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Hydrodynamic theory applied to fourth sound in a moving superfluid

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The hydrodynamic equations of superfluid ^{4}He in a superleak are used to obtain an exact and general expression for the velocity of fourth sound in the presence of a steady superfluid flow. This expression, which depends only on the flow velocity and on static properties of the helium-superleak system, may be simplified when effects due to the finite-pore dimensions are negligible. In that case the fourth-sound velocity may be rewritten in terms of the flow velocity, thermodynamic properties of bulk helium and bulk filler material, and some purely geometric parameters. The expression obtained disagrees with the results of a previous calculation. A relation between the fourth-sound velocity at zero flow and the Doppler shift at finite flow, predicted by the present theory, is in conflict with existing experimental data. Possible explanations for this discrepancy are discussed.

I. INTRODUCTION

Much of the experimental work on the superfluid properties of ^{4}He,[1-6] and more recently, ^{3}He,[7,8] has been carried out on helium in a superleak. While this facilitates experiments by increasing the critical velocity by a large factor and by preventing any normal-fluid flow, it also makes theoretical interpretation more difficult. A growing body of experimental results have accumulated on superfluid flow,[5] persistent currents,[1,2,6,9-11] and fourth sound[3,4,6-11] in such systems. All of these phenomena clearly arise from the existence of a superfluid order parameter in the system, similar to the one known to exist in pure bulk helium. One would therefore like to have a theory that ties these phenomena together and, if possible, relates them to the properties of bulk helium. One would like the theory to be valid at all temperatures below T_λ, and also at finite velocities, thus enabling nonlinear velocity effects such as the Doppler shift to be discussed.

Most existing theories have tended to restrict their discussion to a limited class of phenomena. Thus, Shapiro and Rudnick[3] only discuss fourth sound while Mehl and Zimmermann[5] only consider superfluid flow. More recently, some attempts have been made to relate superfluid flow and fourth sound in the same system. Revzen *et al.*,[12] who were the first to do this, have arrived, we believe, at an erroneous conclusion. On the other hand, Yanof and Reppy[8] have succeeded in relating some of the properties of persistent currents at low temperatures to those of fourth sound in the same system by using a linearized form of the equations of motion.

We will attempt in this paper to give a more complete theoretical discussion of the problem of fourth sound in the presence of superfluid flow. Our treatment will be based on exact hydrodynamic equations for helium in a superleak which were derived by Halperin and Hohenberg.[13] The main result is an exact expression for the velocity of a fourth-sound wave superimposed on a steady superfluid flow. Our expression, appearing in Eq. (3) below, gives that velocity in terms of the superfluid flow velocity and thermodynamic properties of the helium-superleak system. Furthermore, we show that whenever size effects due to the finite pore dimensions are unimportant, all of these thermodynamic properties can be related to properties of bulk helium and bulk filler material, plus some geometrical parameters. In practice, the properties of the filler often have a negligible effect, and one can then use a simplified expression for the fourth-sound velocity, which includes a Doppler shift, at small flow velocities. This expression, given by Eq. (29), depends on only one geometrical parameter g, which is simply the reduction factor for the effective superfluid fraction at zero flow in going from pure bulk helium to helium in the superleak. At very low temperatures an even simpler expression is obtained [Eq. (13)].

Experiments for which this theory should be applicable have already been performed in a series of measurements of the Doppler splitting of fourth-sound resonances in a ring-shaped superleak, in which a superfluid flow was induced by rotation around the ring axis.[6,9-11] The theory can most easily be applied to the region of low ring velocities, where the system exhibits completely reversible behavior, which is believed to indicate that no vortices are present.[9] While the published quantitative data are rather scant, there are indications of a disagreement between theory and experiment. Such a disagreement, if it were borne out by a careful analysis of the data, could imply the existence of large, as yet undiscovered, relaxation processes on the time scale of 10^{-2}–10^{2} sec.

In Sec. II the hydrodynamic theory is introduced and used to derive an exact expression for the fourth-sound velocity in a moving superfluid [Eq. (3)]. Section III discusses the simplifications that can be made in the theory when size effects are unimportant. The case of zero temperature is considered first, and a particularly simple expression is derived for the fourth-sound velocity [Eq. (13)], after which finite-temperature corrections are calculated. The results are compared with previous theoretical work, and an erroneous result[12] previously obtained for the low-temperature limit is pointed out. Higher-order corrections proportional to the square of the flow velocity are briefly discussed. Section IV reviews the experimental consequences of the theory. It is first shown how the exact results can be checked in principle. For comparison with existing experiments, the simplified expressions in Eq. (29) and, below 1.3 °K, Eq. (13) should be accurate enough. An apparent disagreement with available data is pointed out, and possible explanations discussed. The assumptions underlying the hydrodynamic theory are reviewed, and the effects of vortices and finite relaxation times on its validity are estimated. Experiments are proposed to help understand the discrepancies. In Appendix A a more detailed discussion is given of the averaging procedure by which the hydrodynamic theory can be obtained for small flow velocities in the absence of size effects. This discussion is needed in order to justify some of the simplifications made in Sec. III. The important geometrical parameter g is calculated for a simple model of a superleak, and compared with experimental data. The case of a real superleak is discussed qualitatively. In Appendix B we extend the discussion of Appendix A to include higher-order corrections in the flow velocity for the case where size effects are unimportant.

II. EXACT HYDRODYNAMIC THEORY OF FOURTH SOUND

A thermal equilibrium state of a particular superleak containing liquid helium may be characterized by the total helium mass, the total energy, and the phase φ of the superfluid order parameter, which is independent of position in the absence of superfluid flow. Owing to interaction with the porous superleak, the helium momentum is not conserved, and it therefore vanishes at equilibrium in the rest frame of the superleak. A local equilibrium state of the system, in which a superflow may be present, is characterized by a helium mass density $\tilde{\rho}(r)$, an energy density $\tilde{E}(r)$, and a phase $\tilde{\varphi}(r)$, all of which are defined to be averages over a coarse-graining volume which is large compared to the pore size, yet much smaller than the entire system. The quantities $\tilde{\rho}, \tilde{E}, \tilde{\varphi}$ can be used to describe phenomena whose wavelength is much greater than the coarse-graining volume. Note that $\tilde{\rho}$ and $\tilde{E}$ are defined as densities per unit volume of the whole sample—with filler and pores included. A similar convention is used throughout this paper for densities with a tilde.

Working in the reference frame of the superleak, assuming that $v_n \equiv 0$ (i.e., the normal component is locked to the superleak), that the superleak is completely rigid, and neglecting the dissipative terms, one finds the hydrodynamic equations[13]:

$$\frac{d}{dt}\tilde{\rho} + \nabla \cdot \tilde{\rho}_s \tilde{u}_s = 0 \quad , \tag{1a}$$

$$\frac{d}{dt}\tilde{u}_s + \nabla(\tilde{\mu} + \tfrac{1}{2}\tilde{u}_s^2) = 0 \quad , \tag{1b}$$

$$\frac{d}{dt}\tilde{S} = 0 \quad , \tag{1c}$$

where $\tilde{S}$ is the total entropy per unit volume of the helium plus filler, and where

$$\tilde{u}_s \equiv (\hbar/m)\nabla\tilde{\varphi} \quad , \tag{2a}$$

$$\tilde{\rho}_s \tilde{u}_s \equiv \left(\frac{\partial \tilde{E}}{\partial \tilde{u}_s}\right)_{\tilde{S},\tilde{\rho}} \quad , \tag{2b}$$

$$\tilde{\mu} + \tfrac{1}{2}\tilde{u}_s^2 \equiv \left(\frac{\partial \tilde{E}}{\partial \tilde{\rho}}\right)_{\tilde{S},\tilde{u}_s} \quad . \tag{2c}$$

