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Crossover scaling functions and renormalization-group trajectory integrals

David R. Nelson
Baker Laboratory and Materials Science Center, Ithaca, New York 14850
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By expressing the free energy as a line integral along a renormalization-group trajectory, a technique for calculating the crossover scaling functions which describe tricritical systems, anisotropic spin systems, etc., is developed. This formalism is applied to the model recursion relations of Riedel and Wegner, which simulate crossover behavior. A simple mechanism for the breakdown of dimensionality dependent hyperscaling relationships emerges from the analysis. The specific-heat crossover scaling function describing crossovers from Gaussian to Heisenberg critical behavior is constructed to first order in $\epsilon = 4 - d$.

I. INTRODUCTION

Phenomenological scaling ideas[1] have been very successful in describing the thermodynamic singularities associated with critical points in ferromagnets and fluids. Further understanding of these concepts has come from explicit calculations of scaling functions[2] and associated critical exponents[3] within the renormalization-group ϵ expansion.

Riedel and Wegner[4] have formulated a scaling theory which applies when more complicated kinds of critical behavior are present, incorporating the crossover from one type of critical behavior to another into the scaling picture. Their scaling hypothesis has been useful in describing behavior near critical points of systems with tricritical points,[5] systems with anisotropic interactions in spin space,[4] etc. The scaling approach can also be used to treat systems exhibiting spin-flop transitions.[6]

When these more complicated systems are treated by renormalization-group techniques,[7-10] they are found to be characterized by a multiplicity of fixed points. Instead of a single fixed point of interest, as is believed to be the case for a simple ferromagnet,[3] one must deal simultaneously with at least two fixed points, characterized by distinct critical exponents. We present here a renormalization-group formalism for calculating (by ϵ expansions or other means) the *crossover* scaling functions associated with such situations.

The phase diagrams for a metamagnet (with a tricritical point) and for a uniaxial antiferromagnet (with a spin-flop point) are indicated in Fig. 1. Both these systems have critical lines as well as singled-out multicritical points in the (T,H) plane. The critical exponents describing the various thermodynamic singularities are expected to be different on the critical lines from ones at the isolated multicritical points. We will sketch the phenomenological theory applicable to the tricritical system,[5] although one can describe the spin-flop system in a similar fashion.[6] The extended scaling hypothesis for crossover systems, due to Fisher and Jasnow,[11] will be employed.

Deviations from the tricritical point (T_t, H_t) are defined through the reduced variables

$$t=(T-T_t)/T_t\,,\quad h=c_t(T-T_t)/T_t+c_h(H-H_t)/H_t\,, \tag{1.1}$$

where the relation $h=0$ represents the tangent to the critical line at the tricritical point. The extended crossover scaling ansatz[11] is then that the free-energy density behaves asymptotically as

$$F(t,h)\approx t^{2-\alpha}\Phi(h/t^{\phi}) \tag{1.2}$$

near the tricritical point, i.e., as $t,h\to 0$. The exponent ϕ is the crossover exponent, while α is the primary specific-heat exponent appropriate to the tricritical point. The essential difference from the usual scaling formulations[1] is that $\Phi(z)$ is assumed to have a singularity at $\dot{z}$ of the form

$$\Phi(z)=A+B(z-\dot{z})+C|z-\dot{z}|^{2-\dot{\alpha}} \quad \text{as } z\to z^{*}-\,. \tag{1.3}$$

The exponent $\dot{\alpha}$ is the secondary or critical-line specific-heat exponent. The equation of the critical line itself is then given asymptotically by[11]

$$h\approx\dot{z}t^{\phi}\,. \tag{1.4}$$

This paper will discuss the calculation of the crossover scaling function $\Phi(z)$ by renormalization-group methods. Riedel and Wegner[12] have applied a renormalization-group matching procedure to certain rather *ad hoc* "recursion-relation models" and have shown how to calculate, in principle, the analogous crossover scaling function for the susceptibility. However, a straightforward application of their methods leads to difficulties in a direct calculation of the free-energy scaling function. This point will be dis-

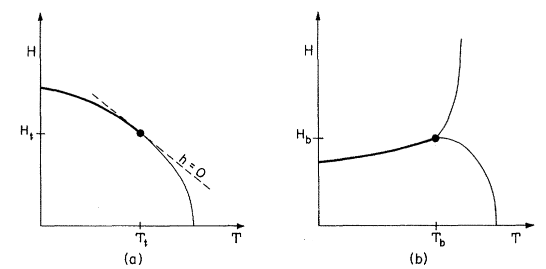

FIG. 1. (a) Schematic phase diagram for a metamagnet in a uniform magnetic field with a tricritical point (T_t, H_t) which marks the joining of the bold line of first-order transitions to a line of second-order transitions. (b) Schematic phase diagram for a uniaxial antiferromagnet in a uniform magnetic field with a bicritical point (T_b, H_b). Field is applied along the direction of anisotropy. Bold line represents the locus of first-order spin-flop transitions.

cussed further in Sec. II. Direct ϵ expansions for *any* crossover scaling functions have not, to our knowledge, been presented before.

We introduce here a formalism which expresses the free energy as a trajectory integral along a renormalization-group flow line. The kernel of the trajectory integral is related to the spin-independent or "constant" term generated with each renormalization-group iteration. Singularities in the crossover scaling functions are seen to arise from the character of the Hamiltonian flows or trajectories, rather then from singularities in the kernel itself. We believe this technique represents a practical method of calculating scaling functions for a variety of crossover problems. A concrete application is given in Sec. V.

Wilson discussed how the free energy could be calculated by summing up the spin-independent terms in one of his first papers on the renormalization group.[3] These terms have been discussed in detail by Wegner,[13] who showed how they could produce the logarithmic singularities in the two-dimensional Ising model. The spin-independent or constant terms resulting from renormalization-group iterations were explicitly summed up for the one-dimensional Ising model by Nelson and Fisher,[14] and have been exploited by Nauenberg and Nienhuis[15] to produce an approximate equation of state for the two-dimensional Ising model. Part of the formalism used here was discussed in a somewhat different form by Rudnick[16] in an alternative approach to renormalization-group calculations.

At this point we summarize the developments presented below. In Sec. II, a formalism is derived which develops the free energy as a trajectory integral along a renormalization-group flow line. In Sec. III, we analyze in detail the application of the formalism to the Gaussian model. Although this model is trivially soluble by various traditional methods,[17] its analysis gives insight into the machinery developed in Sec. II. Two more complicated "recursion-relation models" introduced by Riedel and Wegner[12] are treated in Sec. IV. These models postulate recursion relations which simulate crossover behavior. The analysis produces crossover scaling functions for "specific-heat like" thermodynamic quantities, and suplements Riedel and Wegner's analysis of the susceptibility. A natural mechanism for the breakdown of hyperscaling relationships on crossing a borderline dimensionality appears from a study of the models. Finally, in Sec. V, the utility of this approach for producing an ϵ expansion for crossover scaling functions near four dimensions is discussed. The crossover from the Gaussian to the Heisenberg fixed point induced by the fourth-order spin term is analyzed in detail.

II. FREE ENERGY EXPRESSED AS A TRAJECTORY INTEGRAL

A. Definitions and notation

Although the following discussion can be presented more generally, it is convenient to have a specific set of Hamiltonians in mind. We assume, for simplicity, that the Hamiltonian and all renormalized Hamiltonians depend on only two parameters, $\mathcal{H} = \mathcal{H}(t, h)$, and postulate a set of continuous renormalization-group trajectories in a two-dimensional parameter space, the (t, h) plane.

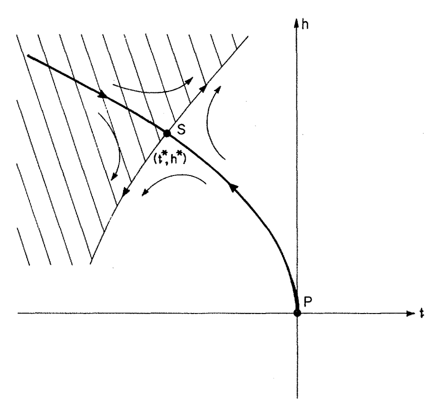

FIG. 2. Hamiltonian flows in (t, h) space. Upon application of the renormalization-group transformation, Hamiltonians cross over from the primary fixed point P to the secondary fixed point S. The bold line indicating the trajectory which connects the fixed points is the separatrix. Note that the principal flow lines for the secondary fixed point need not necessarily be at right angles.

These variables could be those defined in Sec. I for a tricritical system. Within this space there are two nontrivial fixed points, a *primary* fixed point P at the origin and a *secondary* fixed point S located at (t^*, h^*), as in Fig. 2.

If a renormalization-group transformation $\mathfrak{R}_b$ is applied to $\mathcal{H}$, involving a spatial rescaling factor b, recursion relations are generated for t and h. By taking appropriate linear combinations these variables may be chosen so that they behave as

$$\mathfrak{R}_b\begin{bmatrix} t \\ h \end{bmatrix} \equiv \begin{bmatrix} t' \\ h' \end{bmatrix} \approx \begin{bmatrix} b^{\lambda_t} t \\ b^{\lambda_h} h \end{bmatrix} \tag{2.1}$$

under the transformation $\mathfrak{R}_b$ near P. Different linear combinations of t and h, say

$$\bar{t} = c_{11}t + c_{12}h\ , \quad \bar{h} = c_{21}t + c_{22}h\ , \tag{2.2}$$

will satisfy a similar relationship near the secondary fixed point S, namely,

$$\mathfrak{R}_b\begin{bmatrix} \bar{t} - t^* \\ \bar{h} - h^* \end{bmatrix} = \begin{bmatrix} b^{\dot{\lambda}_t}(\bar{t} - t^*) \\ b^{\dot{\lambda}_h}(\bar{h} - h^*) \end{bmatrix} . \tag{2.3}$$

The exponents λ_t, λ_h, $\dot{\lambda}_t$, and $\dot{\lambda}_h$ are related to the critical exponents for the primary and secondary critical behaviors, respectively. For example, if t and $\bar{t}$ are taken to be temperature-like variables, the primary and secondary correlation-length exponents ν and $\dot{\nu}$ are given by

$$\nu = 1/\lambda_t\ , \quad \dot{\nu} = 1/\dot{\lambda}_t\ . \tag{2.4}$$

We will confirm that the crossover exponent, introduced in Eq. (1.2), is given by $\phi = \lambda_h/\lambda_t$.[7]

To obtain the flows illustrated in Fig. 2, the exponents λ_t, λ_h, and $\dot{\lambda}_t$ are required to be positive, with $\dot{\lambda}_h$ negative. There is a singled-out trajectory, the separatrix, which connects the fixed points S and P and is the only trajectory leading into the secondary fixed point. The state of affairs in Fig. 2 represents the situation encountered in virtually every renormalization-group calculation involving crossover[18] and forms the background for the model recursion-relation discussion by Riedel and Wegner.[12]

Further variables which are thermodynamically irrelevant at both fixed points may be introduced, but they turn out not to change the overall picture in an essential way.[19] We will briefly indicate how they can be treated in Sec. IID.

