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General gauge invariance and spin waves in the B phase of superfluid ^{3}He

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Assuming that the B phase of superfluid ^{3}He is described in terms of a spherical triplet state of Balian and Werthamer, the Goldstone bosons associated with the general gauge transformation of the B phase are studied theoretically. Particular attention is paid to spin waves, which are the Goldstone bosons associated with (partially broken) gauge invariance of the original Hamiltonian against the rotation of the spin space. Spin-wave dispersions in the hydrodynamic regime of the B phase are determined.

I. INTRODUCTION

We have shown recently that spin waves exist in the A phase of superfluid ^{3}He.[1,2] These modes can be considered as Goldstone bosons associated with the rotation of the spin space, since the condensate of the A phase, being triplet, breaks the rotational invariance in the spin space in the original Hamiltonian. (Strictly speaking the above invariance is partially broken due to the dipole interaction energy and the resultant spin-wave spectrum acquires a finite energy gap.) A recent experiment on the ringing of the magnetization in the B phase by Webb *et al.*[3] has definitively shown that the condensate of the B phase comprises triplet pairs. Therefore we expect, from the generalized gauge invariance of the theory, the existence of spin waves in the B phase.

This work is devoted to a study of the collective modes (or Goldstone bosons) in the B phase of superfluid ^{3}He. We will pay a particular attention to spin fluctuations, which reflect in details the character of the triplet condensate. We assume here that the condensate of the B phase is described in terms of the spherical triplet state proposed by Balian and Werthamer (BW).[4] Furthermore we limit ourselves to the hydrodynamic regime, which seems to be most likely the case, if we are interested in the low-energy fluctuations (say $\omega\sim 10^5$–10^6 Hz).

In the absence of the dipole energy, the ground-state energy of the BW state is invariant against the following nonAbelian gauge transformations of the triplet order parameter $\vec{\Delta}(\Omega)$:

$$\vec{\Delta}(\Omega)\rightarrow e^{i\phi}R_\sigma(\alpha,\beta,\gamma)R(\alpha',\beta',\gamma')\vec{\Delta}(\Omega), \qquad (1)$$

where ϕ describes the change of the phase of the order parameter, $R\sigma$ the rotation of the spin space, and R the rotation of the coordinate space. Here (α,β,γ) and (α',β',γ') are Eulerian angles describing the rotations of the spin space and the coordinate space, respectively. Therefore the BW state admits three different classes of Goldstone bosons: the zero sound, the spin waves, and the orbital rotational waves. A local phase transformation can be described in terms of phonons or sound waves (i.e., the zero sound) in the superfluid ^{3}He. In a neutral superconductor (with singlet condensate) the corresponding mode is known as the Anderson-Bogoliubov mode.[5] A local rotation of the spin space generates spin waves in the BW state. We will study in details these modes in the following. Finally a local rotation of the coordinate space (i.e., the orbital axis of the condensate) generates the orbital modes, which has been considered by de Gennes[6] previously in the case of the A phase.

We will consider in the following the situation where the orbital part of the condensate is fixed in space, for example, due to the pinning at the wall of a container, and concentrate on a local rotation of the spin coordinate. A local gauge transformation which describes rotation in the spin space introduces a perturbation Hamiltonian, which contains both the linear and the bilinear terms in the derivative of the local Eulerian angles $[\alpha(\vec{r},t), \beta(\vec{r},t), \gamma(\vec{r},t)]$.

A similar transformation has been considered recently by Combescot,[7] in his study of the spin waves in the B phase. The shift of the free energy due to this perturbation is easily calculated. This additional free energy (which is now bilinear in the derivative of the local Eulerian angles) together with the dipole interaction energy determine completely the spin fluctuations in the BW state. In the linear regime and in the absence of the dipole interaction energy, the present result agrees completely with a recent calculation of the spin wave in the hydrodynamic regime by Combescot.[7] In the presence of the dipole interaction energy, the ground state is no longer degenerate against separate rotations of the spin space and the orbital space, although it is still degenerate against a simultaneous rotation of two spaces.[8] When the gauge variables are separable, as in the case of the A phase,[2] the dipole energy introduces simply an energy gap to the spin-wave dispersion equal to the nuclear-magnetic-resonance energy. In general, however, the dipole interaction energy introduces the coupling between different modes and therefore alters the spin dynamics completely. As already pointed out, the dipole interaction energy is not adequate to determine the equilibrium spin-orbital configuration of the B phase.[9] However, in the case of a cylindrical vessel with the symmetry axis parallel to external magnetic fields,[3] two distinct configurations are very likely to be realized; the Leggett configuration and the wall-fixed configuration in the low-field region.[9] Therefore, we will study in some details the spin waves in these particular configurations. In the Leggett configuration the spin wave with longitudinal spin polarization (i.e., $\delta\vec{S} \parallel \vec{H}_0$, where H_0 is a static magnetic field) has rather simple dispersion:

$$\omega^2 = \frac{v_F^2}{3}\frac{N(0)}{\chi_B}\frac{\vec{q}\,\vec{\rho}_s\vec{q}}{\rho} + \Omega_l^2, \tag{2}$$

where

$$(\rho_s)_{xx}/\rho = (\rho_s)_{yy}/\rho = \tfrac{4}{5}(\rho_s/\rho)_{\rm BCS},$$
$$(\rho_s)_{zz}/\rho = \tfrac{2}{5}(\rho_s/\rho)_{\rm BCS}, \tag{3}$$
$$(\rho_s)_{xy} = (\rho_x)_{yz} = (\rho_x)_{zx} = 0,$$

and Ω is the Leggett longitudinal resonance frequency in the B phase,[8] χ_B is the static susceptibility in the BW state, and $N(0)$ is the density of states at the Fermi level. In the absence of the dipole energy (i.e., $\Omega_l = 0$), the above dispersion reduces to the one obtained by Combescot.[7] In the wall-fixed geometry, on the other hand, one of the transverse spin waves has a similar dispersion. However, in general, the spin-wave dispersions are more complex.