While $\tilde{u}_s$ is not in general equal to the spatial average of the microscopic superfluid velocity, it follows from Eq. (1a) that $\tilde{\rho}_s \tilde{u}_s$ is the average helium current or momentum density. (See also the discussion in Appendix A.) Equations of this form were first given by Atkins,[14] but he considered them only for the geometry of a single straight and narrow capillary, where all the quantities $\tilde{\rho}, \tilde{\rho}_s, \tilde{\mu}$ are the same as in pure bulk helium, and where the main effect of the walls is to lock the normal fluid. We, on the other hand, are using these equations to discuss the more general case where the superleak geometry may be disordered and the helium is forced to follow a very winding and random flow pattern. It turns out that, as far as hydrodynamic phenomena are concerned, this complexity only affects the values of the thermodynamic parameters appearing in Eqs. (1),[13] without changing the form of the equations. The assumption of a rigid superleak is a very good one in view of the magnitude of the helium compressibility as compared to that of typical filler materials (e.g., aluminum oxide, CMN). Note that, since $\tilde{E}$ depends on $\tilde{\rho}$, $\tilde{S}$, and $\tilde{u}_s$, both $\tilde{\mu}$ and $\tilde{\rho}_s$ defined in (2) depend on all three of these variables. If we assume that there is a space and time varying superfluid flow $\delta\tilde{u}_s$ superimposed on a steady superfluid velocity $\tilde{u}_s$ as well as a space and time varying density $\delta\tilde{\rho}$ superimposed on the static density $\tilde{\rho}$, linearize the equations in $\delta\tilde{u}_s$ and $\delta\tilde{\rho}$, and take into account that $\tilde{S}$ = const,[13] we find that these equations have plane-wave fourth-sound solutions whose velocity c_4 is given by

$$c_4 = \left\{ \left(\frac{\partial \tilde{\mu}}{\partial \tilde{\rho}}\right)_{\tilde{S},\tilde{u}_s} \left[\tilde{\rho}_s + \tilde{u}_{s\parallel}^2 \left(\frac{\partial \tilde{\rho}_s}{\partial \frac{1}{2}\tilde{u}_s^2}\right)_{\tilde{\rho},\tilde{S}} \right] \right\}^{1/2} + \tilde{u}_{s\parallel} \left(\frac{\partial \tilde{\rho}_s}{\partial \tilde{\rho}}\right)_{\tilde{S},\tilde{u}_s} \quad , \tag{3}$$

where $\tilde{u}_{s\parallel}$ is the component of $\tilde{u}_s$ in the direction of the wave vector $\vec{k}$. In obtaining this expression we used the Maxwell relation

$$\left(\frac{\partial(\tilde{\mu}+\frac{1}{2}\tilde{u}_s^2)}{\partial \tilde{u}_s}\right)_{\tilde{S},\tilde{\rho}} = \left(\frac{\partial(\tilde{\rho}_s \tilde{u}_s)}{\partial \tilde{\rho}}\right)_{\tilde{S},\tilde{u}_s} = \tilde{u}_s \left(\frac{\partial \tilde{\rho}_s}{\partial \tilde{\rho}}\right)_{\tilde{S},\tilde{u}_s} . \quad (4)$$

Since no approximations have been made in deriving (3) from (1) (other than linearizing in the oscillating quantities), Eq. (3) is an exact expression for the velocity of fourth sound, depending only on thermodynamic quantities as mentioned above. It includes finite-temperature and finite-flow-velocity effects as well as size effects. The thermodynamic quantities $\tilde{\rho}$, $\tilde{\rho}_s$, $\tilde{\mu}$ in general depend on the particular superleak under investigation and must be measured or calculated for helium in that superleak.

III. FOURTH SOUND IN THE ABSENCE OF SIZE EFFECTS

Size effects arise from the existence of a helium surface and from the interaction of filler material and adjacent layers of helium. These effects are usually negligible for pure bulk helium. For helium in pores they are still unimportant as long as the pores are large compared to the correlation length of bulk helium. That length is of the order of interatomic distances, except very close to the λ line, where it is increased by a factor $(|T-T_\lambda|/T_\lambda)^{-2/3}$.

If size effects and finite-flow-velocity effects can be neglected, then clearly

$$\tilde{\rho} = \mathcal{P}\rho \quad , \quad (5)$$

where the porosity $\mathcal{P}$ is the free-volume fraction of the superleak, and the local helium density inside the pores is equal to the density of pure bulk helium ρ. In Appendix A we show that in this case we also have

$$\frac{\tilde{\rho}_s}{\tilde{\rho}} = g\frac{\rho_s}{\rho} \quad , \quad (6)$$

where g is a constant factor less than unity, which in general depends on the detailed geometry of the superleak but is independent of the thermodynamic state of the system (i.e., of T and $\tilde{\rho}$). There are indications that in practice g is generally determined by $\mathcal{P}$ for a variety of packing densities.[3,4] Finally, since the energy and entropy of the filler material are independent of $\tilde{\rho}$ at constant T, we find that

$$\tilde{\mu} + \tfrac{1}{2}u_s^2 = \left(\frac{\partial(\tilde{E}_H - T\tilde{S}_H)}{\partial \tilde{\rho}}\right)_{T,\tilde{u}_s} , \quad (7)$$

where $\tilde{E}_H - T\tilde{S}_H$ is the Helmholtz free energy of the *helium only*, per unit of the *total* volume. When $\tilde{u}_s = 0$ and the helium inside the pores is just bulk helium, the right-hand side of (7) can be replaced by the pure-helium derivative and we get

$$\tilde{\mu}(\tilde{u}_s = 0) = \left(\frac{\partial(E-TS)}{\partial \rho}\right)_{T,v_s=0} \equiv \mu(v_s=0) \quad , \quad (8)$$

where E, S, and μ are the energy density, entropy density, and chemical potential of pure bulk helium. From this and Eq. (6) we now get

$$\tilde{\rho}\left(\frac{\partial \tilde{\mu}}{\partial \tilde{\rho}}\right)_{T,\tilde{u}_s=0} = \rho\left(\frac{\partial \mu}{\partial \rho}\right)_{T,v_s=0} = \frac{1}{\rho K_T} \quad , \quad (9)$$

where K_T is the isothermal compressibility of pure bulk helium.

A. Low temperatures

At $T=0$ and low velocities $\tilde{u}_s$ we can now reduce Eq. (3) to a very simple form: In the first term we replace the partial derivative at constant $\tilde{S}$ with a partial derivative at constant $T=0$, thus getting

$$c_{40} = \left[\frac{\tilde{\rho}_s}{\tilde{\rho}}\tilde{\rho}\left(\frac{\partial \tilde{\mu}}{\partial \tilde{\rho}}\right)_{T=0,\tilde{u}_s=0}\right]^{1/2} = \left(\frac{\tilde{\rho}_s/\tilde{\rho}}{\rho K_{T=0}}\right)^{1/2} = \left(\frac{\tilde{\rho}_s}{\tilde{\rho}}\right)^{1/2} c_{10} \quad (10)$$

for the fourth-sound velocity in a stationary superfluid. Here c_{10} is the first-sound velocity at $T=0$ and we used Eq. (9). At the same time since $\rho_s = \rho$, we find from Eq. (6) that

$$\tilde{\rho}_s/\tilde{\rho} = g \quad . \quad (11)$$

The second term of Eq. (3), which represents the Doppler shift of fourth sound, then becomes

$$\tilde{u}_{s\parallel}\left(\frac{\partial \tilde{\rho}_s}{\partial \tilde{\rho}}\right)_{T=0,\tilde{u}_s} = \tilde{u}_{s\parallel}\frac{\tilde{\rho}_s}{\tilde{\rho}} = g\tilde{u}_{s\parallel} \quad . \quad (12)$$

Equations (10) and (12) are still valid for small $T \neq 0$ and for small $\tilde{u}_s \neq 0$, and we can therefore write the following simplified form of Eq. (3):

$$c_4 = g^{1/2}c_{10} + g\tilde{u}_{s\parallel} \quad . \quad (13)$$

In order to compare the above result with previous theoretical treatments, we first note that previous discussions were usually restricted either to fourth sound at zero flow or to steady superfluid flow. Consequently, different and seemingly independent concepts were developed to describe the observed phenomena.