One further idea is needed before analyzing the transformation properties of the free energy under $\mathfrak{R}_b$, namely, Wegner's concept[13] of *nonlinear* scaling fields. With the variables t and h are associated nonlinear functions

$$\begin{aligned} g_t(t,h) &= t + O(t^2, h^2, ht)\ , \\ g_h(t,h) &= h + O(t^2, h^2, ht)\ , \end{aligned} \tag{2.5}$$

as $t, h \to 0$ such that the approximate relation (2.1) becomes exact, that is

$$\mathfrak{R}_b\begin{bmatrix} g_t \\ g_h \end{bmatrix} = \begin{bmatrix} b^{\lambda_t} g_t \\ b^{\lambda_h} g_h \end{bmatrix} . \tag{2.6}$$

Note that the normalizing relations (2.5) indicate that g_t and g_h reduce to t and h near the primary fixed point. The general existence and uniqueness of scaling fields is a complicated problem. Wegner showed how to calculate them as (generalized) power series about, say, the primary fixed point. However, the question of convergence was left quite open. Furthermore, we will need the complete expression for the nonlinear field which, it must be stressed, embraces also the secondary fixed point. Nelson and Fisher[14] demonstrated explicitly that the *nonlinear* scaling fields are physically *nonunique*; i.e., they depend on the choice of $\mathfrak{R}$. However, the nonlinear scaling fields can be calculated explicitly for the various models considered here. Similar scaling fields can be defined for the secondary fixed point, but they will not be needed.

B. Trajectory integrals and the free energy

A renormalization-group transformation $\mathfrak{R}_b$ relates the free energy at a point in parameter space (g_t, g_h) to the free energy at a transformed point according to[3,13]

$$F(g_t, g_h) = b^{-d}F(b^{\lambda_t}g_t,\ b^{\lambda_h}g_h) + b^{-d}G(g_t, g_h;\ b)\ . \tag{2.7}$$

The function $G(g_t, g_h; b)$ represents a spin-independent or "constant" term generated by performing the partial trace operation associated with the transformation $\mathfrak{R}_b$. For small b, say, $b = 2$, $G(x, y; b)$ is expected to be regular in x and y for x and y small, and the thermodynamic singularities associated with $F(x, y)$ should lie in the first term of (2.7). However, as $b \to \infty$, all degrees of freedom are integrated out by $\mathfrak{R}_b$, and G essentially becomes the logarithm of the partition function. Thus singularities are effectively transfered from the first term to the second term of (2.7) as $\mathfrak{R}_b$ acts repeatedly.

Iterating the transformation $\mathfrak{R}_b$ n times, we obtain

$$F(g_b, g_h) = b^{-dn}F(b^{n\lambda_t}g_t,\ b^{n\lambda_h}g_h) + \sum_{n'=1}^{n} b^{-dn'}G(b^{\lambda_h n'}g_t;\ b^{\lambda_h n'}g_h;\ b)\ . \tag{2.8}$$

It is convenient to suppose that one can consider a sequence of infinitesimal transformations corresponding, say, to integrating out modes associated with a thin outer shell in momentum space.[21] Then, writing $b=e^{\delta}$ and $l=n\delta$ and taking the limit $\delta\to 0$, we find

$$F(g_t,g_h)=e^{-dl}F(e^{\lambda_t l}g_t,e^{\lambda_h l}g_h)+\int_0^l e^{-dl'}G_0(e^{\lambda_t l'}g_t,e^{\lambda_h l'}g_h)\,dl', \tag{2.9}$$

where

$$G_0(g_t,g_h)=\left.\frac{\partial G(g_t,g_h;b)}{\partial b}\right|_{b=1}. \tag{2.10}$$

We note that $G(x,y;b=1)=0$. Equations analogous to (2.9) for other thermodynamic quantities can be obtained by differentiation.

If the integral in (2.9) could be neglected, the free energy could then be calculated by a "matching" procedure. Specifically, the recursion relations would be integrated out of the critical region to a regime one could then "match on to" a known noncritical free energy through

$$F(g_t,g_h)\simeq e^{-dl}F(e^{\lambda_t l}g_t,e^{\lambda_h l}g_h). \tag{2.11}$$

This is the technique employed by Riedel and Wegner[12] to calculate model susceptibility scaling functions. It turns out that this matching procedure can be applied to the susceptibility because the spin-independent terms generated by $\mathfrak{R}_b$ cancel out during each iteration, and the integral in (2.5) can be neglected. However, this is only true of *linear* renormalization groups, that is, groups which map the spin-spin correlation function onto itself.[22]

In general, we expect the integral in (2.9) to be as singular as the second term. To see this simply, first set $g_h=0$. Then

$$F(g_t)=e^{-dl}F(e^{\lambda_t l}g_t)+\int_0^l e^{-dl'}G_0(e^{\lambda_t l'}g_t)\,dl'. \tag{2.12}$$

The matching procedure consists of integrating this equation out to $\hat{l}$, where $e^{\lambda_t \hat{l}}g_t$ equals, say, unity. Then, setting $e^{\lambda_t \hat{l}}g_t=e^{\bar{l}}$ in the integral, we find

$$F(g_t)=g_t^{d/\lambda_t}F(1)+\lambda_t^{-1}g_t^{d/\lambda_t}\int_{\ln g_t}^{\hat{l}} e^{-d\bar{l}/\lambda_t}G_0(e^{\bar{l}})\,d\bar{l}. \tag{2.13}$$

If the integral behaves regularly at the lower limit it is clear that it contributes a term just as singular as the first term generated by matching. A discussion of how to analyze integrals of this sort is given in Sec. IIC.

The matching procedure can be bypassed if we let $l\to\infty$ in (2.9). Provided

$$\lim_{l\to\infty}e^{-dl}F(e^{\lambda_t l}g_t,e^{\lambda_h l}g_h)=0, \tag{2.14}$$

the free energy is then just given by

$$F(g_t,g_h)=\int_0^{\infty}e^{-dl}G_0(e^{\lambda_t l}g_t,e^{\lambda_h l}g_h)\,dl. \tag{2.15}$$

This can be thought of as a line integral along a renormalization-group trajectory, determined by $g_t(t,h)$ and $g_h(t,h)$ and parametrized by l. We will assume (2.14) in our discussion here, since it amounts to the assertion that the repeated partial traces associated with $\mathfrak{R}_b$ will eventually generate the partition function. However, if there is some macroscopic occupation of long-wavelength modes (as when an ordering field or spontaneous magnetization are present), one cannot expect (2.14) to hold. This will be demonstrated explicitly for the Gaussian model in Sec. III.

The expression of the free energy as a trajectory integral has certain advantages. The trajectories above and below the separatrix in Fig. 2 exhibit radically different behavior, which is amplified as $l\to\infty$. One might thus hope that singularities across the separatrix stem from different Hamiltonian flows, rather than from singularities in the kernel $G_0(g_t,g_h)$ itself. Model calculations in Sec. IV indicate that this is indeed the case.

C. Crossover scaling functions

From the trajectory integral representation of the free energy, a formal expression for the crossover scaling function $\Phi(z)$ can be constructed. First, however, the problem of splitting the free energy into singular and regular parts will be considered. For simplicity we consider first the case of one variable only by setting $g_h=0$.

Passing to the limit $l\to\infty$ in (2.12) and assuming that the relation (2.14) holds, one obtains

$$F(g_t)=\int_0^{\infty}e^{-dl'}G_0(e^{\lambda_t l'}g_t)\,dl'. \tag{2.16}$$

Upon changing variables, as in the derivation of (2.13), this expression reduces to

$$F(g_t)=g_t^{d/\lambda_t}I(g_t), \tag{2.17}$$

where

$$I(g_t)=\frac{1}{\lambda_t}\int_{\ln g_t}^{\infty}e^{-dl/\lambda_t}G_0(e^l)\,dl. \tag{2.18}$$

If the lower limit in (2.16) could be extended to $-\infty$, the function $I(g_t)$ would be asymptotically independent of g_t and (2.15) would then represent the usual form for the free energy near a critical point with $d/\lambda_t=2-\alpha$.