II. FORMULATION

We will start with a model Hamiltonian[10]

$$H = H_0 + H_L + H_I + H_d, \tag{4}$$

where H_0 describes both the kinetic energy and the BCS-like pairing interaction energy, H_L is the Larmor energy in the presence of a magnetic field along the z axis, H_I is the spin exchange interaction, and finally H_d is the dipole interaction energy between nuclear spins of ^{3}He atoms. In the absence of the dipole interaction energy the above Hamiltonian is invariant against separate rotations of the spin space and the coordinate space. In the superfluid phases of liquid ^{3}He, where the condensates comprise the triplet pairs, the ground state is transformed into a different state by a rotation of spin space or the coordinate (i.e., orbital) space. Thus the original symmetry of the Hamiltonian is broken in the superfluid state. This symmetry is formally restored by the existence of the collective modes (i.e., the Goldstone bosons) which have gapless energy spectra. Therefore it is quite natural to expect the existence of both spin waves and orbital rotational waves in both the A phase and the B phase of superfluid ^{3}He. Until now we neglected the dipole interaction energy. In the presence of the dipole interaction energy, the original Hamiltonian is no longer invariant against separate rotations of the spin and the orbital space. However, the dipole interaction energy does not inhibit the existence of the Goldstone bosons associated with these rotations, but introduces finite energy gaps in the energy spectra of these bosons. In the following we will be concerned with the transformation properties of the Hamiltonian associated with the local rotation of the spin space only. The general transformation describing the above rotation is generated by R (we dropped here the suffix σ indicating the rotation of the spin space).

$$\psi_\mu(\vec{r}) = R_{\mu\nu}\phi_\nu(\vec{r}),$$
$$R = e^{-i\alpha\sigma^z/2}e^{-i\beta\sigma^y/2}e^{-i\gamma\sigma^z/2} \tag{5}$$

where $\psi_\mu(\mathbf{r})$ is the field operator of ^{3}He atoms, α, β, γ, are the local Eulerian angles, and the σ^i are the Pauli spin matrices. Furthermore, repeated suffixes imply the summation over these suffixes. A similar transformation has been considered by Combescot[7] previously (in the linear regime). The transformation (5) introduces additional terms in the Hamiltonian (5)

$$H \to H + \Delta H,$$

where

$$\Delta H = H_1 + H_2, \tag{6}$$

$$H_1 = \frac{1}{2}\int \psi_\mu^\dagger(\vec{r})(\vec{\sigma}_{\mu\nu}\cdot\vec{\omega})\chi_\nu(\vec{r})\,d^3r$$

$$\begin{aligned} H_z = \frac{1}{4m}\int [&\vec{\nabla}\psi_\mu^\dagger(\vec{r})(\vec{A}\vec{\sigma}_{\mu\nu})\psi_\nu(\vec{r}) - \psi_\mu^\dagger(\vec{r})(\vec{\sigma}_{\mu\nu}\vec{A}^t)\vec{\nabla}\psi_\nu(\vec{r}) \\ &+ \tfrac{1}{2}\psi_\mu^\dagger(\vec{r})(\vec{\sigma}_{\mu\lambda}\vec{A}^t)(\vec{A}\,\vec{\sigma}_{\lambda\nu})\psi_\nu(\vec{r})]\,d^3r \\ = j_{s_i}^l&(\vec{r})A_i^l\,d^3r, \end{aligned} \tag{7}$$

$$\omega_1 = -\sin\alpha\,\dot{\beta} + \cos\alpha\,\sin\beta\,\dot{\gamma},$$
$$\omega_2 = \cos\alpha\,\dot{\beta} + \sin\alpha\,\sin\beta\,\dot{\gamma}, \tag{8}$$
$$\omega_3 = \dot{\alpha} + \cos\beta\,\dot{\gamma},$$

$$A_1^l = -\sin\alpha(\beta_l) + \cos\alpha\,\sin\beta(\gamma_l),$$
$$A_2^l = \cos\alpha(\beta_l) + \sin\alpha\,\sin\beta(\gamma_l), \tag{9}$$
$$A_3^l = \alpha_l + \cos\beta(\gamma_l),$$

and $\vec{j}_{s_i}(\vec{r})$ is the spin-current operator:

$$\vec{j}_{s_i} = (1/4m)[\vec{\nabla}\psi_\mu^\dagger(\vec{r})\sigma_{\mu\nu}^i\psi_\nu(\vec{r}) - \psi_\mu^\dagger(\vec{r})\sigma_{\mu\nu}^i\vec{\nabla}\psi_\nu(\vec{r}) + \tfrac{1}{2}\psi_\mu^\dagger(\vec{r})\vec{A}_i\psi_\mu(\vec{r})], \tag{10}$$

and

$$\alpha_l=\frac{\partial}{\partial x_l}\alpha\,,\quad \beta_l=\frac{\partial}{\partial x_l}\beta\,,\quad\text{and}\quad \gamma_l=\frac{\partial}{\partial x_l}\gamma\,, \tag{11}$$

and a dot implies the time derivative.

