PHYSICAL REVIEW B VOLUME 11, NUMBER 11 1 JUNE 1975

Critical dynamics of ferromagnets in 6-ϵ dimensions: General discussion and detailed calculation

Shang-keng Ma*
Department of Physics and Institute for Pure and Applied Physical Sciences, University of California, San Diego, La Jolla, California 92037

Gene F. Mazenko[†‡]
W. W. Hansen Laboratories of Physics, Stanford University, Stanford, California 94304
(Received 6 January 1975)

We report in detail a study of the critical dynamics of a model ferromagnet in $6-\epsilon$ dimensions both above and below T_c, for small ϵ. The precession of spins in the local magnetic field plays a major role in the dynamics. Its effect can be ignored at 6 or more dimensions, as many authors have previously observed. At a dimension slightly less than 6, the effect is nontrivial but tractable by perturbation theory. We carry out a renormalization-group analysis and determine the dynamic exponent $z = 4 - \epsilon/2 = 1 + d/2$, accurate to $O(\epsilon)$. We also solve the equation of motion to obtain the spin response function. The frequency and damping of spin waves below T_c are determined. Forms of equations of motion and the extent of universality in critical dynamics are discussed.

I. INTRODUCTION

The dynamics of an isotropic ferromagnet is characterized by the precession of spins in the local magnetic field and by the conservation of total spin as a result of invariance of interactions under rotation. The role of these characteristics in the dynamics near the critical temperature T_c is of considerable theoretical interest. Important advances in understanding this problem have been made using dynamic scaling and various other ideas and techniques.[1-3] Among many results, several authors found that for a dimension $d > 6$, the dynamics are fairly easily understood, but for $d < 6$, the dynamics are more complicated and the application of dynamic scaling is less straightforward.[3]

In this paper, we study in detail the dynamics using a renormalization-group (RNG) approach for $d = 6 - \epsilon$ with positive and small ϵ. The main results of this study have been reported in a letter.[4] The approach here is parallel to a number of recent applications of RNG to critical dynamics.[5,6]

The model which we use here is defined as follows. We describe the spin configuration by a three-component vector field

$$\vec{S}(x,t) = L^{-d/2} \sum_{k<\Lambda} \vec{S}_k(t)\, e^{ik\cdot x} , \quad (1.1)$$

where $\vec{S}_k(t)$ are the Fourier components of the spin density, L^d is the volume of the system, and Λ is a cutoff wave number. We choose Λ^{-1} to be much larger than the microscopic length (lattice spacing) but still much shorter than the correlation length ξ of spin fluctuations, which is the characteristic length of critical phenomena. Thus $\vec{S}(x,t)$ is the local spin density with variations of wave vectors larger than Λ excluded. The dynamics is based on the equation of motion

$$\frac{\partial \vec{S}}{\partial t} = \lambda \vec{S} \times \vec{H} - \Gamma \nabla^2 \vec{H} + \vec{\zeta} , \quad (1.2)$$

where $\vec{H}(x,t)$ is the local magnetic field; λ, Γ are constants; and $\vec{\zeta}(x,t)$ is a random noise simulating the effect of thermal agitation on the spins. The local field $\vec{H}$ is the sum of the external field $\vec{h}$ plus the field generated by the spins themselves. We assume the latter is given by the derivative of $F[\vec{S}]$, the free energy at the fixed spin configuration $\vec{S}$, and we assume a Ginzburg-Landau form for $F[\vec{S}]$:

$$\vec{H}(x,t) = \vec{h} - \delta F / \delta \vec{S}(x,t) , \quad (1.3)$$

$$F = \frac{1}{2} \int d^d x \left[(\nabla \vec{S})^2 + r_0 S^2 + \tfrac{1}{2} u (S^2)^2 \right] , \quad (1.4)$$

where $r_0 = a(T - T_c)$ and a and u are positive constants. In terms of Fourier components, (1.2) has the form

$$\frac{\partial S_k}{\partial t} = -\lambda L^{-d/2} \sum_{k'} \vec{S}_{k+k'} \times \frac{\partial F'}{\partial \vec{S}_{k'}} - \Gamma k^2 \frac{\partial F'}{\partial \vec{S}_{-k}} + \vec{\zeta}_k \quad (1.5)$$

where

$$F' = F - \int d^d x\, \vec{S}\cdot\vec{h} ,$$

$$\frac{\partial F'}{\partial \vec{S}_{-k}} = (r_0 + k^2)\vec{S}_k - \vec{h}_k - u L^{-d} \sum_{k', k'' < \Lambda} (\vec{S}_{k'} \cdot \vec{S}_{k''}) \vec{S}_{k-k'-k''} . \quad (1.6)$$

Finally, we assume the noise $\vec{\zeta}$ is Gaussian with the property

$$\langle \zeta_{\alpha k}(t)\, \zeta_{\beta k'}(t')\rangle = 2\,\Gamma k^2\, \delta_{\alpha\beta}\, \delta_{-k,k'}\, \delta(t-t') \;, \quad (1.7)$$

$\alpha, \beta = 1, 2, 3$. This completes the definition of the model.

The term with the cross product in (1.2) or (1.5) gives rise to the precession of spins. Note that

$$\vec{S}\times\vec{H} = \vec{S}\times(\nabla^2\vec{S}+\vec{h}) \;, \quad (1.8)$$

since the rest of $\partial F/\partial \vec{S}$ is proportional to $\vec{S}$ and we can use the fact that $\vec{S}\times\vec{S}=0$. The $\vec{S}\times\vec{H}$ term is closely related to the usual interaction term in the Heisenberg-model equation of motion. The conservation of total spin in this model is evident since as $k\to 0$ and $h\to 0$, (1.5) vanishes. If $\lambda=0$, the model reduces to the time-dependent Ginzburg-Landau model studied in Ref. 5.

This model is clearly a semimacroscopic (since Λ^{-1} is much greater than the microscopic lengths) phenomenological model. Ideally, we want it to produce the same critical dynamics as the microscopic Schrödinger equation and quantum statistical mechanics of electrons in the ferromagnet. In practice, it is a difficult task to derive semimacroscopic models from microscopic theories. There is a substantial literature devoted to this task. In Sec. II, we discuss some general criteria and motivations in the choice of semimacroscopic models. The model defined above is the simplest one taking into account the spin precession and conservation, and satisfying various general properties.

Based on the above model, we have carried out a RNG analysis, and determined the fixed points and associated exponents to $O(\epsilon)$. We have also calculated, in the same spirit as the static ϵ expansion of Wilson[7] and Brézin, Wallace, and Wilson,[8] the response function above and below T_c. For completeness, relevant elementary aspects of the renormalization group will be included along with some calculational details. The outline is as follows.

In Sec. II we give a formal discussion of questions involved in setting up semimacroscopic equations of motion. The connections among dissipation, fluctuations, streaming velocities and probability distributions will be studied. In Sec. III we set up the perturbation expansion scheme for solving the equation of motion in powers of λ. A graphic representation and rules of calculation are introduced. Power counting arguments show that for $d<6$ the perturbation expansion is divergent at T_c. In Sec. IV we review general ideas of RNG, associated fixed points and exponents, consequences, and the dynamic scaling hypothesis. Then we work the RNG out in detail and determine the fixed points and exponents to $O(\epsilon)$. The main results are that there are two fixed points, one trivial but unstable with dynamic exponent $z=4$ and crossover exponent $\varphi=\frac{1}{4}\epsilon$, the other nontrivial and stable with $z=4-\frac{1}{2}\epsilon$. In Sec. V we calculate the spin response function to $O(\epsilon)$ for $T\geq T_c$. An appropriate value of λ is chosen to remove confusing logarithms in the calculation. The scaling function is determined. In Sec. VI we calculate the transverse and longitudinal response functions for $T<T_c$. The propagation speed and damping of spin waves below T_c are examined. Results are consistent with the prediction of RNG arguments. In Sec. VII we discuss the apparent violation of the usual form of dynamic scaling below T_c. The effect of the mode coupling terms on the various damping coefficients is discussed. Finally, we make a few comments about dynamic universality and microscopic calculations.

II. MODEL EQUATIONS

A. Motivation

Recently, there has been considerable interest in the study of model equations of motion near the critical point.[5,6,9] Here we want to discuss some of the general background for these semiphenomenological equations.

Dynamic critical phenomena are characterized by the slow variation in time of the order parameter and the conserved variables in the system. Thus, we are interested in finding the equations of motion of these variables, which we call ψ_i. Those variables which are fast varying are expected to be irrelevant except for their combined effect on the slowly varying variables ψ_i. We need to eliminate the fast variables from the microscopic equations of motion to obtain new equations of motion for ψ_i only.

The situation here is similar to that in static critical phenomena.[10] In the static case we do not evaluate the trace over the microscopic canonical distribution directly. Instead we evaluate the partition function in two steps. We first integrate out the short wavelength degrees of freedom. This involves solving a "local" problem, and results in an effective free energy or Hamiltonian for long wavelength degrees of freedom. It is only at this stage, after the short-wavelength degrees of freedom have been eliminated, that RNG techniques are useful.[11,12]

In the dynamical case the problem of the fast variables falls into the same category as the short-wavelength phenomena. The solution of a local dynamical problem allows one to lump the information from these local processes into a set of simple parameters that are important for long wavelengths.

We shall not discuss the problem of eliminating fast variables, but only examine various phenomenological equations of motion for the slow variables ψ_i and criteria for their validity.

B. Van Hove's conventional theory

Van Hove's theory[13] is the dynamic generalization of the static mean field or Landau theory.[14] One assumes $\psi_i(t)$ satisfies a simple Langevin equation[15]

$$\frac{\partial \psi_i(t)}{\partial t} = -L_i \chi_i^{-1} \psi_i(t) + \zeta_i(t) , \tag{2.1}$$

where the L_i are the Onsager or transport coefficients, χ_i is the static susceptibility for ψ_i, and $\zeta_i(t)$ is a Gaussian-distributed noise term satisfying

$$\langle \zeta_i(t)\, \zeta_j(t')\rangle = 2 L_i \delta_{ij}\, \delta(t-t') . \tag{2.2}$$

If ψ_i is conserved, then L_i will be proportional to the wave number squared in the Fourier transform representation.

We can easily calculate the dynamic structure factor from our simple Langevin equation to find

$$C_{ij}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} \langle \psi_j \psi_i(t)\rangle = 2L_i\delta_{ij}[\omega^2 + (L_i\chi_i^{-1})^2]^{-1} . \tag{2.3}$$

In a ferromagnet we choose the Fourier components S_q of the spin configuration as the set of slow variables ψ_i. Noting that the total spin is conserved, $L_q = q^2\Gamma$, we obtain the conventional result

$$C(q,\omega) = 2\Gamma q^2/[\omega^2 + (\Gamma q^2 \chi_q^{-1})^2] . \tag{2.4}$$

The mean-field-theory result for χ_q is the Ornstein-Zernike form

$$\chi_q = (q^2 + \xi^{-2})^{-1} , \tag{2.5}$$

where ξ is the correlation length. The width of $C(q,\omega)$ as a function of ω is given by the characteristic frequency

$$\omega(q) = \Gamma q^2 \chi_q^{-1} = \Gamma q^4 [1 + (q\xi)^{-2}] . \tag{2.6}$$

As $T \to T_c$, $q \to 0$ the width becomes very small. This is the effect of critical slowing down. This characteristic frequency agrees in form with predictions of dynamical scaling as proposed by Halperin and Hohenberg,[2]

$$\omega(q) = q^z f(q\xi) , \tag{2.7}$$

where z is the dynamical critical index. According to (2.6) we find $z=4$. Unfortunately, the experimental results do not agree with this "conventional" result. It is found, from neutron scattering,[16] that $z \approx \frac{5}{2}$.

In the limit $q \to 0$ and for small but nonzero $T-T_c$, hydrodynamic arguments[17] show that the characteristic frequency should be proportional to q^2,

$$\lim_{q\to 0} \frac{\omega(q)}{q^2} = -i\tilde{\Gamma}\chi^{-1} , \tag{2.8}$$

where χ is just the static susceptibility χ_q at $q=0$, and $\tilde{\Gamma}$ is *defined* as the transport coefficient. In the conventional theory (2.1)–(2.6), $\tilde{\Gamma}$ is simply Γ. In other words, the transport coefficient is not affected by the long-wavelength fluctuations and is effectively a constant. Experimentally, $\chi \propto |T-T_c|^{-\gamma} \propto \xi^{\gamma/\nu}$, with $\gamma \approx \frac{4}{3}$, $\nu \approx \frac{2}{3}$. The form (2.7) is consistent with (2.8) only if

$$\tilde{\Gamma} \propto \xi^{2-z+\gamma/\nu} , \tag{2.9}$$

which gives

$$\tilde{\Gamma} \propto |T-T_c|^{-1} \tag{2.10}$$

for the observed values of γ, ν, and z. Thus, $\tilde{\Gamma}$ diverges as $T \to T_c$. The divergence of transport coefficients at critical points of other systems have also been observed. For example, the thermal conductivity for fluids diverges at T_c and this can be measured precisely using light scattering.[18]

It is clear that the conventional model characterized by a linear equation of motion is not sufficient to describe these new and interesting physical effects. The difficulty is that the physical transport coefficient cannot be simply represented by a constant as $T \to T_c$. There are physical processes occurring on the scale of large wavelengths that contribute to $\tilde{\Gamma}$ and therefore the simple hydrodynamical picture, which assumes that all contributions to $\tilde{\Gamma}$ come from very short wavelengths, must break down.

C. Nonlinear models

We must generalize our Langevin equation to include nonlinear couplings between the spins. This will necessitate the inclusion of two new types of terms in our equation of motion.

(i) We must include the effect of nonlinear couplings in the statics. We can accomplish this by writing the equation of motion

$$\frac{\partial \psi_i(t)}{\partial t} = -L_i \frac{\partial F[\psi]}{\partial \psi_i(t)} + \zeta_i(t) , \tag{2.11}$$

where F is the Landau-Ginzburg free-energy functional. Once we include nonlinear couplings, L_i can no longer be interpreted as the measured Onsager coefficient. It must instead be thought of as a "bare" or local approximation for the Onsager coefficient determined by very-short-range interactions. The model given by (2.11) is called the time-dependent Ginzburg-Landau (TDGL) model and was recently analyzed in detail by Halperin, Hohenberg, and Ma.[5] For systems like planar ferromagnets, where the spin is not conserved and L is a constant, this model shows interesting qualitative deviations from the conventional theory. For systems with a conserved order parameter, however, there is no change in z.

(ii) The equation of motion (2.11) is still not general enough. We should also include the "mode coupling" terms discussed by Kawasaki[1,19] and others.[20] These terms have been shown to lead to various transport anomalies near the critical point. Such terms have in general the form of a streaming velocity $V_i[\psi]$ in the space of ψ:

$$V_i[\psi]=\lambda\sum_j\left(\frac{\partial}{\partial\psi_j}Q_{ij}[\psi]-Q_{ij}[\psi]\frac{\partial F[\psi]}{\partial\psi_j}\right). \quad (2.12)$$

This form follows directly from the work of Mori and collaborators.[21] The quantity λ is a constant and $Q_{ij}=-Q_{ji}$ are variables constructed from Poisson brackets or commutators of ψ_i and depend on the particular system of interest. Clearly,

$$\sum_i\frac{\partial}{\partial\psi_i}(V_i e^{-F})=0 . \quad (2.13)$$

This is the statement that the probability current vector (probability density e^{-F})$\times$(streaming velocity V_i) in the ψ space is divergence free. Since a divergence free current does not change the probability density, we conclude that V_i will not change the probability density e^{-F}. Thus the static properties, such as the static susceptibilities, which are determined by e^{-F}, will not depend on λ.

Including V_i, (2.12) becomes

$$\frac{\partial\psi_i(t)}{\partial t}=V_i[\psi(t)]-L_i\frac{\partial F[\psi(t)]}{\partial\psi_i(t)}+\zeta_i(t) . \quad (2.14)$$

If we choose F to be quadratic in the ψ's then this equation reduces to that studied by Kawasaki[1] and others.[20] The quadratic assumption for F is compatible only with Gaussian or mean-field statics. We shall take F to be of the Ginzburg-Landau form to ensure correct static properties.

