bar{u}_0$ is a nonuniversal constant depending on the shape of the cut off. Using the calculations of the graphs in the Appendix (A4), (A11), (A19), (A25), (A26), (A44), (A45), (A48), (A55), and after some tedious but straightforward algebra, we arrive at the result

$$r^{-1}[\Sigma(q) - \Sigma(0)] = 2^5 3^2 u_0^2 \big(\epsilon M_0^2 \tilde{\Sigma}_A^0(Q) + \epsilon^2\{M_0^2\tilde{\Sigma}_A^1(Q) + (2\bar{u}_0 + 1)M_0^2\tilde{\Sigma}_A^0(Q) - 2^2 3u_0 M_0^2[\tilde{\Sigma}_A^0(Q)]^2 + \tfrac{1}{3}\tilde{\Sigma}_B^U(Q)$$
$$+ \tfrac{1}{3}Q^2\sigma_B - 2^4 3u_0 M_0^2 \tilde{\Sigma}_e^U(Q) + 2^6 3^2 u_0^2 M_0^4[\tilde{\Sigma}_D(Q) + \tilde{\Sigma}_E(Q)]\}\big) + O(\epsilon^2)\,, \quad (2.5)$$

where $Q = qr^{-1/2}$,

$$M_0^2 = \rho^2 m^2 r^{-2\beta/\gamma} = \rho^2 m^2 r^{-1+\epsilon/2} + O(\epsilon^2)\,, \quad (2.6)$$

the constant ρ being defined later.

Looking at the explicit expressions of all the Σ (see the Appendix), we note that all cut-off-dependent and nonuniversal quantities have vanished from the problem except for σ_B [Eq. (A20)], which is essential for the eventual disappearance of all explicit r dependence of the scaling function.

Introducing $x = q\xi$, where ξ is the correlation length defined by

$$r^{-2/2-\eta} \equiv a^2\xi^2 \quad (2.7)$$

($\eta = \frac{1}{54}\epsilon^2$ and a being an as yet undetermined nonuniversal cut-off-dependent function of M_0^2) we have

$$Q^2 = x^2 a^2(1 + \tfrac{1}{108}\epsilon^2 \ln r)\,. \quad (2.8)$$

Assuming the scaling relation $\gamma = (2-\eta)\nu$, the scaling function $D^{-1}(x,z) = rG^{-1}(\vec{q}, t, m)$ reads

$$D^{-1}(x,z) = 1 + a^2x^2[1 + \tfrac{1}{54}\epsilon^2(B_4 - \ln L)] - \epsilon(2^7\pi^4/3^2)\big(M_0^2\tilde{\Sigma}_A^0(ax) + \epsilon\{M_0^2\tilde{\Sigma}_A^1(ax) + (2\bar{u}_0 + 1)M_0^2\tilde{\Sigma}_A^0(ax)$$
$$- (2^3\pi^2/3)M_0^2[\tilde{\Sigma}_A^0(ax)]^2 + \tfrac{1}{3}\tilde{\Sigma}_B^U(ax) - (2^5\pi^2/3)M_0^2\tilde{\Sigma}_c^U(ax) + (2^8\pi^4/3^2)M_0^4[\tilde{\Sigma}_D(ax) + \tilde{\Sigma}_E(ax)]\}\big)\,, \quad (2.9)$$

where B_4 is defined by Eq. (A25). We note that all explicit r dependence has now disappeared.

III. NORMALIZATIONS AND PARAMETRIC REPRESENTATIONS

In this section we discuss the normalizations of field and order parameter and calculate the scaling functions for the susceptibility, correlation length, and M_0^2.

The normalization condition (1.2) defines the effective correlation length ξ as the second moment of the correlation function,[2,4] and determines the function $a(M_0^2)$ to be

$$a^2(M_0^2) = 1 + \epsilon a_1(M_0^2) + \epsilon^2 a_2(M_0^2)\,, \quad (3.1)$$

with

$$a_1(M_0^2) = -\tfrac{4}{27}\pi^2 M_0^2,$$
$$a_2(M_0^2) = -\tfrac{1}{54}(B_4 - \ln L) - \tfrac{4}{27}\pi^2 M_0^2[\alpha_1' + (2\bar{u}_0 + 1)\alpha_1 + 4\gamma_1 + \tfrac{4}{27}M_0^2 + (2^5\pi^2/3)M_0^2(\delta_1 + \tfrac{1}{6}e_1)], \tag{3.2}$$

where α_1, etc. are defined by the small-Q expansions of the self-energy diagrams [see Appendix forms (A12) and (A13)].

We define reduced-field and order-parameter variables $h = B_H H$ and $m = B_M M$, where H and M are the physical variables in some units, so that $hm^{-\delta} = 1$ at $t = 0$ and $z = tm^{-1/\beta} = -1$ on the coexistence curve for $t < 0$. The equation of state[22,23] gives us the inverse susceptibility amplitudes $N(z)$ and C as

$$N(z) = |z|^{-\gamma}\{z + 3 + \epsilon[\tfrac{1}{6}(z+9)\ln(z+3) + 1 - \ln 2]\} + O(\epsilon^2) \underset{z\to\infty}{\longrightarrow} 1, \tag{3.3}$$

$$C = (1 + \tfrac{1}{6}\epsilon \ln \tfrac{4}{27})(B_M/B_H),$$

and

$$M_0^2(z) = \rho^2 m^2 \chi^{2\beta/\gamma} = (1/4u_0)[1 - \epsilon(\bar{u}_0 + \tfrac{1}{3}\ln 2)]P(z), \tag{3.4}$$

where

$$P(z) \equiv C^{2\beta/\gamma} m^2 \chi^{2\beta/\gamma} = \frac{1}{z+3}\left(1 + \frac{\epsilon}{z+3}[\tfrac{1}{3}z\ln(z+3) - 1 + \ln 2]\right).$$

With this choice of h and m, the normalization constant ρ, defined in Sec. II is

$$\rho^2 = (1/4u_0)[1 - \epsilon(\bar{u}_0 + \tfrac{1}{3}\ln 2)]C^{2\beta/\gamma}. \tag{3.5}$$