Equation (7) tells us that the time variation of the Eulerian angles introduces spin polarization (i.e., magnetization) and the spatial inhomogeneity of the Eulerian angle generates spin current.

We want now to calculate the change of the free energy due to spatial-temporal variation of the Eulerian angles describing rotation of the spin space. Since the spin polarization and the spin current in the equilibrium configuration in the absence of an external field vanishes identically, we will calculate the correction to the free energy by perturbation. The change in the free energy in the hydrodynamic limit is formally obtained as (see Appendix A)

$$\begin{aligned}\Delta F&=\langle\Delta H\rangle\\&=\tfrac{1}{2}\chi_{ij}\int d^3r[\omega_i(\vec{r},t)\,\omega_j(\vec{r},t)]\\&\quad+\tfrac{1}{2}(\chi^J)^{lm}_{ij}\int d^3r\,A^l_i(\vec{r},t)A^m_j(\vec{r},t)\,,\end{aligned} \tag{12}$$

where $\chi_{ij}=\langle[s_i\,,s_j]\rangle$ is the static spin susceptibility, and $(\chi^J)^{lm}_{ij}=\langle[j^l_{si}\,,j^m_{sj}]\rangle$ is the static spin-current correlation function. In the above derivation we neglected the spin–spin-current correlation function $\langle[j^l_{si}\,,s_j]\rangle$, which vanishes in the static limit. In the BW state (and in the weak-coupling theory) the static susceptibility and the spin-current correlation function (i.e., the superfluid density associated with the spin current) are given by (see Appendix B)

$$\chi_{ij}=\chi_B\delta_{ij}\,,\quad \chi_B=N(0)[1-\tfrac{1}{3}Y(T)]/\{1-\bar{I}\,[1-\tfrac{1}{3}Y(T)]\}, \tag{13}$$

$$(\chi^J)^{lm}_{ij}=(N/m)Y(T)(\delta_{lm}\delta_{ij}-3\,\langle k_lk_mf_if_j\rangle)\,, \tag{14}$$

where

$$Y(T)=2\pi T\sum_{n=0}^{\infty}\frac{\Delta^2}{(\omega_n^2+\Delta^2)^{3/2}}=\left(\frac{\rho_s}{\rho}\right)_{\rm BCS}\,, \tag{15}$$

$\bar{I}=IN(0)$, N is the density of ^{3}He atoms, and m is the mass of ^{3}He atoms. The f_i are obtained from k_i by

$$(\vec{\sigma}\cdot\vec{f}\,)=R(\vec{\sigma}\cdot\vec{k}\,)R^{-1}\,, \tag{16}$$

$\vec{k}$ is the unit vector in the direction of the quasiparticle momentum $\vec{p}$ and finally $\langle A\rangle$ means the average of A over the direction of $\vec{p}$. In Eq. (10) we have taken into account the random-phase-approximation (RPA) correction to the spin susceptibility due to the spin-exchange interaction.[11]

In the absence of the dipole interaction energy, the free energy (12) completely determines spin fluctuations in the BW state. In the presence of the dipole energy a local rotation of the spin space (but with the fixed orbital space) introduces an additional energy (i.e., potential energy). Assuming that we start the rotation from the special BW state with $J=0$ (i.e., the BW state with the spin and the orbital components are parallel to each other),[4] the dipole interaction energy for the present configuration is given by (Appendix C)

$$\langle H_d\rangle=\int d^3r\,E_d(\alpha(\vec{r},\,t),\;\beta(\vec{r},\,t),\;\gamma(\vec{r},\,t))\,, \tag{17}$$

and

$$E_d(\alpha,\,\beta,\,\gamma)=\tfrac{2}{15}\chi_B\Omega_l^2(\{(1+\cos\beta)[1+\cos(\alpha+\gamma)]-\tfrac{3}{2}\}^2-\tfrac{5}{4})\,, \tag{18}$$

where Ω_l is the longitudinal resonance frequency of the B phase.[8] Combining Eq. (12) with Eq. (17), we obtain the effective Hamiltonian which describes the rotation of spin space (i.e., spin components of the condensate) in the hydrodynamic regime:

$$\begin{aligned}H_{\rm eff}&=\int d^3r\left(\frac{1}{2}\chi_B(\vec{\omega})^2+\frac{N}{2m}Y(T)(A^k_iA^k_i-3\,\langle k_lk_mf_if_j\rangle A^l_iA^m_j)+E_d(\alpha,\,\beta,\,\gamma)\right)\\&=\int d^3r\left(\frac{1}{2}\chi_B(\dot\alpha^2+\dot\beta^2+\dot\gamma^2+2\dot\alpha\dot\gamma\cos\beta)+\frac{2N}{5m}Y(T)\{(\vec\nabla\alpha)^2+(\vec\nabla\beta)^2+(\vec\nabla\gamma)^2+2(\vec\nabla\alpha)(\vec\nabla\gamma)\cos\beta\right.\\&\quad\left.-\tfrac{1}{2}[(\tilde{A}^i_i)^2+(\tilde{A}^j_i\tilde{A}^i_j)]\}+E_d(\alpha,\,\beta,\,\gamma)\right),\end{aligned} \tag{19}$$

where

$$\begin{aligned}\tilde{A}^l_1&=-\alpha_l\sin\beta\cos\gamma+\beta_l\sin\gamma\,,\\\tilde{A}^l_2&=\alpha_l\sin\beta\sin\gamma+\beta_l\cos\gamma\,,\\\tilde{A}^l_3&=\gamma_l+\alpha_l\cos\beta\,.\end{aligned} \tag{20}$$