It is easily seen that

$$\frac{\partial I}{\partial g}=\frac{-1}{\lambda_t}g_t^{-d/\lambda_t-1}G_0(g_t). \tag{2.19}$$

Assuming that $G_0(x)$ has the expansion

$$G_0(x) \approx \sum_{n=0}^{\infty} a_n x^n \tag{2.20}$$

about $x=0$, we find

$$\frac{\partial I}{\partial g_t} = -\frac{1}{\lambda_t}\sum_{n=0}^{\infty} a_n g_t^{-d/\lambda_t - 1 + n}, \tag{2.21}$$

and then integrate back up to obtain the formal result

$$I(g_t) = A_0 - \frac{1}{\lambda_t}\sum_{n=0}^{\infty} \frac{a_n g_t^{n-d/\lambda_t}}{n - d/\lambda_t}, \tag{2.22}$$

where A_0 is a constant of the integration. Hence we can see that

$$F = A_0 g_t^{d/\lambda_t} + F_{\text{reg}}(g_t), \tag{2.23}$$

where

$$F_{\text{reg}}(x) = -\frac{1}{\lambda_t}\sum_{n=0}^{\infty} \frac{a_n x^n}{n - d/\lambda_t}. \tag{2.24}$$

The existence and character of the expansion (2.20) can be determined case by case for the various models studied. F_{reg} will play a role above the dimensionalities $d=4$ for critical points and $d=3$ for tricritical points in breaking hyperscaling.[20] The terms $n - d/\lambda_t$ in the denominators may conspire to give logarithms.

One may proceed in a similar fashion to obtain an expression for the crossover scaling function. A change of variables applied to (2.5) gives

$$F(g_t, g_h) = g_t^{d/\lambda_t}\Phi(g_t, g_h/g_t^{\phi}), \tag{2.25}$$

where

$$\Phi(g_t, z) = \frac{1}{\lambda_t}\int_{\ln g_t}^{\infty} e^{-dl/\lambda_t} G_0(e^l, e^{\phi l} z)\, dl \tag{2.26}$$

and $\phi = \lambda_h/\lambda_t$. Only the crossover variable $z = g_h/g_t^{\phi}$ occurs in the integrand, with a residual g_t dependence in the lower limits of integration. As in the case of only one variable, we may calculate

$$\left(\frac{\partial \Phi}{\partial g_t}\right)_z = -\frac{1}{\lambda_t} g_t^{-d/\lambda_t - 1} G_0(g_t, g_t^{\phi} z). \tag{2.27}$$

Again assuming G_0 is expandable in its arguments about the primary fixed point as

$$G_0(x, y) = \sum_{n,m=0}^{\infty} a_{nm} x^n y^m, \tag{2.28}$$

we obtain

$$\left(\frac{\partial \Phi}{\partial g_t}\right)_z = -\frac{1}{\lambda_t}\sum_{n,m=0}^{\infty} a_{nm} g_t^{n+\phi m - d/\lambda_t - 1} z^m. \tag{2.29}$$

The formal results follow

$$\Phi(g_t, z) = \Phi_0(z) - \frac{1}{\lambda_t}\sum_{n,m=0}^{\infty} \frac{a_{nm}}{n + \phi m - d/\lambda_t} g_t^{n-d/\lambda_t} g_h^m, \tag{2.30}$$

that is,

$$F(g_t, g_h) = g_t^{d/\lambda_t}\Phi_0(z) + F_{\text{reg}}(g_t, g_h), \tag{2.31}$$

where

$$F_{\text{reg}}(x, y) = -\frac{1}{\lambda_t}\sum_{n,m=0}^{\infty} \frac{a_{nm}}{n + \phi m - d/\lambda_t} x^n y^m. \tag{2.32}$$

The crossover scaling function is evidently $\Phi_0(z)$, which can now be expressed formally as

$$\Phi_0(z) = \Phi(g_t, z) - g_t^{-d/\lambda_t} F_{\text{reg}}(g_t, z). \tag{2.33}$$

Since the right-hand side must be independent of g_t, we will evaluate (2.33) in the limit $g_t \to 0$. Considering the expression

$$g_t^{-d/\lambda_t} F_{\text{reg}}(g_t, z) = -\frac{1}{\lambda_t}\sum_{n,m=0}^{\infty} \frac{a_{nm}}{n + \phi m - d/\lambda_t} g_t^{n+\phi m - d/\lambda_t} z^m, \tag{2.34}$$

we see that the limit is straightforward except for those pairs (n, m) in a set $\mathfrak{N}$ defined by the condition

$$n + \phi m < d/\lambda_t. \tag{2.35}$$

Rewriting the offending terms as

$$\frac{1}{\lambda_t}\sum_{(n,m)\in\mathfrak{N}} a_{nm} z^m \int_{\ln g_t}^{\infty} e^{-dl/\lambda_t + nl + \phi m l}\, dl, \tag{2.36}$$

(2.31) becomes

$$\Phi_0(z) = \lim_{g_t \to 0} \frac{1}{\lambda_t}\int_{\ln g_t}^{\infty} e^{-dl/\lambda_t} \times \left(G_0(e^l, e^{\phi l} z) - \sum_{(n,m)\in\mathfrak{N}} a_{nm} z^m e^{nl + \phi m l}\right) dl, \tag{2.37}$$

or, taking the limit explicitly,

$$\Phi_0(z) = \frac{1}{\lambda_t}\int_{-\infty}^{\infty} e^{-dl/\lambda_t} \times \left(G_0(e^l, e^{\phi l} z) - \sum_{(n,m)\in\mathfrak{N}} a_{nm} z^m e^{nl + \phi m l}\right) dl. \tag{2.38}$$

In practice the set $\mathfrak{N}$ will normally contain only three or four elements and represents precisely those subtractions needed to yield a convergent extension of the lower limit in (2.24) to $-\infty$.

D. Kernel of the trajectory integral for a momentum-shell-integration renormalization group

Although (2.36) represents a formal expression for the crossover scaling function, it is not particularly useful without knowledge of the kernel function $G_0(e^l, e^{\phi l}z)$. Fortunately, this kernel has a very simple form for the usual continuous spin-Hamiltonian renormalization groups already extensively studied via the ϵ expansion.[3]

Following Wegner and Houghton,[21] we consider a general Hamiltonian expanded in N Fourier components[5] of the continuous spins

$$H_0 = Nv_0 + \frac{1}{2}\sum_k v_2(k) S_k S_{-k} + \frac{1}{4!} N \sum_{k_1\cdots k_4} v_4(k_1,k_2,k_3,k_4) \times S_{k_1}S_{k_2}S_{k_3}S_{k_4}\delta_{k_1+k_2+k_3+k_4} + \cdots . \tag{2.39}$$

The momentum summations run over a spherical Brillouin zone of unit radius. In a perturbation theory in v_4,[3] the inverse propagator will be $v_2(k)$, which is taken to be spherically symmetric. Wegner and Houghton[21] constructed the differential generator of the renormalization group associated with integrating out the spin components in an infinitesimal shell of momentum space, $e^{-\delta} < |k| < 1$. Repeated iterations as in Sec. II B then produce the transformation for finite $b \equiv e^l$. The spin-independent part of the Hamiltonian (per unit spin) is v_0, and it is not hard to show that $\lim_{l\to\infty} e^{-dl} v_0(l)$ is just the free energy per spin F, provided the condition (2.14) holds. The renormalization-group differential equation for $v_0(l)$ is[21]

$$\frac{\partial v_0(l)}{\partial l} = dv_0(l) - \tfrac{1}{2}[2-\eta(l)] + \tfrac{1}{2} d \ln v_2(l,k)\big|_{k=1} . \tag{2.40}$$

The function $\eta(l)$ arises from a spin rescaling used to keep the coefficient of k^2 in $v_2(l,k)$ constant.[3] It will in general be a specified function of l, but must reduce to the correct critical exponent η at all fixed points.

On integrating (2.40) from $l=0$ to $l=\infty$, we obtain

$$\lim_{l\to\infty} e^{-dl} v_0(l) = v_0(0) + \int_0^\infty e^{-dl}\{-\tfrac{1}{2}[1-\eta(l)] + \tfrac{1}{2} d \ln v_2(l,1)\}\, dl . \tag{2.41}$$

The quantity $v_0(0)$ is an initial constant which we choose to be $\frac{1}{2}d$ for convenience. The free energy per spin is then

$$F = \int_0^\infty e^{-dl}[\tfrac{1}{2}\eta(l) + \tfrac{1}{2} d \ln v_2(l,1)]\, dl , \tag{2.42}$$

provided (2.14) holds. Comparison with (2.15) yields the identification

$$G_0(l) = \tfrac{1}{2}\eta(l) + \tfrac{1}{2} d \ln v_2(l,1) . \tag{2.43}$$

Note that $v_2(l,1)$, although independent of $\vec{k}$, is an l-dependent quantity. An expression for the free energy equivalent to (2.42) has been derived by Rudnick.[16]

A procedure for calculating the free energy from the renormalization-group recursion relations is now clear. We expand, as usual,[3] the functions v_2 and v_4 in powers of k:

$$\begin{aligned} v_2(l,k) &= r(l) + ek^2 + O(k^4) , \\ v_4(l,k_1,k_2,k_3,k_4) &= u(l) + O(k^2) . \end{aligned} \tag{2.44}$$

The coefficients of the higher-order powers of k are irrelevant variables, and will, for now, be neglected. Recall that $\eta(l)$ is chosen to keep the coefficient e of k^2 equal to unity. Differential equations for $r(l)$ and $u(l)$ can be constructed by the Wegner-Houghton method or, equivalently, by taking the $b \to e^{\delta}$ limit of the recursion relations in Ref. 3. Constructing the scaling fields $g_r(r,u)$ and $g_u(r,u)$, we obtain

$$G_0(l) = \tfrac{1}{2}\eta(g_r e^{\lambda_r l}, g_u e^{\lambda_u l}) + \tfrac{1}{2} d \ln[1 + r(g_r e^{\lambda_r l}, g_u e^{\lambda_u l})] , \tag{2.45}$$

from which the free energy can be found through (2.15). Here, r and u are equivalent to the t and h variables utilized in Sec. II A. [In practical calculation, we will often take $\eta(l)=0$ as an approximation.]

The simplicity of the result (2.43) is a consequence of taking the continuous limit $\delta \to 0$. The nontrivial graphs arising from a perturbation theory in v_4 are order δ^2 or higher. The only contribution linear in δ is the usual multiplicative term involving the inverse square root of the determinant of the propagator. Some nontrivial graphs which do *not* contribute to the differential equation for the free energy are shown in Fig. 3.