Fourth sound was first discussed in detail by Atkins[14] for the case of a straight capillary, where $g=1$ and $\tilde{\rho}_s/\tilde{\rho} = \rho_s/\rho$. Later discussions of fourth sound in a superleak adopted the equations of Ref. 14 but included the effect of random scattering by the powder grains, which decreases the fourth-sound velocity, by introducing an index of refraction n.[3] This approach differs from our point of view but is of course just as valid. Moreover, from the first term of Eq. (13) it follows that

$$n^{-2} = g \quad , \quad (14)$$

at least for low temperatures. From Eq. (23) below and the subsequent discussion, it will be clear that this relation holds at arbitrary temperatures (neglecting size effects).

Static superfluid flow in superleaks has invariably been studied using circular geometries, i.e., a sphere or a ring-shaped container, packed with superleak material, in which a superfluid current is set up by changing the rotation velocity of the container. For a description of the steady superfluid flow, Refs. 5, 9, and 10 took the somewhat microscopic point view that the superfluid fraction is locally the same as in pure bulk helium but that some of it is dragged along by motion of the superleak. According to this approach, if we begin from a state in which both the helium and the superleak are at rest and then accelerate the superleak to a final velocity v_f, the bulk superfluid fraction will end up moving at an average velocity v_d that lies between 0 and v_f. Taking this point of view, the Doppler shift of fourth sound in the rest frame of the superleak was calculated in terms of these velocities and found to be[9,10]

$$(v_d - v_f)\rho_s/\rho \quad . \tag{15}$$

If size effects can be neglected, the quantity v_d may be identified with the space average of the microscopic superfluid velocity vector. This quantity is *not* in general equal to $\tilde{u}_s$, which we call the (macroscopic) superfluid velocity in the present paper. Note that a change in velocity of the superleak has no effect on $\tilde{u}_s$ [see Eq. (2a)]. Indeed, if no vortex lines are present, the net change of $\tilde{\varphi}$ along the superleak (i.e., the circulation around the ring), as well as the gradient $\nabla\tilde{\varphi}$, are unaffected by this motion. In describing the above-mentioned experiment we would say that the macroscopic superfluid velocity $\tilde{u}_s$ remained zero in the laboratory frame. Any microscopic flows of superfluid around moving obstacles are not considered to affect the superfluid velocity, but are rather considered to decrease the superfluid fraction $\tilde{\rho}_s/\tilde{\rho}$.

While the different points of view are largely a matter of personal taste, it is important to point out that since the Doppler shift and c_{40} both depend on g [see Eq. (13)], v_d is not an independent parameter. In our description we would say that, in the rest frame of the superleak,

$$\tilde{u}_s = -v_f \quad . \tag{16}$$

By comparing the two expressions (15) and (12) for the Doppler shift, and using (14) and (16), we find the following relation between v_d and n:

$$v_d = v_f(1 - n^{-2}) \quad . \tag{17}$$

The connection between c_{40} and the Doppler shift was recognized in the treatment of Ref. 12, whose result may, in our language, be written as

$$c_4 = \mathcal{P}^{-1} g^{1/2} c_{10} + g\tilde{u}_{s\parallel} \quad . \tag{18}$$

As far as we can tell, the additional factor $\mathcal{P}^{-1}$ compared to Eq. (13) is due to the assumption that, excluding the inertial drag due to potential flow around obstacles, the force which accelerates the macroscopic superfluid current in a unit volume of superleak is $F_s = -\nabla P$, where P is the pressure [see Eq. (5) of Ref. 12]. This is incorrect,[15] even though it has been asserted[16] or implied[17] in some well-known texts on theoretical physics and acoustics, in connection with the behavior of the flow of a classical fluid through a porous medium. In reality, this force must be obtained from Eq. (1b). The correct expression for it, neglecting the second-order velocity terms and using the relation $\rho\, d\tilde{\mu} = \rho\, d\mu = dP$, valid at $T=0$ [see Eq. (8)], and Eq. (5), is

$$F_s \equiv \tilde{\rho}_s \frac{d}{dt}\tilde{u}_s = -\tilde{\rho}_s \nabla\tilde{\mu} = -\left(\frac{\tilde{\rho}_s}{\rho}\right)\nabla P = -\mathcal{P}\left(\frac{\tilde{\rho}_s}{\tilde{\rho}}\right)\nabla P \quad . \tag{19}$$

B. Finite temperatures

At finite temperatures, but when $\tilde{u}_s$ is small and size effects are still unimportant, we can again simplify the general expression of Eq. (3). The resulting equation will have a form similar to Eq. (13), and will depend on thermodynamic properties of the filler material as well as pure helium, in addition to the two geometric parameters characteristic of the superleak, g and $\mathcal{P}$.

For the fourth-sound velocity in a stationary superfluid ($\tilde{u}_s = 0$) we find

$$c_{40}^2 = \tilde{\rho}_s\left[\left(\frac{\partial\tilde{\mu}}{\partial\tilde{\rho}}\right)_T - \left(\frac{\partial\tilde{\mu}}{\partial T}\right)_{\tilde{\rho}}\left(\frac{\partial\tilde{S}}{\partial\tilde{\rho}}\right)_T\left(\frac{\partial\tilde{S}}{\partial T}\right)_{\tilde{\rho}}^{-1}\right]$$
$$= \frac{\tilde{\rho}_s}{\tilde{\rho}}\rho\left[\left(\frac{\partial\mu}{\partial\rho}\right)_T + \mathcal{P}\left(\frac{\partial\mu}{\partial T}\right)_\rho^2\left(\frac{\partial\tilde{S}}{\partial T}\right)_\rho^{-1}\right] \quad , \tag{20}$$

where we used $d\tilde{\rho} = \mathcal{P}\, d\rho$ and

$$\left(\frac{\partial\tilde{S}}{\partial\tilde{\rho}}\right)_T = \left(\frac{\partial\tilde{S}_H}{\partial\tilde{\rho}}\right)_T = \left(\frac{\partial S}{\partial\rho}\right)_T = -\left(\frac{\partial\mu}{\partial T}\right)_\rho \quad . \tag{21}$$

Once again, S denotes the entropy per unit volume of pure helium, whereas $\tilde{S}_H$ denotes the entropy of pure helium per unit of the total volume. Let us note that

$$\tilde{C} \equiv T\left(\frac{\partial\tilde{S}}{\partial T}\right)_{\tilde{\rho}} = \mathcal{P}C_v + (1-\mathcal{P})C_f \quad , \tag{22}$$

where C_v and C_f are the constant-volume specific heats per unit volume of pure helium and pure filler material. Moreover, expressing the pure-helium quantities $\partial\mu/\partial\rho$ and $\partial\mu/\partial T$ in terms of measurable parameters, and using (6) and (22), we finally obtain

$$c_{40}^2 = g\frac{\rho_s}{\rho}\,\frac{1}{\rho K_T}\left[1 + TS^2K_T\left(1 - \frac{\alpha}{SK_T}\right)^2\left(C_v + \frac{1-\mathcal{P}}{\mathcal{P}}C_f\right)^{-1}\right], \tag{23}$$

where α is the thermal-expansion coefficient of pure helium. All of the quantities appearing in this equation with the exception of g, $\mathcal{P}$, and C_f are attributes of pure bulk helium. The second term in the square brackets is always very small even if C_f is ignored. Taking C_f into account makes this term even smaller. Therefore, in practice the first term is dominant, and g is the more-important geometric parameter.