Equation (2.14) can be derived from the microscopic equations of motion using the projection operator technique of Zwanzig.[22] A particularly simple discussion is given by Mori and Fujisaka.[23] There are three primary approximations used in deriving these model equations:

(i) Memory effects are neglected in L_i. This means assuming that the processes that contribute to L_i take place on a very-short-time scale. (ii) The functional dependence of L_i on ψ has been neglected. (iii) The noise ζ is assumed to be Gaussian. This approximation neglects the various correlations between the fast variables.

We expect that (i) and (ii) are appropriate near the critical point and for small frequencies and wave numbers. It may be necessary in some cases to treat the ψ dependence of L. The renormalization group should be some help in sorting out the role of these new terms.[24] We suspect that approximation (iii) would be the most difficult to improve upon.

We will be interested in calculating in the presence of an external field. We can introduce a field $h_i(t)$ into our model by making the replacement

$$F\to F'=F-\sum_i h_i(t)\psi_i . \quad (2.15)$$

We see then that our equation of motion takes the final form

$$\frac{\partial\psi_i(t)}{\partial t}=V_i[\psi(t)]-L_i\frac{\partial F[\psi(t)]}{\partial\psi_i(t)}+\sum_j\sigma_{ij}[\psi(t)]h_j(t)+\zeta_i(t) , \quad (2.16)$$

where

$$\sigma_{ij}[\psi]=\lambda Q_{ij}[\psi]+\delta_{ij}L_i . \quad (2.17)$$

D. Isotropic ferromagnets

It is relatively simple to find the correspondence between the model developed above and a particular physical system. We must choose the variables of interest ψ_i, we must specify Q_{ij}, we must specify which variables are conserved (where L_i will then be proportional to the wave number squared) and we must designate the order parameter and thus the effective free energy F.

For an isotropic ferromagnet we study the spin density S_q. This system is unusual in that the order parameter is conserved, and it will be this property and the precession of spins described by a special form of V_i that will lead to the special nature of six dimensions in this problem. In principle, we should also include the energy density in our set of variables since it is a conserved variable. However, Kawasaki[25] has pointed out that the energy density decays on a time scale much shorter than that for the spin. We treat only the spin density here.

The Poisson brackets satisfied by our spin variables follow from the standard spin commutation relations giving

$$Q_{kk',\alpha\beta}=\sum_{\gamma=1}^{3}\epsilon_{\alpha\beta\gamma}S^{\gamma}_{k+k'} . \quad (2.18)$$

We then find using (2.12) that the streaming velocity can be written

$$V_k^\alpha[S] = -\lambda L^{-d/2} \sum_{\beta,\gamma,q<\Lambda} \epsilon_{\alpha\beta\gamma} S_{k+q}^\beta \frac{\partial F[S]}{\partial S_q^\gamma}. \tag{2.19}$$

It is easy to see that this V satisfies the divergence condition (2.13). Since the spin is conserved we write $L_q = \Gamma q^2$, where Γ is the "bare" Onsager coefficient. Finally, we use the standard Ginzburg-Landau form for F given by (1.4).

E. General properties of the model

We now investigate some of the simple properties satisfied by a theory generated by the equation of motion (2.16).

We now show that (2.16) generates the same static correlation functions as given by the static distribution function e^{-F}. Consider the quantity

$$g_\varphi(t) = \prod_i \delta(\varphi_i - \psi_i(t)) \equiv \delta(\varphi - \psi(t)) , \tag{2.20}$$

If we calculate the average of $g_\varphi(t)$ then we have all of the static correlation functions, since

$$\langle \psi_i(t)\,\psi_j(t)\cdots\psi_k(t)\rangle = \int d\varphi\, \varphi_i \varphi_j \cdots \varphi_k \langle g_\varphi(t)\rangle , \tag{2.21}$$

where $d\varphi = \prod_j d\varphi_j$, and $\langle g_\varphi(t)\rangle$ is therefore the distribution function associated with the variables ψ. We can write down the equation of motion satisfied by $g_\varphi(t)$ if we use the chain-rule for differentiation

$$\frac{\partial g_\varphi(t)}{\partial t} = -\sum_i \frac{\partial g_\varphi(t)}{\partial \varphi_i} \frac{\partial \psi_i(t)}{\partial t} = -\sum_i \frac{\partial}{\partial \varphi_i}\left[g_\varphi(t)\left(V_i[\varphi] - L_i \frac{\partial F}{\partial \varphi_i} + \sum_j \sigma_{ij}[\varphi] h_j + \zeta_i(t)\right)\right] . \tag{2.22}$$

If we take the average of (2.22) we see that we have to evaluate $\langle g_\varphi(t)\,\zeta_i(t)\rangle$. We can evaluate quantities like this if we note the identity

$$\frac{\delta\psi_i(t)}{\delta\zeta_j(t')} = \begin{cases} 0, & t' > t \\ \frac{1}{2}\delta_{ij}, & t' = t . \end{cases} \tag{2.23}$$

This result is discussed in Appendix A. It depends on the fact that the field ψ at some time t cannot depend on the value of the noise as some later time t'. Using (2.23) and the Gaussian nature of the noise [see Eq. (A5)], we can show

$$\langle \zeta_i(t)\, g_\varphi(t)\rangle = -L_i \frac{\partial}{\partial \varphi_i} \langle g_\varphi(t)\rangle . \tag{2.24}$$

If we define the generalized Fokker-Planck operator[26]

$$D_\varphi = -\sum_i \frac{\partial}{\partial \varphi_i}\left[V_i - L_i \left(\frac{\partial}{\partial \varphi_i} + \frac{\partial F}{\partial \varphi_i}\right)\right] \tag{2.25}$$

the probability distribution will satisfy the equation of motion

$$\frac{\partial}{\partial t}\langle g_\varphi(t)\rangle = D_\varphi \langle g_\varphi(t)\rangle - \sum_{i,j} \frac{\partial}{\partial \varphi_i} \sigma_{ij}[\varphi] h_j(t) \langle g_\varphi(t)\rangle . \tag{2.26}$$

In the case where $h = 0$, then $\langle g_\varphi(t)\rangle$ is time independent, and we must investigate the solution of

$$D_\varphi \langle g_\varphi(t)\rangle = 0 . \tag{2.27}$$

We easily see that one solution is given by[27]

$$\langle g_\varphi(t)\rangle = e^{-F[\varphi]} \equiv W_\varphi , \tag{2.28}$$

which follows from direct substitution and use of the divergence condition for V. If (2.28) is the only solution, then we can conclude that all static properties generated by the equations of motion (2.16) are given by the distribution function (2.28). We shall not consider the possibility of more than one solution.

It is useful to consider our equation for $\langle g_\varphi(t)\rangle$ for finite h. If we define

$$G_{\varphi\varphi'}(t-t') = \Theta(t-t')\, e^{D_\varphi(t-t')}\, \delta(\varphi - \varphi') , \tag{2.29}$$

then we can write the retarded solution for $\langle g_\varphi(t)\rangle$ as

$$\langle g_\varphi(t)\rangle = W_\varphi + \int_{-\infty}^{+\infty} d\bar{t} \int d\bar{\varphi}\, G_{\varphi\bar{\varphi}}(t-\bar{t}) \times \sum_{i,j} \left(-\frac{\partial}{\partial \bar{\varphi}_i} \sigma_{ij}(\bar{\varphi}) h_j(\bar{t})\right) \langle g_{\bar{\varphi}}(\bar{t})\rangle . \tag{2.30}$$

If we keep only first order terms in h, we obtain

$$\langle g_\varphi(t)\rangle = W_\varphi + \int_{-\infty}^{+\infty} d\bar{t} \int d\bar{\varphi}\, G_{\varphi\bar{\varphi}}(t-\bar{t}) \times \sum_{i,j} \left(-\frac{\partial}{\partial \bar{\varphi}_i} \sigma_{ij}(\bar{\varphi}) h_j(\bar{t})\, W_{\bar{\varphi}}\right) + O(h^2) . \tag{2.31}$$

By multiplying by φ_i and integrating we obtain the linear response of ψ_i to h_j, which we write

$$\langle \psi_i(t)\rangle = \sum_j \int d\bar{t}\, G_{ij}(t-\bar{t}) h_j(\bar{t}) + O(h^2) , \tag{2.32}$$

and defines the linear-response function

$$G_{ij}(t-t')=\int d\bar{\varphi}\,d\varphi\,\varphi_i G_{\varphi\bar{\varphi}}(t-t')\times\left(-\sum_k \frac{\partial}{\partial\bar{\varphi}_k}\sigma_{kj}(\bar{\varphi})W_{\bar{\varphi}}\right). \quad (2.33)$$

We can rewrite our expression for G in a more convenient form if we note

$$\sum_i -\frac{\partial}{\partial\varphi_i}(\sigma_{ij}W_\varphi)=W_\varphi\left(V_j+L_j\frac{\partial F}{\partial\varphi_j}\right)=-D_\varphi\varphi_j W_\varphi, \quad (2.34)$$

so

$$G_{ij}(t-t')=-\int d\varphi\,d\bar{\varphi}\,\varphi_i G_{\varphi\bar{\varphi}}(t-t')D_{\bar{\varphi}}\bar{\varphi}_j W_{\bar{\varphi}}; \quad (2.35)$$

or, using (2.29),

$$G_{ij}(t-t')=-\Theta(t-t')\int d\varphi\,\varphi_i e^{D_\varphi(t-t')}D_\varphi\varphi_j W_\varphi. \quad (2.36)$$

One of the nice properties of the response function is that it is normalized to a constant, independent of the nonlinear terms, at equal times. $G_{ij}(0^+)=L_i\delta_{ij}$, which follows from (2.34) and the divergence condition on V_i.

We now return to the equation for $g_\varphi(t)$ for the case $h=0$. We can rewrite (2.22) in the form[28]

$$\frac{\partial g_\varphi(t)}{\partial t}=D_\varphi g_\varphi(t)+R_\varphi(t), \quad (2.37)$$

where we define

$$R_\varphi(t)=-\sum_i \zeta_i(t)\frac{\partial}{\partial\varphi_i}g_\varphi(t)-\sum_i L_i\frac{\partial^2}{\partial\varphi_i\partial\varphi_i}g_\varphi(t). \quad (2.38)$$

This leads to the formal solution for $g_\varphi(t)$,

$$g_\varphi(t)=e^{D_\varphi t}g_\varphi(0)+\int d\bar{\varphi}\int_0^{+\infty}d\bar{t}\,G_{\varphi\bar{\varphi}}(t-\bar{t})R_{\bar{\varphi}}(\bar{t}). \quad (2.39)$$

If we take the average of (2.39) with $g_{\varphi'}(t=0)$ we have

$$\langle g_\varphi(0)g_\varphi(t)\rangle=\langle g_{\varphi'}(0)\,e^{D_\varphi t}g_\varphi(0)\rangle+\int d\bar{\varphi}\int_0^{+\infty}d\bar{t}\,G_{\varphi\bar{\varphi}}(t-\bar{t})\langle g_{\varphi'}(0)R_{\bar{\varphi}}(\bar{t})\rangle. \quad (2.40)$$

It is easy to show using the results of Appendix A, that

$$\langle g_{\varphi'}(0)R_\varphi(t)\rangle=0 \quad (2.41)$$

for $t>0$, we have therefore

$$\langle g_{\varphi'}(0)g_\varphi(t)\rangle=e^{D_\varphi t}\langle g_{\varphi'}g_\varphi\rangle=e^{D_\varphi t}[\delta(\varphi-\varphi')W_\varphi]. \quad (2.42)$$

In particular, we have

$$\langle\psi_j\,\psi_i(t)\rangle=\int d\varphi\,\varphi_i\,e^{D_\varphi t}\varphi_j W_\varphi\equiv C_{ij}(t),\ t\geq 0. \quad (2.43)$$

The correlation function can be computed without further discussion of the noise term. If we compute the time derivative we find

$$\frac{\partial}{\partial t}C_{ij}(t)=\int d\varphi\,\varphi_i\,e^{D_\varphi t}D_\varphi\varphi_j W_\varphi \quad (2.44)$$

for $t>0$. Since we can show perturbatively that $C_{ij}(t)=C_{ij}(-t)$, we have

$$\frac{\partial}{\partial t}C_{ij}(t)=-G_{ij}(t)+G_{ij}(-t). \quad (2.45)$$

After Fourier transformation and noting that $\mathrm{Re}G(\omega)$ is even, while $\mathrm{Im}G(\omega)$ is odd under $\omega\to-\omega$, we immediately find[29]

$$C_{ij}(\omega)=(2/\omega)\,\mathrm{Im}G_{ij}(\omega), \quad (2.46)$$

which is a fluctuation-dissipation theorem for our model. Since it is more convenient to develop perturbation theory for the response function G than for C we will calculate G and use this theorem to determine the correlation function. This theorem also allows us to calculate the static susceptibility as the $\omega=0$ value of the response function.

Finally, we note that there are two apparently different but equivalent methods for developing perturbation theory in our model. We can use the formal expression given by (2.36) and $D=D_0+D_I$, where $D_0\varphi_j=-L_j\varphi_j$, and we can iterate the response function in powers of the nonlinear couplings in D_I. This type of approach has been developed by Kawasaki[19] and has the advantage that the noise term has been eliminated from the problem. Alternatively, we can iterate the equations of motion (2.16), in an expansion in the nonlinear couplings, and average over the noise term by term. We prefer this second method because it is easier to implement the RNG as we will discuss in Sec. IV.

III. PERTURBATION THEORY

A. Iteration scheme

In this section, we discuss the solution of the equations of motion (1.5) by expansion in powers of λ and u. The discussion is a generalization of that given in Ref. 5. It is self-contained and there is some overlap of the material here and that in Ref. 5.

The linear-response function will be the quantity of interest. It is defined by (2.32) and in the case of a spin density takes the form

$$\langle S^{\alpha}(x,t)\rangle = \int d^d x'\, dt'\, G_{\alpha\beta}(x-x', t-t') h^{\beta}(x',t'), \tag{3.1}$$

where h^{β} is an infinitesimal field and α, β are spin indices. Let $\vec{S}_k(\omega)$ be Fourier components of $\vec{S}(x,t)$,

$$\vec{S}(x,t) = L^{-d/2}\sum_k \int \frac{d\omega}{2\pi}\vec{S}_k(\omega)\, e^{ik\cdot x - i\omega t} \tag{3.2}$$

and similarly for $h_k(\omega)$. Then (3.1) is equivalent to

$$\langle S_k^{\alpha}(\omega)\rangle = G_{\alpha\beta}(k,\omega) h_k^{\beta}(\omega), \tag{3.3}$$

with

$$G_{\alpha\beta}(k,\omega) = \int dt\, d^d x\, e^{i\omega t - ik\cdot x} G_{\alpha\beta}(x,t). \tag{3.4}$$

For $T > T_c$, the system is isotropic and $\langle \vec{S}\rangle$ points in the same direction as $\vec{h}$, and $G_{\alpha\beta} = G\delta_{\alpha\beta}$. For $T < T_c$, there is a finite magnetization $\vec{M}$ which reduces the isotropy to cylindrical symmetry around $\vec{M}$. We then need to distinguish between $G_{\parallel}$ and $G_{\perp}$ for the response to $\vec{h}\parallel\vec{M}$ and $\vec{h}\perp\vec{M}$, respectively. For the moment, we shall restrict our discussion to $T > T_c$. The generalization to $T < T_c$ will be included in Sec. VI.

When λ and u are set to zero, the equations of motion (1.5) reduce to

$$-i\omega\vec{S}_k^0(\omega) = -\Gamma k^2[(k^2+r_0)\vec{S}_k^0(\omega) - \vec{h}_k(\omega)] + \vec{\zeta}_k(\omega). \tag{3.5}$$

The solution is

$$\vec{S}_k^0(\omega) = G_0(k,\omega)[\vec{\zeta}_k(\omega)/\Gamma k^2 + \vec{h}_k(\omega)], \tag{3.6}$$

where

$$G_0(k,\omega) \equiv (r_0 + k^2 - i\omega/\Gamma k^2)^{-1} \tag{3.7}$$

is simply the response function for $\lambda = u = 0$.

