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Skeleton-graph approach to dynamical scaling

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We use the skeleton-graph ϵ-expansion method to discuss the critical dynamics of a Bose liquid in $d = 4 - \epsilon$ dimensions. The treatment is limited to the question of the behavior at the critical temperature of the frequency-dependent order-parameter correlation function (propagator) at zero momentum. We find that a power-law variation with frequency is only possible when the system acquires time-dependent Ginzburg-Landau behavior. Our analysis is incomplete in that the influence of collective modes on the critical behavior is not taken into account in detail. In an appendix, we present a powerful method for evaluating integrals associated with the Feynman graphs which arise in problems in critical phenomena.

Recently, several works[1-7] have appeared in which attempts are made to calculate the critical dynamics of the interacting Bose system. One of the important questions which has so far been unanswered is whether or not scaling is valid in the critical region.

In this note, we address ourselves to the problem of the structure of the frequency-dependent correlation function of the order parameter. Specifically, we look at the behavior of the propagator $G(k=0, \omega, T=T_c)$ at zero momentum and at the critical temperature. We ask the question: Under what circumstances is it possible for $G(\omega)$ to exhibit power-law behavior in ω? By restricting ourselves to $T=T_c$, we not only make the analysis tractable, we also hope thereby to minimize the effect of collective modes, which should be considered in addition to the fluctuations of the order parameter. Any complete discussion of the critical dynamics must take account of these collective modes and their hydrodynamic manifestations.[6,7]

We use a method and a model[8] similar to those which were successful in the analysis of static critical behavior near four dimensions. Again, we use a model microscopic Hamiltonian for liquid helium corresponding to a two-component spin field. The Hamiltonian is

$$\frac{H}{kT}=\int d^d x\,(r_0|\psi|^2+|\nabla\psi|^2+\tfrac{1}{2}u|\psi|^4)\;,$$

where $\psi(x)$ is the complex-order-parameter field and r_0 is the chemical potential. We work in $d=4-\epsilon$ dimensions and look for the dynamical exponent to the lowest nonvanishing order in ϵ.

We conjecture that at the critical temperature T_c the retarded propagator has the form

$$G^R(\vec{k},\omega)=\frac{1}{S\omega^{1-\rho}}f\left(\frac{k^{2-\eta}}{S\omega^{1-\rho}}\right), \tag{1}$$

where S is in general complex and may depend on ϵ. Here ρ is the dynamical exponent we seek. In our choice of propagator, we are guided by the following conditions: (a) The static limit is of the form $1/k^{2-\eta}$ and (b) the sign of the spectral function should change at $\omega=0$. We can then write, first for $k=0$,

$$G^R=1/S\omega^{1-\rho},\quad G^A=1/S^*\omega^{1-\rho}\quad(\omega>0)$$

$$G^R=1/S|\omega|^{1-\rho}e^{i\pi(1-\rho)}\quad G^A=1/S^*|\omega|^{1-\rho}e^{-i\pi(1-\rho)}\quad(\omega<0).$$

These forms correspond to the retarded and advanced continuations of the Matsubara Green's function

$$\begin{aligned}\mathcal{G}&=1/S(i\omega_n)^{1-\rho}\to G^R\quad(n>0)\,,\\ \mathcal{G}&=1/S^*(i\omega_n)^{1-\rho}\to G^A\quad(n<0)\,,\end{aligned} \tag{2}$$

where $\omega_n=2\pi nkT_c$ is the boson Matsubara frequency. The spectral function is $\mathcal{A}=(G^R-G^A)/2\pi i$ and is given by $(S=S_1+iS_2,\ S_2>0)$

$$\mathcal{A}(\omega)=-S_2\omega^{1-\rho}/\pi|S\omega^{1-\rho}|^2\quad(\omega>0)=+(S_2\cos\pi\rho-S_1\sin\pi\rho)|\omega|^{1-\rho}/\pi|S|\omega|^{1-\rho}e^{i\pi(1-\rho)}|^2\quad(\omega<0)\,.$$

