Finite-lattice approximations to renormalization groups*

Thomas L. Bell and Kenneth G. Wilson
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14850
(Received 30 December 1974)

Accurate calculations of the thermodynamic properties of a system near its critical point are now possible using finite-lattice approximations to renormalization groups. In order to investigate some features of such approximations, we introduce a linear-renormalization-group transformation appropriate to a system of continuous spins like the Gaussian model of Berlin and Kac. We solve the renormalization-group equations for the Gaussian model exactly, and then study in detail a finite-lattice approximation to the renormalization group for the three-dimensional case. The renormalization-group transformation contains a parameter b. The exact transformation has fixed points only for $b = 2^{-5/2}$. The approximate transformation has fixed points for a range of values of b around $2^{-5/2}$. Eigenvalues of the transformation, which determine critical exponents, depend on the value of b used in the calculation. We study the problem of identifying the correct value of b.

I. INTRODUCTION

The modern renormalization-group approach[1] provides a very useful framework within which to study the singular behavior of systems near their critical points, but exact renormalization-group equations from which the critical behavior can be derived are usually too difficult to work with. Various approximation schemes have been developed to simplify the equations. These include the approximate recursion formula of Ref. 1, the ϵ expansion,[2,3] and, more recently, what we shall refer to as the *finite-lattice approximation*, of which one of the first and most successful examples was given by Niemeijer and van Leeuwen.[4] Other work in this area has been reported by authors in Refs. 5–10.

These approximation schemes tend to complement one another. For instance, the ϵ expansion has revealed much about the structure and behavior of the renormalization-group equations, and has provided a fundamental explanation for many of the qualitative features of critical phenomena; but with our present computational skills it cannot yield critical exponents to many-digit accuracy. The finite-lattice approximation, on the other hand, allows one to work with a renormalization group tailored to the particular system under consideration, and can determine critical behavior with an accuracy that is very satisfying indeed, even though the structure of the exact renormalization group is obscured in the approximation scheme. There is also considerable pedagogical appeal in these calculations, because they are closer in spirit to the physical picture introduced by Kadanoff[11,12] to explain the scaling laws for critical phenomena.

Before discussing the finite-lattice approximation further, it will be convenient to review the basic ingredients of the renormalization-group approach. We shall assume the physical system which exhibits critical behavior to be a system of spins with ferromagnetic interactions described by a Hamiltonian $\mathcal{H}$, which includes the temperature factor $-1/k_B T$. A renormalization-group transformation $\mathcal{T}$ appropriate to the problem is constructed, which converts the old Hamiltonian $\mathcal{H}$ into a new one,

$$\mathcal{H}' = \mathcal{T}\{\mathcal{H}\} . \tag{1.1}$$

The transformation $\mathcal{T}$ must preserve the partition function of the system: The partition functions calculated from $\mathcal{H}$ and $\mathcal{H}'$ are the same. It is through this connection that one can relate the critical behavior of the original physical system to what one learns about the Hamiltonians generated by $\mathcal{T}$.

The renormalization-group transformation $\mathcal{T}$ determines the critical behavior of the original system if, with successive iterations of the transformation, it carries the original Hamiltonian $\mathcal{H} = \mathcal{H}_c$ (with all parameters such as temperature and magnetic field at their critical values) to a nontrivial fixed point $\mathcal{H}^*$ of the transformation, which satisfies the equation $\mathcal{T}\{\mathcal{H}^*\} = \mathcal{H}^*$. When the parameters of the original system are changed slightly from their critical values, so that $\mathcal{H} = \mathcal{H}_c + \epsilon \mathcal{O}$, the iterated transformation of the Hamiltonian $\mathcal{H}$ may be written, for large k,

$$\mathcal{T}^k\{\mathcal{H}_c + \epsilon \mathcal{O}\} = \mathcal{H}^* + \epsilon[(\Lambda_1)^k \mathcal{R}_1^* + (\Lambda_2)^k \mathcal{R}_2^* + \cdots] + O(\epsilon^2) , \tag{1.2}$$

where $\mathcal{T}^k$ represents k iterations of the transformation, the numbers $\Lambda_1 \geq \Lambda_2 \geq \cdots$ are eigenvalues of the transformation, and $\mathcal{R}_1^*$, $\mathcal{R}_2^*$, etc., are eigenoperators.

