Critical properties of random-spin models from the ϵ expansion

T. C. Lubensky*

Department of Physics and Laboratory for Research in the Structure of Matter, University of Pennsylvania, Philadelphia, Pennsylvania 19174

(Received 2 December 1974)

In quenched random-spin systems, the renormalization group can be used to develop recursion relations for the probability distribution for random potentials. Alternatively, recursion relations for the average values of potentials and their higher cumulants can be obtained. In this paper, the above technique is used to study phase transitions in quenched random n-component classical spin systems using the ϵ expansion to second order in ϵ. It there are long-range correlations in the random potentials (e.g., all potentials along a line are equal), there are no stable physical fixed points within the ϵ expansion. This is interpreted as a smeared transition. If there are no long-range correlations in the random potentials, there is a sharp transition with pure system exponents if the specific-heat exponent α^H of the pure system is negative. If $\alpha^H > 0$ and $n > 1$, there is a sharp transition with new exponents $\eta = [(5n^2 - 8n)/256(n-1)^2]\epsilon^2$ and $\nu = 1/2 + [3n/32(n-1)]\epsilon + [n(127n^2 - 572n - 32)/4096(n-1)^3]\epsilon^2$. For $n = 1$, there is no stable fixed point, which is again interpreted as a smeared transition.

Compositional disorder can be divided into two broad categories: quenched and unquenched or annealed disorder.[1,2] In annealed systems, impurities or vacancies diffuse freely and reach equilibrium with respect to other degrees of freedom. For sufficiently high concentrations of impurities, phase separation can occur in these systems. In quenched systems, on the other hand, impurities are frozen in fixed positions; i.e., they cannot overcome potential barriers for diffusion at the temperatures under study. These systems exhibit the phenomenon of percolation at sufficiently high impurity concentrations.[3]

Phase transitions in annealed systems are rather well understood.[1,2,4] If the impurity chemical potential is held fixed, the critical exponents of the phase transition are the same as those for a pure system. If, on the other hand, the concentration of impurities is held fixed, the exponents undergo a Fisher renormalization[2] if the specific-heat exponent α of the pure system is positive.

Phase transitions in quenched systems are less well understood. Mathematically, the free energy for these systems is obtained by averaging the logarithm of the partition function over impurity configurations, $F = -k_B T\langle\langle \ln Z\rangle\rangle$.[5] As an artifical example of this operation, consider an ensemble of Heisenberg models, each with different uniform exchange J. As pointed out by Harris,[6] an average of the free energy over a probability distribution in J of width δJ will lead to a transition smeared over a temperature range of width $\Delta T \sim \delta J$. McCoy and Wu have solved exactly a somewhat less artifical random Ising model on a two-dimensional lattice.[7] They consider a model in which all horizontal bonds have a fixed value J_1. They then require vertical bonds $J_2(j)$ between rows j and $j+1$ to be equal but allow $J_2(j)$ to be a random function of j with width δJ_2. They find a smeared transition with smearing width $\Delta T \sim (\delta J_2)^2$. A common feature of these two models is that they both have probability distributions with long-range correlations in the bond strengths. Very little is known about the behavior near phase transitions in models with more realistic short-range disorder. Several options are open. The transition could be sharp or smeared. If it is sharp, it could be first or second order.[8] If it is second order, the critical exponents could be those for the pure system or they could be renormalized. (If the exponents are renormalized, there is no reason to expect that they will be Fisher-renormalized.[2]) The only published attempts to distinguish these cases[1,8] involve high-temperature expansions and give inconclusive results.

The renormalization group has been extremely successful in calculating critical exponents in pure systems.[9-12] A pure system is characterized by a set of translationally invariant potentials. The renormalization group develops recursion relations for these potentials. Critical properties are determined by the fixed points of these recursion relations. In random systems, the potentials become random and nontranslationally-invariant with a probability distribution P. Renormalization equations for the probability distribution rather than the potentials can be developed in random systems. The author and Harris[13] have discussed critical properties of a two-dimensional random Ising model using the above idea and the Niemeyer–Van Leeuwen[14] cluster expansion. Critical properties of random systems of n-component classical spins at dimensionality $d = 4-\epsilon$ can also be obtained using the Wilson-Fisher ϵ expansion.[10] An outline of the results to first order in ϵ have been reported elsewhere.[13,15] In this paper, we will present in detail the calculations of critical exponents in random systems to second order in ϵ using the technique of renormalizing the probability distribution.