We are now in a position to find the amplitude of the correlation function

$$\xi = \tilde{f}\,|t|^{-\nu} f(z), \tag{3.6}$$

where, by the definition of $a(z)$ [Eq. (2.7)]

$$\tilde{f} f(z) = a^{-1}(z)[N(z)]^{-\gamma/\nu}. \tag{3.7}$$

Choosing $f(z)$ so that $f(\infty) = 1$, from Eqs. (3.1), (3.4), and (3.7) we find

$$f(z) = |z|^{\nu}(z+3)^{-1/2}\{1 - [\epsilon/2(z+3)] \times [\tfrac{1}{6}(z+9)\ln(z+3) + \tfrac{5}{6} - \ln 2]\}, \tag{3.8}$$

in which case

$$\tilde{f} = C^{-\nu/\gamma}.$$

It is frequently more convenient to express everything in the parametric representation since this avoids the necessity of taking the $z \to \infty$ limit ($t > 0$, $h = 0$) and often the amplitudes, etc., take on a very much simpler form. With our choice of normalizations the equation of state can be written in parametric form using the simple linear model model[20,22]

$$h = AR^{\beta\delta}\theta(1 - \theta^2),$$
$$t = R(1 - B^2\theta^2),$$
$$m = DR^{\beta}\theta,$$

where the coefficients A, B, C are given by

$$A = 2^{-1/2}[1 + (\epsilon/6)\ln \tfrac{4}{27}];$$
$$B^2 = \tfrac{3}{2}[1 - \tfrac{1}{9}\epsilon \ln 2];$$

$D = 2^{-1/2}$. This gives

$$\chi^{-1} = CR^{\gamma}[1 - \epsilon\theta^2(\tfrac{1}{2} + \tfrac{1}{3}\ln 2)], \tag{3.9}$$

$$P[z(\theta)] = \tfrac{1}{2}\theta^2[1 + \epsilon\theta^2(\tfrac{1}{3}\ln 2 - \tfrac{1}{2})], \tag{3.10}$$

$$\xi^{-2} = \tilde{f}^{-2}R^{2\nu}(1 + \tfrac{5}{12}\epsilon\theta^2). \tag{3.11}$$

This last result is in surprisingly good agreement with the estimate of Fisher and Tarko[11] who, using series expansions, find for $\epsilon = 1$

$$\xi^{-2} \propto R^{2\nu}(1 + a_0\theta^2)$$

with $a_0 \approx 0.490$.

IV. ASYMPTOTIC EXPANSIONS OF THE CORRELATION FUNCTION

In this section, we consider the two asymptotic limits (i) $q\xi \ll 1$ and (ii) $q\xi \gg 1$ of the correlation function. The general form for $q\xi \gg 1$ has been previously obtained using the Callen-Symanzik approach for a n-component spin system.[24] In the framework of Wilson-Feynman graph technique, we show that the large-q expansion is consistent with the general prediction (for $n = 1$) and we give the amplitudes in the variable z as well as in the parametric variable θ. Note that our calculation is restricted to $n = 1$, and in this special case we find agreement with previous work where applicable.

(i) $q\xi \ll 1$. It is a straightforward but tedious exercise to find from Eqs. (2.9), (3.4), and the small-Q expansions of the $\tilde{\Sigma}_1(Q)$ [Eqs. (A12), (A13), (A23), (A34), (A47), (A54), and (A55)] that, in this limit,[25]

$$D^{-1}(x, z) = 1 + x^2 - \Sigma_4(z)x^4 + \Sigma_6(z)x^6 + O(x^8), \tag{4.1}$$

where, with our choice of normalizations,

$$\Sigma_4(z) = \epsilon a_1 P(z) + \epsilon^2[a_2 + a_3 P(z) + a_4 P^2(z)] + O(\epsilon^3),$$
$$\Sigma_6(z) = \epsilon b_1 P(z) + \epsilon^2[b_2 + b_3 P(z) + b_4 P^2(z)] + O(\epsilon^3),$$

with $P(z)$ defined by Eqs. (3.4) and (3.10). The coefficients a_i and b_i have the values $a_1 = 1.66 \times 10^{-2}$, $a_2 = 2.78 \times 10^{-4}$, $a_3 = 1.76 \times 10^{-2}$, $a_4 = 3.45$

$\times 10^{-2}$; $b_1 = 2.38\times 10^{-3}$, $b_2 = 7.12\times 10^{-6}$, $b_3 = 2.73 \times 10^{-3}$, $b_4 = 6.65\times 10^{-3}$.

Note that in zero field above T_c, $\theta = 0$ so that $P(z) = 0$, so that the only terms which survive are a_2 and b_2, thereby reproducing the results of Ref. 12. For $\epsilon = 1$, the leading corrections $\Sigma_4(z)$ and $\Sigma_6(z)$ are plotted as functions of θ in Fig. 2, and some special values are presented in Table I and compared to previous estimates. Note that in Tabel I the superscripts +, −, c correspond, respectively, to $t > 0$, $h = 0$; $t < 0$, $h = 0$; and $t = 0$, $h \neq 0$.

We see that the deviations from the Ornstein-Zernike form increase rapidly from a minimum at $h = 0$, $t > 0$ ($\theta = 0$) to a maximum at $h = 0$, $t < 0$ ($\theta = 1$), this latter value being two orders of magnitude greater. This agrees with one's general expectations from considering the analytic properties of the two-point-correlation function.

TABLE I. Comparison between the present theory and the previous results of the deviations from the Ornstein-Zernike theory Σ_4 and Σ_6. The superscripts +, −, c, correspond, respectively, to $t > 0$, $h = 0$; $t < 0$, $h = 0$; and $t = 0$, $h = 0$.

	Present theory	Previous results
Σ_4^+	2.78×10^{-4}	2.78×10^{-4} [a] $(6.5\pm 0.8)\times 10^{-4}$ [b]
Σ_6^+	7.12×10^{-6}	7.12×10^{-6} [a] $(5\pm 2)\times 10^{-6}$
Σ_4^-	2.38×10^{-2}	$(1.2\pm 0.6)\times 10^{-2}$ [b]
Σ_6^-	3.90×10^{-3}	$(7\pm 3)\times 10^{-3}$ [b]
Σ_4^c	1.50×10^{-2}	sc$(3.9\pm 0.5)\times 10^{-2}$ [b] bcc$(1.7\pm 2)\times 10^{-2}$
Σ_6^c	2.36×10^{-3}	

[a] Reference 12.
[b] Reference 11.