The first term in Eq. (19) is the same as that for a spherical top.[9] The equations for α, β, and γ are most easily obtained from the Lagrangian:

$$L=\int d^3r\,\mathfrak{L}(\vec{r})\,,$$

$$\mathfrak{L}(\mathrm{r})=\tfrac{1}{2}\chi_B\,[\dot\alpha^2+\dot\beta^2+\dot\gamma^2+\dot\alpha\dot\gamma\cos\beta-2\omega_0(\dot\alpha+\cos\beta\dot\gamma)+\omega_0^2]$$

$$-(2N/5m)Y\{(\vec{\nabla}\alpha)^2+(\vec{\nabla}\beta)^2+(\vec{\nabla}\gamma)^2+2(\vec{\nabla}\alpha)(\vec{\nabla}\gamma)\cos\beta-\tfrac{1}{2}[(\tilde{A}_i^i)^2+\tilde{A}_i^j\tilde{A}_j^i]\}-E_d(\alpha,\beta,\gamma)\,, \tag{21}$$

as

$$\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\alpha}}\right)+\frac{\partial}{\partial x_k}\left(\frac{\partial\mathcal{L}}{\partial\alpha_k}\right)-\frac{\partial\mathcal{L}}{\partial\alpha}=0,\quad \text{etc.} \tag{22}$$

Here we added an additional Lagrangian linear in ω_0 due to an external field H, where $\omega_0=\gamma H$ is the Larmor frequency.

III. SPIN WAVES

Although Eq. (22), obtained in the preceding section, describes the general motion of the spin configuration in the B phase, it is rather complicated to analyze in details. Therefore in this section we will be concerned with an infinitesimal oscillation of spin components around the equilibrium configurations. These oscillations can be described in terms of spin waves.

Among many possible equilibrium configurations which yield the lowest dipole interaction energy, we will consider two special configurations which are of particular interest. In particular, in the cylindrical geometry as in the recent experiment by Webb *et al.*,[3] we believe that these two configurations are very likely to be realized.[9]

A. Leggett configuration $[\alpha=\cos^{-1}(-\frac{1}{4}),\beta=\gamma=0]$

We will consider small oscillations around this special configuration. This configuration is very likely to be realized in strong magnetic fields. If we retain the lowest-order deviation from the equilibrium configuration we will have

$$\begin{aligned}\omega_1&=\dot{\phi}_1=-\sin\alpha\,\dot{\beta}+\cos\alpha\,\beta\dot{\gamma}\\ \omega_2&=\dot{\phi}_2=\cos\alpha\,\dot{\beta}+\sin\alpha\,\beta\dot{\gamma}\\ \omega_3&=\dot{\phi}_3=\dot{\alpha}+\dot{\gamma}(1-\tfrac{1}{2}\beta^2)\,,\end{aligned} \tag{23}$$

and similar expressions for A_i^l.

Substituting these into the expressions for the Lagrangian density we have

$$\begin{aligned}\mathcal{L}(\mathbf{r})=\tfrac{1}{2}\chi_B[(\dot{\phi}_1-\omega_0)^2+\dot{\phi}_2^2-\omega_0(\dot{\phi}_2\phi_3-\dot{\phi}_3\phi_2)+\dot{\phi}_3^2]-(2N/5m)Y\Big[(\vec{\nabla}\phi_1)^2+(\vec{\nabla}\phi_2)^2+(\vec{\nabla}\phi_3)^2+\tfrac{1}{4}(\phi_2+\sqrt{15}\phi_1)(\vec{\nabla}\phi_1)(\vec{\nabla}\phi_3)\\ -\frac{1}{2}\left(\frac{\partial\phi_1}{\partial z}+\frac{\partial\phi_2}{\partial y}+\frac{\partial\phi_3}{\partial x}-\frac{1}{2}\phi_2\frac{\partial\phi_3}{\partial z}\right)^2\Big]-\tfrac{1}{2}\chi_B\Omega_l^2\phi_1^2\,.\end{aligned} \tag{24}$$

The equations of motion for ϕ_1, ϕ_2, and ϕ_3 are then given as

$$\begin{aligned}\ddot{\phi}_1&=c^2\left[\nabla^2\phi_1-\frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial\phi_1}{\partial z}+\frac{\partial\phi_2}{\partial y}+\frac{\partial\phi_3}{\partial x}\right)\right]-\Omega_l^2\phi_1\,,\\ \ddot{\phi}_2&=-\omega_0\dot{\phi}_3+c^2\left[\nabla^2\phi_2-\frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial\phi_1}{\partial z}+\frac{\partial\phi_2}{\partial y}+\frac{\partial\phi_3}{\partial x}\right)\right],\\ \ddot{\phi}_3&=\omega_0\dot{\phi}_2+c^2\left[\nabla^2\phi_3-\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial\phi_1}{\partial z}+\frac{\partial\phi_2}{\partial y}+\frac{\partial\phi_3}{\partial x}\right)\right],\end{aligned} \tag{25}$$