We emphasize that if a power law is valid, both the real and imaginary parts of G must have the same power law. In what follows, we need only the propagator to the zeroth order in ϵ.[9] For finite k, it is given by

$$\begin{aligned}\mathcal{G}(\vec{k},\omega_n)&=[S(i\omega_n)-k^2]^{-1}\quad(n>0)\\ &=[S^*(i\omega_n)-k^2]^{-1}\quad(n<0)\,.\end{aligned} \tag{3}$$

In the above, the case $S_1=0,\ S_2\neq 0$ corresponds to the time-dependent Ginzburg-Landau model

(TDGL) discussed phenomenologically by Halperin, Hohenberg, and Ma.[10]

A further remark on our choice of propagator is in order. The quantity S arises from the contribution of intermediate states of high momenta and frequencies, which tend to dress the unperturbed propagator even when $\epsilon = 0$. That is, the skeleton-graph expansion that we shall use sums the effects of the critical fluctuations, and the introduction of S is designed to give some account of what is thereby left out. While we cannot compute S by the present method, we give evidence below that it may be driven to the TDGL form by the critical fluctuations as $\omega \to 0$.

We shall use the method of skeleton-graph expansion, details of which are given in Ref. 8. It is first necessary to find the ϵ expansion of $\Gamma(k_i = 0, \omega_i)$, which is the analytic continuation to the real frequency axes of the Matsubara four-point vertex for zero momentum at $T = T_c$. The arguments of Γ are the momenta and frequencies in the three pair channels of Γ.

We have remarked earlier that we are going to neglect the effects of collective modes throughout. We therefore assume that there are no singularities in the various vertex parts other than the critical ones arising from the pair intermediate states in the ϵ expansion (parquet expansion). Then, using the homogeneity form for G given in Eq. (1), we can show, from the structure of the skeleton-graph expansion,[11] that Γ must scale as ω^σ, where

$$\sigma = (\epsilon - 2\eta)(1-\rho)/(2-\eta) = \tfrac{1}{2}\epsilon + O(\epsilon^2) \,.$$

Therefore, to $O(\epsilon)$, just as in the static case,[8] $\Gamma = \Gamma_0 \epsilon$ and is independent of ω.

To actually find Γ_0, we construct a skeleton-graph ϵ expansion similar to that of the static case[8] by considering the derivatives $\partial\Gamma_i/\partial\omega_i$ of those parts of the continued four-point vertex that are reducible in the ith channel. As in Fig. 1, $i=1$ is the particle-particle channel, while $i = 2, 3$ correspond to particle-hole channels. The equations are constructed as in the static case where we considered $\partial\Gamma_i/\partial k_i$. We find

$$\frac{\partial\Gamma_1}{\partial\omega} = -\tfrac{1}{2}\Gamma_0^2\epsilon^2 \frac{\partial}{\partial\omega} \sum_k C \sum_m \mathcal{G}(\vec{k}, \omega_m)\mathcal{G}(\vec{k}, \omega_n - \omega_m) \,,$$

$$\frac{\partial\Gamma_2}{\partial\omega} = -\Gamma_0^2\epsilon^2 \frac{\partial}{\partial\omega} \sum_k C \sum_m \mathcal{G}(\vec{k}, \omega_m)\mathcal{G}(\vec{k}, \omega_n + \omega_m) \,,$$

where the operation C means the continuation $i\omega_n \to \omega + i\delta$. Here we have assumed that to $O(\epsilon)$ on the right-hand side it is allowed to take $\Gamma(\vec{k}, \omega) = \Gamma_0\epsilon$.[11]

It is only necessary to treat the Matsubara sums to $O(\epsilon^0)$. Therefore, we use the unperturbed propagators $(i\omega S - k^2)^{-1}$. We perform the sums and take the high-temperature limit $kT_c/\omega \gg 1$[12]:

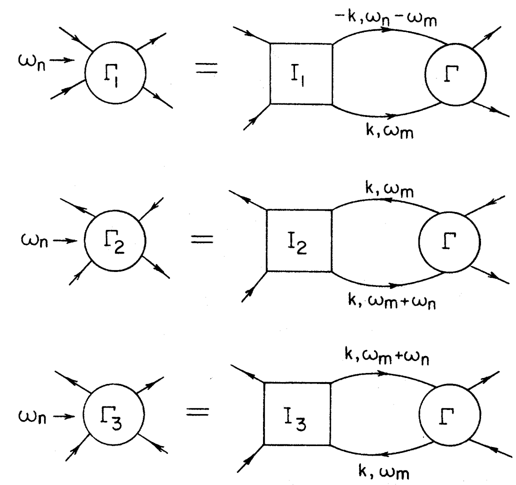

FIG. 1. Decomposition of four-point vertex Γ into its reducible parts.