To calculate the Doppler shift for small $\tilde{u}_s$, at finite T, we note that neglecting terms of order $\tilde{u}_s^2$, we have

$$\left(\frac{\partial\tilde{\rho}_s}{\partial\tilde{\rho}}\right)_{\tilde{S},\tilde{u}_s}=\frac{\tilde{\rho}_s}{\tilde{\rho}}+\tilde{\rho}\left(\frac{\partial(\tilde{\rho}_s/\tilde{\rho})}{\partial\tilde{\rho}}\right)_{\tilde{S},\tilde{u}_s=0} . \tag{24}$$

The last term can be developed as follows:

$$\tilde{\rho}\left(\frac{\partial(\tilde{\rho}_s/\tilde{\rho})}{\partial\tilde{\rho}}\right)_{\tilde{S}}=g\left(\frac{\partial(\rho_s/\rho)}{\partial P}\right)_{\tilde{S}}\left[\frac{1}{\rho}\left(\frac{\partial\rho}{\partial P}\right)_{\tilde{S}}\right]^{-1}$$
$$=\left[g\left(\frac{\partial(\rho_s/\rho)}{\partial P}\right)_T+\left(\frac{\partial(\rho_s/\rho)}{\partial T}\right)_P\left(\frac{\partial T}{\partial P}\right)_{\tilde{S}}\right]$$
$$\times\left[\frac{1}{\rho}\left(\frac{\partial\rho}{\partial P}\right)_T+\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_P\left(\frac{\partial T}{\partial P}\right)_{\tilde{S}}\right]^{-1} . \tag{25}$$

The factor $\partial T/\partial P$ can be evaluated in terms of pure-helium and pure-filler quantities, as well as $\mathcal{P}$:

$$\left(\frac{\partial T}{\partial P}\right)_{\tilde{S}}=\frac{\mathcal{P}(\alpha-SK_T)+(1-\mathcal{P})(\alpha_f-SK_{Tf})}{\mathcal{P}(C_P/T-\alpha S)+(1-\mathcal{P})(C_f/T-\alpha_f S_f)}$$
$$=\frac{\alpha-SK_T}{(C_P/T)-\alpha S+(1-\mathcal{P})C_f/\mathcal{P}T} , \tag{26}$$

where the last form results from using the assumed rigidity of the superleak, i.e., $\alpha_f=K_{Tf}=0$. Quite often one can neglect the filler specific heat in (26) since $(1-\mathcal{P})/\mathcal{P}$ is usually of order 1 while $C_f \ll C_P$ for a filler such as aluminum oxide. We then have

$$\left(\frac{\partial T}{\partial P}\right)_{\tilde{S}}\cong\left(\frac{\partial T}{\partial P}\right)_S , \tag{27}$$

and consequently we can write for the Doppler shift

$$\tilde{u}_{s\parallel}\left(\frac{\partial\tilde{\rho}_s}{\partial\tilde{\rho}}\right)_{\tilde{S},\tilde{u}_s}\cong\tilde{u}_{s\parallel}\frac{\rho_s}{\rho}g\left[1+\frac{\rho^2}{\rho_s}\left(\frac{\partial(\rho_s/\rho)}{\partial\rho}\right)_{S,v_s=0}\right] . \tag{28}$$

The square brackets now contain only pure-helium quantities and there is no dependence on $\mathcal{P}$, only on g. The second term in the brackets can be evaluated by expanding the pure-helium derivative as in (25) and using static thermodynamic data on pure helium. Rough estimates indicate that this term, while no more than a few per cent below 1.3 °K, can become quite significant above 1.4 °K.

We can summarize this discussion by giving the following expression for c_4, valid for all temperatures when size effects are negligible and $C_f(1-\mathcal{P})/\mathcal{P} \ll C_v$:

$$c_4=\left(g\frac{\rho_s}{\rho}\frac{1}{\rho K_T}\right)^{1/2}\left[1+\frac{TK_T}{C_v}\left(S-\frac{\alpha}{K_T}\right)^2\right]^{1/2}$$
$$+g\frac{\rho_s}{\rho}\tilde{u}_{s\parallel}\left[1+\frac{\rho^2}{\rho_s}\left(\frac{\partial(\rho_s/\rho)}{\partial\rho}\right)_S\right]+O(\tilde{u}_s^2) . \tag{29}$$

We would like to point out that while the temperature dependence of c_{40} was derived correctly already by Atkins,[14] a fully correct expression for the temperature dependence of the Doppler shift is given here for the first time. Kojima has made an essentially correct analysis of this quantity[11] but the final result that is quoted in Refs. 10 and 11 is not exact. Moreover, it turns out that one of the terms omitted from his equation is quantitatively quite significant. The fact that the same geometric factor g must appear in both parts of the expression for c_4 does not seem to have been noted before. However, a relation identical to our Eq. (17), connecting n, the index of refraction for fourth sound, with the dragging velocity v_d of the superfluid by the filler material in potential flow, has been obtained before by Yanof and Reppy.[8,18] (See footnote 6 of Ref. 8.)

C. Higher-order flow-velocity corrections

For sufficiently large flow velocities $\tilde{u}_s$, corrections to Eq. (29) will appear due to the velocity dependence of $\tilde{\mu}$ and $\tilde{\rho}_s$, as is apparent from Eq. (3). For the case where size effects are unimportant, these corrections are again obtainable in terms of thermodynamic quantities of pure-helium and pure-filler material, as well as geometric factors of the superleak. A detailed calculation of the $O(\tilde{u}_s^2)$ corrections to the relevant thermodynamic quantities is given in Appendix B. Here we will merely point out that the contribution of $\tilde{\rho}_s$ to the $O(\tilde{u}_s^2)$ corrections in c_4 is enhanced by a factor 3 for fourth-sound waves traveling parallel to $\tilde{u}_s$. This is due to the appearance of the term

$$\tilde{\rho}_s+\tilde{u}_{s\parallel}^2\left(\frac{\partial\tilde{\rho}_s}{\partial\frac{1}{2}\tilde{u}_s^2}\right)_{\tilde{\rho},\tilde{S}}$$

in the first part of Eq. (3), rather than just $\tilde{\rho}_s$. In practice these corrections to c_4 are always expected to be very small, and in fact they have not yet been observed.