where

$$c^2=\tfrac{4}{15}v_F^2[N(0)/\chi_B]Y(T)\,, \tag{26}$$

v_F is the Fermi velocity, $N(0)$ is the density of states at the Fermi level, and $Y(T)$ has already been defined in Eq. (15). Equation (25) implies that these modes are coupled to each other even in the limit of small oscillation. When the spin wave is propagated along the z direction (i.e., along the symmetry axis of the cylinder) with the wave vector q, these modes are decoupled into the longitudinal and the transverse mode and we have:

$$\begin{aligned}\ddot{\phi}_1&=\frac{1}{2}c^2\frac{\partial^2}{\partial z^2}\phi_1-\Omega_l^2\phi_1\,,\\ \ddot{\phi}_\pm&=\pm i\omega_0\dot{\phi}_\pm-c^2\frac{\partial^2}{\partial z^2}\phi_\pm\,.\end{aligned} \tag{27}$$

Or the spin wave dispersions are $\omega^2=\frac{1}{2}c^2q^2+\Omega_l^2$ for the longitudinal mode and

$$\omega(\omega\pm\omega_0)=c^2q^2 \tag{28}$$

for the transverse mode. These modes reduce to those obtained by Combescot[7] in the absence of the dipole energy (i.e., $\Omega_l=0$).

B. Wall-fixed geometry

We will consider another situation where the boundary condition at the wall of the cylindrical vessel plays a prominent role in fixing the equilibrium configuration ($\alpha=\gamma=0$, $\cos\beta=-\frac{1}{4}$). This configuration is very likely to be present at low temperatures in the B phase of superfluid ^{3}He.[9] In this situation we approximate $\vec{\omega}$ by

$$\omega_1 = -\alpha\dot{\beta} + \sin\beta\dot{\gamma}, \quad \omega_2 = \dot{\beta} + \alpha\sin\beta\dot{\gamma}, \quad \omega_3 = \dot{\alpha} + \cos\beta\dot{\gamma}, \quad \text{etc.} \tag{29}$$

The Lagrangian density is then given by

$$\mathcal{L}(\vec{r}) = \tfrac{1}{2}\chi_B[(\dot{\alpha} + \cos\beta\dot{\gamma} - \omega_0)^2 + \dot{\beta}^2 + \sin^2\beta\dot{\gamma}^2] - (2N/5m)Y(T)\{[\vec{\nabla}\alpha + \cos\beta(\vec{\nabla}\gamma)]^2 + (\vec{\nabla}\beta)^2 + \sin^2\beta(\vec{\nabla}\gamma)^2$$
$$- \tfrac{1}{2}[\beta_y + \sin\beta\alpha_x - \cos\beta\alpha_z + 2\sin\beta\cos\beta\gamma_x + (\sin^2\beta - \cos^2\beta)\gamma_z]^2\} - \tfrac{1}{2}\chi_B\Omega_l^2(\beta - \beta_0)^2, \tag{30}$$

where $\cos\beta_0 = -\frac{1}{4}$. Introducing new variables by

$$\phi_1 = \alpha + \gamma, \quad \phi_2 = \alpha - \gamma, \quad \phi_3 = \beta - \beta_0, \tag{31}$$

we can rewrite Eq. (30) as

$$\mathcal{L}(r) = \tfrac{1}{2}\chi_B(\tfrac{1}{2}(1+\cos\beta)\dot{\phi}_1^2 + \tfrac{1}{2}(1-\cos\beta)\dot{\phi}_2^2 + \dot{\phi}_3^3 - \omega_0[(1+\cos\beta)\dot{\phi}_1 + (1-\cos\beta)\dot{\phi}_2] + \omega_0^2)$$
$$- \frac{2N}{5m}Y(T)\Bigg(\tfrac{1}{2}(1+\cos\beta)(\vec{\nabla}\phi_1)^2 + \tfrac{1}{2}(1-\cos\beta)(\vec{\nabla}\phi_2)^2 + (\vec{\nabla}\phi_3)^2 - \frac{1}{2}\Bigg\{\frac{\partial\phi_3}{\partial y} + \frac{(1+\cos\beta)}{2}\left[\sin\beta\left(\frac{\partial\phi_1}{\partial x}\right) - \cos\beta\left(\frac{\partial\phi_1}{\partial z}\right)\right]$$
$$+ \frac{(1-\cos\beta)}{2}\left[\sin\beta\left(\frac{\partial\phi_2}{\partial x}\right) - \cos\beta\left(\frac{\partial\phi_2}{\partial z}\right)\right]\Bigg\}^2\Bigg) - \tfrac{1}{2}\chi_B\Omega_l^2\phi_3^2. \tag{32}$$

