Space-time–dependent spin correlation of the one-dimensional Ising model with a transverse field. Application to higher dimensions

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The correlation functions $\rho_{xx}(R,t)$ and $\rho_{yy}(R,t)$ are calculated for the one-dimensional Ising model with a transverse field h at $T=0$. This model corresponds to the XY model with $\gamma=1$, and is equivalent to the two-dimensional anisotropic ($J_1\rightarrow\infty$, $J_2\rightarrow 0$) Ising model. The additional dimension in the classical model is the imaginary time of the quantum model. For all values of h, $\rho_{yy}(R,t)=-(1/h^2)(\partial^2/\partial t^2)\rho_{xx}(R,t)$. At the critical field $h=h_c=1$, $\rho_{xx}(R,t)\sim(R^2-t^2)^{-1/8}$. For $h<1, h>1$ the results already obtained for the XY model are recovered. We give some consequences from this equivalence in higher dimension, concerning the behavior of the correlation function at the critical field at $T=0$.

The Ising model with a transverse field is the limit $\gamma=1$ of the XY model defined by

$$H=-\sum_{i=1}^{N}[(1+\gamma)S_i^xS_{i+1}^x+(1-\gamma)S_i^yS_{i+1}^y+hS_i^z], \qquad (1)$$

where S_i^v are one-half the Pauli spin matrices. The static properties of the general XY model (1) have been discussed in detail.[1] The correlation functions

$$\rho_{vv}(R,t)=\langle e^{iHt}S_1^ve^{-iHt}S_{R+1}^v\rangle$$

have been calculated at $T=0$, $\rho_{zz}(R,t)$ by Niemeyer,[2] $\rho_{xx}(R,t)$ and $\rho_{yy}(R,t)$ by McCoy[3] for all values of h and γ except in the limit $h=1$. The method and results presented here are valid only in the limit $\gamma=1$ and for all values of h.

As first suggested in the study of the static properties of the Ising model with a transverse field,[4] the XY model at $T=0$ for a chain of N sites is related to the two-dimensional Ising model[5] with horizontal and vertical exchanges J_1 and J_2, on a lattice with $M\rightarrow\infty$ rows and N columns. Susuki[6] has shown that, if h and γ are coupled to $K_1=\beta J_1$ and $K_2=\beta J_2$ (where $\beta=1/kT$) through the two relations

$$\gamma=\tanh 2K_1,\ (1-\gamma^2)^{1/2}/h=\tanh 2K_2, \qquad (2)$$

then the transfer matrix V of the two-dimensional Ising model[5] commutes with the Hamiltonian H [Eq. (1)] of the XY model. The operators H and V have the same eigenvectors but different eigenvalues. The static properties of the two models are equivalent. H and V can be written[3,5] in the form

$$H=H^+P^++H^-P^-, \qquad V=V^+P^++V^-P^-, \qquad (3)$$

where the operator P^+ (P^-) is a projection operator for states of an even (odd) number of c_j excitations,

$$b_j^+=S_j^x+iS_j^y=C_j^+\exp\left[-\pi\sum_{k=1}^{j=1}c_kc_k^+\right],$$

and with

$$H^\pm=\sum_{\phi^\pm}\Lambda(\phi^\pm)\{^+\eta^\pm_{\phi^\pm}\eta^\pm_{\phi^\pm}-\tfrac{1}{2}\}$$
$$V^\pm=(2\sinh 2K_1)^{M/2}\exp\left(-\sum_{\phi^\pm}\lambda(\phi^\pm)\{^+\eta^\pm_{\phi^\pm}\eta^\pm_{\phi^\pm}-\tfrac{1}{2}\}\right) \qquad (4)$$

where the sets $\phi^\pm$, the anticommuting operators $^+\eta^\pm_{\phi^\pm}$, $\eta^\pm_{\phi^\pm}$ and the dispersion relations $\Lambda(\phi^\pm)$ and $\lambda(\phi^\pm)$ are defined in Refs. 3 and 5. In the limit $\gamma=1$ the relations (2) show that the equivalent two-dimensional Ising model has $K_2\rightarrow 0$, $K_1\rightarrow\infty$ with $\exp(-2K_1)/K_2\rightarrow h$. In this limit we find for $\Lambda(\phi)$ and $\lambda(\phi)$ the following expressions:

$$\Lambda(\phi)=(1+h^2-2h\cos\phi)^{1/2}, \qquad \lambda(\phi)\sim(2K_2)\Lambda(\phi),\ K_2\rightarrow 0. \qquad (5)$$

The eigenvalues of H^+ are directly connected through (4) and (5) to the eigenvalues of V^+. The correlation function $\rho_{xx}(R,t)$ is then given by

$$\rho_{xx}(R,t)=\langle S(i,j)S(i+R,j+it/2K_2)\rangle, \qquad (6)$$

where the correlation function on the right-hand side of (6) is taken between two spins at different sites in a two-dimensional Ising model and for the limits $M,N\rightarrow\infty$, and $K_1\rightarrow\infty$, $K_2\rightarrow 0$ with $\exp(-2K_1)\sim K_2h$. Such correlation functions have been calculated for two spins in the same row[7] and for two spins in different rows and columns.[8] From these expressions and relation (6) we first recover the results of McCoy[3] [expression (3.15) of Ref. 3 for $h<1$ and (3.33) of Ref. 3 for $h>1$ where λ_1^{-1} and λ_2^{-1} are replaced by 0 and h].