$$\frac{\partial\Gamma_1}{\partial\omega} = \tfrac{1}{2}\Gamma_0^2\epsilon^2 \frac{\partial}{\partial\omega}\, \omega \sum_k \frac{1}{k^4}\, \frac{1}{\omega - 2k^2/S} \,,$$

$$\frac{\partial\Gamma_2}{\partial\omega} = \Gamma_0^2\epsilon^2 \frac{\partial}{\partial\omega}\, \omega \sum_k \frac{1}{k^4}\, \frac{1}{\omega + 2ik^2 S_2/|S|^2} \,.$$

The k sums are performed in four dimensions $[\sum_k = (\frac{1}{16}\pi^{-2})\int E\,dE,\ E = k^2]$. We find

$$\frac{\partial\Gamma_1}{\partial\omega} = -\frac{1}{16\pi^2}\Gamma_0^2\epsilon^2 \frac{1}{S}\int dE \frac{1}{(\omega - 2E/S)^2} = \frac{1}{32\pi^2}\,\frac{\Gamma_0^2\epsilon^2}{\omega} \,,$$

$$\frac{\partial\Gamma_2}{\partial\omega} = \frac{1}{16\pi^2}\Gamma_0^2\epsilon^2 \frac{2iS_2}{|S|^2}\int dE \frac{1}{(\omega + 2iS_2E/|S|^2)^2} = \frac{1}{16\pi^2}\,\frac{\Gamma_0^2\epsilon^2}{\omega} \,.$$

If we look at the full Γ with the same frequency ω in each channel, we have, just as in the static case,

$$\frac{\partial\Gamma}{\partial\omega} = \frac{\partial}{\partial\omega}(\Gamma_1 + \Gamma_2 + \Gamma_3) = \frac{5}{32\pi^2}\,\frac{\Gamma_0^2\epsilon^2}{\omega} \,.$$

Then, with $\Gamma \approx \Gamma_0\epsilon\omega^{\epsilon/2}$, we find $\Gamma_0 = \frac{16}{5}\pi^2$. Our result is independent of S, provided S has an imaginary part. Of course, we expect the self-energy to have an imaginary part; it already develops one in the second order of perturbation theory.

The procedure to determine the ω exponent of G is also the same as the one we used in the static case for the k exponent η. We construct the vertex $\Lambda = 1 - \partial\Sigma/\partial\omega$, where again the derivative is taken after analytic continuation of the self-energy Σ to the real ω axis. Actually, Λ is an object with two legs and its analytic structure is complicated in the general case; it is described in Appendix A.

We construct a skeleton-graph expansion for Λ by differentiating again with respect to ω. We find [compare Eq. (7) of Ref. 8]

$$\frac{\partial\Lambda}{\partial\omega}=-\frac{\partial}{\partial\omega}C\sum_{km}I_2(0,\vec{k};\omega_n,\omega_m)\mathcal{G}^2(\vec{k},\omega_m)\times\Lambda(\vec{k},\omega_m)\,. \quad (4)$$

Here I_2 is that part of Γ which is irreducible in channel 2; it contains Γ_1 and Γ_3. For the case at hand, where there is no momentum transfer, $\partial\Lambda/\partial\omega$ may be found to $O(\epsilon^2)$ from the two graphs shown in Fig. 2, where the solid dot is $\Gamma_0\epsilon$ and the solid triangle is the zeroth-order $\Lambda(\omega_m)=S_1+iS_2\times(\mathrm{sgn}\,m)$. We reserve the details of the integrals for Appendix B. The result is

$$\frac{\partial\Lambda}{\partial\omega}=-\frac{1}{\omega}3i\eta S_2F(S_1/S_2)\,, \quad (5)$$