Luther and Grinstein[16,17] have independently studied the properties of phase transitions in quenched random systems using a completely different renormalization approach than the one described here. (To facilitate comparison with our work, we have interchanged the role of their n and m.) They consider a model Hamiltonian for an mn-component spin at each lattice site with a quadratic spin coupling that is rotationally invariant in the complete mn-component space and two quartic spin couplings, one of which is rotationally invariant in the mn-component space and one of which is rotationally invariant only in each of the n-component subspaces separately. This was first studied by Brezin *et al.*[18] Grinstein and Luther show that the analytic continuation of this model to $m=0$ is equivalent to a model for a random n-component spin system. An elegant derivation of this equivalence has been given by Emery.[19] The renormalization equations that result from the Grinstein-Luther model are identical to those presented here to second order in the ϵ expansion.[17]

The results of this paper are as follows. If there are long-range correlations in the random potentials such as in the model of McCoy and Wu, the pure-system fixed points (Heisenberg and Gaussian) are unstable with respect to creating a small amount of randomness. In analogy with the exact results of McCoy and Wu, we argue that this instability is the signal of a smeared transition. If the disorder contains no long-range correlations, the randomness can be characterized by a single variable Δ, which behaves like a quartic potential in the renormalization equations. Δ is directly proportional to the variance of fluctuations in the *local* transition temperature. In addition, one has to consider the averaged two-spin and four-spin potentials r and u. As is usually the case in systems with two quartic potentials,[18,20,21] there are four fixed points within the ϵ expansion. There is the unstable Gaussian fixed point, the usual Heisenberg fixed point, and unphysical but always stable fixed point with $u^*=0$ and $\Delta^*<0$, and a random fixed point in which both Δ^* and u^* are nonzero. For n greater than a critical value $n_c(\epsilon)$, the Heisenberg fixed point is stable; for $1<n<n_c(\epsilon)$, the mixed fixed point is stable. We interpret this to mean that for $n>n_c$ there is a smooth second-order transition with pure-system exponents, and for $1<n<n_c$, there is a smooth second-order transition with new exponents. $n_c(\epsilon)$ is determined to second order in ϵ by $\alpha^H(n_c(\epsilon))=0$ where α^H is the specific-heat exponent for the Heisenberg fixed point. For $n>1$, the specific-heat exponent for the stable fixed point is negative, in agreement with heuristic arguments.[6] For $n=1$, the situation is not entirely clear; the mixed fixed point goes to infinity and effectively can be ignored. The Heisenberg fixed point is, however, unstable. We would like to interpret this as a smeared transition in analogy with the instability induced by randomness with long-range correlations. The renormalization group, however, is unable to make any definite statement about systems that have no stable fixed points, and other interpretations are possible. Grinstein and Luther,[17] for example, have argued that the $n=1$ transition may be first order in analogy with the first-order transition that occurs for large n in the hypercubic model[22] or that is believed to occur in superconductors.[23]

We begin with the reduced Hamiltonian in momentum space:

$$\mathcal{H}=\frac{1}{2}\int_{\vec{q}_1,\vec{q}_2} v_2(\vec{q}_1,\vec{q}_2)\vec{S}(\vec{q}_1)\cdot\vec{S}(\vec{q}_2) + \int_{\vec{q}_1,\ldots,\vec{q}_4} v_4(\vec{q}_1,\vec{q}_2,\vec{q}_3,\vec{q}_4)\vec{S}(\vec{q}_1)\cdot\vec{S}(\vec{q}_2)\vec{S}(\vec{q}_3)\cdot\vec{S}(\vec{q}_4)\,, \tag{1}$$

where $\vec{S}(\vec{q})$ is an n-component classical spin of wave vector $\vec{q}$, and $\int_{\vec{q}}=\int_{S_\Lambda} d^dq/(2\pi)^d$, where S_Λ is a sphere of radius Λ. v_2 and v_4 are arbitrary random potentials for an inhomogeneous system. In general, six-spin, eight-spin, etc. potentials v_l, $l=6, 8, \ldots$ should be included in the reduced Hamiltonian. The ensemble of the set of potentials $\{v_l\}$, $l=2, 4, \ldots$ ($\vec{q}$ dependence has been suppressed) is described by a probability distribution $P(\{v_l\})$. The renormalization group operates on the potentials $\{v_l\}$ of a particular member of the ensemble to produce a new set $\{v_l'\}$:

$$\{v_l'\}=R\{v_l\}\equiv R_s R_b\{v_l\}\,, \tag{2}$$

where R_b represents a removal of all spin degrees of freedom with $b^{-1}\Lambda<|\vec{q}|<\Lambda$, $b>1$, and R_s represents a change of scale

$$\vec{q}\to b\vec{q}\,,\quad \vec{S}(\vec{q})\to\xi\vec{S}(b\vec{q}) \tag{3}$$

The operation R then generates a new probability distribution $P'(\{v_l'\})$ via

$$P'(\{v_l'\})=\int \delta(\{v_l'\}-R\{v_l\})P(\{v_l\})\,d\{v_l\}\,, \tag{4}$$

where the integral is over all degrees of freedom in $\{v_l\}$. $P\{v_l\}$ can alternatively be described in terms of its cumulants c_k. The lowest-order cumulants are $\langle v_l\rangle$ and $\langle v_l v_m\rangle-\langle v_l\rangle\langle v_m\rangle$ where $\langle v_l\rangle = \int v_l P(\{v_k\})$. Higher-order cumulants are defined as usual. The recursion relation for P can thus be converted into a recursion relation for the cumulants $\{c_k\}$,

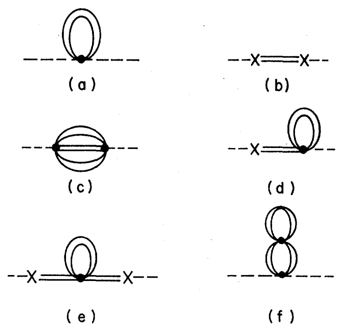

FIG. 1. Diagrams for $v_2(\vec{q},\vec{q}\,')$ to second order in v_4. The double line is the propagator $G^{>}(\vec{q},\vec{q}\,')$. The dot represents v_4 and the cross sepresents Y.

$$\{c_k'\}=\tilde{R}\{c_k\}\,, \qquad (5)$$

where $\tilde{R}$ represents the operation of the renormalization group on $\{c_k\}$.

