Derivation of a low-temperature expansion for the general-spin Ising model*

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The low-temperature finite-field free-energy expansion for the general-spin nearest-neighbor Ising model is derived. A method is presented which calculates the arbitrary spin polynomials directly from the spin-1/2 embeddings. These polynomials are tabulated for all of the common two- and three-dimensional lattices.

I. INTRODUCTION

The Hamiltonian for the spin-s nearest-neighbor Ising model may be written

$$\mathcal{H} = -J\sum_{\langle 12\rangle} S(1)S(2) - h\sum_{1} S(1)\ , \qquad (1.1)$$

where the spin at each site is normalized to unity, $S(1) = -1, -1+1/s, -1+2/s, \ldots, 1-1/s, 1$. Numerical arguments label the sites of a regular lattice. The sum $\langle 12\rangle$ is over nearest-neighbor pairs. The usual low-temperature high-field expansion for the free energy is

$$-\beta f = \frac{1}{N}\ln \mathrm{Tr}\, e^{-\beta\mathcal{H}} = \beta(h + \tfrac{1}{2}zJ) + \sum_{n=1}^{\infty} L_n^{(s)}(u)\mu^n\ , \qquad (1.2)$$

where $u \equiv e^{-\beta J/s^2}$ and $\mu \equiv e^{-\beta h/s}$. z is the coordination number of the lattice and N is the total number of lattice sites. The polynomials $L_n^{(s)}$ are explicitly dependent on both s and the lattice type. They contain integer powers of u, except when n, $2s$, and z are all odd (in which case the powers of u that appear are half odd integer).

The techniques for calculating the low-temperature polynomials $L_n^{(1/2)}$ for the $s=\frac{1}{2}$ Ising model were pioneered by Domb[1] and Sykes[2] and have by now been refined to a high degree.[3] Direct generalization of these techniques to spins $s > \frac{1}{2}$ involves graphs with multiple bonds and multiple occupations which become increasingly complicated as s grows. However, a number of polynomials $L_n^{(s)}$ have recently been calculated by these methods for $s=1$ and $\frac{3}{2}$.[4] A recent derivation[5] of the expansion for $s=\frac{3}{2}$ outlines the general recipe for calculating the higher-spin polynomials, but does not make clear the simple structure of the expansion for general spin.

Our derivation shows that the high-field polynomials may be written,

$$(1/u)^{szn}L_n^{(s)}(u) = D_n^{(s)}(1/u) = \sum_{t=1}^{\min(n,2s)} D_{nt}(1/u)\ , \qquad (1.3)$$

where the polynomials $D_{nt}(1/u)$ are *independent* of s and contain integer powers of $1/u$. Only the polynomials D_{n1} are necessary for $s=\frac{1}{2}$. As s increases in the range $\frac{1}{2} < s < \frac{1}{2}n$, successively more polynomials enter. For $s > \frac{1}{2}n$, $D_n^{(s)}$ becomes independent of s, as shown in Table I. A variety of polynomials $D_{nt}(1/u)$ are presented in Appendix A for the more common lattices. Our method of calculation eliminates all multiple bonds and occupations. The topological information required is the ordinary ($s=\frac{1}{2}$) strong-embedding lattice constants.[1] Because of the structure of the expansion (1.3), the D_{n1} are already available from data for the $s=\frac{1}{2}$ Ising model.[1-3] Furthermore, the actual computation of the polynomials D_{nt} (n fixed, $t>1$) becomes dramatically easier as t increases, and it is, in fact, quite easy to write down closed expressions for $D_{n,n}$, $D_{n,n-1}$ ($n>2$), $D_{n,n-2}$ ($n>4$), etc. (Appendix A). Rearrangement of the μ-grouped expansion (1.2) to give the u-grouped expansion in powers of u follows now-standard lines.[2]

II. DERIVATION

Consider a decomposition of the spin variable into $2s$ distinguishable particle occupation numbers,

$$S(1) = 1 - \frac{1}{s}\sum_{k=1}^{2s} kn_k(1)\ , \qquad (2.1)$$

with $\sum_{k=1}^{2s} n_k(1) = 0,1$, i.e., at most one particle per site. Substitution of (2.1) into (1.1) gives,

$$\mathcal{H} = -N(h + \tfrac{1}{2}zJ) + \frac{1}{s}(h + zJ)\sum_{1,k} kn_k(1) - \frac{J}{s^2}\sum_{\substack{\langle 12\rangle \\ j,k}} jkn_j(1)n_k(2)\ , \qquad (2.2)$$

$$Z = \mathrm{Tr}\, e^{-\beta\mathcal{H}} = Z_0 \sum_{\{n_k(1)\}} (u^{sz}\mu)^{\Sigma_{1,k} kn_k(1)} (1/u)^{\Sigma_{\langle 12\rangle}\Sigma'_{j,k} jkn_k(1)n_k(2)}\ , \qquad (2.3)$$

where $Z_0 = e^{\beta N(h+zJ/2)}$. The sum $\sum_{\{n_k(1)\}}$ runs over

TABLE I. Contributing spin-independent free-energy polynomials for spin s to order μ^n: To calculate the low-temperature free energy for spin s to order μ^n requires all $D_{n't}$ such that $n' \le n$ and $t \le 2s$. The D_{n1} polynomials are simply calculated from the spin-$\frac{1}{2}$ free energy (column to the left of the heavy vertical line). The The $D_{n,n-l}$ $(l < \frac{1}{2}n)$ polynomials are easily calculated for general n (polynomials to the right of the heavy steplike line). Only the shaded D_{nt} must be calculated in detail.

SPIN	1/2	1	3/2	2	5/2	3	7/2	...
n \ t	1	2	3	4	5	6	7	...
1	D_{11}							
2	D_{21}	D_{22}						
3	D_{31}	D_{32}	D_{33}					
4	D_{41}	D_{42}	D_{43}	D_{44}				
5	D_{51}	D_{52}	D_{53}	D_{54}	D_{55}			
6	D_{61}	D_{62}	D_{63}	D_{64}	D_{65}	D_{66}		
7	D_{71}	D_{72}	D_{73}	D_{74}	D_{75}	D_{76}	D_{77}	
⋮					$D_{n,n-3}$	$D_{n,n-2}$	$D_{n,n-1}$	D_{nn}

all states of the system. For small T and large h (2.3) generates an expansion in powers of μ and $u^{1/2}$ about the perfectly ordered ground state [$S(1) = +1$ for all sites 1]. Each state $\{n_k(1)\}$ can be interpreted graphically: A type-k point denotes a site occupied by a type-k particle [$n_k(1) = 1$] and contributes a factor $(u^{sz}\mu)^k$ [Fig. 1(a)]. Two occupied nearest-neighbor sites $n_j(1) = n_k(2) = 1$ are joined by a bond and contribute a factor $(1/u)^{jk}$ [Fig. 1(b)]. We denote by C the linear graph (with labeled vertices) or "point cluster" thus formed. The factors which each portion of the cluster contributes make up the weight of each state $\{n_k(1)\}$ in (2.3).

$$W[C] = (u^{sz}\mu)^{n[C]}(1/u)^{r[C]}, \tag{2.4}$$

where

$$n[C] = \sum_{\text{vertices}} k \quad \text{and} \quad r[C] = \sum_{\text{bonds}} jk, \tag{2.5}$$

i.e., $n[C]$ is just the sum over C of the particle-type number at each vertex and $r[C]$ is the sum over all bonds of the product of the particle-type numbers at the bond endpoints. It is useful to index each graph $C^n_{pml\alpha}$, where n is given in (2.5). p and m are, respectively, the number of vertices and bonds in the graph C:

$$p = \sum_1 \sum_{k=1}^{2s} n_k(1),$$
$$m = \sum_{\langle 12\rangle} \sum_{j=1}^{2s} \sum_{k=1}^{2s} n_j(1) n_k(2). \tag{2.6}$$

l indexes the topologically distinct graphs (unlabeled vertices) corresponding to given p and m. Thus, p, m, l are a complete indexing of topologically distinct graphs with unlabeled vertices. The index α distinguishes between different particle-type designations at the vertices. Figure 2 gives some examples.