IV. EXPERIMENTAL CONSEQUENCES

In order to compare our expression for the fourth-sound velocity with experiments, the static quantities appearing in Eq. (3) must be known. In the general case, when size effects are important, these would have to be determined separately for each particular superleak, either by a measurement or by a microscopic calculation. In order to make contact with measurable quantities, we express the thermodynamic derivatives appearing in (3) as follows:

$$\left(\frac{\partial\tilde{\mu}}{\partial\tilde{\rho}}\right)_{\tilde{S}}=\left(\frac{\partial\tilde{\mu}}{\partial\tilde{\rho}}\right)_T-\left(\frac{\partial\tilde{\mu}}{\partial T}\right)_{\tilde{\rho}}\left(\frac{\partial\tilde{S}}{\partial\tilde{\rho}}\right)_T\left(\frac{\partial\tilde{S}}{\partial T}\right)_{\tilde{\rho}}^{-1}$$

$$=\left(\frac{\partial\tilde{\mu}}{\partial\tilde{\rho}}\right)_T+\left(\frac{\partial\tilde{\mu}}{\partial T}\right)_{\tilde{\rho}}^2\frac{T}{\tilde{C}}\ , \tag{30}$$

$$\left(\frac{\partial\tilde{\rho}_s}{\partial\tilde{\rho}}\right)_{\tilde{S}}=\left(\frac{\partial\tilde{\rho}_s}{\partial\tilde{\rho}}\right)_T-\left(\frac{\partial\tilde{\rho}_s}{\partial T}\right)_{\tilde{\rho}}\left(\frac{\partial\tilde{S}}{\partial\tilde{\rho}}\right)_T\left(\frac{\partial\tilde{S}}{\partial T}\right)_{\tilde{\rho}}^{-1}$$

$$=\left(\frac{\partial\tilde{\rho}_s}{\partial\tilde{\rho}}\right)_T+\left(\frac{\partial\tilde{\rho}_s}{\partial T}\right)_{\tilde{\rho}}\left(\frac{\partial\tilde{\mu}}{\partial T}\right)_{\tilde{\rho}}\frac{T}{\tilde{C}}\ , \tag{31}$$

where all the derivatives are taken at $\tilde{u}_s=$ const. The necessary quantities can be determined, in principle, as follows: $\tilde{\rho}$ is obtained by weighing the filled and empty superleak, and $\tilde{\rho}_s$ can be found be measuring the momentum of a dc superfluid current associated with a known velocity $\tilde{u}_s$. The velocity is presumably known when the flow is generated by carefully bringing to a stop a superleak filled with helium at $T<T_\lambda$ that had been set into motion above T_λ. The chemical potential $\tilde{\mu}$ can be determined by having helium in the superleak in equilibrium with a pure-helium bath. Both $\tilde{\rho}_s$ and $\tilde{\mu}$ can thus be determined as functions of $\tilde{\rho}$ and T. Finally, $\tilde{C}$ is just the total heat capacity of the filled superleak. If all of that information is put into Eq. (3), one obtains a definite prediction for the velocity and Doppler shift of fourth sound.

It is much easier to make the comparison with experiments when size effects are unimportant: We have shown that in that case, and for low velocities $\tilde{u}_s$, c_4 is entirely determined by pure-helium and pure-filler attributes plus the two geometric parameters g and $\mathcal{P}$, and that in most cases only g is important. Using Eq. (29), it is clear that if all of the pure-helium quantities are known, we can investigate the validity of the theory, and also get a value for g or $\tilde{\rho}_s/\tilde{\rho}$, by measuring the velocity of fourth sound and its Doppler shift. Such measurements have actually been performed using a ring-shaped fourth-sound resonator in which superfluid flow was induced by rotating the ring around its axis.[6,9–11,19] In this resonator one observes several regimes of superfluid flow. A complicating factor is that at high velocities vortices are apparently created and we cannot infer the value of $\tilde{u}_s$ from the history of the flow. Furthermore, if the vortices manage to move around during a single period of the fourth-sound wave, this will change the fourth-sound velocity. But if all velocities are kept low enough throughout the experiment, the system exhibits reversible behavior and its state appears to depend only on the present velocity of rotation of the ring and not on previous history. The reversibility is believed to indicate that no vortices are present[9,10] and that the flow is a pure potential flow. From our point of view this means that if the ring were first cooled below T_λ while at rest, $\tilde{u}_s$ vanishes in the laboratory frame throughout the experiment and is always equal to minus the ring velocity in the ring frame of reference.

For this regime there are data available on the temperature dependence of the Doppler shift in the range of temperatures 1.2–2.0 °K.[11] A comparison of these results with Eq. (28) can be meaningful since g is constant, even though its precise value is not known. To make this comparison we must have information about the thermodynamic quantities ρ_s/ρ and $[\partial(\rho_s/\rho)/\partial\rho]_S$. For the latter quantity we turn to Eq. (25), with $\tilde{S}$ replaced by S, and find that the result depends on a partial cancellation between $[\partial(\rho_s/\rho)/\partial P]_T$ and $[\partial(\rho_s/\rho)/\partial T]_P(\partial T/\partial P)_S$. Unfortunately, the data which are available for calculating $[\partial(\rho_s/\rho)/\partial P]_T$ are not sufficiently accurate to permit a meaningful comparison between theory and experiment for the Doppler shift: These data are measurements of the second-sound velocity versus pressure $c_2(P)$ by Peshkov and Zinoveva[20] and by Maurer and Herlin,[21] which seem to be in fairly good agreement with each other on the values of $c_2(P)$. These data, however, allow a range of values for $(\partial c_2/\partial P)_T$ at $P=0$ which leads to a large uncertainty in the temperature-dependent correction factor

$$D(T)\equiv\frac{\rho^2}{\rho_s}\left(\frac{\partial(\rho_s/\rho)}{\partial\rho}\right)_S$$

of Eq. (28). Measurements of the Doppler shift can be used to obtain $1+D(T)$ by taking out the factor ρ_s/ρ and normalizing the result to be unity at low temperatures. Denoting the value of D calculated from the above-mentioned $c_2(P)$ data by D_{calc} and the value of D calculated from the Doppler shift data of Kojima (see Fig. 46 of Ref. 11) by D_{exp}, we find for two representative temperatures:

$$-0.04<D_{\text{calc}}<0.03,\quad D_{\text{expt}}=-0.05\quad\text{at } T=1.3\ ^\circ\text{K}\ ,$$

$$-0.24<D_{\text{calc}}<0.14,\quad D_{\text{expt}}=-0.54\quad\text{at } T=1.7\ ^\circ\text{K}\ ,$$

While D_{expt} seems to be outside the range of values found for D_{calc}, especially at the higher temperature, we feel that the evidence for a discrepancy is not conclusive. More careful measurements of $[\partial(\rho_s/\rho)/\partial P]_T$ are needed at $P=0$ in order to test the theory in this way.

The difficulties encountered above in trying to compare the theory with experiment are alleviated at temperatures below 1.3 °K, where the temperature dependence of all the terms in Eq. (29) becomes unimportant and $\rho_s/\rho\cong 1$, so that we can use Eq. (13) instead of (29). Since c_4 still depends on the unknown geometric factor g, which can vary from superleak to superleak, we need a measurement of both c_{40} and the Doppler shift on the same system in order to make a meaningful comparison. Such measurements have been published for just one ring-shaped fourth-sound resonator (i.e., the resonator FSR IIIa described in Ref. 11). From the frequency of the fundamental resonance in that resonator at $T = 1.2\,^\circ$K we deduce $g = 0.808$, while from the Doppler shift (Fig. 27 of Ref. 11) at low rotation speeds and $T = 1.3\,^\circ$K we deduce $g = 0.65$. Other unpublished data collected in the above-mentioned potential flow regime[22] confirm and strengthen this discrepancy between Eq. (13) and the experimental results: The measured value of g obtained from c_{40} is consistently greater than the value obtained from the Doppler shift. The relative difference varies in magnitude and is sometimes as high as 40%. In view of this discrepancy, we now turn to an examination of the assumptions necessary to obtain the hydrodynamic equations (1).