Our first goal is to develop recursion relations for the potentials in a general inhomogeneous system. To do this, it is convenient to divide $v_2(\vec{q},\vec{q}\,')$ into three parts:

$$v_2(\vec{q},\vec{q}\,')=v_2^{>}(\vec{q},\vec{q}\,')+v_2^{<}(\vec{q},\vec{q}\,')+Y(\vec{q},\vec{q}\,')\,. \qquad (6)$$

$v_2^{>}(\vec{q},\vec{q}\,')$ is zero unless both $|\vec{q}|$ and $|\vec{q}\,'|$ are greater than $b^{-1}\Lambda$; $v_2^{<}(\vec{q},\vec{q}\,')$ is zero unless both are less than $b^{-1}\Lambda$; and $Y(\vec{q},\vec{q}\,')$ is zero unless $|\vec{q}|<b^{-1}\Lambda$ and $|\vec{q}\,'|>b^{-1}\Lambda$ or vice versa. $Y(\vec{q},\vec{q}\,')$ couples spin variables that are eliminated by R_b to those that are not. Following the usual treatment of the ϵ expansion, we develop the operation R_b as a power series in v_4. The diagrams for v_2 to second order in v_4 and those for v_4 to third order in v_4 are shown in Figs. 1 and 2. Weights for these diagrams are given in Table I. The double line represents the inhomogeneous propagator $G^{>}(\vec{q},\vec{q}\,')$ which is the inverse of the submatrix $v_2^{>}(\vec{q},\vec{q}\,')$; the dots, $v_4(\vec{q}_1,\vec{q}_2,\vec{q}_3,\vec{q}_4)$; and the crosses, $Y(\vec{q},\vec{q}\,')$. These diagrams include all contributions from Y for the given order in v_4. To illustrate the use of these diagrams, we show the recursion relations for $v_2(\vec{q},\vec{q}\,')$ to first order in v_4 [Figs. 1(a) and 1(b)]

$$v_2'(\vec{q},\vec{q}\,')=b^{-2d}\zeta^2\left(v_2(b^{-1}\vec{q},b^{-1}\vec{q}\,')+\int_{\vec{q}_1,\vec{q}_2}v_4(b^{-1}\vec{q},b^{-1}\vec{q}\,',\vec{q}_1,\vec{q}_2)\,G^{>}(\vec{q}_1,\vec{q}_2)-\int_{\vec{q}_1,\vec{q}_2}Y(b^{-1}\vec{q},\vec{q}_1)G^{>}(\vec{q}_1,\vec{q}_2)Y(\vec{q}_2,b^{-1}\vec{q}\,')\right), \qquad (7)$$

where the integrals are over $|\vec{q}|\in(b^{-1}\Lambda,\,\Lambda)$.

Recursion relations for $\langle v_2\rangle$, $\langle v_4\rangle$, etc. can be obtained by averaging the recursion relations for the general inhomogeneous potentials over P. The averaging process restores translational invariance; so we can write

$$v_2(\vec{q},\vec{q}\,')=\langle v_2(\vec{q},\vec{q}\,')\rangle+\delta v_2(\vec{q},\vec{q}\,')\,, \qquad (8)$$

where the part of $\delta v_2(\vec{q},\vec{q}\,')$ coupling $|\vec{q}|\in(b^{-1}\Lambda,\,\Lambda)$ to $|\vec{q}|\in(0,\,b^{-1}\Lambda)$ is $Y(\vec{q},\vec{q}\,')$, and where

$$\langle v_2(\vec{q},\vec{q}\,')\rangle=(r+q^2)\,\delta(\vec{q}+\vec{q}\,') \qquad (9)$$

in the long-wavelength limit. In this limit, we also have

$$\langle v_4(\vec{q}_1,\vec{q}_2,\vec{q}_3,\vec{q}_4)\rangle=u\delta(\vec{q}_1+\vec{q}_2+\vec{q}_3+\vec{q}_4)\,. \qquad (10)$$

The spin renormalization coefficient ζ is chosen so that the coefficient of q^2 in Eq. (9) remains unity after each iteration

$$\zeta=b^{1+d/2-\eta/2}\,, \qquad (11)$$

where η is the usual exponent describing order-parameter correlations at the critical point. We are interested in systems which are nearly homogeneous and in developing a perturbation expansion in some parameter characterizing the small amount of disorder. $\delta v_2(\vec{q},\vec{q}\,')$ represents the deviation of $v_2(\vec{q},\vec{q}\,')$ from translational invariance and would be zero in a homogeneous system. We, therefore, expand the renormalization equation in a power series in δv_2 via the matrix equation

$$v_2^{-1}=(\langle v_2\rangle+\delta v_2)^{-1}=\langle v_2\rangle^{-1}-\langle v_2\rangle^{-1}\delta v_2\langle v_2\rangle^{-1}+\cdots\,. \qquad (12)$$