E. Irrelevant variables

We now indicate how an additional irrelevant variable at the primary fixed point may be incorporated into this scheme, using an idea due to Wilson.[23] Suppose we have constructed a set of three coupled differential recursion relation involving the variables t, h, and w, namely,

$$\begin{aligned} dt/dl &= T(t,h,w) , \\ dh/dl &= H(t,h,w) , \\ dw/dl &= W(t,h,w) . \end{aligned} \tag{2.46}$$

These recursion relations are assumed to be diagonal about a primary fixed point P to first order, i.e.,

$$dt/dl \approx \lambda_t t\,,$$
$$dh/dl \approx \lambda_h h\,, \qquad (2.47)$$
$$dw/dl \approx \lambda_w w\,.$$

Note that the primary fixed point has been taken to be at the origin. The eigenvalues λ_t and λ_h are taken to be positive, so t and h correspond to the variables discussed in Sec. II A. The parameter w is an additional variable irrelevant at the primary fixed point, i.e., $\lambda_w = -|\lambda_w|$. As in Sec. II A, it is assumed that there is a secondary fixed point S to which Hamiltonians can flow crossing over from P. We again assume that there is one singled-out trajectory, the separatrix, which connects P to S.

In general, the Hamiltonian flows governed by (2.46) could be quite complicated. However, we will show that there is a two-dimensional manifold $w_0(t,h)$ in (t,h,w) space on which the trajectories behave as depicted in Fig. 2. If an initial (unrenormalized) Hamiltonian has parameters lying in the manifold, it will lie in a crossover scaling regime without any transient effects. Although this manifold will not in general just be the trivial plane $w=0$, this could be taken as an approximation.[3]

The first step is to find the scaling fields $g_t(t,h,w)$, $g_h(t,h,w)$, and $g_w(t,h,w)$ associated with the primary fixed point, which amounts to solving the system (2.46). The functions $g_t(t,h,w)$, $g_h(t,h,w)$, and $g_w(t,h,w)$ are defined in analogy to (2.5). In terms of these variables the solutions of (2.46) may be expressed

$$t(l) = t(g_t e^{\lambda_t l}, g_h e^{\lambda_h l}, g_w e^{-|\lambda_w| l})\,,$$
$$h(l) = h(g_t e^{\lambda_t l}, g_h e^{\lambda_h l}, g_w e^{-|\lambda_w| l})\,, \qquad (2.48)$$
$$w(l) = w(g_t e^{\lambda_t l}, g_h e^{\lambda_h l}, g_w e^{-|\lambda_w| l})\,.$$

Note that all trajectories in the simple two-dimensional parameter space discussed in II A near the primary fixed point originate from P (see Fig. 2). Letting $l \to -\infty$ in (2.48), we find that the only solutions "originating" from P are those with $g_w(t,w,h)=0$. Consequently, we define a manifold $w_0(t,h)$ by the condition

$$g_w(t, w_0, h) = 0\,. \qquad (2.49)$$

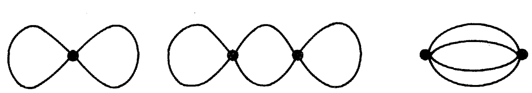

FIG. 3. Feynman graphs which are order δ^2 or higher, and hence do not contribute to the differential equation for the free energy.

On this manifold, because the nonlinear scaling fields must be identical to the linear scaling fields for small arguments,

$$\lim_{l\to -\infty} [t(l), h(l), w(l)] = [0,0,0]\,. \qquad (2.50)$$

Since the separatrix is by definition a trajectory originating from P, it, as well as the secondary fixed point S, will lie in the surface $w_0(t,h)$. At the secondary fixed point, there is already one irrelevant direction singled out by the separatrix. The remaining eigenaxis must be associated with a relevant operator since, if it were irrelevant, there would be more than one separatrix, which we have taken to be unique. (We exclude here the interesting case of a marginal operator.) Note that the prescription given above only gives trajectories on one side of the secondary eigenaxis of S (unshaded portion of Fig. 2). We expect that the remaining trajectories can be obtained by analytically continuing the function $w_0(t,h)$ into the shaded region of Fig. 2. A picture of the Hamiltonian flows we have in mind is shown in Fig. 4.

Formally, the asymptotic crossover scaling function can be extracted by expanding in g_w. Expressing the kernel of the trajectory integral as

$$G_0(g_t e^{\lambda_t l}, g_h e^{\lambda_h l}, g_w e^{-|\lambda_w| l}),$$

the free energy is just

$$F(g_t, g_h, g_w) = \int_0^\infty e^{-dl} G_0(g_t e^{\lambda_t l}, g_h e^{\lambda_h l}, g_w e^{-|\lambda_w| l})\, dl\,. \qquad (2.51)$$

All trajectories approach the manifold $w_0(t,h)$ during the large-l part of the integration. Expanding in the third argument of $G_0(x,y,z)$ and changing

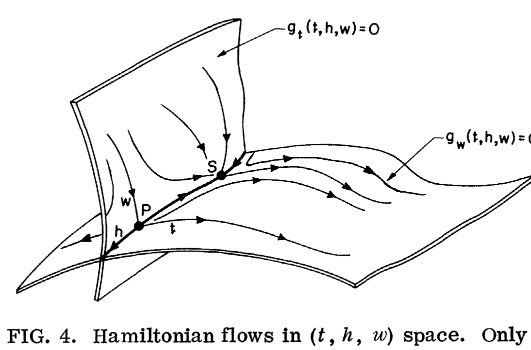

FIG. 4. Hamiltonian flows in (t, h, w) space. Only the trajectories in the manifolds given by the equations $g_w(t,h,w)=0$ and $g_t(t,h,w)$ are shown. The separatrix, which connects the primary fixed point P to the secondary fixed point S, is shown as a bold line.

variables, one obtains corrections to crossover scaling of the sort first discussed by Wegner,[13]

$$F(g_t, g_h, g_w) = \lambda_t^{-1} g_t^{d/\lambda_t} \int_{\ln g_t}^{\infty} e^{-dl/\lambda_t} G_0(e^l, e^{\phi l} z, 0)\, dl + \lambda_t^{-1} g_t^{d/\lambda_t + |\lambda_w|/\lambda_t} \int_{\ln g_t}^{\infty} e^{-(d+|\lambda_w|)l/\lambda_t} G_0^1(e^l, e^{\phi l})\, dl + \cdots, \tag{2.52}$$

where

$$G_0^1(x, y) = [\partial G_0(x, y, z)/\partial z]_{z=0}. \tag{2.53}$$

The first term is the asymptotic crossover scaling function (provided the regular parts of the free energy are subtracted out), while the second is the leading correction term due to the irrelevant variable. It corrects for initial Hamiltonian parameters which are off the "scaling manifold" $w_0(t, h)$.

Although explicit reference to the irrelevant scaling field $g_w(t, h, w)$ has been eliminated in the first term of (2.52), the irrelevant variable w still plays a role since it appears in the arguments of g_t and g_h. The asymptotic crossover scaling prediction from this renormalization-group treatment is

$$F(g_t, g_h, g_w) \approx F(g_t, g_h, 0) = g_t^{2-\alpha} \Phi(g_h / g_t^{\phi}). \tag{2.54}$$

However, the result of an analysis by series expansions or through actual experiments would be of the form

$$F(t, h) \approx t^{2-\alpha} \tilde{\Phi}(h/t^{\phi}), \tag{2.55}$$

where the results are expressed in terms of the linear scaling fields. Now, g_t and g_h must vanish when t and h do, if (2.54) is to give a correct description of the physical system which scales according to (2.55). Thus g_t and g_h must behave like

$$g_t \approx t[1 + c_t w + O(w^2)], \tag{2.56}$$

$$g_h \approx h[1 + c_w w + O(w^2)], \tag{2.57}$$

as $t, h \to 0$, where c_t and c_w are constants. Defining

$$P = \lim_{t, h \to 0} (g_t / t), \quad Q = \lim_{t, h \to 0} (g_h / h), \tag{2.59}$$

we see by equating (2.54) and (2.55) that the connection between the scaling function $\Phi(z)$ and the "observed" scaling function $\tilde{\Phi}(z)$ is

$$\tilde{\Phi}(z) = P^{2-\alpha} \Phi(Qz/P^{\phi}). \tag{2.60}$$

Thus an observed scaling function $\tilde{\Phi}(z)$ must be normalized properly if it is to be identified with the universal crossover scaling function $\Phi(z)$.[11] One could, for example, enforce the conventions $\tilde{\Phi}(0) = 1$ and $\tilde{\Phi}'(0) = 1$ by rescaling t and h properly,[11] thus eliminating the effect of the irrelevant variable w. Although these techniques for treating irrelevant variables will not be exploited in this paper, they have, in fact, been used to obtain the equation of state for systems with dipolar interactions.[19]

III. GAUSSIAN MODEL

Although the Gaussian model is a well known and easily solved model, it is well suited for a first demonstration of the abstract machinery developed above. The Gaussian model is simply described in momentum space by the Hamiltonian

$$H = Nv_0 + \frac{1}{2} \sum_k v(k) S_k S_{-k}. \tag{3.1}$$

On expanding $v(k) = r + k^2 + O(k^4)$, the usual renormalization-group analysis[3] gives simply

$$dr(l)/dl = 2r(l), \quad \eta(l) = 0. \tag{3.2}$$

This choice of η keeps the coefficient of k^2 equal to unity. The coefficients of higher powers of k are irrelevant variables, which could be incorporated into the analysis by the techniques outlined in Sec. II E. They will be neglected here. Thus the nonlinear scaling field is just $g_r = r$ and $r(l) = r(0) e^{2l}$.