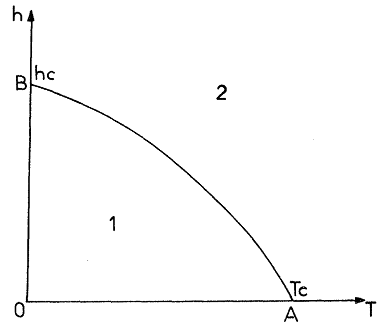

FIG. 1. Schematic phase diagram of an Ising model in a transverse field. (1) Ferromagnetic region; (2) Paramagnetic region.

At the critical field $h=h_c=1$, we expect as for the Onsager relation[5] at $T=T_c$, a qualitative change in the behavior of the correlation functions. From Eq. (6) and from the expression given by Wu[7] for the correlation function between two spins in the same row, we get for the autocorrelation function at $h=1$

$$\rho_{xx}(R,t)\sim e^{1/4}2^{1/12}A^{-3}(it)^{-1/4}, \quad t \text{ large.} \tag{7}$$

From the analysis of the behavior of $\rho_{xx}(R,t)$ in the limit $h\to 1$, we deduce the result for the correlation function at $h=1$

$$\rho_{xx}(R,t)\sim e^{1/4}2^{1/12}A^{-3}(R^2-t^2)^{-1/8}, \tag{8}$$

for R^2-t^2 large. For the Ising model with a transverse field ($\gamma=1$) S_i^Y is the commutator of H with S_i^x and $\rho_{yy}(R,t)$ is then simply given by

$$\rho_{yy}(R,t)=-\frac{1}{h^2}\frac{\partial^2}{\partial t^2}\rho_{xx}(R,t)\ . \tag{9}$$

The relation (9) can be checked for $h\neq 1$ by McCoy's results,[3] replacing λ_1^{-1} and λ_2^{-1} by 0 and h in expressions (3.33) and (4.2) of Ref. 3.

At $h=h_c=1$ the exact result given in Ref. 4:

$$\rho_{yy}(R,0)=\frac{-1}{4R^2-1}\rho_{xx}(R,0)\sim\frac{-1}{4R^2}\rho_{xx}(R,0), \quad R \text{ large} \tag{10}$$

is in fact a direct consequence of Eqs. (8) and (9).

The power-law behavior of the correlation functions for $\gamma=1$ and $h=1$ is related to the qualitative change in the excitation spectrum $\Lambda(\phi)$ when the gap disappears at $h=1$. The behavior of the space-time correlation function at $h=1$ as a function of R^2-t^2 is a consequence of the shape of the excitation spectrum which for small k is a linear function of the wave vector; so any elementary excitation propagates at the same speed.

The results presented here had been already used in the study of a two-dimensional Landau-Ginzburg-type model[9] where the two-dimensional classical problem can be reduced to the one-dimensional quantum Ising model with a transverse field.

We can also make the following remarks. Under the condition described above, there is an equivalence between the statistical mechanics of a d-dimensional Ising model and the space-time behavior of the $(d-1)$-dimensional Ising model in a transverse field at $T=0$. If we consider an Ising model with $d>1$ in a transverse field, we have in the h, T plane the schematic diagram shown in Fig. 1. Region 1 is ferromagnetically ordered while 2 is paramagnetic; the line connecting $A(T_c,0)$ to $B(0,h_c)$ is a line of continuous phase transition. From general considerations of phase change[10] we can deduce the following behavior of the correlation function.

At point A: Ising model: we have from the static correlation function:

$$\rho_{zz}^A(R,\infty)\sim\frac{1}{R^{d-2+\eta_d}}\ ; \tag{11}$$

at point B:

$$\rho_{zz}^B(R^2-t^2)\sim\frac{1}{(R^2-t^2)^{(d-1+\eta_{d+1}/2)}}, \tag{12}$$

with a space-time Fourier transformation of the form

$$\rho_{zz}^B(k^2-\omega^2)\sim\frac{1}{(k^2-\omega^2)^{2-\eta_{d+1}}}\ . \tag{13}$$

From these last expressions, we can easily extract

$$\rho_{zz}^B(R,0)\sim\frac{1}{R^{d-1+\eta_{d+1}}}, \tag{14}$$

$$\rho_{zz}^B(R,\infty)(\text{static correlation function})\sim\frac{1}{R^{d-2+\eta_{d+1}}}\ . \tag{15}$$

We note two results:

$$\rho_{zz}^B(R,0)<\rho_{zz}^B(R,\infty)\ , \tag{16}$$

and for the static correlation function,

$$\rho_{zz}^A(R)\leq\rho_{zz}^B(R)\ . \tag{17}$$

The equality holds for $d>4$ (logarithmic corrections are omitted).

Thus in this case we can expect that asymptotic behavior is the same everywhere on the line.

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