Equation (12) is expressed diagrammatically in Fig. 3. Here the single line represents $\langle v_2\rangle^{-1}$ and the $\times$ represents δv_2. Equation (12) applies either to the full matrix $v_2(\vec{q},\vec{q}\,')$ or to the submatrix $v_2^{>}(\vec{q},\vec{q}\,')$. In the latter case, of course, $\delta v_2^{>}(\vec{q},\vec{q}\,')$

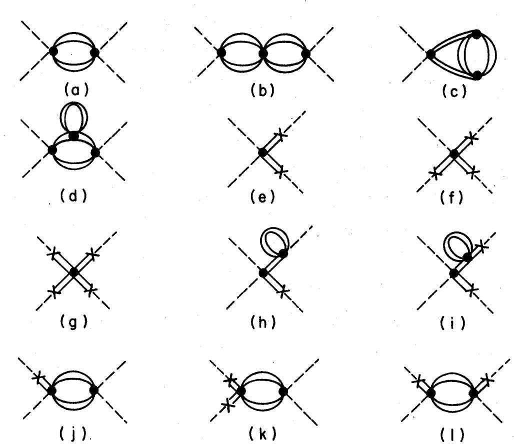

FIG. 2. Diagrams for $v_4(\vec{q}_1,\vec{q}_2,\vec{q}_3,\vec{q}_4)$ to third order in v_4.

TABLE I. Weights for diagrams in Figs. 1 and 2.

1a	$4(n+2)$	1d	$-8(n+2)$
1b	-1	13	$4(n+2)$
1c	$-32(n+2)$	1f	$-16(n+2)^2$
2a	$-4(n+8)$	2g	1
2b	$16(n^2+6n+20)$	2h	$48(n+2)$
2c	$64(5n+22)$	2i	$-48(n+2)$
2d	$32(n+2)(n+8)$	2j	$16(n+8)$
23	6	2k	$-8(n+8)$
2f	-4	2l	$-16(n+8)$

is defined only for $q,\ q' \in (b^{-1}\Lambda,\ \Lambda)$. Using the diagrams in Figs. 1–3, we can obtain recursion relations for v_2 and v_4 in terms of the small parameters δv_2 and v_4. (These equations will, of course, involve cumulants of δv_2 and v_4.) From these, we can obtain recursion relations for $\delta v_l = v_l - \langle v_l \rangle$ for $l=2, 4$. The recursion relations for the second cumulants $\langle \delta v_l\, \delta v_m \rangle$ can then be obtained by taking the product of the recursion relations for δv_l and δv_m and averaging. This process can be continued to obtain recursion relations for all cumulants.

The algorithm described above is, in general, quite complicated. Fortunately, in the case of greatest interest, of all the cumulants, only $\langle \delta v_2\, \delta v_2 \rangle$ is relevant. If the potential $v_2(\vec{x}, \vec{x}')$ dies off more rapidly than $|\vec{x} - \vec{x}'|^{-d}$ for large separation of the spatial coordinates $\vec{x}$ and $\vec{x}'$, and $\langle \delta v_2(\vec{R} + \frac{1}{2}\vec{\rho}, \vec{R} - \frac{1}{2}\vec{\rho}) \delta v_2(\vec{R}' + \frac{1}{2}\vec{\rho}', \vec{R}' - \frac{1}{2}\vec{\rho}') \rangle$ dies off more rapidly than $|\vec{R} - \vec{R}'|^{-d}$, we have

$$\langle \delta v_2(\vec{q}_1, \vec{q}_2)\, \delta v_2(\vec{q}_3, \vec{q}_4) \rangle = \Delta \delta(\vec{q}_1 + \vec{q}_2 + \vec{q}_3 + \vec{q}_4)\ , \tag{13}$$

where Δ is a constant in the long-wavelength limit proportional to $\langle \delta v_2(\vec{x}, \vec{x})\, \delta v_2(\vec{x}, \vec{x}) \rangle$. Since $v_2(\vec{x}, \vec{x})$ determines the mean-field transition temperature, Δ is thus proportional to fluctuations in the local value of the mean-field transition temperature. When there are long-range correlations in δv_2, Eq. (13) is modified. This will be discussed later.

We see from Eq. (13) that Δ has the same order of relevancy as u in the ϵ expansion. Any other cumulants will be a function of more q's and will be irrelevant just as the higher-order potentials v_6, v_8, etc. are irrelevant in the homogeneous problem.[11] This means that within the ϵ expansion, v_4 can be replaced by $\langle v_4 \rangle$ wherever it appears and averages of products of δv_2's can be expressed in terms of products of $\langle \delta v_2\, \delta v_2 \rangle$. Diagrammatically, the last statement means that the average of a string of $\times$'s is obtained by tying together two $\times$'s at a time in all possible ways. The product of four $\times$'s is shown in Fig. 4(a). Figure 4(b) does not contribute because it is irrelevant.