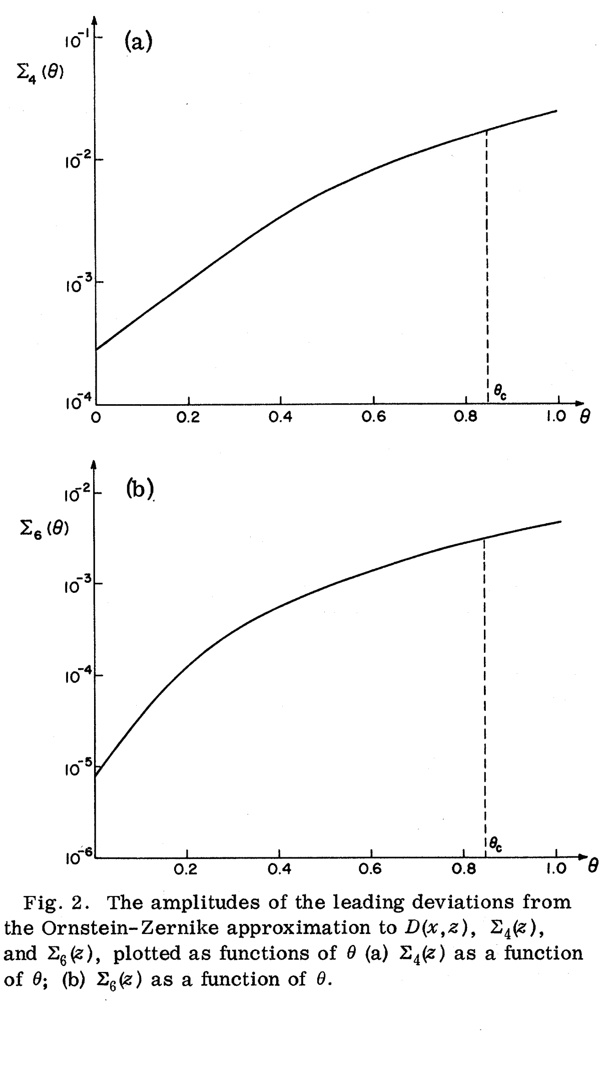

Fig. 2. The amplitudes of the leading deviations from the Ornstein-Zernike approximation to $D(x,z)$, $\Sigma_4(z)$, and $\Sigma_6(z)$, plotted as functions of θ (a) $\Sigma_4(z)$ as a function of θ; (b) $\Sigma_6(z)$ as a function of θ.

According to the Landau conditions for the singularities of a general Feynman diagram,[26] $G(q, t, m)$ must have a simple pole at $q^2 = -\xi^{-2}$ and a set of branch cuts along the $q^2 < 0$ axis beginning at $q^2 = -n^2\xi^{-2}$, where n is the number of intermediate lines.[27] Note that here ξ is the true correlation length which is not quite equal to the correlation length defined by the second moment of the correlation function used elsewhere in this paper. For $q\xi \ll 1$, the branch cuts and pole are sufficiently far from the physical region, $q^2 > 0$, so that a valid expansion in powers of $(q\xi)^2$ can be made. Moreover, the pole dominates so that the corrections to the Ornstein-Zernike form are small. Above T_c in zero field, the first branch cut is due to the three-particle intermediate state, while as we increase θ, corresponding to turning on a field or going below T_c, the magnetization becomes nonzero so that a two-particle intermediate state contributes, and, with increasing magnetization, eventually dominates, the other branch cuts. This behavior is demonstrated in Fig. 2, where we can see that, when $\theta \sim 0.5$ [or $ht^{-\beta\delta} = O(1)$], a change of "regime" in $\Sigma_4(\theta)$ and $\Sigma_6(\theta)$ occurs corresponding to the two-particle cut taking over. In principle, the most favorable places to see the deviations from Ornstein-Zernike are below T_c and above T_c in a field (with θ close to 1).

(ii) $q\xi \gg 1$. In this region, the multiparticle branch cuts accumulate about the origin and each becomes as important as the pole, so that an expansion in powers of $q\xi$ is no longer valid and the Ornstein-Zernike form tends to break down completely. General arguments based on the operator product expansion[6,7] yield the general form[14] for the correlation function

$$G^{-1}(\vec{q}, t, m) \propto q^{2-\eta}[E(z) + F(z) t q^{-1/\nu} + G(z) |t|^{1-\alpha} q^{-(1-\alpha)/\nu}] . \quad (4.2)$$

The simplest way to check this general form using ϵ Feynman-graph expansion is to make an ϵ expansion to second order of $tq^{-1/\nu}$ and $(tq^{-1/\nu})^{1-\alpha}$ in powers of $Q = qr^{1/2}$ and powers of $\log Q$, and write

$$E(z) = E_0(z) + \epsilon E_1(z) + \epsilon^2 E_2(z)$$

and similarly for $F(z)$ and $G(z)$. On the other hand, the large-Q expansion of the $\Sigma_1(Q)$ [see Eqs. (A14), (A15), (A24), (A42), and (A55)] is fed into Eq. (2.9) so that we obtain an ϵ expansion of $G^{-1}(\vec{q}, t, m)$ containing powers of $\log Q$. The last step is to compare separately the coefficients of all the combinations of ϵ, Q, and $\log Q$ appearing in the ϵ expansion of Eq. (4.3) and the large-Q asymptotic expansion of $G^{-1}(q, t, m)$ as calculated from the graphical analysis. This procedure, leads to a number of equations for the $E_i(z)$, etc. It turns out that the zeroth-order amplitudes $E_0(z)$, etc., are overdetermined, leading to two consistency checks, the first-order amplitudes are exactly determined, and the second-order are underdetermined. To calculate these latter terms, an $O(\epsilon^3)$ calculation would have to be performed.

After many pages of algebra one arrives at the conclusion that the perturbation calculation is consistent with the form of Eq. (4.2), with

$$E(z) = 1 + 0.038\epsilon^2 ,$$
$$F(z) = 2 + \epsilon + O(\epsilon^2) ,$$
$$G(z) = |z|^{-(1-\alpha)}\{-z + 3 + \epsilon[-z + \tfrac{1}{6}(z+3)\ln(z+3) - 1 - \ln 2]\} + O(\epsilon^2) , \quad (4.3)$$

in agreement with the calculations of Aharony and Fisher[12] for $z \to \infty (T > T_c,\ h = 0)$ and more generally with Brezin *et al.*[14,15]

From Eq. (4.3) we see that there is a maximum in the scattering intensity in zero field at fixed momentum transfer above T_c, while below T_c the intensity is a smoothly decreasing function of $|t| q^{-1/\nu}$ to a minimum at

$$|t| q^{-1/\nu} = \left(\frac{(1-\alpha) G(-1)}{F(-1)} \right)^{1/\alpha} , \quad t < 0 .$$

However, this minimum is probably spurious since at $O(\epsilon)$, $F(-1) < G(-1) = 2^{2-\alpha} + O(\epsilon^2)$. This minimum therefore occurs at a value of $|t| q^{-1/\nu} \gg 1$, which is outside the range of validity of the asymptotic expansion. This calculation strengthens the interpretation of the scattering data as a smooth curve peaked above T_c. There is, of course a singularity in the slope at T_c, with a discontinuity in the amplitude of the singularity.