where $\eta=\frac{1}{50}\epsilon^2$ is the k exponent of G and F is a complex function:

$$F(x)=F_1+iF_2=\left[\ln\tfrac{4}{3}-\tfrac{1}{2}\ln(1+\tfrac{1}{9}x^2)-\tfrac{1}{6}\ln(1+x^2)+\tfrac{1}{3}x\tan^{-1}\tfrac{1}{3}x+\tfrac{1}{3}x\tan^{-1}x\right]+i\left[-\tfrac{1}{3}x\ln\tfrac{4}{3}+\tfrac{1}{6}x\ln(1+\tfrac{1}{9}x^2)-\tfrac{1}{6}x\ln(1+x^2)+\tan^{-1}\tfrac{1}{3}x-\tfrac{1}{3}\tan^{-1}x\right]\,.$$

Now we can check the possibility of a power law for the propagator at $T=T_c$. Suppose $G^R(k=0,\omega)$ has the form $1/S\omega^{1-\rho}$. Then $\Lambda=(1-\rho)S\omega^{-\rho}$ and

$$\frac{\partial\Lambda}{\partial\omega}=\rho(1-\rho)S\omega^{-(1+\rho)}\cong-\rho S/\omega. \quad (6)$$

Since we have found $\partial\Lambda/\partial\omega=O(\epsilon^2)$, we see that $\rho=O(\epsilon^2)$ and $\Lambda=S+O(\epsilon^2)$. To find ρ, we separate the real and imaginary parts of Eq. (5):

$$\frac{\partial\Lambda_2}{\partial\omega}=-\frac{3\eta S_2}{\omega}F_1\left(\frac{S_1}{S_2}\right)+O(\epsilon^3)\,,$$
$$\frac{\partial\Lambda_1}{\partial\omega}=\frac{3\eta S_2}{\omega}F_2\left(\frac{S_1}{S_2}\right)+O(\epsilon^3)\,. \quad (7)$$

We therefore find power-law behavior only when

$$-S_1F_1(S_1/S_2)=S_2F_2(S_1/S_2)\,,$$

since Λ_1 and Λ_2 must have the same exponent. This condition is only satisfied when $S_1=0$ in which case, by comparing Eqs. (6, 7) we find

$$\rho=3\eta\ln\tfrac{4}{3}\,.$$

In this way, we find a power-law behavior for $G(k=0,\omega)$ at T_c only when the propagator has the TDGL form and in that case the exponent is the same as that of Ref. 10. Whether or not the TDGL form is appropriate in the critical region is not answered by this calculation. However, we can give an argument that once S develops an imaginary part S_2, the order-parameter fluctuations drive the propagator to the TDGL form ($S\rightarrow iS_2$) as the frequency goes to zero. To see this, write Λ in the form $\Lambda=Pe^{i\theta}$, where P and θ are real functions of ω. TDGL corresponds to $\theta=\frac{1}{2}\pi$. From Eq. (5), we see that

$$\frac{\partial\Lambda}{\partial\omega}=\Lambda'=(P'+i\theta'P)e^{i\theta}=-3i\eta(P\sin\theta/\omega)F(1/\tan\theta)\,,$$

where, since the right-hand side is $O(\epsilon^2)$ already because of η, we may take the zeroth-order values of P and θ: $P=|S|$, $\theta=\tan^{-1}(S_2/S_1)$. It follows that $\theta(\omega)$ varies according to

$$\frac{\partial\theta}{\partial\omega}=-\frac{3\eta}{\omega}\frac{xF_1(x)+F_2(x)}{1+x^2}\quad(x=1/\tan\theta). \quad (8)$$

When θ is small, we find that it increases as ω decreases according to $\partial\theta/\partial\omega=-\pi\eta/\omega$, but that as it approaches $\frac{1}{2}\pi$, $\partial\theta/\partial\omega\rightarrow 0$. This behavior gives support to the adoption of the TDGL form for the propagator in the infrared limit.[13] This behavior of S ($S\rightarrow i$ as $\omega\rightarrow 0$) may be thought of as describing an overdamped quasiparticle. From Eq. (8) we can calculate the ω exponent of θ as it approaches $\frac{1}{2}\pi$. We find $\frac{1}{2}\pi-\theta\propto\omega^{\lambda}$, $\lambda=2\eta\ln\frac{4}{3}$.