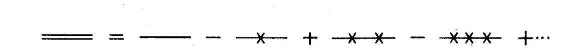

FIG. 3. Diagrammatic expansion of the inhomogeneous propagator. The single line represents the propagator $\langle v_2 \rangle^{-1}$ and the cross represents $\delta v_4(\vec{q}, \vec{q}')$.

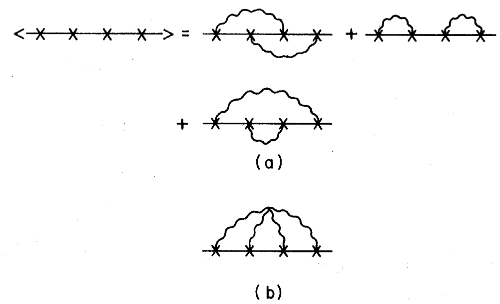

FIG. 4. Example of diagrammatic averaging of a product of crosses: (a) relevant contractions of four crosses and (b) irrelevant contraction of four crosses.

To summarize, recursion relations for the relevant variables r, u, and Δ are obtained by expanding diagrams in Figs. 1 and 2 in terms of the expansion of v_2^{-1} shown in Fig. 3. $\langle v_2 \rangle$ and $\langle v_4 \rangle$ are then calculated by tying together all products of $\times$'s in pairs and associating $\Delta \delta(\vec{q}_1 + \vec{q}_2 + \vec{q}_3 + \vec{q}_4)$ with each pair. From these, one obtains the recursion relations for r and u. Recursion relations for δv_2 are then obtained by subtracting $\langle v_2 \rangle$ from v_2. Finally the recursion relation for δv_2 is squared to obtain the recursion relation for Δ. The diagrams contributing to r, u, and Δ are shown in Figs. 5 and 6, and evaluated with appropriate weight factors in Table II. The circle represents Δ, i.e. a contraction of two $\times$'s. The mass renormalization graphs, Figs. 5(f)–5(h) and Figs. 6(f)–6(k), can be ignored as pointed out by Aharony[21] and Bruce, Droz, and Aharony.[24] (They are included in Table II to provide a useful consistency check for those who are interested.) The recursion relations for r, u, and Δ to second order in $\epsilon = 4 - d$ are thus

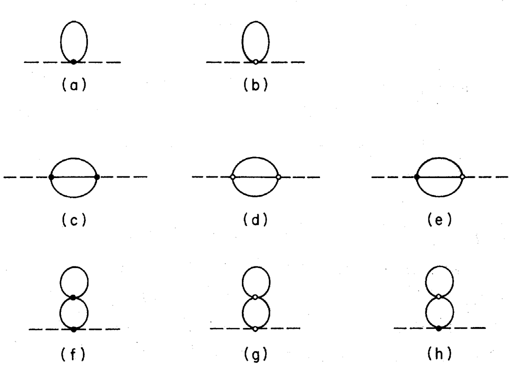

FIG. 5. Diagrams for $\langle v_2 \rangle$. The dots represent u and the open circles represent Δ.

$$r' = b^{2-\eta}\{ r + [4(n+2)u - \Delta] A(r) - [32(n+2)u^2 - 8(n+2)u\Delta + \Delta^2] B(r)\} , \tag{14}$$

$$u' = b^{\epsilon-2\eta}\{u - K_d \ln b(1+\tfrac{1}{2}\epsilon \ln b - \epsilon \ln\Lambda)[4(n+8)u^2 - 6u\Delta] + K_4^2 \ln^2 b\,[16(n^2+6n+20)u^3 - 12(n+8)u^2\Delta + 9u\Delta^2]$$
$$+ K_4^2 \ln b\,(1+\ln b)[32(5n+22)u^3 - 48(n+5)u^2\Delta + 21u\Delta^2]\} , \tag{15a}$$

$$\Delta' = b^{\epsilon-2\eta}\{\Delta - K_d \ln b(1+\tfrac{1}{2}\epsilon \ln b - \epsilon \ln\Lambda)[8(n+2)u\Delta - 4\Delta^2] + K_4^2 \ln^2 b\,[5\Delta^3 - 24(n+2)u\Delta^2 + 48(n+2)^2 u^2\Delta]$$
$$+ K_4^2 \ln b\,(1+\ln b)[11\Delta^3 - 48(n+2)u\Delta^2 + 96(n+2)u^2\Delta]\} , \tag{15b}$$

where

$$K_d^{-1} = 2^{d-1}\Pi^{d/2}\,\Gamma(d/2) ,$$
$$\eta = \tfrac{1}{4}K_d^2[32(n+2)u^2 - 8(n+2)u\Delta + \Delta^2] , \tag{16}$$

and

$$A(r) = \int_{\vec{q}}^{>} (q^2 + r)^{-1} , \tag{17a}$$

$$B(r) = \int_{\vec{q}_1}^{>}\int_{\vec{q}_2}^{>} (q_1^2 + r)^{-1}(|\vec{q}_1 + \vec{q}_2|^2 + r)^{-1}(q_2^2 + r)^{-1} , \tag{17b}$$

where $\int_{\vec{q}}^{>} = \int_{b^{-1}\Lambda}^{\Lambda} d^d q/(2\pi)^d$ and the integrals in $B(r)$ are understood to be restricted to the domain $|\vec{q}_1|$, $|\vec{q}_2|$, and $|\vec{q}_1 + \vec{q}_2| > b^{-1}\Lambda$.