Except for the possible effects of vortex lines, which were assumed to be absent in the hydrodynamic theory, the neglect of normal-fluid motion and other dissipative processes is expected to be exact in the limit of long wavelengths and low frequencies.[23,24] At low temperatures, the relaxation time involved in setting up the complicated flow pattern is expected to be of the order d/c_1, where d is the pore size and c_1 is the first-sound velocity in pure helium. Therefore, if $\omega d/c_1 \approx kd \ll 1$, as is certainly the case in the experiments of interest here, this dissipative process is unimportant. (Note that $k \approx \omega/c_1$ since c_{40} is of the order of c_1.) This conclusion would change if there were a long relaxation time τ_p involved in setting up the complicated potential flow pattern. In that case one would see a different effective superfluid fraction $\tilde{\rho}_s/\tilde{\rho}$ depending on whether $\omega\tau_p$ is greater or less than unity. The long-wavelength limit also assumes that inhomogeneities in the superleak are unimportant; this is expected to be the case whenever the wavelength is much greater than the pore size, i.e., we again have the requirement $kd \ll 1$.

At higher temperatures, we also require the frequency to be sufficiently low so that the characteristic viscous and thermal decay lengths of the bulk normal fluid inside the pores will be greater than the pore size. This condition appears to be well satisfied in the experiments of Refs. 6, 9–11, and 19. Even if this were not the case, however, the bulk ρ_n/ρ is small at the temperatures in question, so that the difference between the "low-frequency" and "high-frequency" fourth-sound velocities would be small.

One may next ask whether the presence of vortex lines could account for a difference between the effective $\tilde{\rho}_s/\tilde{\rho}$ which determines the finite-frequency fourth-sound velocity, and the dc value of $\tilde{\rho}_s/\tilde{\rho}$ which enters in our interpretation of the Doppler shift. Such vortex lines apparently are produced in the ring-shaped superleaks when these are rotated fast enough.[9,10] If, however, the vortices are completely pinned, there should be no effect on the velocity or on the Doppler shift of fourth sound. If the vortices become unpinned, then their motion will reduce the apparent $\tilde{\rho}_s/\tilde{\rho}$. In order for the vortices to become unpinned, a minimum threshold velocity is presumably required, since the Magnus force must be greater than the pinning force. Furthermore, since there must be a relaxation time τ_v associated with the movement of vortices in the superleak, the subsequent effects will depend on the frequency. For high frequencies, $\omega\tau_v \gg 1$, we expect to see no change in $\tilde{\rho}_s/\tilde{\rho}$ and only some additional attenuation of fourth-sound waves. For low frequencies $\omega\tau_v \ll 1$ (including the case of dc flow) we expect to see changes in $\tilde{\rho}_s/\tilde{\rho}$, since the vortex lines have enough time to move across the flow and lower the superfluid current during a single period. The rotating-ring experiments are in qualitative agreement with these expectations: At high rotation velocities the fraction of helium which appears to be dragged by the superleak increases, which is another way to say that the apparent $\tilde{\rho}_s/\tilde{\rho}$ for dc flow decreases. But the superfluid is never dragged along completely—the dissipative mechanism that allows more and more of the helium to be set into motion by the superleak (some of this decay has actually been observed experimentally[19]) ceases to operate when the difference in velocities becomes too small. Presumably the Magnus force is then no longer able to overcome the pinning force. The velocity c_{40}, which is determined from the average of c_4 for a fourth-sound wave traveling upstream and downstream with respect to $\tilde{u}_s$, remains almost constant at these high rotation velocities: It decreases by only 0.3% as $\tilde{u}_s g$ increases from the onset of irreversibility to its saturated maximum value of ≈ 60 cm/sec. (See Fig. 52 of Ref. 11.) There is also an increase in the attenuation.[22]

All of these phenomena, however, take place only at high rotation velocities where the system exhibits a history-dependent and irreversible behavior. They are completely absent from the low-velocity reversible region, and indeed there is no particular evidence for the occurrence of vortex lines in this regime.

An alternative explanation for the discrepancy between the c_{40} and Doppler-shift measurements might be found if for some reason the flow pattern that appears in the system when it carries a persistent dc current is quite different from the flow pattern that is set up by a fourth-sound wave, thus leading to different values of g for the two cases. In this connection it should be mentioned that even at the slowest ring velocities, the dc flow rates are typically much greater than the oscillating superfluid velocities in a fourth-sound wave[22] (this is the opposite of the situation in bulk helium, where the dc critical velocity is often exceeded in an intense sound wave). Moreover, the total excursion of a given element of fluid from its initial position is also much greater in the dc flow. One could therefore imagine that small macroscopic cracks in the filler powder would create enough dissipation to make the flow in the superleak relax towards a pattern which effectively circumvents these cracks. This relaxation might only be effective in the case of the dc flow, owing either to the length of the relaxation time or to the size of the critical velocity associated with the relaxation. Since the lowest rotation speeds for which the Doppler shift was measured correspond to $v_f \cong 4$ cm/sec (see Fig. 2 of Ref. 10), and since g was always found to be constant over the entire regime of reversible flow, the hypothesized critical velocity must be less than 4 cm/sec. To account quantitatively for such a value would require cracks or voids in excess of 20 μm, and it seems doubtful that cracks of that size would have been overlooked.[11] While the detailed mechanism just outlined probably does not operate in practice, there may nevertheless exist other mechanisms for producing a different pattern for the dc and the ac flows.

If a discrepancy between $\bar{\rho}_s/\bar{\rho}$ obtained from c_{40} and the Doppler shift is to be explained in terms of a relaxation effect—arising from either cracks or vortex motion or from any other cause—then the presence of relaxation or of a new critical velocity should be manifest in suitably designed experiments.

In order to estimate the possible magnitude of the relaxation time τ that we are seeking, we note that a lower bound to it can be obtained from the width of the fourth-sound resonances in the ring-shaped superleak. Since in the best cases this width was about 2–4 Hz,[6,10,11] but the hypothesized relaxation involved at most 40% of the superfluid current, we get a lower bound of 0.03 to 0.02 sec for τ. An upper bound on τ is obtained by noting that no decay of a persistent current in the reversible regime was ever found from the Doppler-shift measurements made beginning a few minutes after the superfluid flow was set up.[22] We may thus infer that the relaxation time we are seeking lies somewhere in the region 10^{-2}–10^{2} sec. Part of this range, i.e., $\tau > 1$ sec, is experimentally accessible in the usual ring-shaped fourth-sound resonator by using the technique of observing the beats of a freely oscillating Doppler-split resonant mode to measure the decay of the dc superfluid flow.[11,19] The other part, i.e., $10^{-2} < \tau < 1$ sec, could perhaps be explored by making fourth-sound resonators whose characteristic periods were in the above mentioned range and looking for dispersion and an enhanced dissipation.

A new critical velocity v_c, which must be less than 4 cm/sec, would manifest itself if one did experiments at rotation velocities below v_c. One would then find a new regime of superfluid flow for extremely slow rotation velocities in which the value of g obtained from Doppler shifts would approach the value obtained from c_{40}.

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APPENDIX A: POTENTIAL FLOW OF HELIUM IN A SUPERLEAK IN THE ABSENCE OF SIZE EFFECTS

When size effects are unimportant, the properties of helium flowing through a superleak are determined by a boundary-value problem in the classical theory of potential flow.