Our calculation is internally self-consistent. However, the results are obtained without accounting for any structure in the vertex parts which might be due to collective modes which will in general produce additional singularities which we have ignored. The inclusion of the collective modes is necessary in any model which is supposed to describe the critical dynamics of helium. A general microscopic treatment of the hydrodynamic modes is not available. (See, however, Ref. 7, where a treatment for the spherical model is carried out.) A possible way of including them which is suited to our formulation has been outlined by Polyakov[1]: The hydrodynamic mode is introduced via an extra field which satisfies hydrodynamic equations and it is coupled to the particle field in a way which satisfies Ward identities. In the entirely phenomenological models,[6,17] both the order parameter and hydrodynamic fields are introduced at the outset and coupled to each other in a variety of ways.

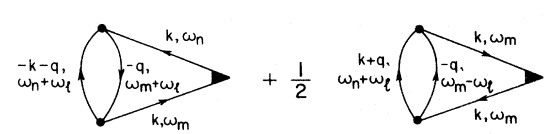

FIG. 2. Graphs for the calculation of $\partial\Lambda/\partial\omega$.

We acknowledge conversations on the subject of this paper with J. Sak and A. Zawadowski. A part of this work was carried out while one of us (T.T.) was a visitor at the University of Illinois. His sincere thanks are due Professor J. Bardeen for his hospitality.

APPENDIX A

We give a brief elucidation of some of the properties of the vertex function $\Lambda(\vec{k}, \vec{q};\ \omega, \nu)$ which is the amputated vertex having two legs and one corner.[14] The incoming momentum and frequency are $\vec{k}$ and ω, while the momentum-frequency transfer at the corner is $\vec{q}, \nu$. The Matsubara frequencies ω and ν are integer multiples of $2\pi kT$. This vertex gives the response to a scalar field $\phi(\vec{q}, t)$ which couples to the density fluctuation of wave vector $\vec{q}$. In what follows, ϕ is understood to contain the density fluctuation operator. In fact, Λ is the amputated Fourier transform of the time-ordered amplitude $L = \langle T\psi(t_1)\phi(t_2)\psi^\dagger(t_3)\rangle$. It has a spectral decomposition by means of which we may study its analytic continuation into the complex frequency space $i\omega \to z,\ i\omega + i\nu \to z'$:

$$\Lambda(\vec{k}, \vec{q};\ z, z') = \iint_{-\infty}^{+\infty} dx_1\, dx_2 \left(\frac{Q_1(\vec{k}, \vec{q}, x_1, x_2)}{(x_1 - z' + z)(x_2 - z)} + \frac{Q_2(\vec{k}, \vec{q}, x_1, x_2)}{(x_1 - z' + z)(x_2 - z')} \right).$$

We see that Λ has cuts when $\mathrm{Im}\, z = 0$ and $\mathrm{Im}(z - z') = 0$. This yields six regions in which Λ is a separate analytic function according to the signs of the imaginary parts of the three relevant frequency variables.[14c]

The spectral representation of Λ simplifies considerably when $q = 0$, the case discussed in the text. In that case, by means of a study of the matrix elements entering L, it can be shown that

$$Q_1(x_1, x_2) = -Q_2(x_1, x_2) = \delta(x_1)\tilde{Q}(x_2),\ \mathrm{Im}\tilde{Q} = 0\ .$$

Then it follows that

$$\Lambda(\vec{k}, q = 0; z, z') = \int_{-\infty}^{+\infty} dy\, \frac{\tilde{Q}(\vec{k}, y)}{(y - z)(y - z')} \qquad \text{(A1)}$$

and there are four regions of analyticity for Λ. Thus as $z \to \omega \pm i\delta,\ z' \to \omega' \pm i\delta$, we can define the four functions Λ^{RR}, Λ^{RA}, Λ^{AR}, Λ^{AA}, where, for example, Λ^{RA} means $z \to \omega + i\delta,\ z' \to \omega' - i\delta$.