The free energy, by (2.15), is

$$F(r) = \int_0^{\infty} e^{-dl} \ln(1 + re^{2l})\, dl. \tag{3.3}$$

Using (2.23), we obtain

$$F(r) = Ar^{d/2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-)^n r^n}{(n - \frac{1}{2}d)n}. \tag{3.4}$$

The amplitude A can be expressed

$$A = \frac{1}{2} \int_{\ln r}^{\infty} e^{-dl/2} \ln(1 + e^l)\, dl - \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-)^n r^{n-d/2}}{(n - \frac{1}{2}d)n}, \tag{3.5}$$

which is equivalent to Eq. (2.31). The right-hand side must be independent of r', so we take the limit $r \to 0$,

$$A = \lim_{r \to 0} \left(\frac{1}{2} \int_{\ln r}^{\infty} dl\, e^{-dl/2} \ln(1 + e^l) + \frac{1}{2} \frac{r^{1-d/2}}{1 - d/2} - \frac{1}{4} \frac{r^{2-d/2}}{2 - d/2} \right). \tag{3.6}$$

As in the discussion following Eq. (2.34), we have explicitly taken the limit $r\to 0$ for those terms in (3.5) which have $n\geq 3$. The two terms retained in (3.6) allow the range of integration to be convergently extended to $-\infty$ for all dimensionalities less than $d=6$. Writing these terms as integrals from $\ln r$ to infinity and explicitly taking the limit we obtain an expression for the amplitude analogous to (2.38), namely,

$$A=\frac{1}{2}\int_{-\infty}^{\infty} e^{-dl/2}[\ln(1+e^{l})-\tfrac{1}{2}e^{l}-\tfrac{1}{4}e^{2l}]\,dl\,. \quad (3.7)$$

That the "trajectory-integral" expression (3.3) is in fact the correct expression for the Gaussian model can be seen by a trivial change of variables. Setting $q=e^{-l}$, and neglecting an unimportant constant, it is easily seen that

$$F(r)=\int_0^1 \ln(r+q^2)q^{d-1}\,dq\,, \quad (3.8)$$

which is the usual expression for the free energy of a Gaussian model in d dimensions defined on a spherical Brillouin zone of unit radius (and neglecting higher powers of q^2).

The Gaussian model in a magnetic field h provides an interesting counterexample to the assertion (2.14). We introduce the field by adding the term NhS_0 to (3.1). The exact free energy in a magnetic field is

$$F(r,h)=\int_0^1 \ln(r+q^2)q^{d-1}\,dq+h^2/K_d r\,,$$
$$K_d^{-1}=2^{d-1}\pi^{d/2}\Gamma(\tfrac{1}{2}d)\,, \quad (3.9)$$

where the factor K_d^{-1} in the second term comes from a normalization convention.[21] However, a naive application of the formalism developed in Sec. II would lead to the result (3.8), *without* any field dependence.

The missing term, in fact, arises from the limit of $e^{-dl}F[r(l)]$ as $l\to\infty$. The recursion relation for r is just (3.2), while for h one has

$$dh(l)/dl=(1+\tfrac{1}{2}d)h(l)\,. \quad (3.10)$$

Substituting the trajectories $r(l)=re^{2l}$ and $h(l)=he^{(1+d/2)l}$ into (3.9) and taking the limit explicitly, we find

$$\lim_{l\to\infty} e^{-dl}F(re^{2l},he^{(1+d/2)l})=h^2/K_d r\,, \quad (3.11)$$

which is, of course, the missing term. That the right-hand side of (3.11) is nonzero suggests that (2.14) does not hold when the modes at $q=0$ are macroscopically occupied. One could conceivably surmount this problem by transforming away the linear term in S by a spin shift, which works for the Gaussian model. More generally, one can hope to determine this limit by a matching procedure near the attracting fixed point ($r\to\infty$). In this case the expansions for large r can easily be found and leads to the same result (3.11). However, in the remainder of this work we will stay above any ordering temperatures and take the h variable defined in Sec. II to be a nonordering field.

IV. MODEL RECURSION RELATIONS

Riedel and Wegner[12] have discussed crossover critical behavior in terms of simple "model" recursion relations. They analyzed the Hamiltonian flows and scaling fields in detail, and calculated a susceptibility crossover scaling function. We will use these models to test the procedures outlined in Sec. II. It is convenient to calculate the scaling function for the "specific heats" of these models, that is, two derivatives of the free energy with respect to a temperature-like parameter. These specific heats would be rather difficult to obtain for realistic systems by the matching procedure, for the reasons outlined in Sec. II B.

A. Recursion-relation model I

We first consider the recursion relations most extensively discussed in Ref. 12. Expressed in terms of the variables of Sec. II A, these recursion relations are

$$dt/dl=\lambda_t t+(\mathring{\lambda}_t-\lambda_t)th\,, \quad (4.1)$$

$$dh/dl=\lambda_h h(1-h)\,. \quad (4.2)$$

The secondary fixed point of the differential equations (4.1) and (4.2) is $(t^*,h^*)=(0,1)$ (see Fig. 4), and the model implies the identity $\mathring{\lambda}_h=-\lambda_h$, (see Sec. II A).

These recursion relations were postulated by Riedel and Wegner without reference to an underlying Hamiltonian or renormalization-group procedure. However, they are similar in structure to the renormalization-group equations derived to first order in ϵ by Wilson and Fisher[24] from a Hamiltonian like (2.39)—this presumably was the motivation of Riedel and Wegner in considering the equations. Thus, it is reasonable to postulate that the function $G_0(t,h)$ associated with the system (4.1) and (4.2) has the form $G_0(t,h)=\ln(1+t+ah)$, where we have set $\eta=0$ and the combination $t+ah$ plays the role of r in Eq. (2.43). The equality $r=t+ah$ results from the introduction of new variables in terms of which the recursion relations are diagonal to first order about the primary fixed point. Although in general a will be nonzero (see Sec. V), we will set it to zero for convenience. Thus we suppose

$$G_0(l)=\ln[1+t(l)]\,. \quad (4.3)$$

Note that, following Riedel and Wegner, we are attempting only to find a reasonable *simulation* of crossover behavior. It will become evident that equations with the structure of (4.1) and (4.2) and the relation (4.3) lead quite naturally to crossover scaling functions with the expected properties. A nonzero value of a turns out to make no significant difference.

The solution of (4.1) and (4.2) may be represented as[12]

$$t(l)=g_t e^{\lambda_t l}/(1+g_h e^{\lambda_h l})^{(\lambda_t-\dot{\lambda}_t)/\lambda_h}, \tag{4.4}$$

$$h(l)=g_h e^{\lambda_h l}/(1+g_h e^{\lambda_h l}), \tag{4.5}$$

where the scaling fields g_t and g_h are given by

$$g_t=t/(1-h)^{(\lambda_t-\dot{\lambda}_t)/\lambda_h}, \tag{4.6}$$

$$g_h=h/(1-h). \tag{4.7}$$

That these solutions mimic crossover behavior can be seen explicitly in the l dependence of the "temperature-like" variable t. Thus $t(l)$ initially behaves like $e^{\lambda_t l}$, with an exponent characteristic of the primary fixed point, while for large l it goes as $e^{\dot{\lambda}_t l}$, which is the behavior appropriate to the secondary fixed point.

Insertion of (4.3) into (2.15) and use of (4.4) and (4.5) then gives

$$F(g_t,g_h)=\int_0^\infty e^{-dl}\ln[1+g_t e^{\lambda_t l}/(1+g_h e^{\lambda_h l})^{(\lambda_t-\dot{\lambda}_t)/\lambda_h}]\,dl.$$

We will analyze explicitly the specific-heat-like quantity

$$C=\frac{-\partial^2 F}{\partial g_t^2}=\int_0^\infty e^{(2\lambda_t-d)l}[(1+g_h e^{\lambda_h l})^p+g_t e^{\lambda_t l}]^{-2}. \tag{4.9}$$

where

$$p=(\lambda_t-\dot{\lambda}_t)/\lambda_h. \tag{4.10}$$

For convenience, we have taken derivatives with respect to the scaling field g_t rather than with respect to the more "physical" variable t. One can go back and forth between the two possibilities using (4.6).

A change of variables applied to (4.9) yields

$$C=\lambda_t^{-1}g_t^{-\alpha}\int_{\ln g_t}^\infty e^{\alpha l}[(1+ze^{\phi l})^p+e^l]^{-2}, \tag{4.11}$$

where $\alpha=-(d/\lambda_t)+2$ and $z=g_h/g_t^\phi$. If one assumes that α is positive (corresponding to a divergent specific heat), there is no problem in extending the lower limit in (4.5) to $-\infty$. This immediately shows that the singular part of C is

$$C_{\text{sing}}=g_t^{-\alpha}\psi(z), \tag{4.12}$$

where the scaling function is given explicitly by

$$\psi(z)=\lambda_t^{-1}\int_{-\infty}^\infty e^{\alpha l}[(1+ze^{\phi l})^p+e^l]^{-2}. \tag{4.13}$$

The difference arising from extending the lower limit has been absorbed into the regular part of C; if $\alpha<0$, subtractions must be made as in the case of the free energy.

A particular Hamiltonian trajectory in (t,h) space can be defined by the condition $z=g_h(l)/g_t(l)^\phi=\text{const}$, and we expect the integral (4.13) to be singular when z equals some constant $\dot{z}$ which locates the separatrix. It is easy to see from Eqs. (4.1) and (4.2) that the separatrix is just the line $t=0$ (see Fig. 5). Since $g_t=t/(1-h)^p$, it is apparent that the value of z indexing this separatrix is $\dot{z}=\infty$.