We consider a macroscopic length L of superleak with a total cross sectional area A. [See Fig. 1(a).] Let $\theta(r)$ be a function which is 1 when r is a point occupied by helium and 0 when r is occupied by filler material. Clearly

$$\mathcal{P} = \frac{1}{LA} \int \theta(r) d^3r \quad . \tag{A1}$$

Let $\varphi(r)$ be the local value of the phase of the helium order parameter, and $v_s(r) = (\hbar/m)\nabla\varphi(r)$ the local value of the superfluid velocity, defined on a scale small compared to the pore size. In a steady state we must have

$$\nabla \cdot [\theta(r)\nabla\varphi(r)] = 0 \tag{A2}$$

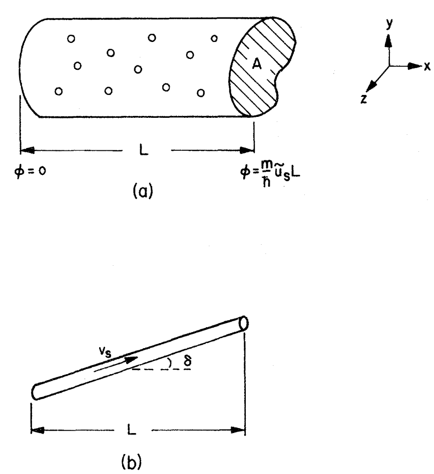

FIG. 1. (a) Schematic drawing of an open-ended, powder-packed, cylindrical superleak with superfluid flow along the cylinder axis in the x direction, and impermeable boundaries at the lateral walls. The length is L and the cross-sectional area is A. (b) A single, obstruction-free tube spanning the system. The volume of the tube is Ω.

inside the superleak. If we impose boundary conditions

$$\begin{aligned} &\varphi(r)=0 && \text{at } x=0 \ , \\ &\varphi(r)=(m/\hbar)\tilde{u}_s L && \text{at } x=L \ , \\ &\frac{\partial\varphi}{\partial n}=0 && \text{at the other boundaries} \ , \end{aligned} \tag{A3}$$

we will find a unique solution of (A2) with $\varphi(r)$ linear in $\tilde{u}_s$.

The excess energy of the superfluid due to the nonzero flow velocity may be written as

$$E[\varphi]=\tfrac{1}{2}\rho_s\left(\frac{\hbar}{m}\right)^2\int \theta(r)(\nabla\varphi)^2 d^3r \ , \tag{A4a}$$

$$\equiv \tfrac{1}{2} g\,\rho_s \mathcal{P} L A \tilde{u}_s^2 \equiv \tfrac{1}{2}\tilde{\rho}_s L A\, \tilde{u}_s^2 \ . \tag{A4b}$$

A function which obeys (A2) and (A3) also minimizes (A4a) subject to the constraint (A3). Hence we may establish an upper bound to E by choosing the simple trial function $\varphi_0(r)=(m/\hbar)\tilde{u}_s x$:

$$E < E[\varphi_0]=\tfrac{1}{2}\rho_s \mathcal{P} L A\, \tilde{u}_s^2 \ . \tag{A5}$$

By comparison with (A4b) it follows that $g<1$.

The momentum carried by the flow is

$$P=\rho_s\left(\frac{\hbar}{m}\right)\int \theta(r)\nabla\varphi\, d^3r \ , \tag{A6}$$

which is obviously linear in $\tilde{u}_s$. With the intention of determining the exact dependence of P on $\tilde{u}_s$ we use (A4a) to calculate the variation of E when $\tilde{u}_s$ is given a small increment $\delta\tilde{u}_s$:

$$\delta E=\rho_s\left(\frac{\hbar}{m}\right)^2\left[\int \nabla\cdot(\delta\varphi\theta\nabla\varphi)d^3r - \int \delta\varphi\nabla\cdot(\theta\nabla\varphi)d^3r\right] . \tag{A7}$$

The second integral obviously vanishes because φ obeys (A2). Since the first integral can be transformed to a surface integral, we may replace $\delta\varphi$ by $(m/\hbar)\delta\tilde{u}_s x$: Both functions are identical at the ends and the integrand vanishes at the other boundaries. The $\nabla\cdot$ operation may now be carried out explicitly in the first integral

$$\delta E=\rho_s\frac{\hbar}{m}\delta\tilde{u}_s\int \theta\frac{\partial\varphi}{\partial x}d^3r \equiv \delta\tilde{u}_s P_x \ . \tag{A8}$$

Therefore, using (A4b) for E, we find

$$P_x=\tilde{\rho}_s\tilde{u}_s LA \ . \tag{A9}$$

The momentum components P_y and P_z will vanish if the superleak is isotropic. They will also vanish even when the superleak is not isotropic but is periodic in the x direction, as in the case of a ring-shaped superleak where the x axis goes around the circumference. In the general case, P_y and P_z do not vanish and the effective superfluid density is then a second-rank symmetric tensor rather than a scalar quantity.

From (A9), (A4b), and (A1) we find that another way to calculate g is

$$g=\frac{[\int\theta(\partial\varphi/\partial x)d^3r]^2}{[\int\theta(\nabla\varphi)^2d^3r](\int\theta d^3r)} \equiv \frac{\langle\partial\varphi/\partial x\rangle^2}{\langle(\nabla\varphi)^2\rangle} \ . \tag{A10}$$

From this equation and the Cauchy-Schwartz inequality, it is again clear that $g<1$ unless $\nabla\varphi$ is a constant and points in the x direction. An expression of this type has recently been given by Yanof and Reppy.[8]

As a simple model of a superleak for which the above discussion is applicable, consider a solid permeated by a large number of nonintersecting straight tubes free of any obstructions, with a certain distribution of cross sectional areas and with completely random orientations. Consider first a single long tube which stretches from one end of the system to the other making an angle δ with the x direction [see Fig. 1(b)]. Given the boundary conditions (A3), the velocity in that tube will be

$$(\hbar/m)|\nabla\varphi|=v_s=\tilde{u}_s\cos\delta \ . \tag{A11}$$

The x component of momentum carried by fluid in the tube is

$$\Omega\rho_s\tilde{u}_s\cos^2\delta \ , \tag{A12}$$

where Ω is the volume of the tube. The total volume of all the tubes is just $\mathcal{P}AL$ and the total momentum carried in them is

$$P_x=\mathcal{P}AL\,\tilde{u}_s\rho_s\langle\cos^2\delta\rangle \ . \tag{A13}$$

Therefore, by comparison with (A9) we find

$$\tilde{\rho}_s=\mathcal{P}\rho_s\langle\cos^2\delta\rangle=\tfrac{1}{3}\mathcal{P}\rho_s$$

or

$$g=\langle\cos^2\delta\rangle=\tfrac{1}{3} \ . \tag{A14}$$

This result should also apply to a system of interconnected tubes if the sections between junctions are straight and long compared to the diameter. From measurements of c_{40}, g is usually found to be much greater than $\frac{1}{3}$ in compressed powders.[4] In fact, these measurements indicate that $g\equiv 1/n^2 \geq 0.65$. The value $\frac{1}{3}$ is also outside the range of values permitted by the empirical formula

$$g=2-\mathcal{P} \ , \tag{A15}$$

which was proposed to describe these experimental results.[4] Indeed, one might expect that the flow pattern is less distorted by the presence of obstacles in the form of a packed powder than for the straight-tube model. The latter model might be better suited to discuss superfluid flow in Vycor glass, in which the channels may have more nearly uniform cross sections and relatively fewer interconnections. Some measurements in Vycor glass 7930 show in fact a large reduction in c_{40}^2 relative to the "bulk value" $(\rho_s/\rho)c_{10}^2$.[25,26] If the observed reduction were entirely due to geometric effects, it would mean that $g \cong \frac{1}{4}$ to $\frac{1}{5}$. However, it is possible that for the narrow tubes (mean diameter of about 40 Å) used in these experiments, size effects are important even at low temperatures. We should also point out that while results from Refs. 25 and 26 are in rough agreement with each other, they disagree with the results of Ref. 27.