Many states $\{n_k(1)\}$ correspond to the same graph C, so each C appears in (2.3) with a multiplicity $\mathfrak{M}[C]$ and

$$Z = Z_0 \sum_C^{(s)} \mathfrak{M}[C] W[C], \tag{2.7}$$

where the sum ranges over all topologically distinct vertex-labeled graphs C (connected *and* disconnected) with particle types up to and including $2s$. The multiplicity $\mathfrak{M}[C]$ depends on lattice type and is an ith-order polynomial in N, where i is the number of disconnected parts of the cluster.[6] $\mathfrak{M}[C]$ may be written

$$\mathfrak{M}[C] = E[C]/g[C]$$

or (2.8)

$$\mathfrak{M}_{pml\alpha} = E_{pml}/g_{pml\alpha},$$

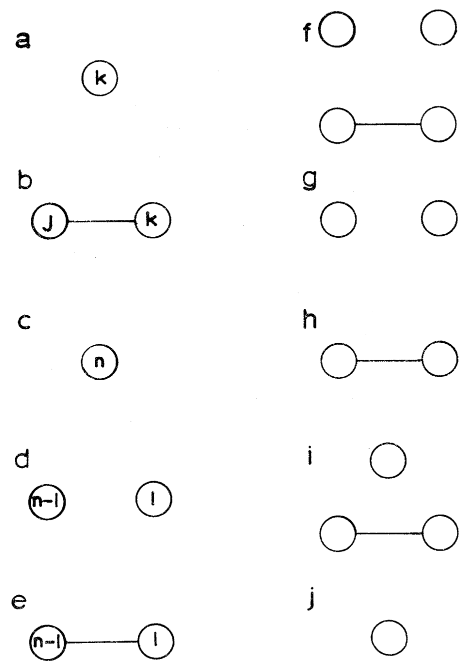

FIG. 1. Various point clusters.

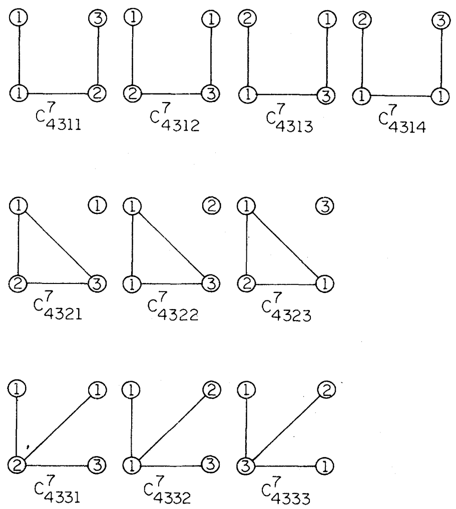

FIG. 2. Decomposition of various four-point topologically distinct clusters. The set of decorations shown contribute to D_{73}.

where $E[C]$ is the number of strong embeddings of C in the lattice—i.e., the number of ways the C can be embedded on a given lattice such that no two points are nearest neighbors unless connected by a bond. $E[C]$ is, therefore, lattice dependent but independent of the α index labeling the vertex decoration. $g[C]$ is the symmetry factor of C—i.e., the number of distinct mappings of C onto itself. $g[C]$ is lattice independent but decoration dependent.

The free energy is extensive, so the process of forming $\ln Z$ picks out the term in $\overline{\mathfrak{M}}[C]$ which is linear in N.[1] Thus

$$-\beta f=\frac{1}{N}\ln Z=\beta(h+\tfrac{1}{2}zJ)+\sum_{C}^{(s)}\mathfrak{M}[C]W[C]+O(1/N)\,, \tag{2.9}$$

where[7]

$$\mathfrak{M}[C]=\frac{d}{dN}\overline{\mathfrak{M}}[C]\bigg|_{N=0}\,. \tag{2.10}$$

In parallel with (2.8), we may also define

$$\mathfrak{M}[C]\equiv\mathcal{E}[C]/g[C]$$

and

$$\mathfrak{M}_{pml\alpha}=\mathcal{E}_{pml}/g_{pml\alpha}\,. \tag{2.11}$$

As with $E[C]$, $\mathcal{E}[C]$ is independent of the vertex decoration and therefore the spin. *It is therefore a point of some significance that the spin-$\frac{1}{2}$ embeddings can be used directly for the general-spin problem with no modifications.*[8]

It is now an easy matter to derive Eqs. (1.2) and (1.3). Classification of the graphs C in (2.9) by n leads to

$$\mu^n L_n^{(s)}(u)=\sum_{C_\alpha^n(\text{fixed } n)}^{(s)}\mathfrak{M}[C^n]W[C^n]\,. \tag{2.12}$$

The spin dependence of (2.12) lies entirely in the restriction of the maximum particle-type number in C to $2s$. Since every graph of type C^n carries a factor $(\mu u^{sz})^n$ from (2.5), the spin-independent polynomials can be readily identified:

$$(\mu u^{sz})^n D_{nt}(1/u)=\sum_{C^n,t}\mathfrak{M}[C^n]W[C^n]\,, \tag{2.13}$$

where the sum now runs over all graphs C^n which have at least one type-t vertex but none of type number higher than t. Thus, the calculation of D_{nt} requires embeddings for graphs with up to $(n-t+1)$. vertices only. For example, the only graph to contribute to D_{nn} is the single n-point [Fig. 1(c)], so $D_{nn}(1/u)=1$ for all lattices. Similarly, the only two graphs contributing to $D_{n,n-1}$ have $\mathcal{E}[C\{\text{Fig. 1(d)}\}]=-z-1$ and $\mathcal{E}[C\{\text{Fig. 1(e)}\}]=z$ so

$$D_{n,n-1}(1/u)=-(z+1)+z(1/u)^{n-1}\,,\quad n>2\,.$$

The calculation of $D_{n,n-2}$ and $D_{n,n-3}$ is given in Appendix A. In fact, as long as $q<\frac{1}{2}n$, the coefficients in $D_{n,n-q}$ (but *not* the powers of $1/u$ which they multiply) are *independent* of n. This is so, because, provided $q<\frac{1}{2}n$, the graphs contributing to $D_{n,n-q}$ have exactly one type-$(n-q)$ vertex. As n increases (at fixed q), the type number of this vertex increases, but, otherwise, all graphs, embeddings, and symmetry factors are unchanged.

For purposes of calculation $D_{nt}(1/u)$ can be expressed in terms of the embeddings

$$D_{nt}(1/u)=\sum_{C^n,t}\frac{\mathcal{E}_{pml}}{g_{pml\alpha}}(1/u)^{r_{pml\alpha}}\,, \tag{2.14}$$

where

$$\sum_1 n_t(1)\geq 1$$

and

$$\sum_1 n_{t'>t}(1)=0\,.$$

The 1–5 point $C^n_{pml\alpha}$, $\mathcal{E}_{pml}$, $g_{pml\alpha}$, and $r_{pml\alpha}$ are presented in Appendix B for all of the common two- and three-dimensional lattices. Through order μ^6, the six-point clusters enter only in the $\alpha=1$ state (all particles equivalent). Therefore all that is required for the spin-s free energy through order μ^6 are the five-point clusters and the $L_6^{(1/2)}$ polynomial. Since only the five-point clusters are available, the calculation of the spin-s free energy through order μ^7 requires the $L_7^{(1)}$ polynomial in addition to $L_7^{(1/2)}$.