In ferromagnetic systems, if the perturbation $\epsilon \mathcal{O}$ is due to a shift in the temperature, then there is ordinarily only one eigenvalue in the expansion (1.2) larger than 1, called the thermal eigenvalue Λ_t. If the perturbation $\epsilon \mathcal{O}$ is instead due to a shift in the magnetic field, then the corresponding eigenvalue larger than 1 in the expansion is called the

magnetic eigenvalue Λ_h. If more than one eigenvalue larger than 1 is found, one must consider the possibility of "redundant" eigenoperators discussed by Wegner,[13] and use the eigenvalue left after these unphysical eigenvalues have been identified and discarded.

The exponents which characterize the behavior of the physical system near its critical point are determined by these eigenvalues. For example, if the scale change in the system caused by the renormalization-group transformation is by a factor of 2, as it will be for the renormalization group discussed in most detail here, then the critical exponents ν and η are given by

$$\nu = \ln 2/\ln\Lambda_t \tag{1.3a}$$

and

$$\eta = d + 2 - 2(\ln\Lambda_h)/\ln 2 \ . \tag{1.3b}$$

It is not necessary to confine oneself to computing the critical exponents of the system. Nauenberg and Nienhuis have described a more ambitious program in which entire thermodynamic functions can be obtained from the renormalization group.[7]

In the finite-lattice approximation, calculations are carried out in coordinate space instead of the Fourier-component space used in the other approaches, and the approximation is based on neglecting the effect of distant regions of the lattice on each other in calculating $\mathcal{K}'$ from $\mathcal{K}$. The validity of the approximation rests on the same foundation as the renormalization-group approach as a whole. One assumes that the coupling strengths in $\mathcal{K}'$ are analytic functions of the coupling strengths in $\mathcal{K}$, because the interactions among the spin variables in $\mathcal{K}'$ in one region of the lattice are not appreciably affected by distant regions of the system, even though the infinite correlation length at the critical point invalidates such an approximation in the calculation of the partition function itself.

The renormalization-group equations are simplified in the finite-lattice approximation, because interactions in $\mathcal{K}'$ that involve spins beyond a certain range are not kept, thereby reducing the infinity of possible interactions to a finite set; and the interactions in $\mathcal{K}'$ that *are* kept are determined by a bounded portion of the lattice, and so can be obtained by graphical methods[4,5,8,10] or by actually working with finite sections of the lattice.[4,6,7,9] If the fixed point $\mathcal{K}^*$ and the interesting eigenoperators $\mathcal{R}_t^*$ and $\mathcal{R}_h^*$ are sufficiently localized, this truncated set of equations will produce very accurate eigenvalues.

It is frequently possible to formulate the renormalization-group transformation so that its action depends on one or more parameters. Indeed, for transformations which relate the new spin variables to the old ones linearly ("linear-renormalization-group transformations"), at least one such parameter is essential, since its value determines whether or not a fixed point will be reached.[1,14] Because of the truncation of the renormalization-group equations in this approximation scheme, one sacrifices certain general properties of the exact renormalization group, the most troublesome of which are related to these parameters. For example, a linear-renormalization-group transformation should carry the critical Hamiltonian $\mathcal{K}_c$ to a fixed point for only one value of the parameter just mentioned; but in the finite-lattice approximation, fixed points will be reached for a variety of values of this parameter, and the critical exponents obtained will depend on what value of the parameter is chosen. Nonlinear renormalization groups have a similar problem. One expects the critical exponents calculated from the nonlinear renormalization group to be independent of the choice of parameters in the transformation, within certain limits.[13,14] But the exponents will depend on these parameters in the finite-lattice approximation.

It is therefore necessary to establish criteria with which to identify a best choice for the parameter values. In order to test some of these criteria for a linear renormalization group, we have investigated a model for which the exact renormalization-group equations can be obtained, and for which we can also obtain equations in a finite-lattice approximation. The model also serves to illustrate many features of the finite-lattice approximation.