For nonzero u and λ, we write the equations of motion (1.5) as

$$\vec{S}_k(\omega) = \vec{S}_k^0(\omega) + G_0(k,\omega)\left(\frac{\lambda}{\Gamma k^2} L^{-d/2}\sum_{k'}\int\frac{d\omega'}{2\pi}\vec{S}_{k-k'}(\omega-\omega') \times[-k'^2\vec{S}_{k'}(\omega') + \vec{h}_{k'}(\omega')] - uL^{-d}\sum_{k'k''}\int\frac{d\omega'}{2\pi}\frac{d\omega''}{2\pi} \times[\vec{S}_{k'}(\omega')\cdot\vec{S}_{k''}(\omega'')]\vec{S}_{k-k'-k''}(\omega-\omega'-\omega'')\right). \tag{3.8}$$

We shall use the symbol $\tilde{\lambda}$ for λ/Γ. To find $G(k,\omega)$, we iterate (3.8) to obtain $\vec{S}_k(\omega)$ as a power series in λ and u, keeping $\vec{h}$ to first order. The result is a sum of products of S^0's and G_0's. Then we take the average. Note that the Fourier transform of (1.7) gives

$$\langle \zeta_k^{\alpha}(\omega)\zeta_{k'}^{\beta}(\omega')\rangle = 2\Gamma k^2\delta_{\alpha\beta}\delta_{-kk'}2\pi\delta(\omega+\omega'). \tag{3.9}$$

It follows from (3.6), (3.7), and (3.9) that, for $h=0$,

$$\begin{aligned}\langle S_k^{0\alpha}(\omega)S_{k'}^{0\beta}(\omega')\rangle &= 2\pi\delta(\omega+\omega')\,\delta_{-kk'}\,\delta_{\alpha\beta}\,2\,\mathrm{Im}G_0(k,\omega)/\omega\\ &\equiv 2\pi\delta(\omega+\omega')\,\delta_{-kk'}\,\delta_{\alpha\beta}\,C_0(k,\omega).\end{aligned} \tag{3.10}$$

Since $\vec{\zeta}$ is a Gaussian noise, $\vec{S}^0$ must also be a Gaussian noise in view of (3.6). Consequently, the average of a product of $\vec{S}^0$'s is the product of pairwise averages, each of which has the form of (3.10). After we obtain $\langle\vec{S}_k(\omega)\rangle$ to first order in $\vec{h}$, we divide it by $\vec{h}_k(\omega)$ and thus obtain $G(k,\omega)$ as a series in powers of λ and u.

B. Graph representation

The terms in the series can be represented by graphs, as illustrated below. Let us set $u=0$ for simplicity of illustration. The λ term of (3.8) can be represented by a thin line, representing G_0, which then branches out into two thick lines, representing the two $\vec{S}$'s, or one $\vec{S}$ and one h, which is represented by a dot [see Fig. 1(a)]. To $O(\lambda)$, the $\vec{S}$'s can be approximated by $\vec{S}^0$'s. By (3.6), $\vec{S}^0$ has two terms, one ends in ζ and the other ends in $\vec{h}$. A thin line ending with a dot means $G_0 h$. The other term in (3.6) is denoted by a line ending without decoration. Figure 1(b) shows the terms to $O(\lambda)$. To $O(\lambda^2)$, the $O(\lambda)$ terms in the thick lines in Fig. 1(a) must be included. The graphs are shown in Fig. 1(c). It is easy to see that further iterations simply generate tree graphs like Fig. 1(d). Note that

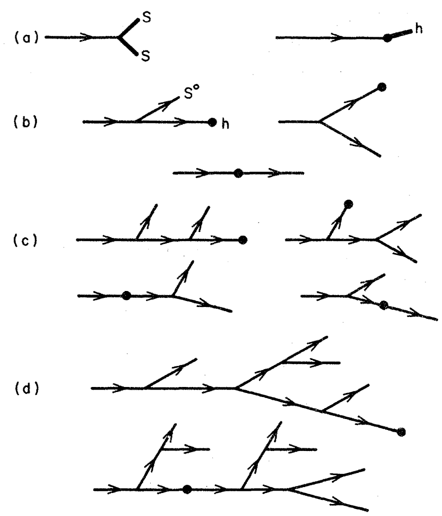

FIG. 1. (a) Graph representation of the λ terms in the equation of motion (3.8). (b) $O(\lambda)$ terms of S to first order in h. (c) $O(\lambda^2)$ terms. (d) Some $O(\lambda^4)$, $O(\lambda^5)$ terms.

the arrows point along the direction of iteration. Note also that we only keep one power of h. Thus, there is just one dot either at an end point, or joining two lines.

The end points (without dot, and not the beginning point) are joined in pairs in all possible ways to represent the averaging of the product of S^0's. Each pair is drawn as a circle joining two ends, thus giving a circled line (see Fig. 2). The construction of graphs for $\langle S_k(\omega)\rangle$ is thus complete. Let us sum up with the following rules for $O(\lambda^n)$ terms.

(a) Draw a tree graph with n vertices and one dot either at an end or joining two lines. Each vertex has one arrow pointing in and two pointing out.

(b) Join all end points in pairs to form circled lines. Label each line with a spin index, a wave vector, and a frequency. The wave vectors and frequencies must follow conservation laws at each vertex, i.e., the incoming one must equal the sum of the outgoing two. Note that there is in general more than one way to join the end point. Each way gives a term separately. All must be accounted for.

(c) Write a factor $G_0(p,\epsilon)$ for each line, or a factor $C_0(p,\epsilon)$ for a circled line. The values of p,ϵ are given for each line by the labels. Write $\tilde{\lambda}\epsilon_{\alpha\beta\gamma}(p'^2-p''^2)/p^2$ $(\tilde{\lambda}\equiv\lambda/\Gamma)$ for each vertex, α and p being, respectively, the component label and the wavevector for the incoming line, and β,p', γ,p'' are those for the outgoing lines. Write $h_k(\omega)$ for a dot at an end, or $\lambda\epsilon_{\alpha\beta\gamma}h_k^\gamma(\omega)$ for a dot not at an end.

(d) Integrate over all wave vectors and frequencies which are not fixed by the conservation laws mentioned in (b). A factor $(2\pi)^{-d-1}$ goes with each wave-vector-frequency integral. Sum over unfixed component labels.

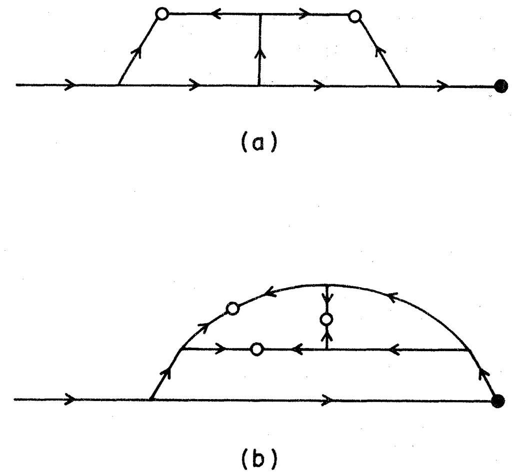

FIG. 2. (a) Average of the first term, and (b) that of the second term of Fig. 1(d).

These form a complete set of rules for constructing graphs and writing down their contributions to $\langle \vec{S}_k(\omega)\rangle$. Figure 3 shows the graphs of $O(\lambda^2)$ as an illustration. For simplicity, take $\vec{h}_k(\omega)$ to point in the one direction. Then $\beta\gamma$ must be either 23 or 32. Following the above rules, we have, for Fig. 3(a),

$$G_a h_k = 2G_0(k,\omega)\left(\frac{\tilde{\lambda}}{k^2}\right)(2\pi)^{-d-1} \times \int d^dq\, d\nu[q^2-(q+k)^2]\,G_0(q+k,\nu+\omega) \times C_0(q,\nu)\left(\frac{\tilde{\lambda}}{(q+k)^2}\right)(k^2-q^2)\,G_0(k,\omega)h_k(\omega)\,. \tag{3.11}$$

The factor 2 in front comes from the fact that $\beta\gamma=23$ and 32 contribute equally. The contribution of Fig. 3(b) is

$$G_b h_k = 2G_0(k,\omega)\left(\frac{\tilde{\lambda}}{k^2}\right)(2\pi)^{-d-1}\int d^dq\, d\nu[q^2-(q+k)^2] \times G_0(q+k,\nu+\omega)\,C_0(q,\nu)\,\tilde{\lambda}h_k(\omega)\,. \tag{3.12}$$

C. Power counting

What we are interested in is the behavior for small k and ω. To get some idea of how the graphs behave in general in the limit $k,\omega\to 0$, we count the powers of wave vectors and frequencies.

We note from the structure of $G_0(k,\omega)$ that every power of ω goes with -4 powers of k. If the perturbation series converges, then we expect this correspondence to hold in $G(k,\omega)$ also. Another way of saying the same thing is that the characteristic frequency exponent z is

$$z=4. \tag{3.13}$$

On the other hand if the series diverges for $k,\omega\to 0$, no such conclusion can be drawn.

In any given graph, each $G_0(q,\nu)$ gives -2 powers of q, each $C_0(q,\nu)$ gives -2 powers of q and -1 power of ν. Each integral (one for each closed loop) gives d powers of q and 1 power of ν. But the number of closed loops is the same as the number of circled lines, i.e., C_0's, because each circled line came from joining two ends of a tree thereby forming a loop. Thus the powers of ν from integrations exactly cancel that from C_0's. We can forget about powers of ν.

The λ vertices give no net power of wave vectors. Whenever two more vertices [i.e., $O(\lambda^2)$ more] are added to a graph, the number of lines increases by 3 and the number of loops increases by 1. This means that the increase in powers of q is

$$d-3\times 2=d-6. \tag{3.14}$$

Thus, for $d<6$, higher-order terms will be more divergent in the $k,\omega\to 0$ limit, and the perturbation series cannot converge. For $d>6$, there is no such problem and $z=4$ should therefore hold. With the aid of the renormalization group ideas, perturbation series can still be useful for $d<6$ but for very small values of $6-d$.

IV. RENORMALIZATION GROUP (RNG)

A. General ideas

The equations of motion (1.4), (1.5) are specified by a set of parameters

$$\mu=(\tilde\lambda,r_0,u,h), \quad \tilde\lambda\equiv\lambda/\Gamma \tag{4.1}$$

and all physical quantities calculated using these equations of motion are functions of this set of parameters. Note that we need not include Γ in μ because, by choosing appropriate unit of time, Γ can be made equal to any positive value.

If the equations of motion have more terms than those of (1.5), then there must be additional parameters to specify these extra terms, and we need to extend the definition of μ to include more entries for the additional parameters. As we mentioned in Sec. II, some of these terms come from the generalization $\Gamma\to\Gamma[S_q]$. Mori, Fujisaka, and Shigematsu[21] have discussed the parameterization needed in the equation of motion in this case. In general, higher powers of S_q will appear in the equation of motion and there will be a more-complicated wave-number dependence.

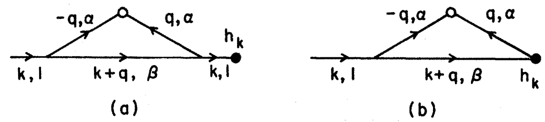

FIG. 3. $O(\lambda^2)$ terms of $\langle S_k'\rangle$.

The RNG is a continuous set of transformations $\{R_b;1\le b<\infty\}$. Given μ, R_b transforms it to $\mu'=R_b\mu$ via the following steps.

(a) Eliminate the variables $\vec S_q$ with $\Lambda/b<q<\Lambda$ from the equations of motion. This means solving the equations of motion for $\vec S_q$, substituting the solution in the remaining equations of motion and than averaging over $\vec\zeta_q$.

(b) Replace the remaining $\vec S_k(t)$, $k<\Lambda/b$, by $b^{1-\eta/2}\vec S_{bk}(tb^{-z})$ in the remaining equations of motion. Also replace L by bL'. The new equations of motion are then written in the old form with modified parameters, which are identified as entries in $\mu'=R_b\mu$. The volume of the system described by the new equations of motion is $L'^d=(L/b)^d$. (The constants η and z will be chosen later.)

Steps (a) and (b) define R_b. They are direct generalizations of those used in static problems. The step (b) changes the unit of length by a factor of b, the unit of $\vec S_k$ by $b^{1-\eta/2}$ and the unit of time by b^{+z}. It can be viewed as a scale transformation. Step (a), which eliminates $\vec S_q$, $\Lambda/b<q<\Lambda$, is a coarse graining procedure. It lowers the wave-vector cutoff and thus downgrades the spatial resolution. Details concerning these S_q are lost.

Step (b) is very easy to carry out but step (a) is more involved. Furthermore, after performing (a), the new equations of motion will in general have additional terms than those of (1.5). This means that more entries are needed in μ as was mentioned earlier. At present our mathematical capability is inadequate in carrying out (a) except when all entries of μ are very small so that perturbation methods can be applied. Consequently, as will be seen, application of the RNG to our equations of motion become accessible to simple calculation only when d is close to 6.

As far as those variables which are not eliminated in step (a) are concerned, the transformation R_b is simply a change of name of the variables and no physical content of the equations of motion is altered. Since $\vec S_k(t)$ is replaced by $b^{1-\eta/2}\vec S_{bk}(tb^{-z})$ we must have, for example,

$$\langle\vec S_k(t)\rangle_\mu=\langle\vec S_{bk}(tb^{-z})\rangle_{\mu'}b^{1-\eta/2}, \tag{4.2}$$

where the subscript μ denotes that the quantity is calculated with the set of parameters μ, and μ' denotes that it is calculated with the transformed parameters. Under R_b, h is transformed to h' with

$$h_k = b^{-1+\eta/2} h'_{bk} . \quad (4.3)$$

Equation (4.3) is strictly correct when h is static and contains no Fourier component with $k > \Lambda/b$. (See Sec. III D, Ref. 12, for example.) As we shall see, (4.3) is consistent with our approximations for slowly varying h as well as for static h. Combining (4.2), (4.3) and the definition of the response function $G_{\alpha\beta}(k, \omega)$, (3.3), we have

$$G_{\alpha\beta}(k, \omega, \mu) = b^{2-\eta} G_{\alpha\beta}(bk, b^z\omega, R_b\mu) . \quad (4.4)$$

This is all the RNG can tell us about the response function. Note that RNG is a set of transformations on μ. It does not solve the model nor does it calculate any physical quantity such as G. It is like the rotation group in atomic physics. It helps but does not provide many details.

The application of the RNG to critical dynamics is along the same line as to statics. It is via the hypothesis that, for T near T_c, the set of parameters $\mu(T)$ describing the physical system is very near a special set of values called a fixed point μ^*, and that $R_b\mu(T_c) \to \mu^*$ for large b, provided that Λ^{-1} is sufficiently large compared to atomic distances. The fixed point μ^* is invariant under R_b,

$$R_b \mu^* = \mu^* . \quad (4.5)$$

The constants η and z are adjusted so that (4.5) has a solution, i.e., one cannot find a fixed point unless η and z take certain values.

We shall find three fixed points consistent with the assumption that all entries of μ are small. There is a trivial fixed point with all entries equal to zero

$$\mu^* = 0 , \quad (4.6a)$$

$$z = 4 , \quad (4.6b)$$

$$\eta = 0 . \quad (4.6c)$$

There are two nontrivial fixed points given by

$$\tilde{\lambda}^* = \pm (96\pi^3\epsilon)^{1/2} , \quad (4.7a)$$

$$r_0^* = h^* = u^* = 0 , \quad (4.7b)$$

$$z = 4 - \tfrac{1}{2}\epsilon , \quad (4.7c)$$

$$\eta = 0 , \quad (4.7d)$$

where $\epsilon = 6 - d$ and (4.7) is valid only for small ϵ to $O(\epsilon)$. The two signs in (4.7a) give the right-handed and left-handed precessions, respectively.