III. CONSISTENCY

The high-temperature finite-field series provides certain constraints[9,10] on the coefficients of the low-temperature polynomials. For spin $\frac{1}{2}$, these constraints were used to calculate the coefficients. For the general-spin case, they may act as a check. In practice these constraints reduce to evaluating both the high- and low-temperature finite-field free energies in the limit of $h\to\infty$. In this limit (where both series are valid), we can compare the coefficients of $(J/T)\mu$.[11] Given our method of derivation, the coefficients of the low-temperature polynomials are completely uncorrelated, therefore agreement with the high-temperature series through order $(J/T)^3\mu^7$ is a strong check.

The low-temperature free-energy polynomials for $S=1$, $\frac{3}{2}$ are also in agreement with Fox and Gaunt.[4] Our series could be easily extended via extant,[3] but unpublished, higher-order spin-$\frac{1}{2}$ lattice constants and clusters.

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APPENDIX A

Spin-independent polynomials $D_{nt}(1/u)$ required for the spin-s low-temperature free energy on all common two- and three-dimensional lattices through μ^7.

(a) Honeycomb:

$D_{nn}(1/u)=1$,

$D_{n,n-1}(1/u)=-4+3/u^{n-1},\quad n>2$

$D_{n,n-2}(1/u)=15-9/u-18/u^{n-2}+6/u^{n-1}+6/u^{2n-4},\quad n>4$

$D_{n,n-3}(1/u)=-64+102/u-42/u^2+84/u^{n-3}-87/u^{n-2}+21/u^{n-1}-42/u^{2(n-3)}+12/u^{2n-5}+6/u^{2(n-2)}+10/u^{3(n-3)},\quad n>6$

$D_{21}(1/u)=-2+1\frac{1}{2}/u$,

$D_{31}(1/u)=6\frac{1}{3}-9/u+3/u^2$,

$D_{41}(1/u)=-24\frac{1}{2}+51/u-33\frac{3}{4}/u^2+7/u^3$,

$D_{51}(1/u)=106\frac{1}{5}-291/u+288/u^2-121/u^3+18/u^4$

$D_{61}(1/u)=-495\frac{1}{6}+1681\frac{1}{2}/u-2212\frac{1}{2}/u^2+1400\frac{1}{2}/u^3-421\frac{1}{2}/u^4+46\frac{1}{2}/u^5+\frac{1}{2}/u^6$,

$D_{71}(1/u)=2428\frac{1}{7}-9831/u+16128/u^2-13647/u^3+6225/u^4-1422/u^5+116/u^6+3/u^7$,

$D_{42}(1/u)=17-9/u-18/u^2+6/u^3+4\frac{1}{2}/u^4$

$D_{52}(1/u)=-79+102/u+60/u^2-87/u^3-15/u^4+12/u^5+7/u^6$,

$D_{62}(1/u)=390\frac{1}{3}-822/u+69/u^2+716/u^3-240/u^4-139\frac{1}{2}/u^5-4/u^6+18/u^7+12/u^8$,

$D_{63}(1/u)=-62+102/u-42/u^2+84/u^3-87/u^4+21/u^5-42/u^6+12/u^7+6/u^8+8\frac{1}{2}/u^9$,

$D_{72}(1/u)=-2007+5853/u-3546/u^2-3874/u^3+4245/u^4+114/u^5-695/u^6-123/u^7-24/u^8+36/u^9+21/u^{10}$,

$D_{73}(1/u)=275-771/u+651/u^2-514/u^3+753/u^4-480/u^5+282/u^6-198/u^7+6/u^8-55/u^9+27/u^{10}+12/u^{11}+12/u^{12}$.

(b) Simple quadratic:

$D_{nn}(1/u)=1$,

$D_{n,n-1}(1/u)=-5+4/u^{n-1},\quad n>2$

$D_{n,n-2}(1/u)=26-16/u-32/u^{n-2}+12/u^{n-1}+10/u^{2n-4},\ n>4$

$D_{n,n-3}(1/u)=-152+236/u-94/u^2+204/u^{n-3}-216/u^{n-2}+52/u^{n-1}-94/u^{2(n-3)}+28/u^{2n-5}+16/u^{2(n-2)}+20/u^{3(n-3)},\quad n>6$

$D_{21}(1/u)=-2\frac{1}{2}+2/u$,

$D_{31}(1/u)=10\frac{1}{3}-16/u+6/u^2$,

$D_{41}(1/u)=-52\frac{1}{4}+118/u-85/u^2+18/u^3+1/u^4$,

$D_{51}(1/u)=295\frac{1}{5}-872/u+926/u^2-400/u^3+43/u^4+8/u^5$,

$D_{61}(1/u)=-1789\frac{5}{6}+6520/u-9144/u^2+5992\frac{2}{3}/u^3-1651/u^4+30/u^5+40/u^6+2/u^7$,

$D_{71}(1/u)=11397\frac{1}{7}-49328/u+85954/u^2-75640/u^3+33609/u^4-5664/u^5-486/u^6+136/u^7+22/u^8$,

$D_{42}(1/u)=28\frac{1}{2}-16/u-32/u^2+12/u^3+8/u^4$,

$D_{52}(1/u)=-178+236/u+142/u^2-216/u^3-32/u^4+28/u^5+20/u^6$,

$D_{62}(1/u)=1172\frac{5}{6}-2498/u+178/u^2+2304/u^3-794/u^4-420/u^5-62/u^6+72/u^7+43/u^8+4/u^9$,

$D_{63}(1/u)=-149\frac{1}{2}+236/u-94/u^2+204/u^3-216/u^4+52/u^5-94/u^6+28/u^7+16/u^8+18/u^9$,

$D_{72}(1/u)=-7996+23464/u-13928/u^2-16708/u^3+18067/u^4+168/u^5-2342/u^6-852/u^7-133/u^8+136/u^9+104/u^{10}+16/u^{11}+4/u^{12}$,

$$D_{73}(1/u) = 911 - 2380/u + 1922/u^2 - 1656/u^3 + 2442/u^4 - 1500/u^5 + 856/u^6 - 588/u^7 - 22/u^8 - 136/u^9 + 76/u^{10} + 36/u^{11} + 39/u^{12} .$$

(c) Triangular:

$$D_{nn}(1/u) = 1 ,$$

$$D_{n,n-1}(1/u) = -7 + 6/u^{n-1} , \quad n > 2$$

$$D_{n,n-2}(1/u) = 51 - 30/u - 60/u^{n-2} + 18/u^{n-1} + 15/u^{2n-4} + 6/u^{2n-3} , \quad n > 4$$

$$D_{n,n-3}(1/u) = -410 + 576/u - 177/u^2 - 24/u^3 + 516/u^{n-3} - 480/u^{n-2} + 78/u^{n-1} + 12/u^n - 177/u^{2(n-3)} - 18/u^{2n-5} + 42/u^{2(n-2)} + 6/u^{2n-3} + 26/u^{3(n-3)} + 12/u^{3n-8} + 18/u^{3n-7} , \quad n > 6$$

$$D_{21}(1/u) = -3\tfrac{1}{2} + 3/u ,$$

$$D_{31}(1/u) = 19\tfrac{1}{3} - 30/u + 9/u^2 + 2/u^3 ,$$

$$D_{41}(1/u) = -129\tfrac{3}{4} + 288/u - 178\tfrac{1}{2}/u^2 + 5/u^3 + 12/u^4 + 3/u^5 ,$$

$$D_{51}(1/u) = 971\tfrac{1}{5} - 2796/u + 2637/u^2 - 680/u^3 - 177/u^4 + 18/u^5 + 21/u^6 + 6/u^7 ,$$