The equations of motion are then given by

$$\ddot{\phi}_1 = -\sqrt{\tfrac{5}{3}}\,\omega_0\dot{\phi}_3 + c^2\left\{\nabla^2\phi_1 - \frac{1}{8}\left(\sqrt{15}\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right)\left[\frac{\partial}{\partial y}\phi_3 + \frac{3}{32}\left(\sqrt{15}\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right)\phi_1 + \frac{5}{32}\left(\sqrt{15}\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right)\phi_2\right]\right\},$$

$$\ddot{\phi}_2 = \sqrt{\tfrac{5}{3}}\,\omega_0\dot{\phi}_3 + c^2\left\{\nabla^2\phi_2 - \frac{1}{8}\left(\sqrt{15}\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right)\left[\frac{\partial}{\partial y}\phi_3 + \frac{3}{32}\left(\sqrt{15}\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right)\phi_1 + \frac{5}{32}\left(\sqrt{15}\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right)\phi_2\right]\right\},$$

and

$$\ddot{\phi}_3 = \frac{\sqrt{15}}{4}\omega_0(\dot{\phi}_1 - \dot{\phi}_2) + c^2\left\{\nabla^2\phi_3 - \frac{1}{2}\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\phi_3 + \frac{3}{32}\left(\sqrt{15}\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right)\phi_1 + \frac{5}{32}\left(\sqrt{15}\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right)\phi_2\right]\right\} - \Omega_l^2\phi_3 \tag{33}$$

where we took $\sin\beta_0 = \sqrt{\frac{15}{4}}$.

In general, these modes couple each other even in the linear approximation. In a special situation where the spin wave is propagated along the z axis (i.e., the symmetric axis of the cylinder) the dispersion is determined by

$$\begin{vmatrix} \omega^2 - c^2q^2(1-3\xi) & 5\xi c^2q^2 & i\sqrt{\frac{5}{3}}\,\omega_0\omega \\ 3\xi c^2q^2, & \omega^2 - c^2q^2(1-5\xi), & -i\sqrt{\frac{3}{5}}\,\omega_0\omega \\ -i\sqrt{\frac{15}{4}}\,\omega_0\omega, & i\sqrt{\frac{15}{4}}\,\omega_0\omega & \omega^2 - \Omega_l^2 - c^2q^2 \end{vmatrix} = 0, \tag{34}$$

or

$$[\omega^2 - c^2q^2(1-8\xi)][(\omega^2 - c^2q^2)(\omega^2 - c^2q^2 - \Omega_l^2) - 2\omega^2\omega_0^2] = 0, \tag{35}$$

where $\xi = (256)^{-1}$. Therefore, in this particular case the spin-wave dispersions are given by

$$\omega^2 = c^2q^2(1-8\xi),$$

for the longitudinal mode, and

$$\omega^2 = c^2q^2 + \tfrac{1}{2}\omega_0^2 + \Omega_l^2 \pm [(\omega_0^2 + \Omega_l^2)^2 + 4c^2q^2\omega_0^2]^{1/2}, \tag{36}$$

for the transverse mode.

In the present configuration the longitudinal mode is gapless, while the transverse modes split into the gapless one and the other with a finite energy gap:

$$\omega^2 = [\Omega_l^2/(\omega_0^2 + \Omega_l^2)]c^2q^2 + O(q^4)$$

and

$$\omega^2 = \omega_0^2 + \Omega_l^2 + [1 + \omega_0^2/(\omega_0^2 + \Omega_l^2)]c^2q^2 \tag{37}$$

IV. CONCLUDING REMARKS

Analyzing the gauge invariance of the original Hamiltonian (in the absence of the dipole interaction energy), we obtain microscopically the effective Hamiltonian which describes the motion (i.e., the rotation) of the spin configurations of the condensate in the BW state. The resulting equations of motion for the local Eulerian angles are rather complicated in general. We have studied a small oscillation in the Eulerian angles around two equilibrium configurations in the B phase of superfluid ^3He, which are very likely to be realized in the high-field region and in the low-field region, respectively. In particular, we found rather simple spin-wave dispersions when the wave vector is parallel to the symmetry axis of the cylinder. The present results on the spin-wave dispersion reduce to those obtained by Combescot,[7] if we neglect the dipole interaction energy term.

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APPENDIX A: DERIVATION OF THE ADDITIONAL FREE ENERGY ASSOCIATED WITH ROTATION OF THE SPIN CONFIGURATION

The shift of the free energy due to ΔH is calculated by perturbation as

$$T^{-1}(\Delta F') = -\int_0^{T^{-1}} d\tau\, \langle \Delta H\rangle_{\text{con}} + \frac{1}{2}\int_0^{T^{-1}} d\tau_1 \int_0^{T^{-1}} d\tau_2 \langle T_\tau\{\Delta H(\tau_1)\,\Delta H(\tau_2)\}\rangle_{\text{con}}\,, \tag{A1}$$

where $\Delta H(\tau)$ is ΔH in the interaction representation; τ, τ_1, and τ_2 are imaginary time; T_τ is Wick's time-ordering operator; T is the temperature. The suffix con implies taking only the connected diagrams. The first term in (A1) vanishes identically, while the second term is simplified as

$$\begin{aligned} T^{-1}(\Delta F') = \frac{1}{2}\int_0^{T^{-1}} d\tau_1 \int_0^{T^{-1}} d\tau_2 \int d^3 r_1 \int d^3 r^2 \{ &\langle T_\tau(s_i(1),\ s_3(2))\rangle\,[\omega_i(1)\,\omega_j(2)] \\ &+ \langle T_\tau(s_i(1),\ j^m_{s_j}(2))\rangle \omega_i(1) A^m_j(2) + \langle T_\tau(j_{s_i}(1),\ s_j(2))\rangle A^l_i(1)\,\omega_j(2) \\ &+ \langle T_\tau(j^l_{s_i}(1),\ j^m_{s_j}(2))\rangle A^l_i(1) A^m_j(2)\}\,, \end{aligned} \tag{A2}$$

where (1) and (2) imply the four coordinate, $\vec{s}(1)$ is the spin operator and $\vec{j}_{\vec{s}}(1)$ is the spin-current operator.