We may determine the form of $\tilde{Q}(k = 0, y)$ from the assumed form of the propagator as follows. We use the Ward identity[15]

$$\Lambda^{RR}(0, 0; \omega, \omega) = \frac{\partial[G^R(k = 0, \omega)]^{-1}}{\partial\omega}$$

and Eq. (A1) to deduce

$$\mathcal{G}^{-1}(k = 0, z) = \int dy \frac{\tilde{Q}(y)}{y - z}\ .$$

The form of $\mathcal{G}^{-1}$ is obtained from Eq. (2). It then follows that

$$\tilde{Q}(y) = [1/G^R(y) - 1/G^A(y)]/2\pi i = S_2 y^{1-\rho}/\pi, \quad y > 0$$
$$= -(S_2 \cos\pi\rho - S_1 \sin\pi\rho)|y|^{1-\rho}/\pi, \quad y < 0\ .$$

The four functions $\Lambda^{R(A)R(A)}$ are now determined from Eq. (A1). The result of the integral in Eq. (A1) is

$$\Lambda(z, z') = \frac{1}{\sin\pi\rho}\left[S_2 e^{i\pi\rho} \left(\frac{z^{1-\rho} - z'^{1-\rho}}{z - z'} \right)_A - (S_2 \cos\pi\rho - S_1 \sin\pi\rho) \left(\frac{z^{1-\rho} - z'^{1-\rho}}{z - z'} \right)_B \right]. \qquad \text{(A2)}$$

Here, the subscripts A, B signify that the cuts which define the phases z, z' in the large parentheses are to be taken along the positive real axis for A and along the negative real axis for B.

For the calculation in the text, the zeroth order in ϵ ($\rho = 0$) values of Λ are required. These are, for $z \to \omega \pm i\delta,\ z' \to \omega' \pm i\delta$,

$$\Lambda^{RR}(\omega, \omega') = \Lambda^{AA*}(\omega, \omega') = S\ ,$$
$$\Lambda^{AR}(\omega, \omega') = \Lambda^{RA*}(\omega, \omega') = S - 2iS_2 \frac{\omega}{\omega - \omega' - i\delta}\ .$$

APPENDIX B

Here, we show how to perform the integrals required in the evaluation of Eq. (4). The method we use is quite powerful for many calculations involving the evaluation of Feynman graphs in the many-body problem. We are going to calculate the graphs of Fig. 2. The internal propagators are $O(\epsilon^0)$ as given in Eq. (3). We therefore have to evaluate the expression

$$\frac{\partial\Lambda}{\partial\omega} = \epsilon^2\Gamma_0^2 \frac{\partial}{\partial\omega} C \sum_{k,q} \sum_{m,l} \mathcal{G}^2(\vec{k}, \omega_m)\Lambda_m\, \mathcal{G}(\vec{k} + \vec{q},\ \omega_n + \omega_l) \times [\mathcal{G}(\vec{q},\ \omega_m + \omega_l) + \tfrac{1}{2}\mathcal{G}(\vec{q},\ \omega_m - \omega_l)], \qquad \text{(B1)}$$

where, as before, C denotes an analytic continuation of ω_n to the real axis. As indicated in the text and at the end of Appendix A, Λ_m, the zeroth-order vertex part, is simply $S_1 + iS_2\, \mathrm{sgn}(m)$. In writing Eq. (B1), we have actually omitted some contributions involving regions in which the two legs entering Λ_m have frequencies on opposite sides of the real axis. These terms would couple to Λ^{AR} and Λ^{RA} (cf. Appendix A). These contributions all cancel in the present case of zero momentum transfer at the vertex.

The first step is the calculation of the frequency sums. The convenient way to proceed is to replace each by its spectral representation

$$\mathcal{G}(\vec{k}, z) = \int dx\, \mathcal{A}(\vec{k}, x)/(x - z).$$

Here[9]

$$\mathcal{A}(\vec{k}, x) = [G^R(\vec{k}, x) - G^A(\vec{k}, x)]/2\pi i = -S_2 x/\pi \left| S(x - U_k)\right|^2, \tag{B2}$$