If μ is very close to a fixed point μ^*, we can linearize the equation $\mu' = R_b\mu$. Write $\delta\mu = \mu - \mu^*$, $\delta\mu' = \mu' - \mu^*$. Then the linear operator R_b^L takes $\delta\mu$ to $\delta\mu'$,

$$\delta\mu' = R_b^L \delta\mu \quad (4.8)$$

neglecting $(\delta\mu)^2$ and higher orders. We shall find that (4.8) is

$$\delta\tilde{\lambda}' = \delta\tilde{\lambda} b^x , \quad (4.9a)$$

$$r_0' = (r_0 + uA)b^2 - uAb^{4-d} , \quad (4.9b)$$

$$u' = ub^y , \quad (4.9c)$$

$$h_k' = h_{k/b} b , \quad (4.9d)$$

where, for the trivial fixed point $\mu^* = 0$,

$$x = \tfrac{1}{2}\epsilon \quad (4.10)$$

and for either of the nontrivial fixed points (4.7),

$$x = -\epsilon . \quad (4.11)$$

The exponents ν and y are

$$\nu = \tfrac{1}{2}, \quad y = -2 + \epsilon , \quad (4.12)$$

and A is a constant, for the trivial and nontrivial fixed points. If we are to approach a fixed point we must set $r_0 + uA = h = 0$ since they would otherwise grow for large b. The variable u however is "irrelevant" and will diminish as b increases since $y < 0$. For the trivial fixed point, $\delta\lambda'$ will grow with b because $x > 0$. For this reason, the trivial fixed point is called "unstable." For the nontrivial fixed point, $\delta\lambda'$ diminishes with increasing b and hence the term "stable." Associated with the instability of the trivial fixed point, there is a "crossover" exponent φ defined by

$$\varphi = x\nu = \tfrac{1}{4}\epsilon . \quad (4.13)$$

According to the hypothesis given above, the critical point is characterized by $R_b\mu(T_c) \to \mu^*$, i.e., $R_b^L \delta\mu(T_c) \to 0$ for large b. Since $1/\nu > 0$, $t_0(T_c) \equiv r_0(T_c) + u(T_c)A$ must vanish. Since parameters in the equations of motion are smooth functions of T, we have

$$t_0(T) \propto T - T_c \quad (4.14)$$

for very small $T - T_c$. Of course, the critical point is defined with $h = 0$. Write

$$\xi = |t_0(T)|^{-\nu} \propto |T - T_c|^{-\nu} . \quad (4.15)$$

Then, setting $b = \xi$ in (4.4), we have

$$G_{\alpha\beta}(k, \omega, \mu(T)) = \xi^2 G_{\alpha\beta}(k\xi, \omega\xi^z, \xi^x\delta\tilde{\lambda} + \tilde{\lambda}^*, \pm 1, u\xi^y) , \quad (4.16)$$

where $\pm = \mathrm{sgn}(T - T_c)$. For $|T - T_c| \to 0$, $\xi \to \infty$. Since $x = -\epsilon$ (stable fixed point), the third argument on the right-hand side becomes simply $\tilde{\lambda}^*$ for sufficiently large ξ, i.e., $\xi^x\delta\tilde{\lambda} \propto |T - T_c|^{\epsilon\nu}\delta\tilde{\lambda}$ becomes negligible. The characteristic frequency $\omega(q)$ defined as the pole of G on the ω plane, must be a function of $k\xi$, $u\xi^y$, $\xi^x\delta\tilde{\lambda} + \tilde{\lambda}^*$ times ξ^{-z}. With $z = 1 + \tfrac{1}{2}d$ and for sufficiently small $|T - T_c|$ we can neglect $\xi^x\delta\tilde{\lambda}$ and $u\xi^y$ on the right-hand side of

(4.16), and the characteristic frequency must be of the form

$$\omega(q)=\xi^{-z}\Omega(q\xi)\,. \tag{4.17}$$

The form (4.17) is of course the statement of dynamic scaling. The neglected variables $\delta\tilde{\lambda}$ and u are "irrelevant variables." However, it might turn out that an irrelevant variable may not be neglected, and (4.17) fails as we shall see, for example, in the case of transverse spin-wave frequency below T_c, where u does not disappear for $|T-T_c|\to 0$.

Clearly, the statement $\lim_{b\to\infty}R_b\mu(T_c)\to\mu^*$ is not possible for the unstable fixed point $\mu^*=0$ unless $\tilde{\lambda}=\delta\tilde{\lambda}=0$. In that case, a similar conclusion like (4.17) with $z=4$ follows from (4.16). If $\tilde{\lambda}$ is not zero but extremely small, then

$$\xi^x\delta\tilde{\lambda}=\xi^{\epsilon/2}\tilde{\lambda}\propto|T-T_c|^{-\varphi}\tilde{\lambda} \tag{4.18}$$

is small for not too small $T-T_c$ and taking $\xi^x\delta\tilde{\lambda}\approx 0$ does not make a big error. However, as $|T-T_c|$ is decreased, $\xi^x\delta\tilde{\lambda}$ grows as (4.18) indicates and eventually becomes important for sufficiently small $|T-T_c|$. The crossover exponent $\varphi=\frac{1}{4}\epsilon$ [given by (4.13)] tells us how the effect of $\delta\tilde{\lambda}$ increases as $|T-T_c|$ decreases.

B. Calculation to $O(\epsilon)$

Assuming that all entries of $\mu=(\tilde{\lambda},r_0,u,h)$ are small, we now consider the detailed calculations of $R_b\mu$. Arguments of Sec. III pointed out that the smallness assumption would lead to useful information only for $d=6-\epsilon$ with ϵ small. Here we shall determine the fixed points and related properties to $O(\epsilon)$. Complications of higher orders will be discussed qualitatively.

Initially, we demonstrate the method for the case $u=h=0$ and show how Γ is changed under step (a) of R_b. We divide the wave vectors into two sets, $\{q\}$ and $\{k\}$, with

$$\Lambda/b<q<\Lambda\,,\quad k<\Lambda/b\,. \tag{4.19}$$

Equations (3.8) are then divided into two sets of coupled equations, one for $\vec{S}_q$ and one for $\vec{S}_k$. If we label the lines representing $\vec{S}_q$ by a line with a slash and $\vec{S}_k$ without a slash, we obtain Fig. 4(a) representing these two sets of equations. We then solve the equations for $\vec{S}_q$ to obtain $\vec{S}_q$ to $O(\lambda^2)$ as shown in Fig. 4(b). Next we insert this solution into the equations for $\vec{S}_k$, average over $\vec{S}_q$, and then obtain $\vec{S}_k$ as given by Fig. 4(c). The first two terms in Fig. 4(c) are just those which we would obtain by ignoring the $\vec{S}_q$'s all together. The fourth term can be ignored for small wave numbers which cannot add up to a wavenumber larger than Λ/b. It will turn out that we can write the contribution from the bubble in the third term as $k^2\Delta\Gamma(k,\omega)/\Gamma$. So we can write Fig. 4(c) as

$$\begin{aligned}\vec{S}_k(\omega)&=G_0(k,\omega)\vec{\zeta}_k(\omega)/\Gamma k^2\\&\quad-G_0(k,\omega)k^2[\Delta\Gamma(k,\omega)/\Gamma]\vec{S}_k(\omega)\\&\quad+\text{the term proportional to two }\vec{S}_k\text{'s}\,.\end{aligned} \tag{4.20}$$

Thus, ignoring the third term in (4.20) for the moment,

$$\begin{aligned}\vec{S}_k(\omega)&=\left[\left(G_0^{-1}(k,\omega)+k^2\frac{\Delta\Gamma(k,\omega)}{\Gamma}\right)\Gamma k^2\right]^{-1}\vec{\zeta}_k(\omega)\\&=\frac{1}{-i\omega+k^4(\Gamma+\Delta\Gamma(k,\omega))}\vec{\zeta}_k(\omega)\,,\end{aligned} \tag{4.21}$$

which gives the change in Γ to order λ^2 as $\Delta\Gamma(0,0)$. It is also easy to see that the effect of the $\Delta\Gamma$ terms on the third term in Fig. 4(c) is to simply replace $\Gamma\to\Gamma+\Delta\Gamma$. Another way of looking at this result is as follows. Multiply both sides of (4.20) by $k^2\Gamma G_0^{-1}(k,\omega)=-i\omega+\Gamma k^4$ and move $\Gamma k^4\vec{S}_k(\omega)$ on the left to the right, we then get

$$-i\omega\vec{S}_k(\omega)=\vec{\zeta}_k(\omega)-k^4[\Gamma+\Delta\Gamma(k,\omega)]\vec{S}_k(\omega)+\cdots\,. \tag{4.21$'$}$$

This is of course just the Fourier transform of an equation of motion of the form (1.5) with $\Gamma+\Delta\Gamma$

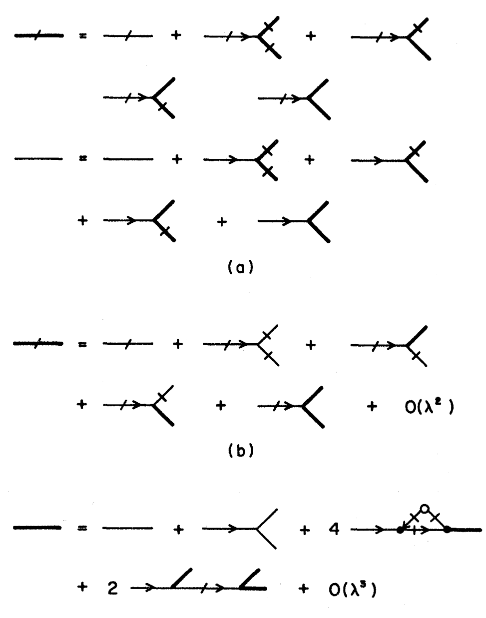

FIG. 4. (a) Simultaneous equations for S_q and S_k. The lines with a slash represent S_q's or response functions in the shell $\Lambda/b<q<\Lambda$. (b) $O(\lambda)$ terms for S_q. (c) $O(\lambda^2)$ terms for S_k.

replacing Γ [ignoring (k, ω) dependence of $\Delta\Gamma$]. Note that $\partial F'/\partial \vec{S}_{-k} = k^2 \vec{S}_k$ since we set $r_0 = u = h = 0$.

In carrying out the analysis in more generality we shall solve the equation of motion (1.5) for $\vec{S}_q$ to $O(\lambda^3)$, substitute into the equation for $\vec{S}_k$ and average over ζ_q.[30] The results in terms of graphs are shown in Fig. 5. The three graphs in Fig. 5(a) are proportional to $\vec{S}_{k+k'} \times \vec{S}_{-k'}$ and hence give a correction $\Delta\lambda$ to λ. The three in (b) have the form $\vec{S}_{k+k'} \times \vec{h}_{k'}$ and hence give $\Delta\lambda$ also. For consistency, the results of $\Delta\lambda$ from (a) and (b) must agree. Figure 5(c) is proportional to $\vec{S}_k$ and hence makes a correction to Γ. Figure 5(d) is proportional to $\vec{h}_k$ and must make the same correction to Γ. Let us look at the details.

It is sufficient to consider the spin component 1, frequency component ω of $\vec{S}_k$; 2, $\omega' + \omega$ of $\vec{S}_{k'+k}$; and 3, ω' of $\vec{S}_{k'}$ as labeled in Fig. 5(a). Following the rules of Sec. III, Fig. 5(a) gives

$$\Delta\lambda[(k+k')^2 - k'^2] = \lambda\tilde{\lambda}^2(2\pi)^{-d-1} \times \int d^dq\, d\nu[(k+q)^2 - q^2](A+B+C), \tag{4.22}$$

where A, B, C are the contributions from the three graphs in Fig. 5(a), respectively,

$$A = (k+q)^{-2}[(k+k')^2 - (k'-q)^2](q-k')^{-2}(k'^2 - q^2) \times G_0(k+q, \omega+\nu) G_0(q-k', \nu-\omega') C_0(q, \nu), \tag{4.23a}$$

$$B = (q+k)^{-2}[(k+k')^2 - (k'-q)^2] q^{-2}[(q-k')^2 - k'^2] \times G_0(-q, -\nu) G_0(q+k, \nu+\omega) C_0(q-k', \nu-\omega'), \tag{4.23b}$$

$$C = q^{-2}[(q-k')^2 - k'^2](q-k')^{-2}[(q+k)^2 - (k+k')^2] \times G_0(-q, -\nu) G_0(k'-q, \omega'-\nu) C_0(q+k, \nu+\omega). \tag{4.23c}$$

In view of (4.22) and (4.23), it is clear that $\Delta\lambda$ will depend on k', ω', and k, ω. We can expand $\Delta\lambda$ in powers of these variables. The transformed equations of motion thus contain extra terms of higher space and time derivatives in addition to the original form $\vec{S} \times \nabla^2 \vec{S}$. As we mentioned before, this means that more parameters (and hence more entries in μ) are needed to specify these extra terms. We shall see later that these extra terms will not affect the results of interest to $O(\epsilon)$.

The frequency integral over ν in (4.22) is elementary. The result is that

$$\Delta\lambda = 0 \tag{4.24}$$

as far as the constant term is concerned.

Similarly, the graphs in Fig. 5(b) can be calcu-

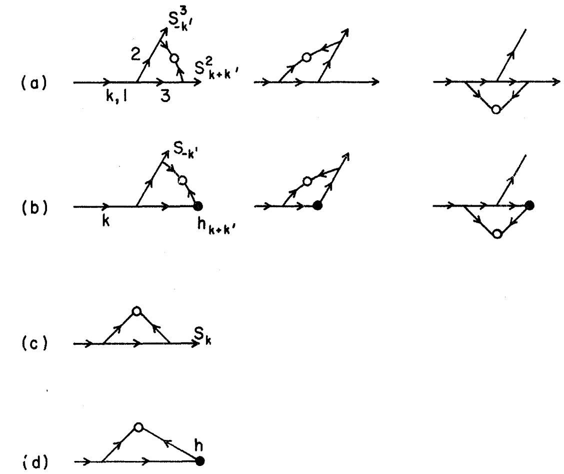

FIG. 5. (a) and (b): $O(\lambda^3)$ terms in $\Delta\lambda$ [see (4.20)]. (c) and (d): $O(\lambda^2)$ terms in $\Delta\Gamma$.

lated. We find $\Delta\lambda = 0$, consistent with (4.24).

Figure 5(c) gives

$$\frac{\Delta\Gamma}{\Gamma} = -k^{-4}\, 2\tilde{\lambda}^2 (2\pi)^{-d-1} \int d^dq\, d\nu [q^2 - (q+k)^2] \times G_0(q+k, \nu+\omega) C_0(q, \nu)(q+k)^{-2}(k^2 - q^2)$$
$$= k^{-2}\tilde{\lambda}^2(2\pi)^{-d} \int d^dq [q^2 - (q+k)^2]^2 q^{-2}(q+k)^{-2} \times [(q+k)^4 + q^4 - i\omega]^{-1}. \tag{4.25}$$

In view of (4.25), $\Delta\Gamma$ depends on ω and k. Thus the modified G_0 gets extra ω and k dependence which means higher derivatives in the transformed equations of motion requiring more parameters. Again we shall argue that these complications will not affect our results to $O(\epsilon)$. Setting k and ω to zero in (4.25), we obtain, for setting $d = 6$ in the integral,

$$\frac{\Delta\Gamma}{\Gamma} = \tilde{\lambda}^2(2\pi)^{-6} \int d^6q \frac{(2q\cdot\hat{k})^2}{2q^8} = \tilde{\lambda}^2(192\pi^3)^{-1} \int_{\Lambda/b}^{\Lambda} \frac{dq}{q} = \tilde{\lambda}^2(192\pi^3)^{-1}\ln b. \tag{4.26}$$

Figure 5(d) gives a term proportional to $k^2\Gamma\vec{h}_{-k}\cdot\vec{S}_k$ with a coefficient

$$-k^{-2} 2\tilde{\lambda}^2(2\pi)^{-d-1} \int d^dq\, d\nu [q^2 - (q+k)^2] \times G_0(q+k, \nu+\omega) C_0(q, \nu)(q+k)^{-2}$$
$$= k^{-2}\tilde{\lambda}^2(2\pi)^{-d} \int d^dq [q^2 - (q+k)^2]^2 q^{-2}(q+k)^{-2} \times [(q+k)^4 + q^4 - i\omega]^{-1}, \tag{4.27}$$

which is identical to (4.25). This means a correction to Γ as implied by (4.25) and no correction to h. To sum up, step (a) of R_b gives no effect on λ, but modifies Γ to

$$\Gamma_a = \Gamma + \Delta\Gamma = \Gamma[1 + \tilde{\lambda}^2(192\pi^3)^{-1}\ln b]. \quad (4.28)$$

Now we apply step (b) of R_b, i.e., replacing $S_k(t)$ by $b^{1-\eta/2}S_{bk}(tb^{-z})$ and L and bL', in the equation of motion. We obtain the transformed equations of motion

$$\frac{\partial}{\partial t'}\vec{S}_p(t') = \lambda' L'^{-d/2} \times \sum_{p'} \vec{S}_{p+p'} \times (-p'^2\vec{S}_{-p'} + \vec{h}'_{-p'}) + b^{z-4}\Gamma_a p^2(-p^2\vec{S}_p + \vec{h}'_p), \quad (4.29)$$

where

$$t' = tb^{-z}, \quad p = bk, \quad p' = bk'. \quad (4.30)$$

The wave vectors p, p', $p+p'$ are now ranging from 0 to Λ. The transformed parameters λ' and h' are given by

$$\lambda' = \lambda b^{z-1-\eta/2-d/2}, \quad (4.31a)$$

$$h'_p = h_{p/b} b^{1+\eta/2}. \quad (4.31b)$$

The educated reader would see that the sign in front of η in (4.31b) is not consistent with that given by the analysis in statics, which gives

$$\vec{h}'_p = \vec{h}_{p/b} b^{1-\eta/2}. \quad (4.32)$$

The reason is that a nonzero η is necessary for a fixed point with a nonzero uS^4 term. Here $u^* = 0$ and we can remove this inconsistency by setting $\eta = 0$.