If we set $g_h=0$ and then approach the primary fixed point along the path (i) in Fig. 5, the specific heat varies as

$$C=A/g_t^\alpha, \tag{4.14}$$

where the amplitude is given by

$$A=\psi(0)=\lambda_t^{-1}\int_{-\infty}^\infty e^{\alpha l}(1+e^l)^{-2}. \tag{4.15}$$

If, however, we approach the separatrix at a nonzero value of g_h, such as path (ii) in Fig. 5, we must use the result

$$\psi(z)\approx\lambda_t^{-1}\int_{-\infty}^\infty e^{\alpha l}(e^l+z^p e^{\phi p l})^{-2}$$

$$\approx\text{const}\left(\frac{1}{z}\right)^{d(\lambda_t-\dot{\lambda}_t)/\dot{\lambda}_t\lambda_h},\quad z\to\infty. \tag{4.16}$$

For fixed g_h then, as $t\to 0$ we obtain

$$C(g_t,g_h)\sim(g_t^\phi/g_h)^{d(\lambda_t-\dot{\lambda}_t)/\dot{\lambda}_t\lambda_h}g_t^{-\alpha}$$

$$\sim g_t^{-\dot{\alpha}},\quad \dot{\alpha}=2-d/\dot{\lambda}_t, \tag{4.17}$$

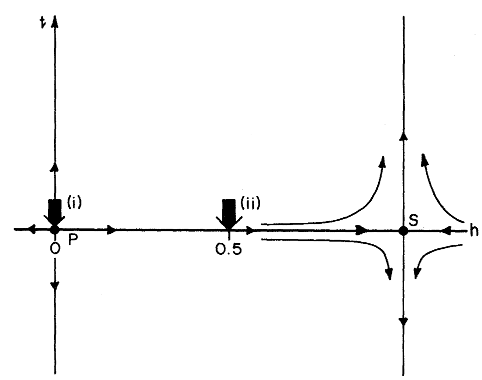

FIG. 5. Hamiltonian flows for recursion-relation model I. Two different paths of approach to the separatrix, (i) and (ii), are indicated.

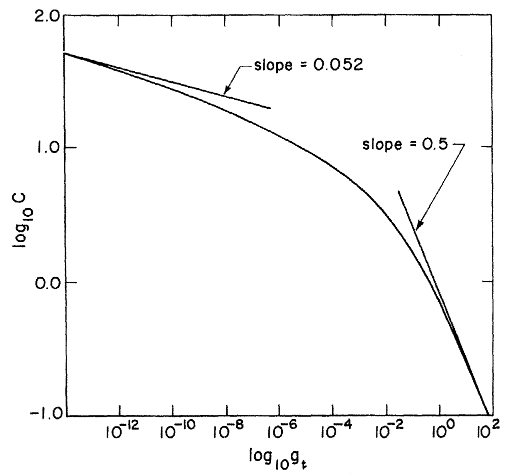

FIG. 6. Plot of the logarithm of the specific heat vs the logarithm of the scaling field g_t for recursion-relation model I.

where $\dot{\alpha}$ is the secondary specific-heat exponent for the critical line. We note that (4.16) can be written

$$\psi(z) \sim z^{(\dot{\alpha}-\alpha)/\phi}, \quad z \to \infty, \tag{4.18}$$

and hence satisfies the predicted[11] double scaling law of phenomenological crossover scaling theories. A logarithmic plot of C along the path (ii) is shown in Fig. 6 for a choice of parameter values which mimics the crossover from a "tricritical" specific index $\alpha = 0.5$ to a "λ-line" specific-heat index $\dot{\alpha} = 0.052$. The expression (4.13) was evaluated numerically with $\lambda_t = 2.0$, $\lambda_h = 1.6$, $\dot{\lambda}_t = 1.54$, and $d = 3$. In Fig. 7, a plot of the *effective* specific-heat index,[12] defined through

$$\alpha_{\text{eff}}(t) = (\partial \ln C / \partial \ln g_t)_{g_h}, \tag{4.19}$$

is shown.

B. Breakdown of hyperscaling

The model recursion relations described above provide a convenient arena in which to investigate one source of the breakdown of the dimensionality-dependent hyperscaling[25] exponent relationships. Specifically, we have in mind a breakdown of hyperscaling above *three* dimensions for *tricritical* systems which can be described by recursion relations similar to those of model I, but with explicitly calculated coefficients depending on d.

For concreteness, we will consider a tricritical point like that occurring in He^3-He^4 mixtures.[26] The appropriate experimental parameters for this system are the temperature T and the mole fraction x of He^3. In the notation introduced in Sec. II A the field H is the difference between the chemical potentials of the two species, $H = \mu_3 - \mu_4$. The renormalization-group scaling prediction[7] for the free energy of such systems is

$$F(t, h) \approx t^{d\nu} \Phi(h/t^{\phi}) + F_{\text{reg}} . \tag{4.20}$$

A derivative with respect to the field h then gives the prediction that the discontinuity $\Delta x = x_3 - x_4$ in the He^3 and He^4 densities should vary as

$$\Delta x \sim t^{d\nu - \phi} \equiv t^{\beta} . \tag{4.21}$$

The exponent β describing the discontinuity obeys the dimensionality-dependent hyperscaling relationship

$$d\nu - \phi = \beta . \tag{4.22}$$

However, this dimensionality-dependent relationship is expected to break down for $d > 3$. For $3 < d < 4$, a renormalization-group analysis[7] shows that $\nu = \frac{1}{2}$ and $\phi = \frac{1}{2}d - 2$ *exactly*. The relation (4.21) then gives $\beta = d - 2$. Thus for $3 < d < 4$ the "naive" β calculated from (4.21) exceeds unity, and one expects the temperature dependence of the concentration discontinuity to be dominated by a linear term from the "regular part" of the free energy. This is in accord with the idea[7] that tricritical exponents should lock into their classical values for $d > 3$. Below $d = 3$, expansions of β in powers of $3 - d$ indicate[27] that $\beta < 1$; so the hyperscaling relationship (4.21) should hold in this case.

Taking model I as a description of He^3 – He^4 mixtures and using the identifications suggested above, we can calculate the discontinuity in density Δx by taking one derivative of (4.8) with respect to g_h:

$$\Delta x \approx \left(\frac{\partial F}{\partial g_h}\right)_{g_h = 0} = \frac{p}{\lambda_t} g_t^{d/\lambda_t - \phi} \int_{\ln g_t}^{\infty} \frac{e^{(\phi - d/\lambda_t) l}}{1 + e^{-l}} \, dl , \tag{4.22}$$

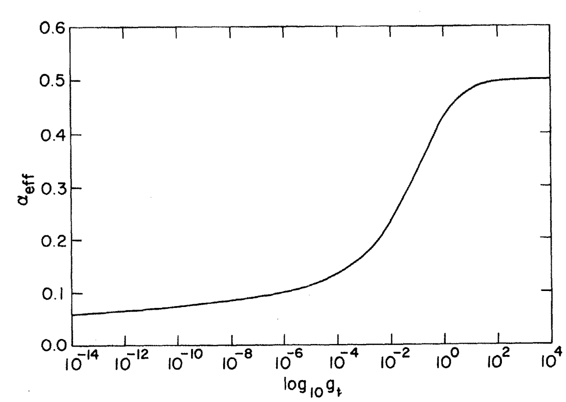

FIG. 7. Plot of the effective critical exponent α_{eff} vs $\log_{10} g_t$ for recursion-relation model I.

where we have made the usual change of variables in the integral. Calculating the density discontinuity associated with model I involves allowing negative values of h. In a real physical system, this could correspond to an unbounded Hamiltonian[7] and would require the introduction of a third stabilizing irrelevant variable to get a genuine density discontinuity. We will, however, in this illustrative example of the breakdown of hyperscaling, ignore such difficulties. Setting

$$d/\lambda_t - \phi = d\nu - \phi \equiv \overline{\beta}\,, \tag{4.23a}$$

one can evaluate (4.22) exactly to find

$$\begin{aligned}\Delta x &= p\lambda_t^{-1}\pi\csc(\pi\overline{\beta})g_t^{\overline{\beta}} - p\lambda_t^{-1}\frac{g_t}{1-\overline{\beta}}\,{}_2F_1(1;1-\overline{\beta};2-\overline{\beta},-z)\\ &= p\lambda_t^{-1}\left(\pi\csc(\pi\overline{\beta})g_t^{\overline{\beta}} - g_t\sum_{n=0}^{\infty}\frac{(-g_t)^n}{n-\overline{\beta}+1}\right),\end{aligned} \tag{4.23b}$$

where ${}_2F_1$ is a hypergeometric function.[28] Assuming $\overline{\beta}$ is less than unity for $d<3$, greater than unity for $d>3$, and unity for $d=3$ we find that Δx behaves asymptotically as

$$\Delta x \approx \begin{cases} p\lambda_t^{-1}\pi\csc(\overline{\beta}\pi)g_t^{\overline{\beta}}\,, & d<3\\ p\lambda_t^{-1}g_t/(\overline{\beta}-1)\,, & d>3\\ p\lambda_t^{-1}g_t|\ln g_t|\,, & d=3\,.\end{cases} \tag{4.24}$$

Thus below $d=3$ hyperscaling holds; above $d=3$ a term from the regular part of Δx dominates; and at $d=3$ the regular and scaling parts conspire to give a logarithmic correction. The form of the logarithmic correction is not that predicted by a more complete analysis of tricritical points in three dimensions,[29] but we attribute this to the neglect of a marginally irrelevant variable (at $d=3$)[29] in the formulation of model I. An example of such a variable would be the coefficient of the S^6 term in an expression like (2.39).

The mechanism for the breakdown of hyperscaling is seen to be the finite lower limit of (4.22). If this could be ignored (say, by extending it to $-\infty$), the hyperscaling relation $\overline{\beta}_u = \beta_u$ would be inescapable. The lower limit produces a regular part of x which "locks in" a critical exponent at its classical value above the changeover dimensionality $d=3$. Apparently, below $d=3$, it is the large-l or long-wavelength modes of (4.22) which are important, while for $d>3$ small-l or short-wavelength modes yield the dominant behavior. A similar mechanism seems to destroy the usual hyperscaling relationship $2-\alpha = d\nu$ above four dimensions.[30] We do not expect that these ideas are merely an artifact of the model.