Equations (15) have the following fixed points for the sharp cutoff employed here:

(i) Gaussian:

$$u^G = \Delta^G = 0 ; \tag{18}$$

(ii) Heisenberg:

$$\Delta^H = 0 , \quad u^H = \frac{\epsilon\Lambda^\epsilon}{4K_d(n+8)}\left(1 + \frac{3(3n+14)}{(n+8)^2}\,\epsilon\right) + O(\epsilon^3) ; \tag{19}$$

(iii) unphysical:

$$u^U = 0 , \quad \Delta^U = -\frac{\epsilon}{4K_d}\,\Lambda^\epsilon(1 + \tfrac{21}{32}\epsilon) + O(\epsilon^3) ; \tag{20}$$

(iv) random:

$$u^R = \frac{\epsilon\Lambda^\epsilon}{16K_d(n-1)}\left(1 + \frac{25n^2 - 248n + 64}{128(n-1)^2}\,\epsilon\right) + O(\epsilon^3) ,$$

$$\Delta^R = \frac{\epsilon\Lambda^\epsilon}{8K_d(n-1)}$$
$$\times\left((4-n) - \frac{105n^3 - 364n^2 + 992n - 256}{128(n-1)^2}\,\epsilon\right) + O(\epsilon^3) . \tag{21}$$

The fixed points are cutoff dependent. All cutoff dependence disappears from the critical exponents calculated from Eqs. (14) and (15), and these fixed-point values for u and Δ,[21,25] Eqs. (18)–(21). Note that Gaussian and unphysical fixed points are independent of n. The exponents λ_u and λ_Δ for the first three fixed points[25] are

Gaussian:

$$\lambda_u^G = \lambda_\Delta^G = \epsilon ; \tag{22}$$

Heisenberg:

$$\lambda_u^H = -\epsilon + \frac{9n+42}{(n+8)^2}\,\epsilon^2 + O(\epsilon^3) ,$$
$$\lambda_\Delta^H = \frac{4-n}{n+8}\,\epsilon - \frac{(n+2)(13n+44)}{(n+8)^3}\,\epsilon^2 + O(\epsilon^3) ; \tag{23}$$

unphysical:

$$\lambda_u^U = -\tfrac{1}{2}\epsilon + \tfrac{19}{24}\epsilon^2 ,$$
$$\lambda_\Delta^U = -\epsilon + \tfrac{23}{32}\epsilon^2 . \tag{24}$$

The stability of the random fixed point is studied by linearizing Eqs. (15) about the random fixed point and diagonalizing the 2×2 matrix of coefficients. We quote here only the results to first order in ϵ as the expressions to second order in ϵ are extremely complicated and probably uninteresting:

$$\lambda_1^R = -\epsilon + O(\epsilon^2)$$
$$\lambda_2^R = \frac{n-4}{4(n-1)}\,\epsilon + O(\epsilon^2) \tag{25}$$

The exponents η, ν, and α (calculated using $d\nu = 2 - \alpha$) for the three non-Gaussian fixed points are listed in Table III. Note that λ_Δ^H is equal to α^H/ν^H

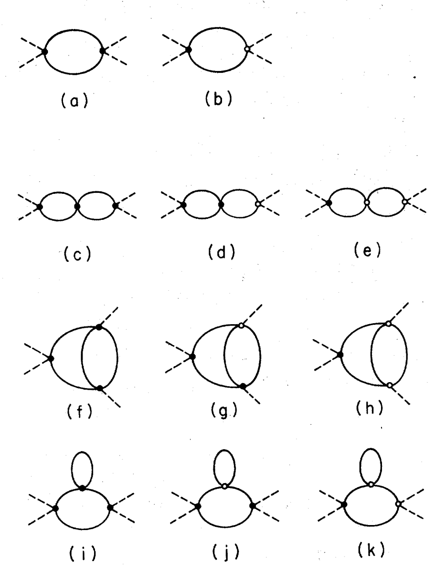

FIG. 6. Diagrams for u. The dots represent u and the open circles represent Δ. To obtain the diagrams for Δ, merely interchange the dots and open circles.

TABLE II. Weight factors for diagrams in Figs. 5 and 6. The column labeled 6 refers to diagrams for u from Fig. 6 and the column labeled 6′ refers to diagrams for Δ from Fig. 6, with circles and dots interchanged.