$$D_{61}(1/u) = -7796\tfrac{2}{3} + 27555/u - 34920/u^2 + 16807/u^3 - 136\tfrac{1}{2}/u^4 - 1320/u^5 - 278/u^6 + 33/u^7 + 42/u^8 + 14/u^9 ,$$

$$D_{71}(1/u) = 65718\tfrac{1}{7} - 275184/u + 437997/u^2 - 307476/u^3 + 63870/u^4 + 21168/u^5 - 3007/u^6 - 2682/u^7 - 564/u^8 + 24/u^9 + 105/u^{10} + 30/u^{11} + 1/u^{12} ,$$

$$D_{42}(1/u) = 54\tfrac{1}{2} - 30/u - 60/u^2 + 18/u^3 + 12/u^4 + 6/u^5 ,$$

$$D_{52}(1/u) = -461 + 576/u + 399/u^2 - 504/u^3 - 78/u^4 - 6/u^5 + 44/u^6 + 18/u^7 + 12/u^8 ,$$

$$D_{62}(1/u) = 4096\tfrac{5}{6} - 8100/u - 516/u^2 + 7780/u^3 - 1947/u^4 - 936/u^5 - 515/u^6 - 18/u^7 + 6/u^8 + 81/u^9 + 45/u^{10} + 18/u^{11} + 5/u^{12} ,$$

$$D_{63}(1/u) = -406\tfrac{1}{2} + 576/u - 177/u^2 + 492/u^3 - 480/u^4 + 78/u^5 - 165/u^6 - 18/u^7 + 42/u^8 + 29/u^9 + 12/u^{10} + 18/u^{11} ,$$

$$D_{72}(1/u) = -37587 + 101832/u - 43227/u^2 - 86324/u^3 + 67551/u^4 + 6492/u^5 - 2860/u^6 - 4404/u^7 - 996/u^8 - 732/u^9 - 147/u^{10} + 126/u^{11} + 120/u^{12} + 90/u^{13} + 48/u^{14} + 12/u^{15} + 6/u^{16} ,$$

$$D_{73}(1/u) = 3415 - 7812/u + 5076/u^2 - 4694/u^3 + 7413/u^4 - 3672/u^5 + 1605/u^6 - 756/u^7 - 525/u^8 - 170/u^9 - 48/u^{10} - 60/u^{11} + 99/u^{12} + 48/u^{13} + 45/u^{14} + 24/u^{15} + 12/u^{16} .$$

(d) Diamond:

$$D_{nn}(1/u) = 1 ,$$

$$D_{n,n-1}(1/u) = -5 + 4/u^{n-1} , \quad n > 2$$

$$D_{n,n-2}(1/u) = 26 - 16/u - 32/u^{n-2} + 12/u^{n-1} + 10/u^{2n-4} , \quad n > 4$$

$$D_{n,n-3}(1/u) = -156 + 244/u - 98/u^2 + 212/u^{n-3} - 232/u^{n-2} + 60/u^{n-1} - 98/u^{2(n-3)} + 36/u^{2n-5} + 12/u^{2(n-2)} + 20/u^{3(n-3)} , \quad n > 6$$

$$D_{21}(1/u) = -2\tfrac{1}{2} + 2/u ,$$

$$D_{31}(1/u) = 10\tfrac{1}{3} - 16/u + 6/u^2 ,$$

$$D_{41}(1/u) = -53\tfrac{1}{4} + 122/u - 91/u^2 + 22/u^3 ,$$

$$D_{51}(1/u) = 311\tfrac{1}{5} - 944/u + 1054/u^2 - 512/u^3 + 91/u^4 ,$$

$$D_{61}(1/u) = -1971\tfrac{5}{6} + 7442/u - 11058/u^2 + 8066\tfrac{2}{3}/u^3 - 2877/u^4 + 396/u^5 + 2/u^6 ,$$

$$D_{71}(1/u) = 13215\tfrac{1}{7} - 59640/u + 110586/u^2 - 107608/u^3 + 57749/u^4 - 16072/u^5 + 1746/u^6 + 24/u^7 ,$$

$$D_{42}(1/u) = 28\tfrac{1}{2} - 16/u - 32/u^2 + 12/u^3 + 8/u^4 ,$$

$$D_{52}(1/u) = -182 + 244/u + 146/u^2 - 232/u^3 - 28/u^4 + 36/u^5 + 16/u^6 ,$$

$$D_{62}(1/u) = 1246\tfrac{5}{6} - 2710/u + 254/u^2 + 2576/u^3 - 1014/u^4 - 508/u^5 + 46/u^6 + 72/u^7 + 37/u^8 ,$$

$$D_{63}(1/u) = -153\tfrac{1}{2} + 244/u - 98/u^2 + 212/u^3 - 232/u^4 + 60/u^5 - 98/u^6 + 36/u^7 + 12/u^8 + 18/u^9 ,$$

$$D_{72}(1/u) = -8932 + 26936/u - 17236/u^2 - 18876/u^3 + 22907/u^4 - 704/u^5 - 4042/u^6 - 364/u^7 + 31/u^8 + 192/u^9 + 88/u^{10} ,$$

$$D_{73}(1/u) = 979 - 2588/u + 2142/u^2 - 1872/u^3 + 2802/u^4 - 1820/u^5 + 1032/u^6 - 772/u^7 + 102/u^8 - 168/u^9 + 96/u^{10} + 36/u^{11} + 31/u^{12} .$$

(e) Simple cubic:

$$D_{n,n}(1/u) = 1 ,$$

$$D_{n,n-1}(1/u) = -7 + 6/u^{n-1} , \quad n > 2$$

$$D_{n,n-2}(1/u) = 57 - 36/u - 72/u^{n-2} + 30/u^{n-1} + 21/u^{2n-4} , \quad n > 4$$

$$D_{n,n-3}(1/u) = -530 + 810/u - 315/u^2 + 738/u^{n-3} - 828/u^{n-2} + 216/u^{n-1} - 315/u^{2(n-3)} + 126/u^{2n-5} + 42/u^{2(n-2)} + 56/u^{3(n-3)} , \quad n > 6$$

$$D_{21}(1/u) = -3\tfrac{1}{2} + 3/u\ ,$$

$$D_{31}(1/u) = 21\tfrac{1}{3} - 36/u + 15/u^2,$$

$$D_{41}(1/u) = -162\tfrac{3}{4} + 405/u - 328\tfrac{1}{2}/u^2 + 83/u^3 + 3/u^4,$$

$$D_{51}(1/u) = 1406\tfrac{1}{5} - 4608/u + 5532/u^2 - 2804/u^3 + 426/u^4 + 48/u^5,$$

$$D_{61}(1/u) = -13150\tfrac{2}{3} + 53370/u - 84738/u^2 + 64574/u^3 - 22144\tfrac{1}{2}/u^4 + 1575/u^5 + 496/u^6 + 18/u^7,$$

$$D_{71}(1/u) = 129919\tfrac{1}{7} - 628236/u + 1240035/u^2 - 1261904/u^3 + 674652/u^4 - 157380/u^5 - 1360/u^6 + 3888/u^7 + 378/u^8 + 8/u^9\ ,$$

$$D_{42}(1/u) = 60\tfrac{1}{2} - 36/u - 72/u^2 + 30/u^3 + 18/u^4,$$

$$D_{52}(1/u) = -587 + 810/u + 495/u^2 - 828/u^3 - 78/u^4 + 126/u^5 + 62/u^6,$$

$$D_{62}(1/u) = 6075\tfrac{5}{6} - 13419/u + 1149/u^2 + 13534/u^3 - 5550/u^4 - 2544/u^5 + 61/u^6 + 474/u^7 + 207/u^8 + 12/u^9,$$

$$D_{63}(1/u) = -526\tfrac{1}{2} + 810/u - 315/u^2 + 738/u^3 - 828/u^4 + 216/u^5 - 315/u^6 + 126/u^7 + 42/u^8 + 53/u^9,$$

$$D_{72}(1/u) = -65493 + 199656/u - 126729/u^2 - 150140/u^3 + 182112/u^4 - 8742/u^5 - 27691/u^6 - 5916/u^7 + 429/u^8 + 1650/u^9 + 756/u^{10} + 96/u^{11} + 12/u^{12},$$

$$D_{73}(1/u) = 5206 - 13014/u + 10365/u^2 - 9836/u^3 + 14877/u^4 - 9414/u^5 + 5175/u^6 - 3870/u^7 + 405/u^8 - 644/u^9 + 432/u^{10} + 174/u^{11} + 144/u^{12}\ .$$