We have thus completed $R_b\mu$ for the case $u = r_0 = 0$. Since we did not get a $k^2\vec{S}_k$ term [Fig. 5(c) is proportional to $k^4\vec{S}_k$] nor a $k^2S_{k'}S_{k''}S_{k'''}$ term, r_0 and u were not generated to the order calculated. Thus, zero is the suitable fixed point value for r_0 and u. Together with (4.32), we have

$$r_0^* = u^* = h^* = 0. \quad (4.33)$$

Now we collect the transformation formulas for λ and Γ from (4.31) and (4.29), (4.28):

$$\lambda' = \lambda b^{z-1-d/2}, \quad (4.34a)$$

$$\Gamma' = \Gamma_a b^{z-4} = \Gamma b^{z-4}[1 + \tilde{\lambda}^2(192\pi^3)^{-1}\ln b]. \quad (4.34b)$$

From these formulas, we obtain the formula for $\tilde{\lambda}' = \lambda'/\Gamma'$:

$$\tilde{\lambda}' = \tilde{\lambda} b^{3-d/2}[1 - \tilde{\lambda}^2(192\pi^3)^{-1}\ln b]. \quad (4.34c)$$

To obtain $\tilde{\lambda}^*$, we simply set $\tilde{\lambda}' = \tilde{\lambda} = \tilde{\lambda}^*$ in (4.34c). There is a trivial solution $\tilde{\lambda}^* = 0$. In this case z must be 4 to keep $\Gamma' = \Gamma$ in (4.34b). There are two nontrivial solutions given by

$$\tfrac{1}{2}\epsilon = \lambda^{*2}(192\pi^3)^{-1}, \quad (4.35)$$

since $b^{3-d/2} = b^{\epsilon/2} \approx 1 + (\frac{1}{2}\epsilon)\ln b$. We then get two nontrivial fixed points as summarized by (4.7).

We now examine the case $r_0 \neq 0$, $u \neq 0$. We must then analyze the full equations given by (1.5) and (1.6). We shall go through the steps (a) and (b) of R_b keeping $\delta\lambda = \lambda - \lambda^*$, r_0 and u to the first order, i.e., we shall determine R_b^L, the linearized R_b.

For the trivial fixed point, the answer is easily found. The only contribution of step (a) is the modification of r_0 from averaging a pair of S_q's in the u term as shown in Fig. 6. We have

$$\Delta r_0 = 5u(2\pi)^{-d-1}\int_{\Lambda/b}^{\Lambda} d^dq \int_{-\infty}^{+\infty} d\nu\, C_0(q,\nu) = uA(1 - b^{-d+2}) \quad (4.36)$$

where A is a constant. The rest of R_b is just step (b), which gives

$$\lambda' = \lambda b^{\epsilon/2}, \quad (4.37a)$$

$$h'_p = h_p b, \quad (4.37b)$$

$$r'_0 = t'_0 - u'A, \quad (4.37c)$$

$$t'_0 = t_0 b^2 \equiv (r_0 + uA)b^2, \quad (4.37d)$$

$$u' = ub^{4-d}. \quad (4.37e)$$

The first two equations follow from (4.31) with $z = 4$, $\eta = 0$.

Now we work out R_b^L for μ near the non-trivial fixed point keeping u and r_0 to first order. Consider step (a).

First we look at Fig. 5(c), which produced a term $-\Delta\Gamma k^4 S_k$ when we calculated with $r_0 = u = 0$. [See (4.25) and (4.26).] If we include a nonzero r_0 in the calculation of Fig. 5(c), we get simply

$$-\Delta\Gamma(r_0 + k^2)k^2 S_k. \quad (4.38)$$

Thus Fig. 5(c) still can be interpreted as affecting a correction to Γ.

Figure 7(a) gives some corrections to Fig. 5(c) to first order in u. They are obtained by inserting Fig. 6, i.e., $-\Delta r_0$ of (4.36), into Fig. 5(c). Since

$$G_0^2 = \frac{-\partial G_0}{\partial r_0}, \quad (4.39)$$

these three graphs of Fig. 7(a) simply give

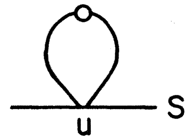

FIG. 6. $O(u)$ contribution to r'_0.

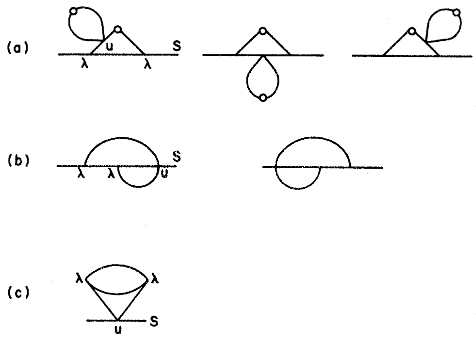

FIG. 7. $O(u\lambda^2)$ contribution to the transformation of $r_0\Gamma$. In (b) and (c), circles on some of the lines are understood.

$$(-\Delta r_0)\frac{-\partial}{\partial r_0}[\text{Fig. 4(c)}] = -\Delta\Gamma(\Delta r_0)k^2 S_k\,, \qquad (4.40)$$

i.e., adding Δr_0 to r_0 in (4.38). The contribution of Fig. 7(b) and 7(c) vanish. Thus step (a) of R_b simply changes r_0 to $r_0+\Delta r_0$ as expected. Step (b) is easy. One thus obtains again (4.37c) and (4.37d) for the transformation of r_0 under R_b.

To obtain u', we collect modifications to the equation of motion which are proportional to k^2 times three powers of S. The graphs are given in Fig. 8. These in Fig. 8(a) are just those in Fig. 7(a) with uS^2 replacing Δr_0. They thus should be interpreted like (4.40), a correction of Γ which multiplies the u term in the equation of motion, not a correction of u. The rest of the graphs in Fig. 8 vanish. Thus step (a) of R_b does not change u. Step (b) again is trivial. We obtain u with the same formula as (4.37e).

Finally, we come to the transformation of $\tilde{\lambda}$ taking nonzero r_0 and u into account to first order. When r_0 is included in the calculation of $\Delta\lambda$ given earlier, one still gets $\Delta\lambda=0$. The $O(u)$ terms are shown in Fig. 9. They all vanish. Thus the transformation formula (4.34) is not perturbed. We obtain the linearized equation of (4.34c) as

$$\delta\tilde{\lambda}' = b^{-\epsilon}\,\delta\tilde{\lambda} \qquad (4.41)$$

around the nontrivial fixed point with $\tilde{\lambda}^* = (\epsilon/96\pi^3)^{1/2}$.

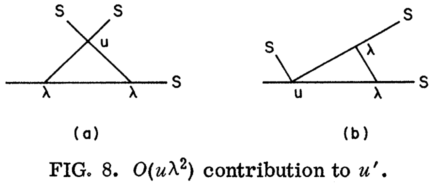

FIG. 8. $O(u\lambda^2)$ contribution to u'.

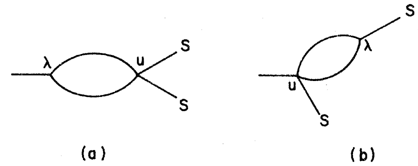

FIG. 9. $O(u\lambda)$ contribution to λ'.

C. Higher-order terms

So far we have worked out R_b only to $O(\epsilon)$. When higher orders are included, many complications arise. The transformation of λ and Γ will have more terms, like those in Fig. 10(a), for example. The simple form of our equations of motion will no longer be sufficient. For example, Fig. 10(b) shows that a term proportional to 4 powers of spin variables in the equation of motion will be generated by R_b. Figure 10(c) shows that one has to introduce more complicated coupling between the noise and the spin. The transformed noise would not be purely Gaussian, but has nontrivial higher moments. All these complications are not unexpected. The step (a) of R_b, i.e., the elimination of S_q with $\Lambda/b < q < \Lambda$, summarizes all effects of the eliminated variables on the remaining ones. Since the former have complicated behaviors, their effect on the latter should not be very simple.

We shall not attempt to explore the higher-order calculations here but only to show evidence that higher orders can be consistently ignored as far

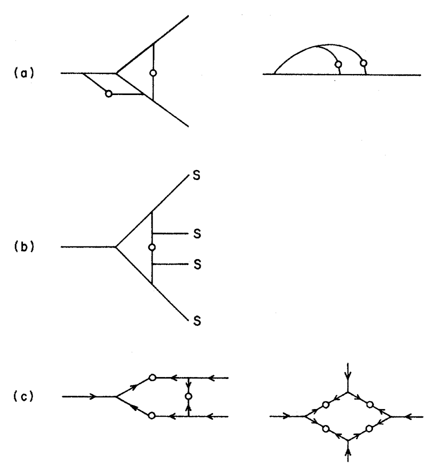

FIG. 10. (a) $O(\lambda^4)$ contribution to λ' and Γ', respectively. (b) New term generated by R_s proportional to four powers of spin. (c) These two graphs are joints of three and four ends of lines, respectively. They are possible parts of more complicated graphs. They can be regarded as the average of three and four powers of noise, respectively.

as the $O(\epsilon)$ results already calculated above are concerned. That is, if there are other terms in the equation of motion, we want to make sure that they will not alter our results to $O(\epsilon)$, *assuming* that these other terms are of higher order in ϵ than $\lambda = O(\epsilon^{1/2})$. The consistency of this assumption is not obvious because we did have to calculate λ' to $O(\lambda^3) = O(\epsilon^{3/2})$ in order to obtain $\tilde{\lambda}^*$ and exponents to $O(\epsilon^{1/2})$ and $O(\epsilon)$, respectively.

The simplest term that can ruin this consistency is a term proportional to four powers of spin with a coupling constant λ_5 of $O(\epsilon^{3/2})$. Figure 11(a) represents such a term. Under R_b, a term shown in Fig. 11(b) is a contribution to λ' of $O(\epsilon^{3/2})$, the same order as $O(\lambda^3)$. To save our results obtained earlier, we then have to assume that λ_5 is of higher order than $O(\epsilon^{3/2})$. The assumption that $\lambda_5 = O(\epsilon^{5/2})$ turns out to be consistent. Figure 10(b) shows that λ_5' generated by λ is of $O(\lambda^5) = O(\epsilon^{5/2})$, not $O(\epsilon^{3/2})$. Similar arguments apply to more complicated terms. Thus, as far as the determination of $\tilde{\lambda}^*$ and exponents to $O(\epsilon)$ is concerned, there is no inconsistency. These arguments are parallel to those used in showing the static calculation to $O(4\text{-}d)$ (see Refs. 11 and 12).

V. CALCULATION OF THE RESPONSE FUNCTION BY PERTURBATION THEORY, $T \geq T_c$

A. Discussion

Understanding of the RNG can only tell us the transformation property (4.16) of the response function G, but not its explicit form. To obtain more information, we shall calculate G as an expansion in powers of λ. Assuming $\lambda = O(\epsilon^{1/2})$, we shall calculate to $O(\epsilon)$ for $T \geq T_c$. The case $T < T_c$ will be considered in Sec. VI.

If one blindly carries out perturbation theory to second order in λ one obtains a number of logarithmic terms. We note from our renormalization group analysis of the response function that among these logarithms will be terms contributing to [see (4.16)],

$$\tilde{\lambda}^* + \xi^{-\epsilon}\,\delta\tilde{\lambda} \approx \tilde{\lambda}^* + (1 - \epsilon \ln\xi)(\delta\tilde{\lambda}) + O(\epsilon^2). \tag{5.1}$$

Thus, instead of getting $\xi^{-\epsilon}$ which goes to zero as $\xi \to \infty$, one sees $\ln\xi$ which gets mixed up with other logarithms of interest. To remove such

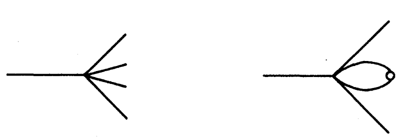

FIG. 11. (a) Term proportional to four powers of spin in the equation of motion. (b) R_s will generate this contribution to λ' from (a).

confusing $\ln\xi$ terms we simply set $\delta\tilde{\lambda} = 0$, i.e., $\tilde{\lambda} = \tilde{\lambda}^*$ in performing the perturbation expansion to $O(\epsilon)$. This choice of $\tilde{\lambda}$ is in the same spirit as Wilson's choice of u_0 in computing static exponents by perturbation theory.[7]

B. Second-order self-energy

For $T \geq T_c$, the system is isotropic apart from the small effect of the external field h. The response function, defined by (3.4), is thus proportional to $\delta_{\alpha\beta}$,

$$G_{\alpha\beta} = G\delta_{\alpha\beta}. \tag{5.2}$$

Let the self-energy $\Sigma(k,\omega)$ be defined by

$$G^{-1}(k,\omega) = G_0^{-1}(k,\omega) - \Sigma(k,\omega). \tag{5.3}$$

The $O(\lambda^2)$ terms for $G(k,\omega)$ are given by Fig. 3. We can identify the self-energy as

$$\Sigma = + G_0^{-2}(G_a + G_b), \tag{5.4}$$

where G_a and G_b are given by (3.11) and (3.12).

After performing the ν integral and making some simple rearrangements, we find

$$\Sigma(k,\omega) = -(i\omega/\Gamma k^2)Q(k,\omega), \tag{5.5}$$

where

$$Q(k,\omega) = \frac{\tilde{\lambda}^2}{k^2}\int \frac{d^6q}{(2\pi)^6}\frac{[(k-q)^2 - q^2]^2\chi(q)\chi(k-q)}{D(k,q,\omega)}, \tag{5.6}$$

$$D(k,q,\omega) = -i\omega/\Gamma + q^2\chi^{-1}(q) + (q-k)^2\chi^{-1}(q-k), \tag{5.7}$$

and χ^{-1} is given by (2.5) ($r_0 = \xi^{-2}$). We note that the inverse response function can then be written

$$G^{-1}(k,\omega) = \chi^{-1}(k) - (i\omega/\Gamma k^2)[1 - Q(k,\omega)], \tag{5.8}$$

preserving the identity $G^{-1}(k,0) = \chi^{-1}(k)$ to the order calculated.