	5	6	6′
a	$4(n+2)$	$-4(n+8)$	4
b	-1	$+6$	$-8(n+2)$
c	$-32(n+2)$	$16(n^2+6n+20)$	5
d	-1	$-12(n+8)$	$-24(n+2)$
e	$8(n+2)$	9	$48(n+2)^2$
f	$-16(n+2)^2$	$64(5n+22)$	22
g	-1	$-96(n+5)$	$-96(n+2)$
h	$8(n+2)$	42	$192(n+2)$
i		$32(n+2)(n+8)$	8
j		$-8(7n+20)$	$48(n+2)$
k		12	$64(n+2)^2$

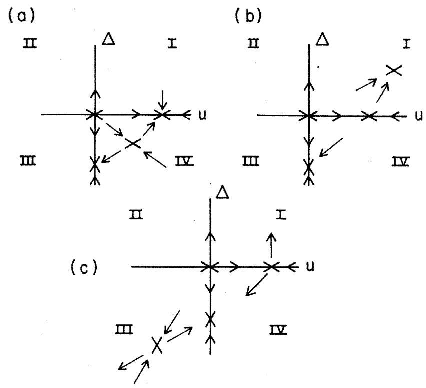

FIG. 7. Flow diagrams for the four fixed points represented by crosses. (a) $n>n_c$, the Heisenberg fixed point is stable; (b) $1<n<n_c$, the random fixed point is in the physical quadrant and is stable; (c) $n<1$, the random fixed point has moved into the unphysical third quadrant and there is no stable fixed point in the physical quadrant. For $n=1$ the random fixed point is at infinity.

to second order in ϵ. Thus, the crossover exponent to the random fixed point is $\varphi=\alpha^H$. The Heisenberg fixed point is stable as long as $\lambda_\Delta^H<0$ (λ_u^H is always negative). Thus, it becomes unstable at a critical value n_c of the spin dimensionality determined by $\alpha^H(n_c)=0$. To lowest order in ϵ, $n_c=4$. For $n<0$ to first order in ϵ.

The flow diagrams for the four fixed points are shown in Fig. 7. In giving physical interpretations to these diagrams, it is important to remember that Δ is by definition a positive definite quantity. In this paper, we are only interested in phase transitions that can be second order in the pure systems; so we will restrict u to be positive. The physical region in Fig. 7 is, therefore, quadrant I. The flow lines form a ridge along $u=0$ and $\Delta=0$ making it impossible for a point that starts in quadrant I to flow after renormalization into the other quadrants. Thus, the unphysical fixed point is inaccessible to physical systems.

We now consider the relative stability of the Heisenberg and random fixed points. For $n>n_c$, the Heisenberg fixed point is stable and the random fixed point is in quadrant IV. Thus, when α^H is negative, there is a sharp phase transition in the random system with exponents of the pure system. When $1<n<n_c$, the random fixed point moves into the first quadrant and becomes stable whereas the Heisenberg fixed point is unstable. Thus, in this regime there is a sharp phase transition with renormalized but not Fisher-renormalized exponents. Note that α^R is negative in this regime in agreement with heuristic arguments.[6] When $n=1$, one can return to Eqs. (14)–(16) and recalculate the fixed point. It is easy to verify that there is no solution corresponding to the random fixed point (both u^* and Δ^* nonzero). This is consistent with allowing the random fixed point to go to infinity as n tends to unity as indicated by Eq. (21). [It is possible that Eq. (21) is the small-ϵ expansion of a function of the form $g=(\epsilon/(n-1))f(\epsilon/(n-1)^2)$ that does not become infinite as n tends to 1. In this case the $n=1$ and the $n=1^+$ values of the fixed points would differ. We have no way of verifying this behavior at the moment.] Thus, there is no stable fixed point in the physical quadrant for $n=1$. We will discuss this further shortly.

We now return to the question of what happens when there are long-range correlations in the random potential. Consider a partition of the d-

TABLE III. The exponents ν, η, and α for the three non-Gaussian fixed points.

	Heisenberg	Random	Unphysical
ν	$\frac{1}{2}+\frac{n+2}{4(n+8)}\epsilon+\frac{(n+2)(n^2+23n+60)}{8(n+8)^3}\epsilon^2$	$\frac{1}{2}+\frac{3n}{32(n-1)}\epsilon+\frac{n(127n^2-572n-32)}{4096(n-1)^3}\epsilon^2$	$\frac{1}{2}+\frac{1}{16}\epsilon-\frac{27}{512}\epsilon^2$
η	$\frac{n+2}{2(n+8)}\epsilon^2$	$\frac{(5n-8)n}{256(n-1)^2}\epsilon^2$	$\frac{1}{64}\epsilon^2$
α	$\frac{4-n}{2(n+8)}\epsilon-\frac{(n+2)(n^2+30n+56)}{4(n+8)^3}\epsilon^2$	$\frac{n-4}{8(n-1)}\epsilon-\frac{n(31n^2-380n-128)}{1024(n-1)^3}\epsilon^2$	$\frac{1}{4}\epsilon+\frac{35}{128}\epsilon^2$