(f) Body-centered cubic:

$$D_{nn}(1/u) = 1\,,$$

$$D_{n,n-1}(1/u) = -9 + 8/u^{n-1},\quad n > 2$$

$$D_{n,n-2}(1/u) = 100 - 64/u - 128/u^{n-2} + 56/u^{n-1} + 36/u^{2n-4},\quad n > 4$$

$$D_{n,n-3}(1/u) = -1256 + 1896/u - 724/u^2 + 1768/u^{n-3} - 2000/u^{n-2} + 520/u^{n-1} - 724/u^{2(n-3)} + 296/u^{2n-5} + 104/u^{2(n-2)} + 120/u^{3(n-3)}\ ,\ n > 6$$

$$D_{21}(1/u) = -4\tfrac{1}{2} + 4/u\ ,$$

$$D_{31}(1/u) = 36\tfrac{1}{3} - 64/u + 28/u^2,$$

$$D_{41}(1/u) = -366\tfrac{1}{4} + 948/u - 798/u^2 + 204/u^3 + 12/u^4,$$

$$D_{51}(1/u) = 4174\tfrac{1}{5} - 14184/u + 17592/u^2 - 9072/u^3 + 1262/u^4 + 216/u^5 + 12/u^6,$$

$$D_{61}(1/u) = -51444\tfrac{1}{2} + 216036/u - 353640/u^2 + 275021\tfrac{1}{3}/u^3 - 92992/u^4 + 4312/u^5 + 2368/u^6 + 312/u^7 + 27/u^8\ ,$$

$$D_{71}(1/u) = 669438\tfrac{1}{7} - 3344712/u + 6798900/u^2 - 7072736/u^3 + 3795726/u^4 - 833064/u^5 - 36348/u^6 + 17616/u^7 + 4404/u^8 + 704/u^9 + 72/u^{10},$$

$$D_{42}(1/u) = 104\tfrac{1}{2} - 64/u - 128/u^2 + 56/u^3 + 32/u^4,$$

$$D_{52}(1/u) = -1356 + 1896/u + 1172/u^2 - 2000/u^3 - 168/u^4 + 296/u^5 + 160/u^6\ ,$$

$$D_{62}(1/u) = 18709\tfrac{5}{6} - 41604/u + 3324/u^2 + 43136/u^3 - 17964/u^4 - 7816/u^5 - 164/u^6 + 1616/u^7 + 690/u^8 + 72/u^9\ ,$$

$$D_{63}(1/u) = -1251\tfrac{1}{2} + 1896/u - 724/u^2 + 1768/u^3 - 2000/u^4 + 520/u^5 - 724/u^6 + 296/u^7 + 104/u^8 + 116/u^9\ ,$$

$$D_{72}(1/u) = -268390 + 821592/u - 517704/u^2 - 639592/u^3 + 773372/u^4 - 44040/u^5 - 107360/u^6 - 30440/u^7 + 1652/u^8 + 6672/u^9 + 3476/u^{10} + 624/u^{11} + 138/u^{12},$$

$$D_{73}(1/u) = 16694 - 40656/u + 31692/u^2 - 31320/u^3 + 47600/u^4 - 29664/u^5 + 16076/u^6 - 11960/u^7 + 1036/u^8 - 1728/u^9 + 1276/u^{10} + 512/u^{11} + 442/u^{12}\ .$$

(g) Face-centered cubic:

$$D_{nn}(1/u) = 1\,,$$

$$D_{n,n-1}(1/u) = -13 + 12/u^{n-1},\quad n > 2$$

$$D_{n,n-2}(1/u) = 198 - 120/u - 240/u^{n-2} + 84/u^{n-1} + 54/u^{2n-4} + 24/u^{2n-3},\quad n > 4$$

$$D_{n,n-3}(1/u) = -3368 + 4644/u - 1362/u^2 - 200/u^3 + 4404/u^{n-3} - 4368/u^{n-2} + 780/u^{n-1} + 120/u^n - 1362/u^{2(n-3)} - 36/u^{2n-5} + 336/u^{2(n-2)} + 48/u^{2n-3} + 140/u^{3(n-3)} + 120/u^{3n-8} + 96/u^{3n-7} + 8/u^{3(n-2)},\quad n > 6$$

$$D_{21}(1/u) = -6\tfrac{1}{2} + 6/u\,,$$

$$D_{31}(1/u) = 70\tfrac{1}{3} - 120/u + 42/u^2 + 8/u^3,$$

$$D_{41}(1/u) = -944\tfrac{1}{4} + 2322/u - 1653/u^2 + 126/u^3 + 123/u^4 + 24/u^5 + 2/u^6\ ,$$

$$D_{51}(1/u)=14303\tfrac{1}{5}-45792/u+49290/u^2-16296/u^3-2871/u^4+792/u^5+448/u^6+96/u^7+30/u^8,$$

$$D_{61}(1/u)=-234103\tfrac{1}{6}+922152/u-1329240/u^2+771272/u^3-64224/u^4-65070/u^5-6904/u^6+3930/u^7+1212/u^8+776/u^9+168/u^{10}+30/u^{11}+1/u^{12},$$

$$D_{71}(1/u)=4044119\tfrac{1}{7}-18902160/u+34148478/u^2-28260664/u^3+8356041/u^4+1503912/u^5-622498/u^6-281400/u^7+1038/u^8-1160/u^9+9036/u^{10}+3528/u^{11}+1350/u^{12}+336/u^{13}+36/u^{14}+8/u^{15},$$

$$D_{42}(1/u)=204\tfrac{1}{2}-120/u-240/u^2+84/u^3+48/u^4+24/u^5,$$

$$D_{52}(1/u)=-3566+4644/u+3282/u^2-4568/u^3-504/u^4+84/u^5+380/u^6+168/u^7+72/u^8+8/u^9,$$

$$D_{62}(1/u)=65920\tfrac{5}{6}-135054/u-7722/u^2+142112/u^3-42924/u^4-18528/u^5-7594/u^6+96/u^7+1383/u^8+1228/u^9+720/u^{10}+264/u^{11}+86/u^{12}+12/u^{13},$$

$$D_{63}(1/u)=-3361\tfrac{1}{2}+4644/u-1362/u^2+4204/u^3-4368/u^4+780/u^5-1242/u^6-36/u^7+336/u^8+182/u^9+120/u^{10}+96/u^{11}+8/u^{12},$$

$$D_{72}(1/u)=-1265364+3551232/u-1606836/u^2-3229668/u^3+2812233/u^4+134376/u^5-193950/u^6-152700/u^7-57561/u^8-7968/u^9+396/u^{10}+6480/u^{11}+4180/u^{12}+3072/u^{13}+1356/u^{14}+480/u^{15}+210/u^{16}+24/u^{17}+8/u^{18},$$

$$D_{73}(1/u)=60607-132732/u+83862/u^2-87016/u^3+139224/u^4-69972/u^5+28456/u^6-15612/u^7-7626/u^8-1136/u^9-1236/u^{10}+228/u^{11}+1267/u^{12}+672/u^{13}+648/u^{14}+216/u^{15}+126/u^{16}+24/u^{17}.$$

APPENDIX B

Clusters $C^n_{pml\alpha}$, symmetries $g_{nml\alpha}$, powers $r_{nml\alpha}$, and embeddings $\mathcal{E}_{pml}$ for the spin-independent polynomials through μ^5. The addition clusters allowing us to extend the series through μ^7, given the spin-$\frac{1}{2}$ and -1 low-temperature polynomials, are also presented.