C. Dispersion relation to order λ^2 for $T > T_c$

We will be primarily interested in the poles of the response function which gives the dispersion relation for the collective modes in the system. We therefore look for solutions of $G^{-1}(k,\omega) = 0$. Solutions occur for small k and ω at the characteristic frequency $\omega(k)$ which we can expand in a power series in k and ω. We find to lowest order in k and ω that

$$\omega(k) = -i\Gamma k^2(k^2 + \xi^{-2})[1 + Q(k,0)]. \tag{5.9}$$

A detailed analysis of $Q(k,0)$ requires one to extract the explicit dependence on the cutoff Λ. This can be accomplished by writing

$$Q(k,0) = Q(0,0) + Q(k,0) - Q(0,0). \tag{5.10}$$

We easily find

$$Q(0,0)=\tfrac{1}{2}\epsilon\ln(\Lambda\xi)\,, \tag{5.11}$$

where we have used (4.7a) for $\tilde{\lambda}^*$. The quantity $Q(k,0)-Q(0,0)$ is finite as $\Lambda\to\infty$ and depends only on the variable $x\equiv k\xi$. If we define

$$\epsilon\Delta(x)=\lim_{\Lambda\to\infty}\left[Q(k,0)-Q(0,0)\right], \tag{5.12}$$

we find, for example,

$$\lim_{x\to 0}\frac{\Delta(x)}{x^2}=-\frac{13}{128} \tag{5.13}$$

and, for large x,

$$\Delta(x)=-\tfrac{1}{2}\ln x+\Delta_\infty+O(1/x)\,, \tag{5.14}$$

where Δ_∞ is given numerically as 0.287. We evaluated $\Delta(x)$ numerically and plotted the results as a function of x in Fig. 12. We then see that the characteristic frequency has the form, to $O(\epsilon)$,

$$\omega(k)=-i\Gamma' k^{4-\epsilon/2}f(x)\,, \tag{5.15}$$

where Γ' is a renormalized "bare" Onsager coefficient

$$\Gamma'=(1+\epsilon\Delta_\infty)\Gamma(1+\tfrac{1}{2}\epsilon\ln\Lambda) \tag{5.16}$$

and we write

$$f(x)=\frac{1+x^{-2}}{1+\epsilon\Delta_\infty}\{\Theta(1-x)x^{\epsilon/2}[1+\epsilon\Delta(x)] +\Theta(x-1)[1+\tfrac{1}{2}\epsilon\ln x+\epsilon\Delta(x)]\}, \tag{5.17}$$

where $\Theta(x)$ is the unit step function. We have normalized $f(x)$ such that $f(\infty)=1$. We must introduce the step functions if we are to satisfy the asymptotic conditions $f(\infty)=1$, and $f(x)\propto x^{\epsilon/2-2}$ for $x\ll 1$. Note that $f(x)$ is continuous at $x=1$, but will have discontinuous derivatives at $x=1$. We have plotted $f(x)$ in Fig. 13 for various values of ϵ. The qualitative behavior of $f(x)$ is in agreement with that found by Resibois and Piette.[31] We cannot, however, take the result for $\epsilon=3$ very seriously. Clearly, the matching condition at $x=1$ is responsible for the unphysical kink that appears near $x=1$ for $\epsilon=3$.

The physical Onsager coefficient can be obtained from (5.15) and (5.17) using (2.8). We find

$$\Gamma=\Gamma'\xi^{\epsilon/2}/(1+\epsilon\Delta_\infty) \tag{5.18}$$

and we see the strong temperature dependence discussed in Sec. II.

VI. CALCULATION OF THE RESPONSE FUNCTIONS BY PERTURBATION THEORY, $T<T_c$

A. Equations of motion below T_c

The major modification in the theory below T_c is that the z component of the spin has a finite average in zero external magnetic field

$$\langle S_z(x,t)\rangle=M. \tag{6.1}$$

It is useful therefore to define a new set of variables

$$\varphi_z=S_z-M \tag{6.2}$$

and

$$\varphi_\pm=(2)^{-1/2}(S_x\pm iS_y)\,, \tag{6.3}$$

$$h_\pm=(2)^{-1/2}(h_x\pm ih_y)\,, \tag{6.4}$$

$$\zeta_\pm=(2)^{-1/2}(\zeta_x\pm i\zeta_y)\,. \tag{6.5}$$

Using (1.7) we see that these noise sources satisfy

$$\langle\zeta_+(k,\omega)\zeta_-(k',\omega')\rangle=\langle\zeta_-(k,\omega)\zeta_+(k',\omega)\rangle =2\Gamma k^2\delta_{k,-k'}2\pi\delta(\omega+\omega')\,, \tag{6.6a}$$

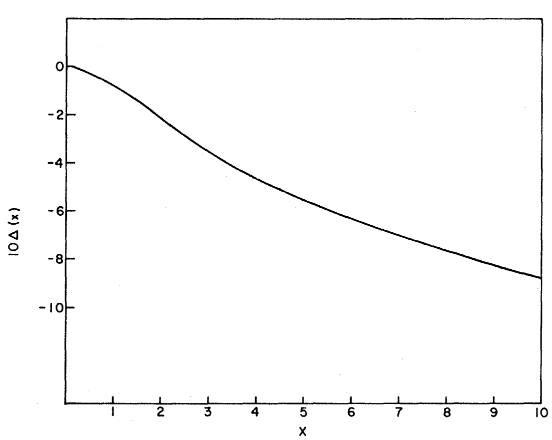

FIG. 12. Function $10\Delta(x)$ defined by (5.12), (5.11), and (5.7), where $x=k\xi$.

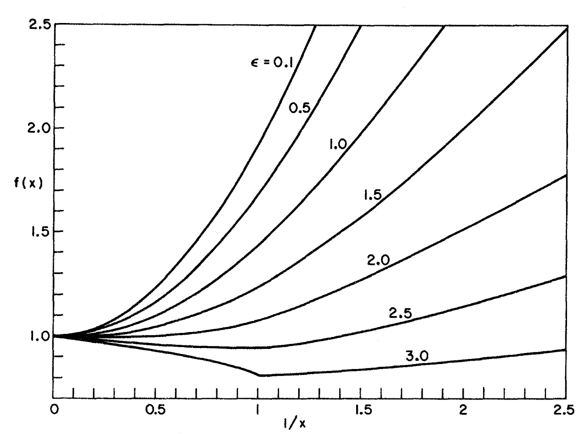

FIG. 13. Function $f(x)$, defined by (5.17), plotted vs $1/x=1/k\xi$, for various values of $\epsilon=6-d$.

$$\langle \zeta_+(k,\omega)\zeta_+(k',\omega')\rangle = \langle \zeta_-(k,\omega)\zeta_-(k',\omega')\rangle = 0 . \tag{6.6b}$$

If we allow the index α to take on the values ± and $0(=z)$, then our equations of motion (1.2) can be written

$$\frac{\partial \varphi_\alpha(x,t)}{\partial t} = i\alpha M\lambda\nabla^2\varphi_\alpha(x,t) + (i\alpha\lambda M - \Gamma\nabla^2)h_\alpha(x,t) + \zeta_\alpha(x,t) + \Gamma\nabla^2\left(\int d^dx'[\chi_\alpha^{-1}(x-x')\varphi_\alpha(x',t)] + u\varphi_\alpha(x,t)\varphi^2(x,t)\right)$$
$$+\int d^dx'\,d^dx'' \sum_{\beta,\gamma} W_{\alpha\beta\gamma}(x;x'x'')\varphi_\beta(x',t)\varphi_\gamma(x'',t) + \sum_{\beta,\gamma} K_{\alpha,\beta\gamma}\,\varphi_\beta(x,t)h_\gamma(x,t) , \tag{6.7}$$

where

$$\varphi^2 = \varphi_0^2 + \varphi_+\varphi_- + \varphi_-\varphi_+ , \tag{6.8}$$

$$\chi_0^{-1}(x-x') = r_0'\delta(x-x') - \nabla^2\delta(x-x') ,$$
$$\chi_0^{-1}(q) = r_0' + q^2 , \tag{6.9}$$

$$\chi_\pm^{-1}(x-x') = -\nabla^2\delta(x-x') , \quad \chi_\pm^{-1}(q) = q^2 , \tag{6.10}$$

$$r_0' = r_0 + 3uM^2 , \tag{6.11}$$

$$W_{\pm,\beta\gamma}(x;\bar{x}\bar{x}') = \delta_{\beta,\pm}\delta_{\gamma,0}\{\pm i\lambda[\delta(x-\bar{x})\nabla^2 \times \delta(x-\bar{x}') - \delta(x-\bar{x}')\nabla^2\delta(x-\bar{x})] + 2\Gamma uM\nabla^2\delta(x-\bar{x}')\delta(x-\bar{x})\} , \tag{6.12}$$

$$W_{0,\beta\gamma}(x;\bar{x}\bar{x}') = \tfrac{1}{2}(i\lambda)(\delta_{\beta,-}\delta_{\gamma,+} - \delta_{\beta,+}\delta_{\gamma,-}) \times [\delta(x-\bar{x})\nabla^2\delta(x-\bar{x}') - \delta(x-\bar{x}')\nabla^2\delta(x-\bar{x})] + 2\Gamma uM\nabla^2\delta(x-\bar{x}')\delta(x-\bar{x})\delta_{\beta,+}\delta_{\gamma,-} + 6\Gamma uM\delta_{\beta,0}\delta_{\gamma,0}\nabla^2\delta(x-\bar{x}')\delta(x-\bar{x}) , \tag{6.13}$$

$$K_{\pm,\beta\gamma} = \pm i\lambda(\delta_{\beta,\pm}\delta_{\gamma,0} - \delta_{\beta,0}\delta_{\gamma,\pm}) , \tag{6.14}$$

$$K_{0,\beta\gamma} = + i\lambda(\delta_{\beta,-}\delta_{\gamma,+} - \delta_{\beta,+}\delta_{\gamma,-}) . \tag{6.15}$$

We note that the result for the transverse susceptibility, (6.10), follows from the use of the relation

$$r_0 + uM^2 = 0 , \tag{6.16}$$

which is valid for the determination of M to lowest order in u. We expect, however, from the work of Brézin, Wallace, and Wilson,[8] that the inverse transverse susceptibility will vanish for $q\to 0$ to all orders in u.

If we ignore the nonlinear terms in (6.7) we obtain, after Fourier transformation, the zeroth-order response functions

$$G_\alpha^0(k,\omega) = (\Gamma k^2 + i\alpha\lambda M) \times[-i(\omega - \alpha M\lambda k^2) + \Gamma k^2\chi_\alpha^{-1}(k)]^{-1} , \tag{6.17}$$

and the correlation functions

$$\langle \varphi_\alpha(k,\omega)\varphi_\beta(k',\omega')\rangle = 2\pi\delta(\omega+\omega')\delta_{k,-k'}C_{\alpha\beta}^0(k,\omega) , \tag{6.18a}$$

with

$$C_{00}(k,\omega) = 2\Gamma k^2/\{\omega^2 + [\Gamma k^2\chi_0^{-1}(k)]^2\} , \tag{6.18b}$$

$$C_{++}(k,\omega) = C_{--}(k,\omega) = C_{0,\pm}(k,\omega) = 0 , \tag{6.18c}$$

$$C_{+-}(k,\omega) = C_{-+}(k,-\omega) = \frac{2\Gamma k^2}{(\omega - \lambda Mk^2)^2 + (\Gamma k^4)^2} . \tag{6.18d}$$

Our model leads, in the absence of nonlinear couplings, to spin waves with frequency $M\lambda k^2$ in the transverse response and correlation functions for $T < T_c$. These spin waves are damped by a Γk^4 term. The longitudinal correlation function looks very much like the correlation function for $T > T_c$. The only change is to replace $\chi(k)\to\chi_0(k)$. However there is a nonlinear coupling between the transverse and longitudinal modes which will lead to modification of the longitudinal spectrum.

Starting with the equation of motion (6.7) we can set up our perturbation calculation just as for $T > T_c$ except that we must treat the K and W vertices defined by (6.12)–(6.15) and keep track of the α indices on the response and correlation functions. We assume that λ and u are small but $uM^2 = \text{const}$. Thus, $M \sim u^{-1/2}$ and we must keep more terms than for $T > T_c$. Note that $uM^2 \propto T_c - T$.

B. Longitudinal self-energy

As in the case for $T > T_c$ we can define the self-energies Σ_α as

$$G_\alpha(k,\omega) = G_\alpha^0(k,\omega) + G_\alpha^0(k,\omega)\Sigma_\alpha(k,\omega)G_\alpha(k,\omega) . \tag{6.19}$$

Keeping all terms in the expansion of G_α to order $(\lambda^2, u, (uM)^2)$ we obtain graphs like Figs. 3 and 6 except there are indices associated with each line and the vertices are given by (6.12)–(6.15). The longitudinal self-energy is given by

$$\Sigma_0(k,\omega)=\int\frac{d^6q}{(2\pi)^6}\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}[4(3uM)^2D_0(k)(-k^2)C_0^0(k-q,\omega-\omega')G_0^0(q,\omega')D_0(k)(-k^2)$$
$$+4D_0(k)I(k,q)C^0_{+-}(k-q,\omega-\omega')G^0_-(q,\omega')D_-(q)I(q,k)+4D_0(k)I^*(k,q)C^0_{-+}(k-q,\omega-\omega')$$
$$\times G^0_+(q,\omega')D_+(q)I^*(q,k)+2D_0(k)I(k,q)C^0_{+-}(k-q,\omega-\omega')G^0_-(q,\omega')D_-(q)(-i\lambda)(G_0^0)^{-1}(k,\omega)$$
$$+2D_0(k)I^*(k,q)C^0_{-+}(k-q,\omega-\omega')G^0_+(q,\omega')D_+(q)(i\lambda)(G_0^0)^{-1}(k,\omega)]+\Sigma_{H,0}, \tag{6.20}$$

where

$$I(k,q)=\tfrac{1}{2}(i\lambda)[q^2-(k-q)^2]-uM\Gamma k^2, \tag{6.21}$$

$$D_\alpha(k)=(\Gamma k^2+i\alpha\lambda M)^{-1}, \tag{6.22}$$

and

$$\Sigma_{H,0}=-u\int\frac{d^6q}{(2\pi)^6}[3\chi_0(q)+2\chi_+(q)] \tag{6.23}$$

is the contribution from a Hartree term similar to Fig. 6. We note that $I(k,q)$ is just the Fourier transform of $\frac{1}{2}W_{++0}$ [see (6.12)]. After doing the frequency integrals, which are simple contour integrals, we obtain, after replacing $q\to k-q$ in the third and fifth terms in (6.20), and noting

$$I(k,q)=I^*(k,k-q), \tag{6.24}$$

$$\Sigma_0(k,\omega)=\int\frac{d^6q}{(2\pi)^6}\frac{4(3uM)^2\chi_0(k-q)i\Gamma q^2}{D_L(\omega,k,q)}+\frac{1}{\Gamma k^2}\int\frac{d^6q}{(2\pi)^6}\frac{2iI(k,q)}{D_T(\omega,k,q)}$$
$$\times[\chi_+(k-q)2I(q,k)+\chi_+(q)2I^*(k-q,k)+\chi_+(k-q)(-i\lambda)(G_0^0)^{-1}(k,\omega)+\chi_+(q)(i\lambda)(G_0^0)^{-1}(k,\omega)]+\Sigma_{H,0}, \tag{6.25}$$

where

$$D_L(\omega,k,q)=\omega+i[\Gamma_L(k-q)+\Gamma_L(q)], \tag{6.26}$$

$$D_T(\omega,k,q)=\omega-\lambda M[q^2-(k-q)^2]+i[\Gamma_T(q)+\Gamma_T(k-q)], \tag{6.27}$$

$$\Gamma_L(q)=\Gamma\chi_0^{-1}(q)q^2, \tag{6.28}$$

$$\Gamma_T(q)=\Gamma\chi_+^{-1}(q)q^2. \tag{6.29}$$

If we concentrate on the term in square brackets in (6.25) we find after considerable manipulation that

$$\chi_+(k-q)2I(q,k)+\chi_+(q)2I^*(k-q,k)+[\chi_+(k-q)-\chi_+(q)][-i\lambda(G_0^0)^{-1}(k,\omega)]$$
$$=-(\lambda\omega/\Gamma k^2)[\chi_+(k-q)-\chi_+(q)]-i\omega u2M\chi_+(k-q)\chi_+(q)+2iuM\chi_+(k-q)\chi_+(q)D_T(\omega,k,q).$$

Using this result it is easy to see that the self-energy can be written in the form

$$\Sigma_0(k,\omega)=\Sigma_0(k,0)-\omega\Sigma_{0,\omega}(k,\omega), \tag{6.30}$$

where

$$\Sigma_0(k,0)=\int\frac{d^6q}{(2\pi)^6}[(3uM)^2\chi_0(q)\chi_0(k-q)+(2uM)^2\chi_+(q)\chi_+(k-q)]+\Sigma_{H,0} \tag{6.31}$$

and

$$\Sigma_{0,\omega}(k,\omega)=2(3uM)^2\int\frac{d^6q}{(2\pi)^6}\frac{1}{D_L(\omega,k,q)}+\int\frac{d^6q}{(2\pi)^6}\frac{\{i\lambda[q^2-(k-q)^2]-2uM\Gamma k^2\}^2\chi_+(q)\chi_+(k-q)}{(\Gamma k^2)^2D_T(\omega,k,q)}. \tag{6.32}$$

We can then write the inverse response function

$$G_0^{-1}(k,\omega)=(i\omega/\Gamma k^2)+\chi_0^{-1}(k)-\Sigma_0(k,0)+\omega\Sigma_{0,\omega}(k,\omega)$$
$$=-(i\omega/\Gamma k^2)[1+i\Gamma k^2\Sigma_{0,\omega}(k,\omega)]+\bar\chi_0^{-1}(k), \tag{6.33}$$

where

$$\bar\chi_0^{-1}(k)=\chi_0^{-1}(k)-\Sigma_0(k,0) \tag{6.34}$$

is the static susceptibility to $O(u^2M^2,u)$.