C. Recursion-relation model II

We now discuss briefly a second model considered by Riedel and Wegner[12] in an appendix. It is more sophisticated than model I, in that it allows for a curved nonanalytic critical line in the vicinity of the primary fixed point. For appropriate values of the model parameters, its recursion relations are those discovered by Wilson and Fisher[24] for the crossover from a Gaussian to an Ising-model fixed point. The recursion relations in question are[12]

$$dt/dl = \lambda_t t(1-h) + \mathring{\lambda}_t(t-t^*h)h + d_0 h(1-h)\,, \tag{4.25}$$

$$dh/dl = \lambda_h h(1-h)\,, \tag{4.26}$$

which have a primary fixed point at (0, 0) and a secondary fixed point at $(t^*, 1)$. The solutions of these differential equations may be written[12] in terms of nonlinear scaling fields as

$$\begin{aligned}t(l) = {}&\frac{g_t e^{\lambda_t l} - Cg_h^{1/\phi}e^{\lambda_t l}}{1+g_h e^{\lambda_h l}} + \frac{t^* g_t e^{\lambda_t l}}{1+g_h e^{\lambda_t l}}\,{}_2F_1\!\left(\frac{1}{\phi},\,1;\,1+\frac{\mathring{\lambda}_t}{\lambda_h};\,-\frac{1}{g_h e^{\lambda_h l}}\right)\\ &-\frac{d_0}{\lambda_h(1+\mathring{\lambda}_t/\lambda_h)(1+g_h e^{\lambda_h l})}\,{}_2F_1\!\left(\frac{1}{\phi},\,1;\,2+\frac{\mathring{\lambda}_t}{\lambda_h};\,-\frac{1}{g_h e^{\lambda_h l}}\right)\end{aligned} \tag{4.27}$$

$$h(l) = g_h e^{\lambda_h l}/(1+g_h e^{\lambda_h l})\,, \tag{4.28}$$

where

$$C = \frac{\Gamma(1+\mathring{\lambda}_t/\lambda_h)\Gamma(1-1/\phi)}{\Gamma(2+\mathring{\lambda}_t/\lambda_h - 1/\phi)}\left[t^*\left(1-\frac{1}{\phi}\right) - \frac{d_0}{\lambda_h}\right]. \tag{4.29}$$

The (l-independent) nonlinear scaling fields g_t and g_h are defined implicitly through Eqs. (4.28) and (4.29) evaluated at $l=0$.

When coupled with an expression for the kernel of the trajectory integral (2.15) such as (4.3), these equations determine the free energy,

$$F(g_t, g_h) = \int_0^\infty e^{-dl} \ln[1 + t(g_t e^{\lambda_t l}, g_h e^{\lambda_h l})]\, dl$$
$$= -\lambda_t^{-1} g_t^{d/\lambda_t} \int_{\ln g_t}^\infty e^{-dl/\lambda_t} \times \ln[1 + t(e^l, e^{\phi l} z)]\, dl . \qquad (4.30)$$

As $l \to \infty$ the result (4.28) simplifies to

$$t(l) \approx t^* + g_h^{\mathring{\lambda}_t/\lambda_h} e^{\mathring{\lambda}_t l}(z^{1/\phi} - C) . \qquad (4.31)$$

If the lower limit of (4.30) is unimportant, we can substitute (4.31) to obtain

$$F_{\text{sing}}(g_t, g_h) \approx g_t^{d/\lambda_t} \int_L^\infty dl\, e^{-dl/\lambda_t} \times \ln[1 + e^{\mathring{\lambda}_t l/\lambda_t} z^{\mathring{\lambda}_t/\lambda_h}(z^{1/\phi} - C)] , \qquad (4.32)$$

where L is some lower limit such that the approximation (4.31) is good for $l \geq L$. Inspection reveals that the integral in (4.32) will be singular at $z = \mathring{z} = C^\phi$, which is the equation of the separatrix associated with the system (4.26)-(4.27). In fact, it is not hard to show that as $z \to \mathring{z}$ the crossover scaling function associated with singular part of $F(g_t, g_h)$ behaves as

$$\Phi(z) \approx \int_L^\infty dl\, e^{-dl/\lambda_t} \ln[1 + e^{\mathring{\lambda}_t l/\lambda_t} z^{\mathring{\lambda}_t/\lambda_h}(z^{1/\phi} - C)]$$
$$\sim (z - C^\phi)^{d/\mathring{\lambda}_t} . \qquad (4.33)$$

This agrees with (1.3) provided we set $d/\lambda_t = 2 - \alpha$ and $d/\mathring{\lambda}_t = 2 - \mathring{\alpha}$. Of course, possibly dominant terms which are regular in $z - \mathring{z}$, could also be present in an expansion of $\Phi(z)$ about $\mathring{z}$.

V. GAUSSIAN TO HEISENBERG CROSSOVER NEAR FOUR DIMENSIONS

A simple example of a crossover problem treated via the epsilon expansion is the crossover from Gaussian to Heisenberg critical behavior. Analyzing an Ising-like model with renormalization-group techniques, Wilson and Fisher[24] found that below four dimensions a fixed point with Ising-like critical exponents is stable relative to the Gaussian fixed point. An analogous fixed point for n-component spins was subsequently investigated by Fisher and Pfeuty[8] and by Wegner.[9] The Hamiltonian flows look roughly like those in Fig. 2, exhibiting a crossover from the primary Gaussian fixed point to the secondary Heisenberg fixed point.

The Hamiltonian is given by (2.39), and the relevant variables which change under iteration are the r and u variables defined in (2.44). They obey the differential recursion relations[21]

$$\frac{dr}{dl} = 2r + \frac{2(n+2)}{n} u - \frac{2(n+2)}{n} ur , \qquad (5.1)$$

$$\frac{du}{dl} = \epsilon u - \frac{2(n+8)}{n} u^2 . \qquad (5.2)$$

The right-hand side of these equations are expansions in r and u carried to second order in ϵ; both r and u are taken to be of order $\epsilon = 4 - d$. We have evaluated the coefficients of the higher-order terms in the equations to lowest order in ϵ; this will not affect our result for the scaling function to first order in ϵ. Note that there is no term proportional to u^2 in (5.1), although such a term would be present in a recursion relation constructed for a *finite* spatial rescaling factor b.[3] It is not hard to see that the renormalization-group eigenvalues for this system are

$$\lambda_t = 2 , \quad \mathring{\lambda}_t = 2 - [(n+2)/(n+8)]\,\epsilon$$
$$\lambda_h = \epsilon , \quad \mathring{\lambda}_h = -\epsilon . \qquad (5.3)$$

The results for $\mathring{\lambda}_t$ and $\mathring{\lambda}_h$ are valid to first order in ϵ, while λ_t and λ_h are exact as given between three and four dimensions.[7]

These equations have the same form as the system (4.25) and (4.26), so their solutions are essentially given by (4.28)–(4.30). However, these solutions break down when evaluated at $\epsilon = 1$. It will be convenient therefore to display the solutions in a form more suitable for calculations. As a first step, we define new variables

$$t = r + [(n+2)/n]\, u \qquad (5.4)$$

$$h = [2(n+8)/n\epsilon]\, u , \qquad (5.5)$$

which obey the differential equations

$$dt/dl = \epsilon\bar{\phi} t - \epsilon a h t - \epsilon b h^2 , \qquad (5.6)$$

$$dh/dl = \epsilon h(1-h) , \qquad (5.7)$$

where

$$a = (n+2)/(n+8), \quad b = 3[(n+2)/(n+8)^2]\,\epsilon , \qquad (5.8)$$

and $\bar{\phi}$ is the inverse crossover exponent

$$\bar{\phi} = 2/\epsilon . \qquad (5.9)$$

The solution of (5.7) is given immediately by (4.5) and (4.7) with $\lambda_h = \epsilon$. To solve (5.6), we first form the ratio of (5.6) and (5.7),[12] namely,

$$h(1-h)\, dt/dh = \bar{\phi} t - a h t - b h^2 . \qquad (5.10)$$

Integrating this equation to find $t(h)$, and then substituting the expression (4.5) for $h(l)$ finally yields

$$t(l)=\frac{C_0 g_h^{\bar\phi} e^{2l}}{(1+g_h e^{\epsilon l})^a}+\frac{b}{\bar\phi-a}\frac{g_h^a e^{\epsilon a l}}{(1+g_h e^{\epsilon l})^a}\times{}_2F_1(2-a,\bar\phi-a;\bar\phi-a+1;-1/g_h e^{\epsilon l}), \tag{5.11}$$

where C_0 is a constant of integration.

The hypergeometric function in (5.11) appears in the form ${}_2F_1(\alpha,\beta;\gamma;-1/g_h e^{\epsilon l})$; it appears desirable to transform this to hypergeometric functions which have expansions about $g_h=0$ rather than $g_h=\infty$. [In fact, this is how one may derive (4.28)–(4.30).] However, it is not possible to do this in a simple way when $\beta-\alpha$ is an integer, i.e., whenever $\bar\phi=2/\epsilon$ assumes integer values.[28] Difficulties occur at $\epsilon=1$ and for a countably infinite set of other dimensionalities between $d=3$ and 4. At these special dimensionalities logarithmic terms are found in the small-g_h expansion of (5.10).[7]

For large l one finds the asymptotic behavior

$$t(l)\approx C_0 g_h^{\bar\phi-a} e^{\lambda_t l}+b/(\bar\phi-a) \tag{5.12}$$

Hence, $t(l)$ diverges as $l\to\infty$ with an exponent characteristic of the secondary fixed point, unless $C_0=0$. By setting $C_0=0$ in (5.11), one obtains the equation of the separatrix $t_0(l)$, namely,

$$t_0(l)=\frac{b}{\bar\phi-a}\frac{g_h^a e^{\epsilon a l}}{(1+g_h)^a}\,{}_2F_1\left(2-a,\bar\phi-a,\bar\phi-a+1,\frac{-1}{g_h e^{\epsilon l}}\right). \tag{5.13}$$

In general one has $t_0\approx h^{\bar\phi}$ near the origin. However, when $\bar\phi$ is an integer,[12] the separatrix is given asymptotically by $t_0=h_0^{\bar\phi}\ln h_0$ near the Gaussian fixed point.[31] The value of t at the secondary fixed point is

$$t^*=t_0(\infty)=b/(\bar\phi-a), \tag{5.14}$$

in agreement with (5.5).