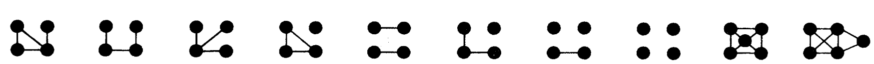
Key: ● = ①, ▲ = ②, ■ = ③, ○ = ⓝ, △ = ($n-1$), □ = ($n-2$), ◇ = ($n-3$).

$D_{p1}(1/u)$

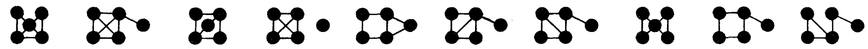
$C^p_{pm\ell 1}$

$g_{pm\ell 1}$	1	2	2	6	2	2	6	24	4	8
$r_{pm\ell 1}$	0	1	0	3	2	1	0	6	5	4
$\mathcal{E}_{pm\ell}$ Honeycomb	1	3	-4	0	6	-18	38	0	0	0
Simple Quadratic	1	4	-5	0	12	-32	62	0	0	8
Triangle	1	6	-7	12	18	-60	116	0	12	0
Diamond	1	4	-5	0	12	-32	62	0	0	0
Simple Cubic	1	6	-7	0	30	-72	128	0	0	24
Body Centered Cubic	1	8	-9	0	56	-128	218	0	0	96
Face Centered Cubic	1	12	-13	48	84	-240	422	48	96	24

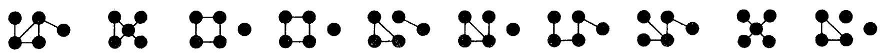
$C^p_{pm\ell 1}$



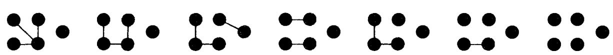
$g_{pm\ell 1}$	2	2	6	6	8	2	4	24	8	4
$r_{pm\ell 1}$	4	3	3	3	2	2	1	0	8	8
$\mathcal{E}_{pm\ell}$ Honeycomb	0	12	6	0	-78	-48	204	-588	0	0
Simple Quadratic	0	28	24	0	-184	-124	472	-1254	0	0
Triangle	24	54	12	-144	-492	-234	1152	-3114	0	0
Diamond	0	36	24	0	-200	-132	488	-1278	0	0
Simple Cubic	0	126	120	0	-684	-486	1620	-3906	0	0
Body Centered Cubic	0	296	336	0	-1616	-1192	3792	-8790	0	0
Face Centered Cubic	240	564	264	-1200	-4248	-2244	9288	-22662	48	96

$C^p_{pm\ell 1}$ 

		1	2	3	4	5	6	7	8	9	10
$g_{pm\ell 1}$		2	6	12	24	2	2	2	8	2	2
$r_{pm\ell 1}$		7	7	6	6	6	6	6	6	5	5
$\mathcal{E}_{pm\ell}$	Honeycomb	0	0	0	0	0	0	0	0	0	0
	Simple Quadratic	0	0	0	0	0	0	0	0	16	0
	Triangle	12	0	0	0	0	12	24	24	0	72
	Diamond	0	0	0	0	0	0	0	0	0	0
	Simple Cubic	0	0	0	0	0	0	0	0	96	0
	Body Centered Cubic	0	0	144	0	0	0	0	0	432	0
	Face Centered Cubic	144	144	0	-1344	48	336	480	576	96	584

$C^p_{pm\ell 1}$ 

		1	2	3	4	5	6	7	8	9	10
$g_{pm\ell 1}$		2	4	4	8	12	2	2	2	24	12
$r_{pm\ell 1}$		5	5	5	4	4	4	4	4	4	3
$\mathcal{E}_{pm\ell}$	Honeycomb	0	0	0	0	0	0	24	12	0	0
	Simple Quadratic	0	0	0	-96	0	0	68	40	24	0
	Triangle	48	0	-168	0	-1152	-360	162	36	0	3024
	Diamond	0	0	0	0	0	0	108	72	24	0
	Simple Cubic	0	0	0	-480	0	0	534	408	360	0
	Body Centered Cubic	0	0	0	-2400	0	0	1640	1344	1680	0
	Face Centered Cubic	1152	384	-2880	-720	-20736	-7584	3804	1656	216	51936

$C^p_{pm\ell 1}$ 

		1	2	3	4	5	6	7
$g_{pm\ell 1}$		6	2	4	8	4	12	120
$r_{pm\ell 1}$		3	3	3	2	2	1	0
$\mathcal{E}_{pm\ell}$	Honeycomb	-60	-120	-204	1032	636	-3492	12744
	Simple Quadratic	-288	-360	-688	3216	2096	-10464	35424
	Triangle	-192	-864	-1872	10800	5148	-33552	116544
	Diamond	-336	-504	-816	3632	2400	-11328	37344
	Simple Cubic	-2352	-2580	-4488	18552	12852	-55296	168744
	Body Centered Cubic	-8496	-8016	-14592	58272	41232	-170208	500904
	Face Centered Cubic	-8688	-18888	-38928	192048	101136	-549504	1716384

$D_{nn}(1/u)$

$C^n_{pm\ell\alpha}$

	1
$g_{nm\ell\alpha}$	1
$r_{nm\ell\alpha}$	0

$D_{n,n-1}(1/u)$, $n > 2$

$C^n_{pm\ell\alpha}$ 

	1	2
$g_{nm\ell\alpha}$	1	1
$r_{nm\ell\alpha}$	n-1	0

$D_{n,n-2}(1/u)$, $n>4$

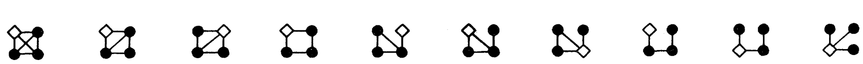
$C^n_{pm\ell\alpha}$ 

	1	2	3	4	5	6	7	8
$g_{nm\ell\alpha}$	1	1	2	2	1	2	1	2
$r_{nm\ell\alpha}$	2n-4	0	2n-3	2n-4	n-1	1	n-2	0

$D_{n,n-3}(1/u)$, $n > 6$

$C^n_{pm\ell\alpha}$ 

	1	2	3	4	5	6	7	8	9	10
$g_{nm\ell\alpha}$	1	1	1	1	1	1	1	1	1	1
$r_{nm\ell\alpha}$	3(n-3)	0	3n-7	3(n-3)	2(n-2)	n-1	2	n-3	2(n-3)	0

$C^n_{pm\ell\alpha}$ 

	1	2	3	4	5	6	7	8	9	10
$g_{nm\ell\alpha}$	6	2	2	2	2	1	2	1	1	6
$r_{nm\ell\alpha}$	3(n-2)	2n-3	3n-7	2(n-2)	n	2(n-2)	3n-8	n-1	2n-5	3(n-3)

$C^n_{pm\ell\alpha}$

$g_{nm\ell\alpha}$	2	2	6	2	1	2	2	2	2	6
$r_{nm\ell\alpha}$	n-1	2n-5	3	n-2	n-2	2(n-3)	2	1	n-3	0