The correlation function, expressed in the parametric representation[20] which obviates the necessity of taking the $z \to \infty$ limit of Eq. (4.3) is found to be

$$G^{-1}(\vec{q}, t, m) \propto q^{2-\eta}[E + \tilde{F}(\theta)\Upsilon + \tilde{G}(\theta)\Upsilon^{1-\alpha} + \cdots] , \quad (4.4)$$

with $\Upsilon = Rq^{-1/\nu}$ and

$$\tilde{F}(\theta) = 2 + \epsilon - \theta^2\{3 + \epsilon[\tfrac{3}{2} - \tfrac{1}{3}\ln 2]\} + O(\epsilon^2) ,$$
$$\tilde{G}(\theta) = -(1+\epsilon) + \theta^2\{3 + \epsilon[1 - \tfrac{2}{3}\ln 2]\} + O(\epsilon^2) . \quad (4.5)$$

V. DISCUSSION

We have extended the Feynman-graph-expansion procedure of Fisher and Aharony to general fields and temperatures in the critical region and have developed a powerful tool for evaluating the diagrams. The extra complications appear whenever the order parameter is nonzero, irrespective of whether one is above or below T_c. This determines only which solution of the equation of state one takes for the variable z. The main new result is that the deviations from the Ornstein-Zernike form for $q\xi \gg 1$ in a field or below T_c are $O(\epsilon)$ and at $\epsilon = 1$ are two order of magnitude larger below T_c than above. For $q\xi \gg 1$, the form of the scaling function is found as predicted by the operator product expansion. The amplitudes are now functions of z and agrees with previous partial results.[15]

In principle, one could carry through the same calculation to third order in ϵ, although the technical complexity may be prohibitive. Such a procedure would give $G(z)$ to second order. Studies of expansion of this calculation to the Heisenberg model and corrections to scaling are left for the future.

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APPENDIX

In this appendix, we discuss the evaluation of Feynman diagrams in 4-ϵ dimensions by using a bare propagator in real space rather than the conventional momentum space. This technique is extremely useful since, when working in momentum space, the angular integrals even in a relatively simple graph either become very tedious. Fourier transforming to real space makes the angular integrations trivial and, in the graphs we have considered, the problem rapidly simplifies to, at worst, a double integral over a product of Bessel functions. It turns out that it is fairly easy to extract asymptotic behaviors and to do numerical work. Paradoxically, using this technique, a calculation for arbitrary $d>0$ requires very little more work than for $d=4$ because the properties of Bessel functions are well known for noninteger order.

For any $d>0$ we have the identity

$$(q^2+r)^{-1}=(2\pi)^{-\nu-1}r^{\nu/2}\int d^d x\, e^{i\vec{q}\cdot\vec{x}}x^{-\nu}K_\nu(x\sqrt{r}) , \tag{A1}$$

where

$$\nu=\tfrac{1}{2}(d-2), \quad x=|\vec{x}|$$

and $K_\nu(x)$ is a modified Bessel function of the second kind. Equation (A1) is easily verified by first performing the angular integral to give

$$(q^2+r)^{-1}=r^{\nu/2}q^{-\nu}\int_0^\infty dx\, xJ_\nu(qx)K_\nu(x\sqrt{r})$$

where $J_\nu(x)$ is a Bessel function of the first kind.[28] This can then be integrated[29a] to give Eq. (A1).

More generally the propagator $G_0(\vec{q})$ may be written as

$$G_0(\vec{q})=(2\pi)^{-\nu-1}r^{\nu/2}\int d^d x\, e^{i\vec{q}\cdot\vec{x}} \times x^{-\nu}K_\nu(x\sqrt{r})f_L(x,r) , \tag{A2}$$

where the function $f_L(x,r)$ defines the cut off. The conventional choice for $f_{\Lambda^{-1}}(x,r)$ is such that

$$G_0(\vec{q})=(q^2+r)^{-1}\Theta(q-\Lambda) ,$$

where

$$\Theta(z)=1, \quad z>0$$
$$=0, \quad z<0 .$$

Since in the asymptotic scaling region $r\to 0$, $q\to 0$ with $q\xi$ fixed, the scaling function is expected to be independent of the form of the cut off, we may choose arbitrarily the function $f_L(x)$, and a convenient choice is

$$f_L(x,r)=\Theta(x-L) ,$$

where $L\approx\Lambda^{-1}$ is of the order of a lattice spacing.

Surprisingly, with the above choice of $f_L(x,r)$, the integral in Eq. (A2) can be evaluated exactly[29b] but it is easy to see that, in the region of interest $qL\ll 1$

$$G_0(\vec{q})=(q^2+r)^{-1}[1+O(q^2L^2)]$$

as expected, while in the opposite limit the propagator falls off as

$$G_0(\vec{q}) \underset{qL\to\infty}{\simeq} (q^2+r)^{-1}(qL)^{-\nu+1/2} \times\sin(qL-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi) .$$

The unphysical oscillations in $G_0(\vec{q})$ for $qL\gg 1$ are caused by the sharp cut off in real space but, since we are restricting ourselves to the asymptotic critical region, the large-qL behavior of the propagator does not contribute to the scaling function, but manifests itself only in certain nonuniversal amplitudes.

The prescription for calculating any diagram is to write it down with the usual rules,[13] with the propagators given by Eq. (A2), carry out the momentum and as many of the real-space integrals as possible. In general, the resultant expression will diverge without the cut-off function $f_L(x,r)$, so one moderates the integral by adding and subtracting the divergent behavior of the integrals. This results in a universal part that converges when $f(x)=1$ and a nonuniversal part, depending on the choice of $f_L(x,r)$.

The nonuniversal part coming from the moderating terms may be evaluated with a sharp cut off in either momentum or real space. Note that when Eq. (A2) is used, the nonuniversal parts of any graph (both L-dependent and constant parts) will be different from those calculated with a sharp cut off in momentum space. In particular, the critical value of the coupling constant u_c is different [see Eq. (2.2)].

(a) As a simple example, let us consider in some detail the single-bubble self-energy graph $\Sigma_A(q)$ (see Fig. 1)

$$\Sigma_A(q)=(2\pi)^{-d}\int d^d p\, G_0(p)[G_0(p+q)-G_0(p)] . \tag{A3}$$

Since this integral is convergent for $p\to\infty$, it is sufficient to use Eq. (A1) for $G_0(p)$. Use of Eq. (A2) will introduce corrections of $O(q^2L^2)$ which are ignored. Making a change of variables $\vec{p}\to\vec{p}r^{-1/2}$, introducing $\vec{Q}=\vec{q}r^{-1/2}$, carrying out the momentum and one of the $\vec{x}$ integrations leads to

$$\Sigma_A(q)=r^{-\epsilon/2}\int d^d x\, x^{-2\nu}K_\nu^2(x)(e^{i\vec{Q}\cdot\vec{x}}-1) = r^{-\epsilon/2}\tilde{\Sigma}_A(Q) . \tag{A4}$$

The angular integration is trivial so that

$$\Sigma_A(q)=(2\pi)^{-\nu-1}r^{-\epsilon/2}\int_0^\infty dx\, xK_\nu^2(x) \times\left(\frac{J_\nu(Qx)}{(Qx)^\nu}-\frac{1}{2^\nu\Gamma(\nu+1)}\right) . \tag{A5}$$