It is convenient to express (5.11) entirely in terms of nonlinear scaling fields. Any renormalization-group trajectory can be written $g_t/g_h^{\bar\phi}=\text{const}$. We are therefore led to define the scaling field for t through

$$g_t=C_0 g_h^{\bar\phi}. \tag{5.15}$$

Equation (5.10) becomes

$$t(l)=t(g_t e^{2l},g_h e^{\epsilon l})=\frac{g_t e^{2l}}{(1+g_h e^{\epsilon l})^a}+\frac{b}{\bar\phi-a}\frac{(g_h e^{\epsilon l})^a}{(1+g_h e^{\epsilon l})^a}\times{}_2F_1(2-a,\phi-a;\phi-a+1;-1/g_h e^{\epsilon l}). \tag{5.16}$$

It is easy to show that $g_t\approx t$ for small t and h.

Note that scaling fields are only useful in the trajectory integral formalism when they appear as dependent variables giving the l dependence of t and h. Their usefulness does not really depend on the invertibility of the transformation giving t and h as a function of g_t and g_h, nor does it depend crucially on the analyticity properties of the transformation. In fact, it is possible to show that the function $t(g_t,g_h)$ defined by Eq. (5.15) evaluated at $l=0$ contains a term which is an analytic function times $g_h^{\bar\phi}$. One can circumvent this manifest nonanalyticity by redefining g_t, but the circumvention breaks down when $\bar\phi=2/\epsilon$ is evaluated at $\epsilon=1$. This point is discussed further in the Appendix. We note that such nonanalyticies will modify the analysis of the regular part of the free energy given in Sec. II C.

To treat n-component spins via the renormalization group, the expression (2.43) for the kernel of the line integral must be replaced by

$$G_0(l)=\tfrac12 n\eta(l)+\tfrac12 dn\ln v_2(1). \tag{5.17}$$

The approximations made in deriving (5.1) and (5.2) enforce $\eta=0$, so

$$G_0(l)=\tfrac12 dn\ln[1+r(l)]=\tfrac12 dn\ln\{1+t(l)-[(n+2)/(n+8)]\tfrac12\epsilon h(l)\}. \tag{5.18}$$

The free energy is

$$F(g_t,g_h)=g_t^{d/\lambda_t}\frac{nd}{2\lambda_t}\int_{\ln g_t}^{\infty}e^{-dl/\lambda_t}\ln[1+K(e^l,e^{\phi l}z)], \tag{5.19}$$

where

$$K(x,y)=\frac{x}{(1+y)^a}+\frac{b}{\bar\phi-a}\frac{y^a}{(1+y)^a}\times{}_2F_1(2-a,\bar\phi-a;\bar\phi-a+1;-1/y). \tag{5.20}$$

The specific heat, which we define by taking two derivatives with respect to g_t, is

$$C\equiv-\frac{\partial^2F}{\partial^2 g_t}=g_t^{d/\lambda_t}\frac{nd}{2\lambda_t}\int_{\ln g_t}^{\infty}e^{(2-d/\lambda_t)l}\times\{(1+e^{\phi l}z)^a[1+K(e^l,e^{\phi l}z)]\}^{-2}\,dl. \tag{5.21}$$

Using (5.16) evaluated at $l=0$, we see that the quantity C is related to the usual specific heat (defined as two derivatives with respect to t) according to

$$\partial^2F/\partial t^2=C\partial^2 g_t/\partial t^2=C(1+g_h)^{2a}. \tag{5.22}$$

Since by (5.3) we have $\lambda_t = 2$, for $3<d<4$, there are no problems in extending the lower limit of (5.21) to $-\infty$. Thus the singular part of the specific heat is

$$C_{\text{sing}} = g_t^{d/\lambda_t}\,\psi(g_h/g_t^{\phi}) , \qquad (5.23)$$

where

$$\psi(z) = \frac{dn}{2\lambda_t}\int_{-\infty}^{\infty} e^{(2-d/\lambda_t)l} \times\{(1+e^{\phi l}z)^a[1+K(e^l, e^{\phi l}z)]\}^{-2}\,dl \ . \qquad (5.24)$$

Figure 8 shows a logarithmic plot of C_{sing} against $\log_{10}g_t$, for fixed $g_h=0.5$, with $n=1$ and evaluated at $\epsilon=1$. A crossover from the "tricritical" specific-heat exponent $\alpha=0.5$ to an Ising-like index $\dot\alpha=0.2$ is clearly indicated. In order to evaluate (5.24) *numerically*, we have set both $d=4-\epsilon$ and λ_t to their values at $\epsilon=1$, which gives $\dot\alpha=-d/\lambda_t+2=0.2$. The "correct" value of $\dot\alpha$ to first order in ϵ at $\epsilon=1$ is $\frac{1}{6}$, corresponding to expanding both numerator and denominator in ϵ of the fraction in the equation $\dot\alpha=-d/\lambda_t+2$. A plot of $\alpha_{\text{eff}}(g_t)$ as defined by Eq. (4.19) is given in Fig 9.

An improved treatment of the crossover scaling functions for this problem would require the introduction of a third irrelevant variable when $\epsilon=1$. This variable couples to an S^6 term in the Hamiltonian (2.39), and is only marginally irrelevant at the Gaussian fixed point.[7] The effect of this marginality is to introduce complicated logarithmic corrections to the primary critical exponents.[7] This variable can be neglected when $3<d<4$. We note that the truncation scheme of solving the differential recursion relations leads to spurious terms of $O(\epsilon^2)$ in (5.24). However, the essential features of crossover are preserved, as indicated by Figs. 8 and 9.

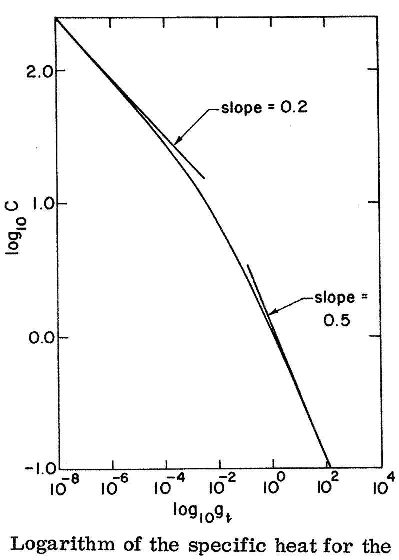

FIG. 8. Logarithm of the specific heat for the Gaussian-to-Heisenberg crossover problem plotted against $\log_{10}g_t$.

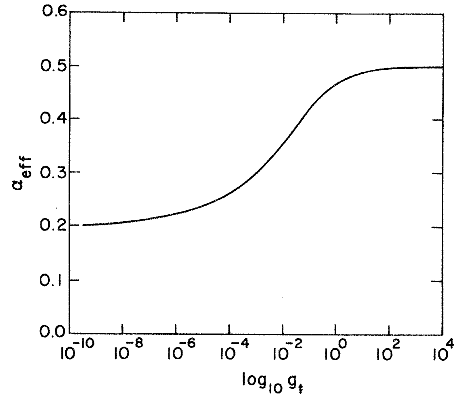

FIG. 9. Effective critical exponent α_{eff} for the Gaussian-to-Heisenberg crossover problem plotted against $\log_{10}g_t$.

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APPENDIX: NONANALTICITIES IN NONLINEAR SCALING FIELDS

In order to examine the question of nonanalyticities in the nonlinear scaling fields of Sec. V, we transform the solution (5.11) of the differential equation (5.10). Using the standard formulas for transforming hypergeometric functions,[28] this solution may be written

$$t(l) = C_1\frac{g_h(l)^{\bar\phi}}{[1+g_h(l)]^a} + \frac{b}{\bar\phi-2}\frac{g_h(l)^2}{[1+g_h(l)]^a} \times {}_2F_1[2-a, 2-\bar\phi; 3-\bar\phi; -g_h(l)] , \qquad (A1)$$

where

$$C_1 = C_0 + bB(\bar\phi - a, 2-\bar\phi) , \qquad (A2)$$

$B(x, y)$ is the beta function, and

$$g_h(l) = g_h e^{\epsilon l} \ . \qquad (A3)$$

The transformation which led to (A1) is only valid when $\bar\phi = \lambda_t/\lambda_h$ does not assume integer values.

When $\bar{\phi}$ is an integer, the appropriate transformation formula is much more complex; and involves logarithmic terms in g_h.[28] These logarithmic terms were predicted by Wegner in his original discussion of the corrections to scaling.[9]

It is clear from (A1) that the definition (5.15) of the scaling field g_t in terms of C_0 will leave a residual nonanalytic term in $t(g_t, g_h)$ proportional to $g_h^{\bar{\phi}}$. The only way to define a scaling field which makes $t(g_t, g_h)$ analytic is to set

$$g_t(l) \equiv C_1 g_h(l)^{\bar{\phi}} , \tag{A4}$$

where

$$g_t(l) = g_t \epsilon^{2l} . \tag{A5}$$

The defining equation (A4) insures that the constant C_1 is, in fact, independent of l, as required. Thus, the analyticity requirement has determined the scaling field g_t up to a multiplying constant.

Of course, when $\epsilon = 1$, $\bar{\phi} = 2$; so the above procedure will not work. The scaling field g_t defined by Eq. (5.15) in the text was constructed so that it would be useful for calculations at $\epsilon = 1$. In this case, the resulting function $t(g_t, g_h)$ is then analytic except for terms logarithmic in the scaling fields, in agreement with the results of Wegner.[9]

Note added in proof. After completion of this work, we received a report of work by M. Nauenberg (Max Plack Institute report) developing a formalism similar to that discussed here, but for ordinary critical points described by discrete recursion relations.

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