where $U_k = k^2/S^*$. We obtain

$$\frac{\partial \Lambda}{\partial \omega} = \epsilon^2 \Gamma_0^2 \frac{\partial}{\partial \omega} \int \prod_{i=1}^{4} dx_i \sum_{k,q} \mathcal{A}(\vec{k}, x_1)\, \mathcal{A}(\vec{k}, x_2)\, \mathcal{A}(\vec{k}+\vec{q}, x_3)\, \mathcal{A}(\vec{q}, x_4)$$

$$\times C \sum_{ml} \frac{1}{x_1 - i\omega_m} \frac{1}{x_2 - i\omega_m} \frac{1}{x_3 - i\omega_n - i\omega_l} \Lambda_m \left(\frac{1}{x_4 - i\omega_m - i\omega_l} + \frac{1}{2} \frac{1}{x_4 - i\omega_m + i\omega_l} \right). \tag{B3}$$

The leading term of the frequency sum on l is obtained by expanding the summand in a series of single partial fractions and setting the total Matsubara frequency in each equal to zero.[16] The remaining sum on m may be written

$$\frac{1}{x_4 x_3} \frac{1}{4\pi i} \int_C dy \coth\frac{y}{2} \frac{1}{(x_1 - y)(x_2 - y)} \Lambda_y \left(\frac{x_4 - x_3}{x_4 - x_3 - y + i\omega_n} + \frac{1}{2} \frac{x_4 + x_3}{x_4 + x_3 - y - i\omega_n} \right), \tag{B4}$$

where, in the upper-half complex y plane $\Lambda_y = S$, and $\Lambda_y = S^*$ in the lower-half. [The contour C reduces to pieces above and below the various cuts in the integrand (see Fig. 3).] We perform the y integral in Eq. (B4) and the integrals over the x_i in Eq. (B3) by contours, using Eq. (B2) for the spectral functions $\mathcal{A}$. The result is

$$\frac{\partial \Lambda}{\partial \omega} = \epsilon^2 \Gamma_0^2 \frac{\partial}{\partial \omega} C \sum_{k,q} \frac{1}{k^2 (\vec{k}+\vec{q})^2 q^2} \left(\frac{U_k^* + U_{k+q}^* - U_q}{(i\omega - U_{k+q}^* + U_q - U_k^*)^2} + \frac{1}{2} \frac{U_k - U_{k+q}^* - U_q^*}{(i\omega - U_{k+q}^* - U_q^* + U_k)^2} \right), \tag{B5}$$

where, we recall, $U_k = k^2/S^*$.

It is now convenient to calculate the remaining integrals in configuration space. This is accomplished by the transformation

$$\sum_{\vec{k},\vec{q}} f(k, q, |\vec{k}+\vec{q}|) = (2\pi)^4 \sum_{\vec{k}_1\vec{k}_2\vec{k}_3} f(k_1 k_2 k_3)\, \delta^4(\vec{k}_1 + \vec{k}_2 + \vec{k}_3)$$

$$= \int d^4 r \sum_{\vec{k}_1\vec{k}_2\vec{k}_3} f(k_1 k_2 k_3)\, e^{i(\vec{k}_1+\vec{k}_2+\vec{k}_3)\cdot\vec{r}}. \tag{B6}$$

In the present case, $f(k_1 k_2 k_3)$ is given by $(n > 0)$

$$f(k_1 k_2 k_3) = \frac{i}{2} \int_0^\infty dt\, (1 - \omega_n t)\, e^{i(i\omega_n - W)t}, \tag{B7}$$

where $W = U_3^* - U_2 + U_1^*$, $\operatorname{Im} W > 0$.

We then have, from Eqs. (B5)–(B7),

$$\frac{\partial \Lambda}{\partial \omega} = \frac{i}{2} \epsilon^2 \Gamma_0^2 \frac{\partial}{\partial \omega} C \int d^4 r \int_0^\infty dt\, e^{-\omega_n t} (1 - \omega_n t) (L^*)^2 L, \tag{B8}$$

where

$$L^*(r, t) = \sum_k \frac{1}{k^2} e^{i\vec{k}\cdot\vec{r}} e^{-i(k^2/S)t}$$

$$= \frac{1}{4\pi^2 r^2} (1 - e^{iSr^2/4t}). \tag{B9}$$

We use Eq. (B9) in Eq. (B8) and the value $\Gamma_0 = \frac{16}{5}\pi^2$ from the text. We find