Assuming now that the spatio-temporal variation of $\vec{\omega}(1)$ and $\overleftrightarrow{A}(1)$ are much slower than the relaxation time of the quasiparticles (i.e., in the hydrodynamic regime), we can simplify (A2);

$$\Delta F' = \frac{1}{2}\int d^3 r_1 \int d^3 r_2 \{\chi_{ij}(0)\,[\omega_i(1)\omega_j(2)] + (\chi^J)^{lm}_{ij}(0) A^l_i(1) A^m_j(2)\}\,, \tag{A3}$$

where

$$\chi_{ij}(0) = \int_0^{T^{-1}} d\tau\,\langle T_\tau(s_i(\vec{r}_1,\ i\tau),\ s_j(\vec{r}_2,\ 0))\rangle$$

and

$$(\chi^J)^{lm}_{ij}(0) = \int_0^{T^{-1}} d\tau\,\langle T_\tau(j^l_{s_i}(\vec{r}_1,\ i\tau),\ j^m_{s_j}(\vec{r}_2,\ 0))\rangle \tag{A4}$$

are the static susceptibility and the static spin-current correlation function, respectively. Here we made use of the fact

$$\int_0^{T^{-1}} d\tau \langle T_\tau(s_i(\vec{r}_1,\ i\tau),\ j^m_{s_j}(\vec{r}_2,\ 0))\rangle = 0\,. \tag{A5}$$

Furthermore, since in the hydrodynamic limit $\chi_{ij}(0)$ and $(\chi^J)^{lm}_{ij}$ are approximated by local correlation functions

$$\chi_{ij}(0) = \chi_{ij}\delta(\vec{r}_1 - \vec{r}_2)$$

and

$$(\chi^J)^{lm}_{ij}(0) = (\chi^J)^{lm}_{ij}\delta(\vec{r}_1 - \vec{r}_2)\,, \tag{A6}$$

we can reduce (A3) to Eq. (12) in the text.

APPENDIX B: CALCULATION OF χ_{ij} AND $(\chi^J)^{lm}_{ij}$ IN THE BW STATE

Since χ_{ij} has been already discussed by Balian and Werthamer[4] in their original paper, we will not repeat here a similar calculation. We note only that the effect of the exchange interaction is easily incorporated into the theory[11] yielding

$$\chi_{ij} = \chi_B \delta_{ij} \tag{B1}$$

and

$$\chi_B = N(0)[1 - \tfrac{1}{3}Y(T)]/\{1 - \bar{I}\,[1 - \tfrac{1}{3}Y(T)]\}, \tag{B2}$$

where

$$Y(T) = 2\pi T \sum_{n=0}^{\infty} \frac{\Delta^2}{(\omega_n^2 + \Delta^2)^{3/2}}$$

and

$$\bar{I} = IN(0). \tag{B3}$$

The spin-current correlation function is calculated as follows:

$$(\chi^J)_{ij}^{lm} = \int_0^{T^{-1}} d\tau \,\langle T_\tau(j_{s_i}^l(i\tau),\, j_{s_j}^m(0))\rangle$$
$$= T\sum_n \int \frac{d^3p}{(2\pi)^3}\,\mathrm{Tr}\left(\alpha_i \frac{p_l}{m} G(\vec{p},\,\omega_n)\alpha_j \frac{p_m}{m} G(\vec{p},\,\omega_n)\right), \tag{B4}$$

where the α_i are the Pauli spin matrices in the four-dimensional representation, [4] and $G(\vec{p},\,\omega_n)$ is the Green's function describing the BW state;

$$G^{-1}(\vec{p},\,\omega_n) = i\omega_n - \xi\rho_3 - \sigma^2\rho_2\Delta(\vec{\alpha}\cdot\vec{f}\,). \tag{B5}$$

$\xi = (1/2m)p^2 - \mu$, ω_n is the Matsubara frequency, ρ_i and σ^i are the Pauli spin matrices operating on the particle-hole space and on the ordinary spin-space, respectively, and Δ is the order parameter. Finally $\vec{\alpha}\cdot\vec{k}$ is obtained from $\vec{\alpha}\cdot\vec{k}$ by a rotation R defined in Eq. (5) [σ^i in Eq. (5) have to be replaced by α^i].

$$\vec{\alpha}\cdot\vec{f} = R(\vec{\alpha},\,\vec{k})R^{-1}, \tag{B6}$$

where $\vec{k}$ is the unit vector parallel to $\vec{p}$, the quasiparticle momentum. The integral over d^3p in (B4) is easily carried out by replacing d^3p by $N(0)(d\Omega/4\pi)\,d\xi$ and we have

$$(\chi^J)_{ij}^{lm} = \frac{N(0)}{m^2}\rho_0^2\pi T\sum_n \left\langle \frac{1}{(\omega_n^2+\Delta^2)^{1/2}}\,\mathrm{Tr}[\alpha^i\alpha^j k_l k_m - \alpha^i k_l(i\omega_n + \Delta\hat{0})\alpha^j k_m(i\omega_n + \Delta\hat{0})/(\omega_n^2+\Delta^2)]\right\rangle,$$