C. Dispersion relation for the longitudinal mode

Since we must consistently drop terms of order $(uM)^2 \sim u$ compared to 1, we should replace $\tilde{\chi}_0^{-1}$ with χ_0^{-1} and ignore the $2uM\Gamma k^2$ terms in $\Sigma_{0,\omega}$. We then have the dispersion relation

$$\omega_0(k) = -i\Gamma k^2(k^2 + 2\xi^{-2})[1 + Q_0(k)] \tag{6.35}$$

and

$$Q_0(k) = \frac{\tilde{\lambda}^2}{k^2} \int \frac{d^6q}{(2\pi)^6} \frac{[q^2-(k-q)^2]^2\chi_+(q)\chi_+(k-q)}{q^4 + (k-q)^4 + i\lambda M[q^2-(q-k)^2]} . \tag{6.36}$$

We can analyze $Q_0(k)$ in much the same manner used in evaluating $Q(k)$ for $T \geq T_c$. We find then that we can write

$$\omega_0(k) = -i\Gamma' k^{4-\epsilon/2}(1 + 2x^{-2})f_L(\alpha) , \tag{6.37}$$

where Γ' is the same as for $T > T_c$ [see (5.16) and $x = k\xi$]. The parameter α is defined by

$$\alpha = \tilde{\lambda}^* M/k^2 \tag{6.38}$$

and to lowest order in u

$$\alpha = \tilde{\lambda}^*/(k\xi)^2(u\xi^{-2})^{1/2} . \tag{6.39}$$

In the limit $\alpha \to 0$ we have $T \to T_c$ and we find

$$\omega_0(k, \alpha=0) = -i\Gamma' k^{4-\epsilon/2} \tag{6.40}$$

in agreement with (5.15) as $T \to T_c$. In the opposite limit of large α, $M\tilde{\lambda}^*$ fixed and $k \to 0$, we find

$$f_L(\alpha) = 1 + \frac{\tilde{\lambda}^{*2}}{192\pi^3} \ln\alpha^{-1/3} + A\tilde{\lambda}^{*2} + O\left(\frac{1}{\alpha^{2/3}}\right)$$
$$= \alpha^{-\epsilon/6}\left[1 + A\tilde{\lambda}^{*2} + O\left(\frac{1}{\alpha^{2/3}}\right)\right] , \tag{6.41}$$

where A is a constant. This leads to a characteristic frequency for large α, [32]

$$\omega_0(k) = -i\Gamma' k^{4-\epsilon/6}(1 + 2x^{-2})(M\tilde{\lambda}^*)^{-\epsilon/6} \times [1 + A\tilde{\lambda}^{*2} + O(\alpha^{-2/3})] . \tag{6.42}$$

Consequently, in the hydrodynamical limit, $k \to 0$, the transport coefficient giving the damping of the longitudinal mode becomes infinite. Just as $\tilde{\Gamma}$ developed an anomalous temperature dependence for $T > T_c$, the longitudinal damping develops an anomalous wave number dependence for small wave numbers

$$\tilde{\Gamma}_0(k) = \Gamma'(M\tilde{\lambda}^* k)^{-\epsilon/6}(1 + A\tilde{\lambda}^{*2}) . \tag{6.43}$$

D. Self-energy for the transverse mode

The contributions to the transverse response function to $O(u, uM, \lambda^2)$ are similar to those in the longitudinal case except for the α indices. The corresponding self-energy is given by

$$\Sigma_+(k,\omega) = \int \frac{d^6q}{(2\pi)^6} \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} [4D_+(k)I(k,q)C^0_{+-}(k-q, \omega-\omega')G_0(q,\omega')D_0(q)I(q,k)$$
$$+ 4D_+(k)I^*(k,q)C^0_0(k-q,\omega-\omega')G^0_+(q,\omega')D_+(q)I^*(q,k) + 2D_+(k)I(k,q)C^0_{+-}(k-q,\omega-\omega')$$
$$\times G^0_0(q,\omega')D_0(q)(i\lambda)(G^0_+)^{-1}(k,\omega) + 2D_+(k)I^*(k,q)C^0_0(k-q,\omega-\omega')G^0_+(q,\omega')D_+(q)(-i\lambda)(G^0_+)^{-1}(k,\omega)]$$
$$+ k^2\Gamma D_+(k)\Sigma_{H,+} , \tag{6.44}$$

where

$$\Sigma_{H,+} = -u\int \frac{d^6q}{(2\pi)^6} [\chi_0(q) + 3\chi_+(q)] , \tag{6.45}$$

$I(k,q)$ is given by (6.21) while the D's are given by (6.22). After performing the frequency integrals and using (6.24), we obtain

$$\Sigma_+(k,\omega) = 2D_+(k)i\int \frac{d^6q}{(2\pi)^6} \frac{I^*(k,q)}{D'_T(\omega,k,q)} \{2\chi_0(k-q)I^*(q,k) + 2\chi_+(q)I^*(k-q,q)$$
$$+ [\chi_+(q) - \chi_0(k-q)](-i\lambda)(G^0_+)^{-1}(k,\omega)\} + k^2\Gamma D_+(k)\Sigma_{H,+} , \tag{6.46}$$

where

$$D'_T(\omega,k,q) = \omega - M\lambda q^2 + i[\Gamma_T(q) + \Gamma_L(k-q)] , \tag{6.47}$$

Γ_T and Γ_L are given by (6.28) and (6.29),

$$\Gamma_L(q) = \Gamma\chi_0^{-1}(q)q^2, \quad \Gamma_T(q) = \Gamma\chi_+^{-1}(q)q^2 . \tag{6.48}$$

After considerable rearrangement we can rewrite (6.46) in the form

$$\Sigma_+(k,\omega) = \Sigma_+(k,0) - i\omega Q_+(k,\omega)/(\Gamma k^2 + i\lambda M), \tag{6.49}$$

where

$$\Sigma_+(k,0)=(2uM)^2\left(\pi(k)-\pi(0)+\frac{\Gamma k^2\pi(0)}{\Gamma k^2+i\lambda M}\right)+\frac{\Gamma k^2\Sigma_{H,+}}{\Gamma k^2+i\lambda M}\,, \tag{6.50}$$

$$\pi(k)=\int\frac{d^6q}{(2\pi)^6}\,\chi_0(k-q)\chi_+(q)\,, \tag{6.51}$$

and

$$Q_+(k,\omega)=-\frac{i}{\Gamma k^2+i\lambda M}\int\frac{d^6q}{(2\pi)^6}\,\frac{\chi_+(q)\chi_0(k-q)\{i\lambda[(k-q)^2-q^2]-2uM\Gamma k^2\}^2}{D_T'(\omega,k,q)}\,. \tag{6.52}$$

The inverse response function is given by

$$G_+^{-1}(k,\omega)=\frac{1}{\Gamma k^2+i\lambda M}[-i\omega+k^2(i\lambda M+k^2)+\Gamma k^2(r_0+uM^2)]-\Sigma_+(k,\omega)=-\frac{i\omega}{\Gamma k^2+i\lambda M}[1-Q_+(k,\omega)]$$
$$+k^2\left(1-\frac{(2uM)^2}{k^2}[\pi(k)-\pi(0)]\right)+\frac{\Gamma k^2}{\Gamma k^2+i\lambda M}[r_0+uM^2-(2uM)^2\pi(0)-\Sigma_{H,+}]\,. \tag{6.53}$$

The quantity multiplying $\Gamma k^2(\Gamma k^2+i\lambda M)^{-1}$ in the last term is simply the equation determining M in terms of r_0 and u evaluated to order u and therefore vanishes to this order. Therefore, we have the result that

$$G_+^{-1}(k,0)=\tilde{\chi}_+^{-1}(k)=k^2\left(1-\frac{(2uM)^2}{k^2}[\pi(k)-\pi(0)]\right) \tag{6.54}$$

and the transverse susceptibility does go as k^{-2} to $O(u)$ as speculated above. We have then

$$G_+^{-1}(k,\omega)=-\frac{i\omega[1-Q_+(k,\omega)]}{\Gamma k^2+i\lambda M}+\tilde{\chi}_+^{-1}(k)\,. \tag{6.55}$$

E. Dispersion relation for the transverse mode

The dispersion relation for the transverse mode can be written

$$\omega_+(k)=-i(\Gamma k^2+i\lambda M)\tilde{\chi}_+^{-1}(k)[1+Q_+(k,0)]\,. \tag{6.56}$$

We can rewrite this as

$$\omega_+(k)=\lambda M\tilde{\chi}_+^{-1}(k)-i\Gamma k^2\tilde{\chi}_+^{-1}(k)[1+Q_+'(k)]\,, \tag{6.57}$$

where

$$Q_+'(k)=[(\Gamma k^2+i\lambda M)/\Gamma k^2]\,Q_+(k,0)\,. \tag{6.58}$$

The spin-wave frequency can now be written

$$\omega_s(k)=\lambda Mk^2b(k), \tag{6.59}$$

where

$$b(k)=1-[(2uM)^2/k^2][\pi(k)-\pi(0)]\,. \tag{6.60}$$

We can easily show that

$$b(0)=1+\tfrac{1}{192}u\xi^{-2}\ln(\Lambda\xi)\,. \tag{6.61}$$

Clearly, the correction of $O(u)$ is irrelevant near the critical point. For consistency we must set $u=0$ in the scaling region while keeping M finite. Thus we set $b=1$, replace $\chi_+^{-1}(k)$ with k^2 and drop the $2uM\Gamma k^2$ terms in Q_+. We can evaluate $Q_+'(k)$ in much the same manner as $Q(k)$ in Sec. IV. We can extract the wave-number cutoff dependence of Q_+' by calculating

$$Q_+'(0)=\tfrac{1}{384}\tilde{\lambda}^2\ln\Lambda^2/M\tilde{\lambda}\,. \tag{6.62}$$

Then $Q_+'(k)-Q_+'(0)$ is finite in the limit $\Lambda\to\infty$, and depends only on the parameter α defined by (6.38),

$$\Delta_+(\alpha)=Q_+'(k)-Q_+'(0)\,. \tag{6.63}$$

Putting this together we can write the characteristic frequency

$$\omega_+(k)=\lambda Mk^2-i\Gamma'k^{4-\epsilon/2}f_\perp(\alpha)\,, \tag{6.64}$$

where Γ' is given by (5.16),

$$f_\perp(\alpha)=(1+\epsilon\Delta_\infty)^{-1}\{\Theta(1-\alpha)[1-\tfrac{1}{4}\epsilon\ln\alpha+\Delta_+(\alpha)]$$
$$+\Theta(\alpha-1)\alpha^{-\epsilon/4}[1+\Delta_+(\alpha)]\} \tag{6.65}$$

and we have normalized such that

$$f_\perp(0)=1\,. \tag{6.66}$$

This is just the limit $T\to T_c$, and we find agreement for $\omega(k)$ at T_c independent of whether we approach T_c from above or below. In the "hydrodynamical" limit M fixed, $k\to 0$ we have for large α,

$$\Delta_+(\alpha)=O[(1/\alpha)\ln\alpha]\,, \tag{6.67}$$

so

$$\omega_+(k)=\lambda Mk^2-i\Gamma'k^4(\tilde{\lambda}M)^{-\epsilon/4}$$
$$\times(1+\epsilon\Delta_\infty)^{-1}\{1+O[(1/\alpha)\ln\alpha]\}\,. \tag{6.68}$$

The spin-wave damping coefficient is given in this case by

$$\tilde{\Gamma}_+ = \frac{\Gamma'}{1+\epsilon\Delta_\infty}(\tilde{\lambda}M)^{-\epsilon/4} = \frac{\Gamma'}{1+\epsilon\Delta_\infty}\left(\frac{\tilde{\lambda}}{u^{1/2}}\right)^{-\epsilon/4}|r_0|^{-\epsilon/8} \quad (6.69)$$

and diverges as $|r_0| \propto |T-T_c| \to 0$.

F. Scaling behavior

We note here that these characteristic frequencies scale in agreement with the predictions of the renormalization group. If we note, under the RNG,

$$M \to M' = b^{2-\epsilon/2}M, \quad (6.70)$$

then

$$\alpha = M\tilde{\lambda}^*/k^2 \to \alpha' = b^{-\epsilon/2}\alpha, \quad (6.71)$$

so $\alpha' = \alpha$ to zeroth order in ϵ. It is then easy to see that the characteristic frequencies $\omega_+(k)$ and $\omega_D(k)$ both transform as

$$\omega' = b^{4-\epsilon/2}\omega$$

to first order in ϵ.

VII. DISCUSSION

To sum up, we have studied the transformation properties of the equation of motion (1.5) under the RNG and then we have solved the equation for a special value of the coupling constant λ. Since the statics are trivial for $d>4$, all we have illustrated is the role of the $\lambda\vec{S}\times\vec{H}$ term, which describes the precession of spins. The conclusion is that this term plays a decisive role. Qualitatively, its effect is very strong. As d increases its effect diminishes. For d approaching 6 it becomes sufficiently weak to be treated by perturbation theory.

Although the calculation we went through is very complicated, the main qualitative results are simple. We have shown that the characteristic frequencies of long-wavelength modes as $|T-T_c| \to 0$ have form

$$\omega(k) = \xi^{-z}\Omega(k\xi, u\xi^{-2+\epsilon}), \quad z = 4 - \tfrac{1}{2}\epsilon = 1 + \tfrac{1}{2}d, \quad (7.1)$$

where $\xi \propto |T-T_c|^{-\nu}$, $\nu = \frac{1}{2}$. The details of the functions Ω differ for the different types of modes. For example, for $T>T_c$, Ω has been computed in Sec. V [see (5.15) and Fig. 13]. In that case we could safely set the second argument $u\xi^{-2+\epsilon}$ equal to zero. Below T_c we have for the longitudinal mode $O(\epsilon)$ comparing (6.42) and (7.1) for small k and to $O(\epsilon)$,

$$\Omega_0 = -i\Gamma'(k\xi)^{4-\epsilon/6}(u\xi^{-2+\epsilon})^{\epsilon/12}(\tilde{\lambda}^*)^{-\epsilon/6} \times(1+A\tilde{\lambda}^{*2})[1+2(k\xi)^{-2}], \quad (7.2)$$

while from (6.68) we find, for the transverse mode,

$$\Omega_+ = \tilde{\lambda}^*(k\xi)^2(u\xi^{-2+\epsilon})^{-1/2} - i\Gamma'(k\xi)^4\tilde{\lambda}^{*2}(u\xi^{-2+\epsilon})^{\epsilon/8}. \quad (7.3)$$

We see that the second argument $u\xi^{-2+\epsilon}$ of Ω cannot be set to zero in these cases.