APPENDIX B: HIGHER-ORDER FLOW-VELOCITY CORRECTIONS IN THE ABSENCE OF SIZE EFFECTS

In the absence of size effects, higher-order corrections depending on $\tilde{u}_s$ arise from the velocity dependence of the thermodynamic properties of pure bulk helium. The discussion of these corrections is facilitated by a proper choice of the independent variables used to describe the bulk helium inside the pores. We will use the variables T, $\overline{\mu} \equiv \mu + \frac{1}{2}v_s^2$, and v_s. The advantage of using T and $\overline{\mu}$ rather than S and ρ is that in a state of steady flow in the superleak, the former are constant everywhere whereas the latter vary from place and have to be averaged over when passing from the "microscopic" description of bulk helium inside the pores to a "macroscopic" description of the homogeneous system of helium in the superleak. In terms of these variables, the basic thermodynamic equality becomes

$$d\Gamma = -S\,dT - \rho\, d\overline{\mu} + \rho_s \mathbf{v}_s \cdot d\mathbf{v}_s \quad , \tag{B1}$$

where

$$\Gamma \equiv E - TS - \rho\overline{\mu} \tag{B2}$$

is the grand canonical potential per unit volume of pure bulk helium.

The total grand potential of helium in the superleak per unit of the *total* volume $\tilde{\Gamma}_H$ is obtained by averaging Γ over all the pores containing helium,

$$\tilde{\Gamma}_H \equiv \frac{1}{AL}\int d^3r\, \theta(r)\, \Gamma(T, \overline{\mu}, v_s(r)) \tag{B3a}$$

$$= \mathcal{P}\langle \Gamma\rangle \quad , \tag{B3b}$$

where the averaging notation here and below is used in the following sense:

$$\langle f\rangle \equiv \frac{\int d^3r\,\theta(r) f(r)}{\int d^3r\,\theta(r)} \quad . \tag{B4}$$

Since $\mathbf{v}_s = (\hbar/m)\nabla\varphi$, and since ρ_s now depends on r through v_s, φ satisfies a modified form of Eq. (A2) in the steady state,

$$\nabla \cdot [\theta \rho_s(v_s)\nabla\varphi] = 0 \quad , \tag{B5}$$

as well as the boundary conditions (A3). Using these facts and the technique employed in deriving (A8) from (A4a) in order to average the last term of (B1), we can show that

$$d\tilde{\Gamma}_H \equiv \frac{1}{AL}\int d^3r\,\theta\,\delta\Gamma$$

$$= -\tilde{S}_H\,dT - \tilde{\rho}\,d\overline{\mu} + \tilde{\rho}_s \tilde{u}_s\,d\tilde{u}_s \quad , \tag{B6}$$

where $\tilde{S}_H$ and $\tilde{\rho}$ are defined by analogy with $\tilde{\Gamma}_H$, and

$$\tilde{\rho}_s\tilde{u}_s \equiv \frac{1}{AL}\int d^3r\,\rho_s(v_s)v_{sx} = \mathcal{P}\langle \rho_s v_{sx}\rangle \quad . \tag{B7}$$

As observed in Appendix A, there exist alternative methods for calculating $\tilde{\rho}_s$. In this case the best one seems to be to first calculate $\tilde{\Gamma}_H$ and then use

$$\tilde{\rho}_s\tilde{u}_s = \left(\frac{\partial\tilde{\Gamma}_H}{\partial\tilde{u}_s}\right)_{T,\overline{\mu}} \quad . \tag{B8}$$

The advantage of using this expression to calculate $\tilde{\rho}_s$ stems from the fact that any solution of (B5) and (A3) leads to a stationary value for the integral in (B3a). Therefore, even if one knows φ only to lowest order in $\tilde{u}_s$, one obtains $\tilde{\Gamma}_H$ and consequently $\tilde{\rho}_s$ to the next higher order in $\tilde{u}_s$. Since ρ_s includes only even powers of v_s, the solution of (A2) and (A3) thus enables us to get $\tilde{\rho}_s$ correctly up to $O(\tilde{u}_s^2)$ by using (B8).

Introducing the notation

$$\varphi(r) = (m/\hbar)\tilde{u}_s\varphi_1(r) + O(\tilde{u}_s^3) \quad , \tag{B9}$$

$$\rho_s(T,\overline{\mu},v_s) = \rho_{s0}(T,\overline{\mu}) - \tfrac{1}{2}\rho_{s2}(T,\overline{\mu})v_s^2 + O(v_s^4) \tag{B10}$$

for the leading orders in the velocity expansions of φ and ρ_s, we get the following expressions for the thermodynamic functions of helium in the superleak, including velocity corrections:

$$\tilde{\Gamma}_H(T,\overline{\mu},\tilde{u}_s) = \mathcal{P}[\Gamma_0(T,\overline{\mu}) + \tfrac{1}{2}g\rho_{s0}\tilde{u}_s^2 - \tfrac{1}{8}g_2\rho_{s2}\tilde{u}_s^4] + O(\tilde{u}_s^6) \quad , \tag{B11}$$

$$\tilde{\rho}_s = \mathcal{P}(g\rho_{s0} - \tfrac{1}{2}g_2\rho_{s2}\tilde{u}_s^2) + O(\tilde{u}_s^4) \quad , \tag{B12}$$

$$\tilde{S} = \mathcal{P}\left(S_0(T,\overline{\mu}) - \tfrac{1}{2}g\frac{\partial\rho_{s0}}{\partial T}\tilde{u}_s^2\right) + O(\tilde{u}_s^4) \quad , \tag{B13}$$

$$\tilde{\rho} = \mathcal{P}\left(\rho_0(T,\overline{\mu}) - \tfrac{1}{2}g\frac{\partial\rho_{s0}}{\partial\overline{\mu}}\tilde{u}_s^2\right) + O(\tilde{u}_s^4) \quad , \tag{B14}$$

where

$$g \equiv \langle(\nabla\varphi_1)^2\rangle \tag{B15}$$

is the geometric factor originally defined in Eq. (6) and discussed in Appendix A, and

$$g_2 \equiv \langle(\nabla\varphi_1)^4\rangle \tag{B16}$$

is a new geometric factor. Γ_0, S_0, ρ_0 are the bulk helium quantities at zero flow.

Note that whereas we showed before that $g < 1$, all we can show in general about g_2 is that $g_2 > g^2$ (this follows from the Cauchy-Schwartz inequality). For the randomly-oriented-straight-tubes model of the superleak discussed in Appendix A we find

$$g_2 = \langle \cos^4\delta \rangle = \tfrac{1}{5} \quad . \tag{B17}$$

In practice, we usually expect to find g_2 of order unity, although one can imagine situations where it is much greater than that (e.g., a straight capillary made of alternating thin and thick segments).

Note also that Eq. (B6) shows that $\overline{\mu}$ is equivalent to the variable $\tilde{\mu} + \frac{1}{2}\tilde{u}_s^2$, which we have used throughout this article.

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$$\tilde{c}^2 = \frac{\partial P}{\partial \tilde{\rho}} = \frac{\rho}{\tilde{\rho}}\frac{\partial P}{\partial \rho} = \frac{1}{\mathcal{P}} c^2 \quad ,$$

where c is the sound velocity in the pure liquid. Note that this becomes infinite as $\mathcal{P} \to 0$. The correct result for this geometry is $\tilde{c} = c$.

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