where

$$\hat{0} = \sigma^2\rho_2(\vec{\alpha}\cdot\vec{f}\,) \tag{B7}$$

and

$$\vec{k} = \vec{p}/p_0, \tag{B8}$$

and p_0 is the Fermi momentum. We can further reduce (B7) to

$$(\chi^J)_{ij}^{lm} = \frac{N}{m}\pi T\sum_n \frac{\Delta^2}{(\omega_n^2+\Delta^2)^{3/2}}\,(\delta_{ij}\delta_{lm} - 3\,\langle k_l k_m f_i f_j\rangle)$$
$$= (N/m)Y(T)(\delta_{ij}\delta_{lm} - 3\langle k_l k_m f_i f_j\rangle), \tag{B9}$$

where

$$Y(T) = 2\pi T\sum_{n=0}^{\infty}\frac{\Delta^2}{(\omega_n^2+\Delta^2)^{3/2}} = \left(\frac{\rho_s}{\rho}\right)_{\mathrm{BCS}}. \tag{B10}$$

Here we made use of the identity

$$N = \tfrac{2}{3}[N(0)/m]p_0^2,$$

and N is the density of ^{3}He atoms.

APPENDIX C: CALCULATION OF THE DIPOLE INTERACTION ENERGY

The dipole interaction energy in the BW state is calculated as the ground-state expectation value of H_d.

$$E_d(\alpha,\,\beta,\,\gamma) = \frac{\gamma_0^2}{8}\int d^3r_1\int d^3r_2\,\langle\langle\psi_\mu^\dagger(\vec{r}_1)\alpha_{\mu\nu}^i\psi_\nu(\vec{r}_1))(\psi_\lambda^\dagger(\vec{r}_2)\alpha_{\lambda\rho}^j\psi_\rho(\vec{r}_2))\rangle_0(\delta_{ij} - 3e_ie_j)/|\vec{r}_1 - \vec{r}_2|^3, \tag{C1}$$

where $\vec{e} = (\vec{r}_1 - \vec{r}_2)/|\vec{r}_1 - \vec{r}_2|$ and γ_0 is the gyromagnetic ratio of the ^{3}He nucleus. The above expectation value (i.e., $\langle\cdots\rangle_0$) can be expressed in terms of the Green's function (B5) as

$$E_d(\alpha,\,\beta,\,\gamma) = -\frac{\pi\gamma_0^2}{6}T^2\sum_{nm}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3}\,\mathrm{Tr}[G(\vec{p},\,\omega_n)\alpha^iG(\vec{p}\,',\,\omega_m)\alpha^j]\Lambda_{ij}(\vec{k},\,\vec{k}\,'), \tag{C2}$$

where $\Lambda_{ij}(\vec{k},\,\vec{k}\,') = 3n_in_j - \delta_{ij}$ is the Fourier transform of $(3/4\pi)(3e_ie_j - \delta_{ij})r_{ij}^{-3}$, and $\vec{n} = (\vec{k} - \vec{k}\,')/|\vec{k} - \vec{k}\,'|$. Substituting the Green's function in (C2) and carrying out the integrations over d^3p and d^3p', we have

$$E_d(\alpha,\,\beta,\,\gamma) = -\frac{2\pi\gamma_0^2}{3}[\pi TN(0)]^2\sum_{nm}\sum_{ij}\int\!\!\int\frac{d\Omega\,d\Omega'}{(4\pi)^2}\,\frac{\Delta_i^*(\Omega)\Lambda_{ij}(k,\,k')\Delta_j(\Omega')}{(\omega_n^2+\Delta^2)^{1/2}(\omega_m^2+\Delta^2)^{1/2}} = -\frac{2\pi\gamma_0^2}{3g^2}\Delta^2\sum_{ij}\langle f_i|\Lambda_{ij}|f_j\rangle, \tag{C3}$$

where

$$\langle f_i | \Lambda_{ij} | f_j \rangle = \int\int \frac{d\Omega\, d\Omega'}{(4\pi)^2} f_i^*(\Omega')\Lambda_{ij}(\Omega', \Omega) f_j(\Omega)$$

and

$$\Delta_i(\Omega) = \Delta f_i(\Omega), \quad \text{etc.} \tag{C4}$$

In deriving (C3), we made use of the approximate gap equation

$$1 = gN(0)\pi T \sum_n \frac{1}{(\omega_n^2 + \Delta^2)^{1/2}} . \tag{C5}$$

Substituting $\vec{f}$ defined by (B6), we obtain after a straightforward but lengthy calculation

$$E_d(\alpha, \beta, \gamma) = \tfrac{2}{5}\pi(\gamma_0^2/g^2)\Delta^2(\{(1+\cos\beta)[1+\cos(\alpha+\gamma)] - \tfrac{3}{2}\}^2 - \tfrac{5}{4}) . \tag{C6}$$

This result agrees with that obtained by Takagi,[12] who used a slightly different method. So far we considered uniform rotation of the spin configuration in space. If the spatial variation of the Eulerian angle is sufficiently slow, the corresponding dipole energy is easily obtained from (C6) by replacing the uniform Eulerian angles by their local values. (C6) then reduces Eq. (18) in the text, if we identify

$$\Omega_l^2 = (3\pi\gamma_0^2/g^2\chi_B)\Delta^2(T) . \tag{C7}$$

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