It is now easy to obtain the limiting behaviors $Q \ll 1$ and $Q \gg 1$ of $\Sigma_A(q)$ (subject always to $qL \ll 1$). For $Q \ll 1$, one simply expands $J_\nu(Qx)$ in a power series in Qx and integrates term by term to obtain

$$\Sigma_A(q) = \frac{\Gamma(1+\epsilon/2)r^{-\epsilon/2}}{2(4\pi)^{d/2}}\left(\frac{Q^2}{3} + \frac{1+\epsilon/2}{2\cdot 3\cdot 5}Q^4 - \frac{(1+\epsilon/2)(2+\epsilon/2)}{2^2 3\cdot 5\cdot 7}Q^6 + O(Q^8)\right) . \quad \text{(A6)}$$

To obtain the asymptotic expansion for $Q \gg 1$ one proceeds as follows. Provided $\nu < 1$ (which is true for $\epsilon > 0$), both integrals in Eq. (A5) converge, so we may consider them separately. The main contribution comes from values of $x \sim Q^{-1}$, so we split the integration as

$$\begin{aligned} I_A &\equiv \int_0^\infty dx\, x K_\nu^2(x) J_\nu(Qx)(Qx)^{-\nu} \\ &= \int_0^{Q^{-n}} dx\, x \frac{\Gamma^2(\nu)}{4}(2/x)^{2\nu} J_\nu(Qx)(Qx)^{-\nu} \\ &+ \int_0^{Q^{-n}} dx\, x\left[K_\nu^2(x) - \frac{\Gamma^2(\nu)}{4}(2/x)^{2\nu}\right] J_\nu(Qx)(Qx)^{-\nu} \\ &+ \int_{Q^{-n}}^\infty dx\, x K_\nu^2(x) J_\nu(Qx)(Qx)^{-\nu} , \end{aligned} \quad \text{(A7)}$$

where n is still to be chosen. A wily choice is such that the last two integrals in Eq. (A7) vanish as $Q \to \infty$ (the second vanishes as $Q^{+n(3\nu-7/2)-\nu-1/2}$ and the third as $Q^{3n(\nu-1/2)-\nu-1/2}$). Remembering that $\nu = 1 - \frac{1}{2}\epsilon$, we can always choose n so that both exponents are negative. Finally

$$I_A \underset{Q\to\infty}{\sim} \frac{\Gamma^2(\nu)2^{2(\nu-1)}}{Q^{2(1-\nu)}} \int_0^{Q^{1-n}} dx\, x^{1-3\nu} J_\nu(x) . \quad \text{(A8)}$$

The upper limit can be extended to infinity introducing errors $O(Q^{(3/2-3\nu)(1-n)})$ so that finally[29]

$$\Sigma_A(q) \underset{Q\to\infty}{\sim} r^{-\epsilon/2}\Gamma(\tfrac{1}{2}\epsilon)(4\pi)^{-d/2} \times \left(Q^{-\epsilon}\frac{\Gamma^2(1-\frac{1}{2}\epsilon)}{\Gamma(2-\epsilon)} - 1\right) . \quad \text{(A9)}$$

Expanding in powers of ϵ, we have

$$\Sigma_A(q) \equiv r^{-\epsilon/2}\tilde{\Sigma}_A(Q) , \quad \text{(A10)}$$

with

$$\tilde{\Sigma}_A(Q) = \tilde{\Sigma}_A^0(Q) + \epsilon\tilde{\Sigma}'_A(Q) , \quad \text{(A11)}$$

$$\tilde{\Sigma}_A^0(Q) = (1/2^5 3\pi^2)(-\alpha_1 Q^2 + \alpha_2 Q^4 - \alpha_3 Q^6) + O(Q^8) , \quad \text{(A12)}$$

$$\tilde{\Sigma}'_A(Q) = (1/2^5 3\pi^2)(-\alpha_1^1 Q^2 + \alpha_2^1 Q^4 - \alpha_3^1 Q^6) + O(Q^8) , \quad \text{(A13)}$$

with $\alpha_1 = 1$, $\alpha_2 = \frac{1}{10}$, $\alpha_3 = \frac{1}{70}$, $\alpha'_1 = \frac{1}{2}(\ln 4\pi - c)$, $\alpha'_2 = \frac{1}{20}(\ln 4\pi - c)$, $\alpha'_3 = \frac{1}{140}(\frac{3}{2} + \ln 4\pi - c)$;

$$\tilde{\Sigma}_A^0(Q) \underset{Q\to\infty}{\sim} (1/8\pi^2)(1 - \ln Q) + O(Q^{-1}) , \quad \text{(A14)}$$

$$\tilde{\Sigma}'_A(Q) \underset{Q\to\infty}{\sim} (1/16\pi^2)\{\ln^2 Q - (2 + \ln 4\pi - c)\ln Q + (\ln 4\pi - c + 2 - \pi^2/12)\} + O(Q^{-1}) . \quad \text{(A15)}$$

At this point, it is convenient to define some integrals which are useful in the following:

$$M_{abc} = \int_0^\infty dx\, x^a K_0^b(x) K_1^c(x) , \quad \text{(A16)}$$

$$R_n = -\int_0^\infty dx\, x K_1^2(x)[x^n K_n(x) - (n-1)!\,2^{n-1}] . \quad \text{(A17)}$$

The above have been evaluated numerically (see Table II). Further, we have

$$\begin{aligned} B_1 &= \int_0^\infty \frac{dx}{x^3}\left(\frac{J_1(x)}{x} - \frac{1}{2} + \frac{x^2}{16(x+1)}\right) \\ &= \tfrac{1}{16}(c - \ln 2 - \tfrac{5}{4}) , \\ B_2 &= \int_0^\infty \frac{dx}{x}\ln x\left(\frac{J_1(x)}{x} - \frac{1}{2(x+1)} + \frac{x^2}{16(x+1)^3}\right) , \\ &= -\tfrac{1}{32} - \tfrac{1}{12}\pi^2 + \tfrac{1}{4}(x - \tfrac{1}{2} - \ln 2)^2 , \\ B_3 &= \int_0^\infty dx\left(K_1^3(x) - \frac{1}{x^3} - \frac{3}{2(x+1)}(\ln x + c - \tfrac{1}{2} - \ln 2)\right) \\ &\approx -5.66 , \\ B_4 &= \int_0^\infty dx\, x^2\left(K_1^3(x) - \frac{1}{x^3(x+1)}\right) \approx -0.683 , \\ B_5 &= \int_0^\infty \frac{dx}{x}\left(\frac{J_1(x)}{x} - \frac{1}{2(x+1)} + \frac{x^2}{16(x+1)^3}\right) \\ &= \tfrac{5}{32} + \tfrac{1}{2}\ln 2 - \tfrac{1}{2}c . \end{aligned}$$

TABLE II. Numerical value of the integrals R_n and M_{abc} defined by (A16) and (A17).