The result $z = 1 + \frac{1}{2}d$ has been obtained before by many authors via different arguments. The simplest argument seems to be the following. From hydrodynamics we know that the long-wavelength spin waves must have a frequency proportional to Mk^2. But statics tell us that $M \propto \xi^{-\beta/\nu}$ and $\beta/\nu = \frac{1}{2}(d-2+\eta)$. Thus, we have

$$\omega(k) \propto \xi^{-(d-2+\eta)/2}k^2 \propto \xi^{-(1+d/2+\eta/2)}(\xi k)^2, \quad (7.4)$$

i.e., $z = 1 + \frac{1}{2}d + \frac{1}{2}\eta$. Whenever $\eta = 0$ or is sufficiently small, we get $z = 1 + \frac{1}{2}d$. This derivation actually applies only to $d<4$ because, for $d>4$, $\beta/\nu = 1$, leading to $z = 3$ instead. We see that the powers of $u\xi^{-2+\epsilon}$ in (7.2) are just what is needed to keep the result $z = 1 + \frac{1}{2}d$ intact, and at the same time maintaining $\beta/\nu = 1$. Recall that our derivation of z is very different. The value $1 + \frac{1}{2}d$ followed from the fact that step (a) of the RNG (i.e., the elimination of S_q, $\Lambda/b<q<\Lambda$) does not affect λ.

Note that the statement of the dynamic scaling hypothesis is that characteristic frequencies have the form $\omega(k) = \xi^{-z}\times$(function of $k\xi$ alone). The appearance of $u\xi^{-2+\epsilon}$ for $T<T_c$ then violates this hypothesis. Such a violation is entirely within the framework of the RNG approach and is not difficult to understand.

We have also obtained several interesting results for the damping coefficients for the various modes. In our zeroth-order models $\tilde{\Gamma}$, $\tilde{\Gamma}_0$, $\tilde{\Gamma}_+$, the damping for $T>T_c$, and of the longitudinal and transverse modes, respectively, were constants as $T\to T_c$, $k\to 0$. They were all equal to Γ. The effect of the mode coupling on these terms is to give,

$$\tilde{\Gamma} = \Gamma'\xi^{\epsilon/2}/(1+\epsilon\Delta_\infty),$$

$$\tilde{\Gamma}_0(k) = \Gamma'(1+A\tilde{\lambda}^{*2})(M\tilde{\lambda}^*k)^{-\epsilon/6},$$

$$\tilde{\Gamma}_+(k) = [\Gamma'/(1+\epsilon\Delta_\infty)](\tilde{\lambda}^*M)^{-\epsilon/4},$$

where

$$\Gamma' = (1+\epsilon\Delta_\infty)\Gamma(1+\tfrac{1}{2}\epsilon\ln\Lambda).$$

There are, therefore, substantial qualitative changes in the damping coefficients due to mode-coupling effects. Of some interest is that $\tilde{\Gamma}_0$ develops an intrinsic wave-number dependence for small wave numbers. Thus the longitudinal damping will have a power dependence less than the hydrodynamical prediction of k^2. In the case of the spin-wave damping we find that the hydrodynamical form $\sim k^4$ persists in the presence of λ, but the coefficient develops a strong temperature dependence $\sim (T-T_c)^{-\epsilon/8}$ as $T \to T_c$.

Extrapolation of our results here to $d=3$ by setting $\epsilon=3$ is not warranted. This is not only because $O(\epsilon)$ results are unlikely to make sense for $\epsilon=3$, but also because the role of the uS^4 term in F for $d<4$ is not well understood in this model. However, we can make some reasonable guesses for ϵ not small.

(i) We expect that the stable fixed point found above will remain when ϵ is not small and z will be given by $1+\frac{1}{2}d+\frac{1}{2}\eta$. (ii) Equation (7.1) remains for $\epsilon<2$. The detailed form of the function Ω may vary as ϵ increases. (iii) When ϵ reaches 2 (i.e., d reaches 4), $u\xi^{-2+\epsilon}$, the second argument of Ω, becomes a constant. As ϵ increases further the fixed point value for u will no longer be zero. We therefore expect that the second argument of Ω will remain a constant independent of ξ for $\epsilon>2$, i.e., for $d<4$. There will then be no violation of dynamic scaling.

A basic question is to what extent critical behaviors are universal. In the language of the RNG, critical behaviors are given by properties of fixed points. If two different systems are at their critical points and they are, respectively, described by the two sets of parameters μ_1 and μ_2, and if R_b drives both μ_1 and μ_2 to the same fixed point for $b\to\infty$, then the two systems share the same critical behaviors. Therefore fixed points must be classified. For the static RNG fixed points are classified by symmetries, number of components of the order parameters and spatial dimensionality. (We will not worry about long-range forces here.) These are not enough to classify fixed points for the dynamic RNG. The analysis in Ref. 5 showed that under the same static classification, different conservation laws can lead to different fixed points. The results of Ref. 6 and the present work show that different forms of the equations of motion can lead to different fixed points under the same conservation law. In view of the discussion in Sec. II, we are tempted to say that the only relevant feature, besides conservation laws and static properties, of the equations of motion is the form of the "streaming velocity" V_i [see (2.12) and (2.14)], which is in turn specified by Q_{ij}, obtained via the Poisson brackets of slow variables. That is, systems with the same kind of statics, conservation laws and Q_{ij} will share the same dynamic critical behavior.[33]

The equations of motion in Sec. II are not derived rigorously from microscopic theories, but are semiheuristic. The ideas used are quite general since they rely on our intuitive notions about symmetry, conservation laws and irreversibility. Thus while the above statement on the classification of dynamic critical behavior is not on a rigorous foundation, it does seem to be strongly supported by our understanding of generalized hydrodynamics and mode-coupling theory. The alternative to these model calculations is an analysis starting from the microscopic equations of motion. There have been a number of such calculations recently. These calculations are usually carried out for the case of helium where the generalization of the static Landau-Ginzburg theory to dynamics is straightforward. There have been a number of direct calculations in this model where the four-point interaction u has been assumed to be of order $4-d$, $d\approx 4$,[34] or $1/N$,[35] where N is the number of components of the order parameter. These calculations give explicit expressions for the index z. However, as Halperin[36] has pointed out, the difficulty with these microscopic calculations is that they do not treat the conservation laws properly. In order to include the transport processes that dominate at long wavelengths, one must eventually discuss a transport equation. Diagrammatically, this usually involves a discussion of the poles of the Bethe-Salpeter equation.[37] In the case of the $1/N$ expansions one is led, to order zero in $1/N$, to such a transport equation.[38] Thus, in the microscopic theory one has to carry out a very nontrivial analysis to make contact with the relevant physics.

A second difficulty with the microscopic approach is the lack of RNG methods available to investigate the existence of new fixed points or the existence of slow transients which contaminate the perturbation theory expansion. As we have seen here and in Refs. 5 and 6, dynamics introduce fixed points other than the static fixed point near $d=4$. We have also seen that for the isotropic ferromagnet there is a fixed point form of our equation of motion correct to order ϵ. It is the existence of this fixed point form for the equation of motion that gives the most support to the validity of our model.

In conclusion, we remark that there are still model equations which can be studied by a perturbation expansion like the work here and in Refs. 5 and 6. We could also calculate to higher orders in ϵ or $1/N$. While this new work may be interesting in certain cases, it seems more urgent now to develop nonperturbative methods to study the dynamic RNG. It is hoped this will lead to a better understanding of the dynamic RNG and the associated fixed point equations of motion.[39]

ACKNOWLEDGMENT

We are grateful to D. Lublin for his help in numerical calculations.

APPENDIX A

Starting from the equation of motion for $\psi_i(t)$, we can perform a partial integration to obtain

$$\psi_i(t)=\sum_j \int_{-\infty}^{+\infty} dt'\, G^0_{ij}(t-t')L_j^{-1}\eta_j(t') +\sum_j \int_{-\infty}^{+\infty} dt'\, G^0_{ij}(t-t')h_j(t') +\sum_j \int_{-\infty}^{+\infty} dt'\, G^0_{ij}(t-t')N_j(t';\psi)\,, \qquad \text{(A1)}$$

where N_j contains all of the nonlinear terms in the equation of motion, and

$$G^0_{ij}(t-t')=\Theta(t-t')e^{-L_i\chi_i^{-1}(t-t')}\delta_{ij}L_i \qquad \text{(A2)}$$

is the zeroth-order response function. We can compute directly, using the chain rule for differentiation

$$\frac{\delta\psi_i(t)}{\delta\zeta_j(t')}=L^{-1}G^0_{ij}(t-t') +\sum_{k,l}\int_{-\infty}^{+\infty} d\bar{t}\, G_{il}(t-\bar{t})\,\frac{\delta N_l(\bar{t})\delta\psi_k(\bar{t})}{\delta\psi_k(\bar{t})\delta\zeta_j(t')}\,. \qquad \text{(A3)}$$

All ζ's that appear in $\psi(\bar{t})$ will occur at a time less than or equal to $\bar{t}$, so we always end up with a factor $\Theta(t-\bar{t})\Theta(\bar{t}-t')$ in the second term, and it vanishes for $t\geq t'$. Equation (A3) then reduces to (2.23) is we note that $G^0_{ij}(t=t')=\frac{1}{2}L_i\delta_{ij}$.

Next we observe, since the noise is Gaussianly distributed, that

$$\langle A\rangle=\int d\eta \exp\left(-\int_{-\infty}^{+\infty}dt\sum_i \frac{\zeta_i^2(t)}{4L_i}\right)A \times\left[\int d\eta\exp\left(-\int_{-\infty}^{+\infty}dt\sum_j\frac{\zeta_j^2(t)}{4L_j}\right)\right]^{-1}. \qquad \text{(A4)}$$

It then follows that

$$\langle\zeta_i A\rangle=2L_i\langle(\delta/\delta\zeta_i)A\rangle\,. \qquad \text{(A5)}$$

---

*Alfred P. Sloan Foundation Fellow. Research supported in part by the National Science Foundation under Grant No. GP38627X.

†Research supported by the Stanford Synchrotron Radiation Project, the National Science Foundation Grant No. GH39525, and the U. S. Army Research Office, Durham.

‡Present address: The James Franck Institute and Dept. of Physics, The University of Chicago, Chicago, Ill. 60637.

[1]K. Kawasaki, Ann. Phys. (N.Y.) 61, 1 (1970).

[2]B. I. Halperin and P. C. Hohenberg, Phys. Rev. 177, 952 (1969).

[3]P. C. Hohenberg, M. DeLeener, and P. Resibois, Physica (Utr.) 65, 505 (1973).

[4]S. Ma and G. F. Mazenko, Phys. Rev. Lett. 33, 1384 (1974).

[5]B. I. Halperin, P. C. Hohenberg, and S. Ma, Phys. Rev. Lett. 29, 1548 (1972); Phys. Rev. B 10, 139 (1974); and unpublished.

[6]B. I. Halperin, P. C. Hohenberg, and E. Siggia, Phys. Rev. Lett. 32, 1289 (1974).

[7]K. G. Wilson, Phys. Rev. Lett. 28, 548 (1972).

[8]E. Brezin, D. J. Wallace, and K. G. Wilson, Phys. Rev. B 7, 232 (1973).

[9]Y. Kuramoto, Prog. Theor. Phys. 51, 1712 (1974); M. Suzuki and G. Igarashi, *ibid.* 49, 1070 (1973); M. Suzuki and F. Tanaka, *ibid.* 52, 722 (1974); H. Yahata, *ibid.* 52, 871 (1974).

[10]K. G. Wilson, Phys. Rev. Lett. 28, 548 (1972).

[11]K. G. Wilson and J. Kogut, Phys. Rev. 12, 75 (1974).

[12]S. Ma, Rev. Mod. Phys. 45, 589 (1973).

[13]L. van Hove, Phys. Rev. 95, 1374 (1954).

[14]See, for an elementary introduction, L. P. Kadanoff *et al.*, Rev. Mod. Phys. 39, 615 (1967).

[15]This equation reduces to that of linearized hydrodynamics if we replace the inverse susceptibility with small-wave-number limit. See, for example, H. Mori, Prog. Theor. Phys. 33, 423 (1965).

[16]J. Als-Nielsen, Phys. Rev. Lett. 25, 730 (1970).

[17]This formula comes from an analysis of the poles of the response function. See, for example, L. P. Kadanoff and P. C. Martin, Ann. Phys. (N.Y.) 24, 419 (1963).

[18]J. V. Sengers, in *Critical Phenomena*, edited by M. S. Green (Academic, New York, 1971).

[19]K. Kawasaki, in Ref. 18.

[20]L. P. Kadanoff and J. Swift, Phys. Rev. 166, 89 (1968); R. Zwanzig, University of Maryland Institute for Fluid Dynamics and Applied Mathematics technical report, 1971 (unpublished); R. A. Ferrell, Phys. Rev. Lett. 24, 1169 (1970).

[21]H. Mori, H. Fujisaka, and H. Shigematsu, Prog. Theor. Phys. 51, 109 (1974).

[22]R. W. Zwanzig, Phys. Rev. 124, 983 (1961). Zwanzig's work was built upon the pioneering, but more intuitive work of M. S. Green, J. Chem. Phys. 20, 1281 (1952); 22, 398 (1954).

[23]H. Mori and H. Fujisaka, Prog. Theor. Phys. 49, 764 (1973).

[24]Y. Kuramoto, see Ref. 9 has discussed how some of these new terms enter in a discussion of the TDGL theory.

[25]See p. 13 of Ref. 1.

[26]This operator has been discussed by many people. See, for example, Ref. 19.

[27]We include in F a constant term which normalizes the integral over W_φ to 1.

[28]This equation has been derived from the microscopic equations of motion by Mori and Fujisaka, Ref. 23. Their "coarse grained" expression for R_φ, see their Eq. (A17), should be replaced by our expression (2.38) if one is to avoid certain inconsistencies.

[29]There has been some debate in the literature about these classical fluctuation-dissipation theorems. See, for example, Ref. 43 in P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A 8, 423 (1973). The proof of the fluctuation-dissipation theorem for the linear Langevin equation (2.1) is well known. However we are not aware of any proof of this theorem for these nonlinear Langevin equations.

[30]We require all internal wave-number integrations in a graph to be restricted to the shell $\Lambda/b < q < \Lambda$.

[31]P. Résibois and C. Piette, Phys. Rev. Lett. 24, 514 (1970).

[32]Equation (12.c) in Ref. 4 is incorrect. It should be replaced by (6.40) and (6.42) here.

[33]In Ref. 6 the authors consider a special case of the general model, (2.14), discussed in Sec. II. Their choice of the streaming velocity V_i for the binary mixture follows from earlier work by Kawasaki (Ref. 1). This V_i is a simplified form of what would follow from the Poisson bracket expression. The important point is that their V_i satisfies the divergence condition (2.131).

[34]M. Suzuki and G. Igarashi, Phys. Lett. A 47, 361 (1974); I. Kondor and P. Szepfulusy, Institute for Theoretical Physics Eötvos University, Report, 1974 (unpublished); Y. Yamashita and T. Tsuneto, Prog. Theor. Phys. 51, 949 (1974); T. Tsuneto and E. Abrahams, report (unpublished); G. Igarashi, report (unpublished).

[35]I. Kondor and P. Szepfalusy, Phys. Lett. A 47, 393 (1974); and report (unpublished); R. Abe and S. Hikami, Phys. Lett. A 47, 341 (1974).

[36]B. I. Halperin, report (unpublished).

[37]A discussion of the relationship of hydrodynamics and the Bethe-Salpeter equation in a nontrivial approximation is given in G. F. Mazenko, Phys. Rev. A 3, 212 (1971).

[38]S. Ma and L. Senebetu, report (unpublished).

[39]One of us has discussed the inclusion of mode-coupling terms from a microscopic point of view. See G. F. Mazenko, Phys. Rev. A 9, 360 (1974). One can see that a direct perturbation expansion in the bare interaction u would be extremely difficult.