$D_{42}(1/u)$

$C^n_{pm\ell\alpha}$

$g_{nm\ell\alpha}$	2	2	2	2	1	2	1	2
$r_{nm\ell\alpha}$	4	0	5	4	3	1	2	0

$D_{52}(1/u)$

$C^n_{pm\ell\alpha}$

$g_{nm\ell\alpha}$	2	2	1	2	1	2	6	2	2	2
$r_{nm\ell\alpha}$	8	4	6	4	2	0	9	8	7	6

$C^n_{pm\ell\alpha}$

$g_{nm\ell\alpha}$	2	1	2	1	1	6	2	2	6	2
$r_{nm\ell\alpha}$	5	6	7	4	5	6	4	5	3	3

$C^n_{pm\ell\alpha}$

$g_{nm\ell\alpha}$	1	2	2	2	2	6
$r_{nm\ell\alpha}$	3	4	2	1	2	0

$D_{62}(1/u)$

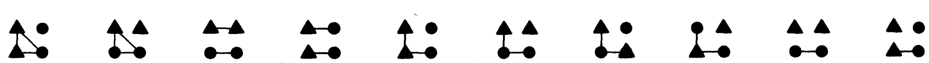
$C^n_{pm\ell\alpha}$

$g_{nm\ell\alpha}$	6	2	2	6	4	1	4	4	2	4
$r_{nm\ell\alpha}$	12	8	4	0	13	11	12	9	9	8

$C^n_{pm\ell\alpha}$

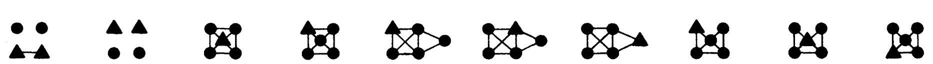
$g_{nm\ell\alpha}$	1	1	2	2	2	1	2	1	2	2
$r_{nm\ell\alpha}$	7	10	9	9	5	7	8	6	8	5

$C^n_{pm\ell\alpha}$



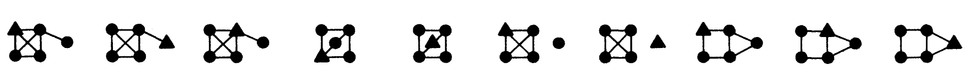
$g_{nm\ell\alpha2}$	2	2	4	2	1	1	2	2	4	1
$r_{nm\ell\alpha}$	8	5	5	4	6	3	4	4	1	2

$C^n_{pm\ell\alpha}$



$g_{nm\ell\alpha}$	4	4	8	2	2	2	4	1	2	1
$r_{nm\ell\alpha}$	4	0	12	11	11	12	10	9	11	10

$C^n_{pm\ell\alpha}$



$g_{nm\ell\alpha}$	2	6	6	6	4	6	24	1	1	2
$r_{nm\ell\alpha}$	10	8	11	9	8	9	6	8	9	8

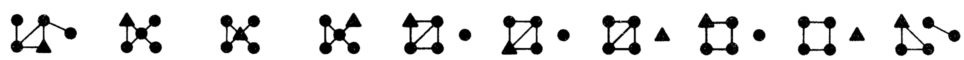
$c^n_{pm\ell\alpha}$

$g_{nm\ell\alpha}$	1	2	2	2	1	2	2	2	2	8
$r_{nm\ell\alpha}$	8	10	7	9	9	7	8	9	8	10

$c^n_{pm\ell\alpha}$

$g_{nm\ell\alpha}$	1	2	2	2	1	2	2	2	1	1
$r_{nm\ell\alpha}$	7	8	6	7	7	8	7	6	6	8

$c^n_{pm\ell\alpha}$

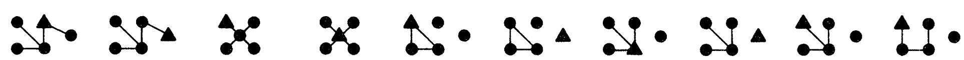


$g_{nm\ell\alpha}$	2	2	4	2	2	2	4	2	8	4
$r_{nm\ell\alpha}$	7	7	9	6	7	8	5	6	4	6

$c^n_{pm\ell\alpha}$

$g_{nm\ell\alpha}$	6	1	2	2	2	1	1	2	1	2
$r_{nm\ell\alpha}$	5	6	7	5	4	5	6	6	5	7

$c^n_{pm\ell\alpha}$

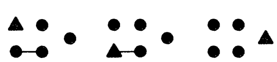


$g_{nm\ell\alpha}$	2	2	6	24	4	6	6	6	2	1
$r_{nm\ell\alpha}$	6	5	5	8	5	3	6	3	4	4

$c^n_{pm\ell\alpha}$

$g_{nm\ell\alpha}$	1	2	2	4	2	2	8	2	2	4
$r_{nm\ell\alpha}$	5	3	4	5	4	3	2	2	3	4

$c^n_{pm\ell\alpha}$



$g_{nm\ell\alpha}$	4	6	24
$r_{nm\ell\alpha}$	1	2	0

$D_{63}(1/u)$

$c^n_{pm\ell\alpha}$

$g_{nm\ell\alpha}$	2	2	1	1	1	1	1	1	1	1
$r_{nm\ell\alpha}$	9	0	11	9	8	5	6	3	2	0

$c^n_{pm\ell\alpha}$

$g_{nm\ell\alpha}$	6	2	2	2	2	1	2	1	1	6
$r_{nm\ell\alpha}$	12	11	9	8	6	8	10	5	7	9

$c^n_{pm\ell\alpha}$

$g_{nm\ell\alpha}$	2	2	6	2	1	2	2	2	2	6
$r_{nm\ell\alpha}$	5	7	3	4	4	6	2	1	3	0

$D_{73}(1/u)$

$c^n_{pm\ell\alpha}$

$g_{nm\ell\alpha}$	2	2	2	1	2	1	2	1	2	1
$r_{nm\ell\alpha}$	15	16	6	12	12	10	9	3	4	6

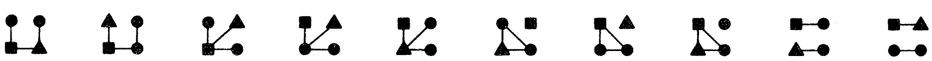
$c^n_{pm\ell\alpha}$

$g_{nm\ell\alpha}$	2	2	2	1	2	2	1	1	2	1
$r_{nm\ell\alpha}$	0	0	17	15	16	11	14	12	10	12

$c^n_{pm\ell\alpha}$

$g_{nm\ell\alpha}$	1	1	2	1	2	1	1	1	1	1
$r_{nm\ell\alpha}$	13	9	13	14	11	8	6	7	9	8

$c^n_{pm\ell\alpha}$

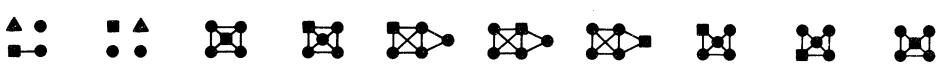


$g_{nm\ell\alpha}$	1	1	2	1	2	2	2	1	1	2
$r_{nm\ell\alpha}$	11	10	12	6	10	5	7	11	5	7

$c^n_{pm\ell\alpha}$

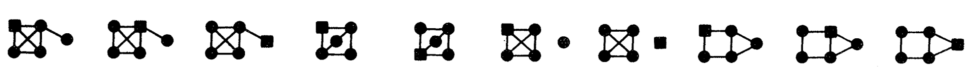
$g_{nm\ell\alpha}$	1	1	1	2	1	1	2	2	1	2
$r_{nm\ell\alpha}$	4	5	8	6	9	3	4	1	2	6