R_2	R_3	R_4	R_5	R_6	R_7
0.232	0.558	2.40	14.9	121	1230

M_{312}	M_{403}	M_{421}	M_{512}	M_{603}	M_{621}	M_{712}	M_{821}
0.333	0.361	0.153	0.241	0.443	0.260	0.590	1.10

The other graphs are some what more complicated, and further tricks are used to perform some of the angular integrations. In the following we evaluate all the graphs of Fig. 1 only to the required order in ϵ, so that all calculations are performed for $d=4$.

(b)

$$\Sigma_B(\vec{q})=\frac{1}{(2\pi)^8}\int d^4p\, d^4k\, G_0(\vec{k})G_0(\vec{p}+\vec{k})\times[G_0(\vec{p}+\vec{q})-G_0(\vec{p})]\equiv r\tilde{\Sigma}_B(Q)\ . \tag{A18}$$

This graph in particular is much more straightforward in real than in momentum space.[12] Using Eq. (A2) we easily find

$$\tilde{\Sigma}_B(Q)=\frac{1}{(2\pi)^4}\int_0^\infty dx\, K_1^3(x)\times\left(\frac{J_1(Qx)}{Qx}-\frac{1}{2}\right)f_L(x)\ . \tag{A19}$$

Since this diverges at $x=0$ with $f_L(x)=1$, we must moderate it and the most convenient way is to use the next term in the expansion of[30a] $J_1(Qx)$ so that

$$\tilde{\Sigma}_B(Q)=\tilde{\Sigma}_B^u(Q)+Q^2\sigma_B\ , \tag{A20}$$

with

$$\tilde{\Sigma}_B^u(Q)=\frac{1}{(2\pi)^4}\int_0^\infty dx\, K_1^3(x)\times\left(\frac{J_1(Qx)}{Qx}-\frac{1}{2}+\frac{Q^2x^2}{16}\right), \tag{A21}$$

and

$$\sigma_B=\frac{1}{(4\pi)^4}\int_0^\infty dx\, x^2K_1^3(x)f_L(xr^{-1/2})\ . \tag{A22}$$

The small-Q expansion is easily found by the methods described previously:

$$\tilde{\Sigma}_B^u(Q)=\frac{1}{(4\pi)^4}(\beta_2Q^4-\beta_3Q^6)+O(Q^8)\ , \tag{A23}$$

where $\beta_2=M_{403}/24$ and $\beta_3=M_{603}/9.2$.[7]

The large-Q expansion is also evaluated in a straightforward fashion by splitting up the integral in Eq. (A21) as

$$(2\pi)^4\tilde{\Sigma}_B^u(Q)=\int_0^{Q^{-n}}dx\left(K_1^3(x)-\frac{1}{x^3}\right)\left(\frac{J_1(Qx)}{Qx}-\frac{1}{2}+\frac{Q^2x^2}{16}\right)+\int_0^{Q^{-n}}dx\, x^{-3}\left(\frac{J_1(Qx)}{Qx}-\frac{1}{2}+\frac{Q^2x^2}{16}\right)$$
$$+\int_{Q^{-n}}^\infty dx\, K_1^3(x)\frac{J_1(Qx)}{Qx}-\frac{1}{2}\int_{Q^{-n}}^\infty dx\, K_1^3(x)+\frac{Q^2}{16}\int_{Q^{-n}}^\infty dx\, x^2K_1^3(x)\ .$$

For $n<\frac{3}{7}$, the third term vanishes as $Q\to\infty$. Now it is a matter of moderating the remaining integrals in such a way that as $Q\to\infty$, all the nonvanishing Q dependence is contained in the moderating terms and the Q-independent contributions are given in closed form. After some manipulations the result follows[12]:

$$\tilde{\Sigma}_B^u(Q)\underset{Q\to\infty}{\simeq}[1/(2\pi)^4][(B_1+\tfrac{1}{16}B_4)Q^2+\tfrac{1}{16}Q^2\ln Q+\tfrac{3}{8}\ln^2Q-\tfrac{3}{4}(c-\tfrac{1}{2}-\ln 2)^2-\tfrac{3}{64}-\tfrac{1}{2}B_3+\tfrac{3}{2}B_2]+O(Q^{-1})\ . \tag{A24}$$

With the sharp cutoff in real space, the nonuniversal part σ_B is

$$\sigma_B=(\ln L\sqrt{r}-B_4)/(4\pi)^4\ . \tag{A25}$$

(c)

$$\Sigma_c(q)\equiv\frac{1}{(2\pi)^8}\int d^4p\, d^4k\, G_0(\vec{p})G_0(\vec{k})G_0(\vec{p}+\vec{k})\times[G_0(\vec{p}+\vec{q})-G_0(\vec{p})] \tag{A28'}$$
$$\equiv\tilde{\Sigma}_c^u(Q)+\sigma_c\tilde{\Sigma}_A(Q)\ , \tag{A26}$$

where

$$\sigma_c=\frac{1}{(2\pi)^4}\int d^4p\, G_0^2(\vec{p})\ , \tag{A27}$$

and the universal part of $\Sigma_c(q)$ is

$$\tilde{\Sigma}_c^u(Q)=\frac{1}{(2\pi)^8}\int d^4\vec{p}\, d^4\vec{k}\, G_0(\vec{p})G_0(\vec{k})\times[G_0(\vec{p}+\vec{k})-G_0(\vec{k})][G_0(\vec{p}+\vec{q})-G_0(\vec{p})] \tag{A28}$$

which is convergent without the help of the cut off so that (A1) may be used for the propagators.

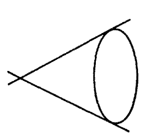
T (0) = 

FIG. 3. One of the diagrams in the evaluation of u_c.

We consider first the small-Q behavior of $\tilde{\Sigma}_c^u(Q)$. Since the angular integrations are rather complicated, we must resort to further tricks. We use the identity

$$\frac{1}{(2\pi)^4}\int d^4k\,\frac{1}{k^2+1}\left(\frac{1}{(\vec{p}+\vec{k})+1}-\frac{1}{k^2+1}\right)=\frac{1}{4\pi^2}\int_0^\infty dx\,xK_1^2(x)\left(\frac{J_1(px)}{px}-\frac{1}{2}\right)\,, \quad \text{(A29)}$$

and then perform the remaining angular integrals with the aid of[29d]

$$\int_0^\pi d\theta\,\frac{\sin^2\theta}{a+b\cos\theta}=\pi[a-(a^2-b^2)^{1/2}]b^2 \quad \text{(A30)}$$

to obtain

$$\tilde{\Sigma}_c^u(Q)=\frac{1}{(2\pi)^4}\int_0^\infty dx\,xK_1^2(x)\int_0^\infty dp\,\frac{p^3}{p^2+1}\left(\frac{J_1(px)}{px}-\frac{1}{2}\right)F(p,Q)\,, \quad \text{(A31)}$$

where

$$F(p,Q)=\frac{1}{4}\left(\frac{1}{p^2Q^2}\{p^2+Q^2+1-[(p^2+Q^2+1)^2-4p^2Q^2]^{1/2}\}-\frac{2}{p^2+1}\right)\,.$$