dimensional space S_d into p- and $(d-p)$-dimensional spaces, S_p and S_{d-p}. Let $\vec{R}=(\vec{R}',\vec{R}'')$ be any point in S_d with $\vec{R}'\in S_p$ and $\vec{R}''\in S_{d-p}$. If $v_2(\vec{R}+\vec{x}/2,\vec{R}-\vec{x}/2)$ is independent of $\vec{R}$ for $\vec{R}\in S_p$, then

$$\langle \delta v_2(\vec{q}_1,\vec{q}_2)\,\delta v_2(\vec{q}_3,\vec{q}_4)\rangle = A\delta^p(\vec{q}_1{}'+\vec{q}_2{}')\,\delta^d(\vec{q}_1+\vec{q}_2+\vec{q}_3+\vec{q}_4) \tag{26}$$

where δ^p and δ^d are, respectively, p- and d-dimensional δ functions and A is a constant in the long-wavelength limit. It is easy to verify that under renormalization, A will grow as

$$A' \sim b^{\epsilon+p-2\eta} A \tag{27}$$

to lowest order in u. Thus, within the ϵ expansion there are only the Gaussian and Heisenberg fixed points, and both are unstable with respect to long-range disorder of the above type. This is the same type of disorder considered by McCoy and Wu.[7] Since they found a smeared transition, it is appealing to identify the instability of the Heisenberg fixed point with respect to disorder with long-range correlations as a signal of a smeared transition.

In this context, it also seems reasonable to identify the instability at $n=1$ with a smeared transition. In general, if there are no stable fixed points accessible to an initial set of potentials, independent calculations are needed to identify the nature of any transition that occurs. In hypercubic systems[22] and superconductors,[23] there exists independent evidence that the inaccessibility of stable fixed points corresponds to a first-order transition. The starting point of the Grinstein-Luther approach to the random problem is an effective translationally invariant Hamiltonian for mn- component spins. Any phase transitions predicted by this Hamiltonian should, therefore, be those of a homogeneous system. In particular, if there are no stable fixed points, there should be a first-order transition. Grinstein and Luther then argue that the $n=1$ instability could correspond to a first-order transition.[17] We feel that there is no compelling reason for this to be so since the analytic continuation to $m=0$ could lead to behavior not found for physical values of m. The question of what happens at $n=1$, therefore remains open, though we feel we have presented reasons why there might be a smeared transition.

The dividing line between the stability of the random and the Heisenberg fixed point within the ϵ expansion is $\alpha^H=0$. In three dimensions, the best experimental value for $\alpha^H(n=2)$ in three dimensions is negative.[26] It is, therefore, unlikely that the new random fixed point will be experimentally observable. Even if $\alpha^H(n=2)$ is positive, the crossover region from the Heisenberg to the random fixed point is of order $\Delta^{1/\alpha}$, which is extremely small if Δ is less than unity. The $n=1$ instability may be observable in three dimensions, and it would be very interesting to determine experimentally whether the transition is first or second order. The instability in systems with long-range correlations in the disorder has a larger crossover exponent $(\varphi=\lambda_\Delta \nu)$ than the $n=1$ short-range instability and should be experimentally easier to detect. It would, therefore, be of some interest to find a material with such long-range correlations in the disorder.

We close with some observations about the results we obtained in Ref. 13. There, we applied the Niemeyer–Van Leeuwen cluster expansion to a two-dimensional random Ising model and found the Ising fixed point to be stable with respect to short-range randomness. To second order in ϵ, we find in this paper that the crossover exponent to the random fixed point is equal to α^H. If this relation persists down to two dimensions, it would be in contradiction with the results of Ref. 13 since $\alpha=0$ for the two-dimensional Ising model. There are two possibilities. The results of Ref. 12 could be wrong because we did not treat large enough clusters. This is a very reasonable possibility since disorder would be a marginally relevant variable if the relation $\varphi=\alpha^H$ persists, and presumably very large clusters would be needed to determine that φ equals zero. Another alternative is that the relation $\varphi=\alpha^H$ breaks down. This also is a reasonable possibility. The heuristic argument that predicts $\varphi=\alpha^H$ assumes that the disorder can be characterized completely by a fluctuating local transition temperature. Between three and four dimensions within the ϵ expansion the disorder is in fact characterized by a single variable Δ, which can be identified with fluctuations in the local transition temperature. Below three dimensions, other disorder variables become relevant, and a more complicated heuristic argument may be needed. In view of these uncertainties, it would be interesting to see if any exact calculation of the size of the crossover regime similar to that for an annealed system[6] could be made for a quenched system. This question is currently under investigation.

The author is indebted to A. B. Harris for bringing this problem to his attention, for checking the first-order calculations presented in this paper, and for providing constant information about random systems. The author would also like to thank K. G. Wilson for pointing out the structure of the renormalization group in terms of probability functions, G. Grinstein for communicating results prior to publication, and A. Aharony for clarifying the relation between the crossover exponent and the specific-heat exponent.

Note added in proof. Recently Aharony[27] has presented a general argument that the crossover exponent φ to the random fixed point is α^H in all dimensions. Thus, the two-dimensional cluster calculation of Ref. 13 is probably incorrect owing to insufficient cluster size.

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*This work supported in part by the National Science Foundation.

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