$$\frac{\partial \Lambda}{\partial \omega} = 8 i \eta \frac{\partial}{\partial \omega} C \int_0^\infty dt \int_0^\infty \frac{1}{r^3} dr\, e^{-\omega_n t}$$

$$\times (1 - \omega_n t)(1 - e^{iSr^2/4t})^2 (1 - e^{-iS^* r^2/4t}),$$

where $\eta = \frac{1}{50}\epsilon^2$ is the k exponent of the propagator. The r integral is performed first. The result is

$$\frac{\partial \Lambda}{\partial \omega} = 3 i \eta\, S_2 F\left(\frac{S_1}{S_2}\right) \frac{\partial}{\partial \omega} C \int_\tau^\infty dt\, e^{-\omega_n t} (1 - \omega_n)/t, \tag{B10}$$

where F is the complex function defined in connection with Eq. (5) of the text. We have replaced the lower limit of the t integral by τ, which corresponds to the high momentum cutoff k_c, that was implicit in all the original integrals. Here $\tau = 0(|S|/k_c^2)$. However, as in the static-skeleton-graph approach,[7] the derivative $\partial/\partial\omega$ eventually eliminates the cut-

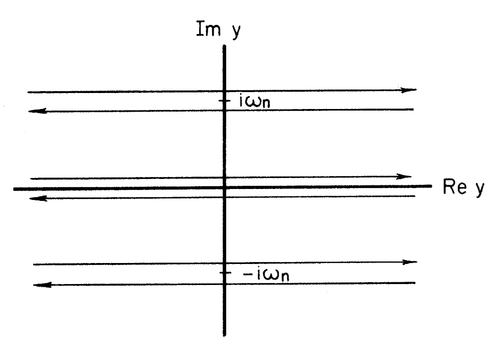

FIG. 3. Contour for the integration of (B4).

off. The integral in Eq. (B10) is approximated by its most divergent term: $-\ln|\tau\omega_n| + i(\frac{1}{2}\pi)\,\mathrm{sgn}(\omega_n)$. We can now perform the continuation operation C, and for $i\omega_n \to \omega + i\delta$, we find

$$\frac{\partial\Lambda}{\partial\omega} = -3i\eta S_2 F\left(\frac{S_1}{S_2}\right)\frac{1}{\omega + i\delta},$$

a result which appears in Eq. (5) of the text.

Note added in proof. The phase of the inverse TDGL propagator in the critical region is actually $(\frac{1}{2})\pi(1+\rho)$, not $(\frac{1}{2})\pi$. In the leading term of the ϵ expansion [Eq. (6)] this ϵ^2 correction is lost. However, it is easy to see that if one generates the skeleton-graph expansion with a Matsubara propagator which is real on the imaginary axis, such as that of Eq. (3) with $S=i$, then the resulting propagator must have the same property. We thank C. DeDominicis and P. Nozières for discussion on this point.

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[9]In the calculation which follows, it is unnecessary to normalize the propagator spectral function, as the normalization cancels in the expression for the dynamical exponent.

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[11]The argument is identical to that for the static case as described in Ref. 8.

[12]In the calculation of Matsubara sums, the statistical factor $\coth(\omega/2kT)$ is always replaced by $2kT_c/\omega$.

[13]A. Zawadowski (private communication) has used a renormalization-group argument to show that the critical fluctuations drive the phase of S to $\frac{1}{2}\pi$ in the region $T=T_c$, $k_c^2 > k^2 > |S|\omega$ as $k^2 \to |S|\omega$.

[14]Discussions of a similar nature may be found in the following: (a) V. Ambegaokar and L. Tewordt, Phys. Rev. 134, A805 (1964); (b) A. A. Abrikosov, L. P. Gor'kov, and I. Ye. Dzyaloshinskii, *Quantum Field Theoretical Methods in Statistical Physics*, 2nd ed. (Pergamon, New York, 1968), Chap. 8; (c) Ref. 7, Appendix A.

[15]See, for example, Ref. 14(b), Chap. 4.

[16]This procedure is equivalent to calculating the full sum and then using the approximation mentioned earlier (Ref. 12).

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