Making the small-Q expansion of $F(p,Q)$ leads to

$$\tilde{\Sigma}_c^u(Q)=\frac{1}{2(2\pi)^4}\int_0^\infty dx\,xK_1^2(x)\int_0^\infty dp\,\frac{p^3}{p^2+1}\left(\frac{J_1(px)}{px}-\frac{1}{2}\right)\times\left(-\frac{Q^2}{(1+p^2)^3}+Q^4\frac{1-p^2}{(1+p^2)^5}-Q^6\frac{1-3p^2+p^4}{(1+p^2)^7}\right)+O(Q^8)\,. \quad \text{(A32)}$$

The momentum integrals can now be performed using[30e]

$$\int_0^\infty dp\,\frac{p^2}{(p^2+1)^{n+1}}J_1(px)=\frac{x^n}{2^n n!}K_{n-1}(x) \quad \text{(A33)}$$

so that, finally,

$$\tilde{\Sigma}_c^u(Q)=\frac{1}{(4\pi)^4}(\gamma_1Q^2-\gamma_2Q^4+\gamma_3Q^6)+O(Q^8) \quad \text{(A34)}$$

where

$$\gamma_1=\frac{R_2}{3!}\;;$$

$$\gamma_2=\frac{1}{2}\left(\frac{R_4}{5!}-\frac{R_3}{4!}\right)\;;$$

$$\gamma_3=\frac{1}{4}\left(\frac{R_4}{5!}-\frac{5R_5}{2\cdot 6!}+\frac{5R_6}{4\cdot 7!}\right)\,.$$

The large-Q expansion is found by working mostly in coordinate space. Taking Fourier transforms and carrying out one angular integration

$$\tilde{\Sigma}_c^u(Q)=\int_0^\infty dx\,x^2K_1(x)h(x)\left(\frac{J_1(Qx)}{Qx}-\frac{1}{2}\right)\,, \quad \text{(A35)}$$

where

$$h(x)=\int d^4y\,g_0^2(\vec{y})[g_0(\vec{x}+\vec{y})-g_0(\vec{y})] \quad \text{(A36)}$$

and $g_0(\vec{y})$ is defined by

$$G_0(\vec{q})=\int d^d\vec{y}\,e^{i\vec{q}\cdot\vec{x}}g_0(\vec{y})\,. \quad \text{(A37)}$$

To carry out the angular integrals in $h(x)$, we return to momentum space for $g_0(\vec{x}+\vec{y})-g_0(\vec{y})$ and find

$$h(x)=\frac{1}{(2\pi)^4}\int_0^\infty dy\,yK_1^2(y)\times\int_0^\infty dp\,\frac{p^3}{p^2+1}\,\frac{J_1(py)}{py}\left(\frac{J_1(px)}{px}-\frac{1}{2}\right). \quad \text{(A38)}$$

The momentum integral may be performed with the aid of the identity[29f]

$$\int_0^\infty dx\,\frac{x}{x^2+c^2}J_\nu(ax)J_\nu(bx)=\begin{cases}I_\nu(bc)K_\nu(ac), & a>b\\ I_\nu(ac)K_\nu(bc), & a<b\,.\end{cases} \quad \text{(A39)}$$

Where $I_\nu(z)$ is a modified Bessel function of the first kind,[28] so that

$$h(x)=\frac{1}{(2\pi)^4}\,\frac{1}{x}\left(K_1(x)\int_0^x dy K_1^2(y)[I_1(y)-\tfrac{1}{2}y]+\int_x^\infty dy K_1^2(y)[K_1(y)I_1(x)-\tfrac{1}{2}yK_1(x)]\right)\,. \tag{A40}$$

We are now in a position to extract the large-Q behavior of $\tilde{\Sigma}_c^u(Q)$ since, in this case, the main contribution comes from the small-x behavior of $h(x)$, which is easily found to be[30b]

$$h(x)=\frac{1}{2(2\pi)^4}\,\frac{1}{x^2}(\ln x+1+c-\ln 2)+O(\ln x)\,. \tag{A41}$$

Using this form of $h(x)$, we apply the method described previously to obtain

$$\tilde{\Sigma}_c^u(Q)_{Q\to\infty}\sim[1/(2\pi)^4](\tfrac{1}{8}\ln^2 Q-\tfrac{3}{8}\ln Q+C_1)+O(Q^{-1})\,, \tag{A42}$$

where $C_1\approx-4.53$.

The factor σ_c in the nonuniversal part of $\Sigma_c(q)$ is

$$\sigma_c=\frac{1}{(2\pi)^4}\int d^4p\, G_0^2(\vec{p})\,, \tag{A43}$$

which, using the form (A2) for $G_0(\vec{p})$ is

$$\sigma_c=-(1/16\pi^2)[\ln(\tfrac{1}{4}rL^2)+1+2c]+O(rL^2)\,. \tag{A44}$$

(d)

$$\begin{aligned}\Sigma_D(q)&=\frac{1}{(2\pi)^8}\int d^4p\,d^4k\,G_0^2(\vec{p})G_0(\vec{k})\\&\quad\times[G_0(\vec{p}+\vec{k})-G_0(\vec{k})][G_0(\vec{p}+\vec{q})-G_0(\vec{p})]\\&\equiv r^{-1}\tilde{\Sigma}_D(Q)\,.\end{aligned} \tag{A45}$$

It is easy to see by naive power counting that, to within logarithms,

$$\tilde{\Sigma}_D(Q)_{Q\to\infty}\sim O(Q^{-2})\,. \tag{A46}$$

We see that the only difference between $\tilde{\Sigma}_D(Q)$ and $\tilde{\Sigma}_c^u(Q)$ is the presence of an extra $G_0(\vec{p})$, so that we may use identical methods, with the result

$$\tilde{\Sigma}_D(Q)=\frac{1}{(4\pi)^4}(\delta_1Q^2-\delta_2Q^4+\delta_3Q^6)+O(Q^8)\,, \tag{A47}$$

where

$$\delta_1=\frac{R_3}{2\cdot 4!}\,;$$

$$\delta_2=\frac{1}{4}\left(\frac{R_5}{6!}-\frac{R_4}{5!}\right)\,;$$

$$\delta_3=\frac{1}{8}\left(\frac{R_5}{6!}-\frac{5R_6}{2\cdot 7!}+\frac{5R_7}{4\cdot 8!}\right)\,.$$