$c^n_{pm\ell\alpha}$



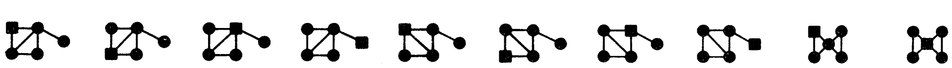
$g_{nm\ell\alpha}$	1	2	8	2	2	2	4	1	1	2
$r_{nm\ell\alpha}$	3	0	16	14	14	16	12	11	13	15

$c^n_{pm\ell\alpha}$

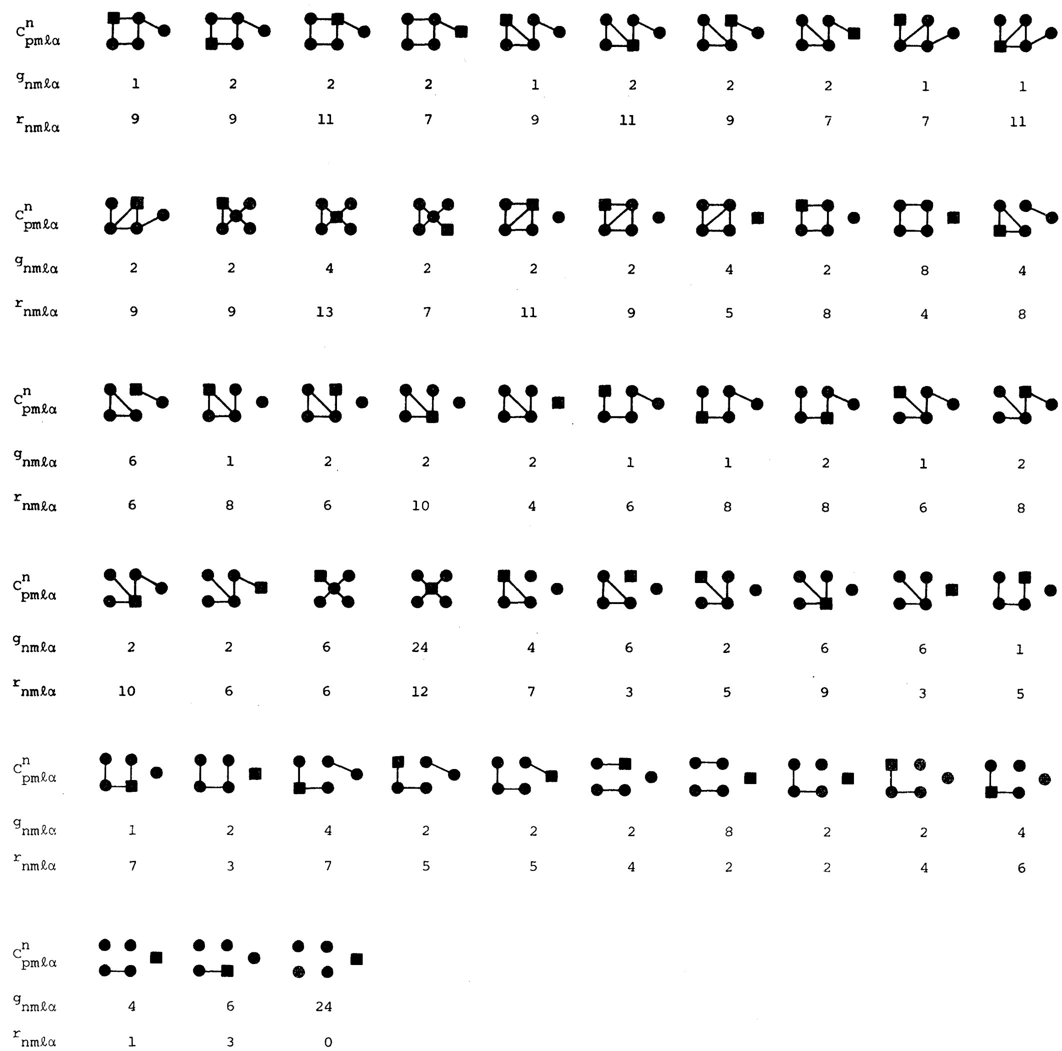


$g_{nm\ell\alpha}$	2	6	6	4	6	6	24	1	1	2
$r_{nm\ell\alpha}$	13	15	9	10	12	12	6	10	12	10

$c^n_{pm\ell\alpha}$



$g_{nm\ell\alpha}$	1	2	2	2	1	2	2	2	2	8
$r_{nm\ell\alpha}$	10	12	14	8	12	10	12	8	10	14



$g_{nm\ell\alpha}$	1	2	2	2	1	2	2	2	1	1
$r_{nm\ell\alpha}$	9	9	11	7	9	11	9	7	7	11
$g_{nm\ell\alpha}$	2	2	4	2	2	2	4	2	8	4
$r_{nm\ell\alpha}$	9	9	13	7	11	9	5	8	4	8
$g_{nm\ell\alpha}$	6	1	2	2	2	1	1	2	1	2
$r_{nm\ell\alpha}$	6	8	6	10	4	6	8	8	6	8
$g_{nm\ell\alpha}$	2	2	6	24	4	6	2	6	6	1
$r_{nm\ell\alpha}$	10	6	6	12	7	3	5	9	3	5
$g_{nm\ell\alpha}$	1	2	4	2	2	2	8	2	2	4
$r_{nm\ell\alpha}$	7	3	7	5	5	4	2	2	4	6
$g_{nm\ell\alpha}$	4	6	24							
$r_{nm\ell\alpha}$	1	3	0							

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[6]The constants $\overline{\mathfrak{M}}[C]$ evaluated at $N=1$ are the lattice constants of Domb tabulated in Ref. 1.

[7]The constants $\mathfrak{M}[C]$ are the lattice constants of Sykes tabulated in Ref. 2.

[8]In practice the $\mathcal{S}[C]$ are calculated from the spin-$\frac{1}{2}$ $\mathfrak{M}[C]$ (when available) by multiplying by the spin-$\frac{1}{2}$ symmetry factor (i.e., all particles identical): $s_{pml}=\mathfrak{M}_{pml1}g_{pml1}$. In cases where only the $\overline{\mathfrak{M}}[C](N=1)$ are available one can calculate $\mathfrak{M}[C]$ by diagrammatically taking the log of of Z for the disconnected clusters, e.g.,

$$\mathfrak{M}[C\{\text{Fig. 1(f)}\}]=\overline{\mathfrak{M}}[C\{\text{Fig. 1(f)}\}](N=1)$$

$$-\tfrac{1}{2}(\overline{\mathfrak{M}}[C\{\text{Fig. 1(g)}\}](N=1)\overline{\mathfrak{M}}[C\{\text{Fig. 1(h)}\}]\ (N=1)$$

$$+\overline{\mathfrak{M}}[C\{\text{Fig. 1(i)}\}](N=1)\overline{\mathfrak{M}}[C\{\text{Fig. 1(j)}\}](N=1))$$

$$+\tfrac{1}{3}\overline{\mathfrak{M}}[C\{\text{Fig. 1(j)}\}]^2(N=1)\overline{\mathfrak{M}}[C\{\text{Fig. 1(h)}\}]\ (N=1)\,.$$

For the connected clusters $\mathfrak{M}[C]=\overline{\mathfrak{M}}[C](N=1)$. Finally if only the polynomials are available, each coefficient must be decomposed graphically into the contributions from the various clusters:

$$D_{nm}^{(1/2)}=\sum_{l}\mathfrak{M}[C_{nml1}^{n}],$$

where

$$L_n^{(1/2)}(u)=\sum_{m=0}^{n(n-1)/2}D_{nm}^{(1/2)}u^{nzs-m}$$

(n points and m lines fixed).

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