(e)

$$\begin{aligned}\Sigma_E(Q)&\equiv\frac{1}{(2\pi)^8}\int d^4\vec{p}\,d^4\vec{k}\,G_0(\vec{p})G_0(\vec{k})G_0(\vec{p}-\vec{k})\\&\quad\times[G_0(\vec{p}+\vec{q})G_0(\vec{k}+\vec{q})-G_0(\vec{p})G_0(\vec{k})]\\&\equiv r^{-1}\tilde{\Sigma}_E(Q)\,.\end{aligned} \tag{A48}$$

The asymptotic behavior of $\tilde{\Sigma}_E(Q)\sim O(Q^{-2})$ and again there are no ultraviolet divergences. In this case, we have found that the easiest way of performing the angular integrals is by a mixture of real space and the Feynman trick in momentum space. Defining $f(\vec{x},\vec{Q})$ by

$$\frac{1}{(p^2+1)[(\vec{p}+\vec{Q})^2+1]}=\int d^4x\,e^{-i\vec{p}\cdot\vec{x}}f(\vec{x},\vec{Q})\,, \tag{A49}$$

$$\begin{aligned}\tilde{\Sigma}_E(Q)&=\int d^4x\,g_0(\vec{x})\{[f(\vec{x},\vec{Q})]^2-[f(\vec{x},0)]^2\}\\&\equiv I(Q)-I(0)\,.\end{aligned} \tag{A50}$$

Using the Feynman trick for $f(\vec{x},\vec{Q})$, we find

$$f(\vec{x},\vec{Q})=\frac{1}{(2\pi)^4}\int d^4p\int_0^1 d\alpha\,e^{i(\vec{p}-\alpha\vec{Q})\cdot\vec{x}}(p^2+\lambda^2)^{-1}\,, \tag{A51}$$

where

$$\lambda^2=1+\alpha(1-\alpha)Q^2\,.$$

The momentum integrals may now be done with the aid of Eq. (A33) so that

$$f(\vec{x},\vec{Q})=\frac{1}{8\pi^2}\int_0^1 d\alpha\,e^{i\alpha\vec{Q}\cdot\vec{x}}K_0(\lambda x)\,. \tag{A52}$$

Substituting this into Eq. (A50) the angular integrations may now be easily performed, leading to

$$I(Q)=\frac{1}{4(2\pi)^4}\,\frac{1}{Q}\int_0^\infty dx\,xK_1(x)\int_0^1\int_0^1\frac{d\alpha\,d\alpha'}{\alpha-\alpha'}K_0(\lambda x)K_0(\lambda' x)J_1[(\alpha-\alpha')Qx]\,, \tag{A53}$$

where λ' is defined as λ but with α replaced by α'. Making the small-Q expansion of the integrand of Eq. (A53) reduces all the integrals over α and α' to elementary ones, so that we finally obtain

$$\tilde{\Sigma}_E(Q)=\frac{1}{32^9\pi^4}(-e_1Q^2+e_2Q^4-e_3Q^6)+O(Q^8)\,, \tag{A54}$$

where

$$e_1 = M_{312} + \tfrac{1}{4}M_{421} ,$$

$$e_2 = \frac{1}{2}\left(\frac{M_{621}}{120} + \frac{M_{512}}{15} + \frac{2M_{312}}{5} + \frac{M_{421}}{5} + \frac{M_{403}}{6}\right),$$

$$e_3 = \frac{1}{8}\left(\frac{M_{821}}{2688} + \frac{M_{712}}{280} + \frac{M_{621}}{84} + \frac{M_{603}}{120} + \frac{8M_{312}}{35} + \frac{4M_{421}}{35} + \frac{2M_{403}}{15} + \frac{5M_{512}}{42}\right) .$$

(f)

$$\Sigma_F(q) \equiv \frac{1}{(2\pi)^8}\int d^4p\, d^4k G_0(\vec{p})G_0(\vec{k})[G_0(\vec{p}+\vec{q})G_0(\vec{k}+\vec{q}) - G_0(\vec{p})G_0(\vec{k})]$$

$$= [\tilde{\Sigma}_A(Q)]^2 + 2\sigma_c\tilde{\Sigma}_A(Q) , \tag{A55}$$

so that no further evaluation of $\Sigma_F(q)$ is required.

(g) The graphs involved in the calculation of u_c are mostly very simple, the only one requiring any discussion is that of Fig. 3. This may be written as

$$T(0) = \frac{1}{(2\pi)^8}\int d^4p\, d^4k\, G_0^2(\vec{p})G_0(\vec{k})G_0(\vec{p}+\vec{k})$$

$$= \int d^4x\, d^4y\, g_0^2(\vec{x})g_0(\vec{y})g_0(\vec{x}+\vec{y}) . \tag{A56}$$

The structure of this is very similar to that of $\Sigma_c(q)$ so that very similar methods can be applied. Returning to momentum space for $g_0(\vec{x}+\vec{y})$ all the angular integrations can be done

$$T(0) = 2\pi^2\int_0^\infty\int_0^\infty dx\,dy\, x^2y^2g_0^2(x)g_0(y)\int_0^\infty dk\,kG_0(k)J_1(kx)J_1(ky) . \tag{A57}$$

An adequate approximation for $G_0(k)$ in Eq. (A61) is $G_0(k) = (k^2+r)^{-1}$, since this will introduce errors of, at worst, $O(L\sqrt{r})$. Such an approximation permits an exact evaluation of the integral with aid of Eq. (A40) and we find

$$T(0) \approx \frac{1}{2(2\pi)^4}\int_{L\sqrt{r}}^\infty dx K_1^2(x)\left(K_1(x)\int_{L\sqrt{r}}^x dy\, yK_1(y)I_1(y) + I_1(x)\int_x^\infty dy\, yK_1^2(y)\right) . \tag{A58}$$

Using the now familiar device of adding and subtracting the small-x and -y behavior of the integrands, it is easy to see that

$$T(0) \underset{r\to 0}{\simeq} \frac{1}{2(2\pi)^4}\left(\int_{L\sqrt{r}}^\infty dx\, x^{-3}\int_{L\sqrt{r}}^x dy\,\tfrac{1}{2}y - \frac{1}{4}\int_{L\sqrt{r}}^\infty dx\, x^{-1}(1+2c+2\ln\tfrac{1}{2}x)\right) , \tag{A59}$$

where we have used[30g]

$$\int_x^\infty dy\, yK_1^2(y) = -\tfrac{1}{2}x^2[K_1^2(x) - K_0(x)K_2(x)] \tag{A60}$$

and the small-x expansion of Eq. (A60). Whence

$$T(0) \underset{r\to 0}{\approx} \frac{1}{2^7\pi^4}[\ln^2 L\sqrt{r} + 2(c-\ln 2)\ln L\sqrt{r}] + (\text{const}) . \tag{A61}$$

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