It may help to summarize our results before entering into a more detailed discussion in the following sections. In Sec. II we introduce a linear-renormalization-group transformation suitable for a system of continuous spins on a lattice. We indicate how a parameter in it must be properly chosen if the transformation is to carry a critical Hamiltonian to a fixed point, and show that the existence of one fixed point implies the existence of a line of fixed points of the transformation (with the same value of the parameter). In Sec. III we study the Gaussian model of Berlin and Kac,[15] and derive the exact renormalization-group equations and their fixed points. In Sec. IV we obtain the finite-lattice equations analytically, and show that the most interesting eigenvalues depend on the parameter in the transformation.

In Sec. V some criteria for selecting the parameter are discussed. It is noted that one criterion for choosing the parameter—requiring that a relation between the eigenvalue Λ_h and the parameter, which is known to hold for the exact renormalization group, be satisfied in the approximation—fails to single out the correct value of the parameter, no matter how large a lattice is used in the approx-

imation. In Sec. VI it is found by numerical means that the finite-lattice equations in three dimensions have fixed points for a range of values of the parameter, but that searching for a vestige of the line of fixed points—a marginal eigenoperator—provides an unambiguous, and correct, choice of the parameter. Further discussion of the results appears in Sec. VII.

II. THE TRANSFORMATION

The renormalization group for which a finite-lattice approximation will be studied is defined for a system of spins $\sigma_{\vec{n}}$ on a d-dimensional lattice. The vector $\vec{n}$ labels the lattice sites, and the classical spin variables $\sigma_{\vec{n}}$ range over all real numbers. We shall assume the system to have only spin-spin interactions, like the Gaussian model of Berlin and Kac,[15] and to be governed by a Hamiltonian of the form

$$\mathcal{H}[\sigma] = -\frac{1}{2}\sum_{\vec{r}}\sum_{\vec{n}}\rho(\vec{r})\sigma_{\vec{n}}\sigma_{\vec{n}+\vec{r}} + h\sum_{\vec{n}}\sigma_{\vec{n}} \,. \tag{2.1}$$

The quantity h is proportional to an external magnetic field, and both h and $\rho(\vec{r})$ include a factor of $-1/k_BT$ in their definitions. The vectors $\vec{r}$ and $\vec{n}$ run over all lattice sites. The partition function for the system is

$$Z = \int_\sigma e^{\mathcal{H}[\sigma]} \,, \tag{2.2}$$

where $\int_\sigma$ denotes integration over all spin variables:

$$\int_\sigma \equiv \prod_{\vec{n}} \int_{-\infty}^{\infty} d\sigma_{\vec{n}} \,.$$

We shall define the renormalization-group transformation for the case of a simple-cubic lattice. The lattice sites are divided up into cubic blocks, each block containing 2^d sites. An illustration of the two-dimensional case appears in Fig. 1. As in Kadanoff's picture,[11] with each block is associated a new block-spin variable $S_{\vec{n}'}$, the vector $\vec{n}' = (n_1', n_2', \ldots, n_d')$ labels the sites of the block-spin lattice. (Quantities defined on the block-spin lattice will always be distinguished by primes.) The renormalization-group transformation is written as

$$e^{\mathcal{H}'[S]} = T_{a,b}[S]\, e^{\mathcal{H}[\sigma]} \,, \tag{2.3a}$$

with

$$T_{a,b}[S]\, e^{\mathcal{H}[\sigma]} \equiv \int_\sigma e^{\mathcal{T}_{a,b}[S,\sigma]+\mathcal{H}[\sigma]} \,, \tag{2.3b}$$

$$\mathcal{T}_{a,b}[S,\sigma] \equiv -\tfrac{1}{2}a\sum_{\vec{n}'}\left(S_{\vec{n}'} - b\sum_{\vec{j}\in\vec{n}'}\sigma_{\vec{j}}\right)^2 . \tag{2.3c}$$

The term $\sum_{\vec{j}\in\vec{n}'}\sigma_{\vec{j}}$ in Eq. (2.3c) means the sum of the spins $\sigma_{\vec{j}}$ included within the block labeled by $\vec{n}'$.

Many of the properties of this transformation can be deduced without specific knowledge about the Hamiltonian $\mathcal{H}$. First, the transformation preserves the partition function of the system:

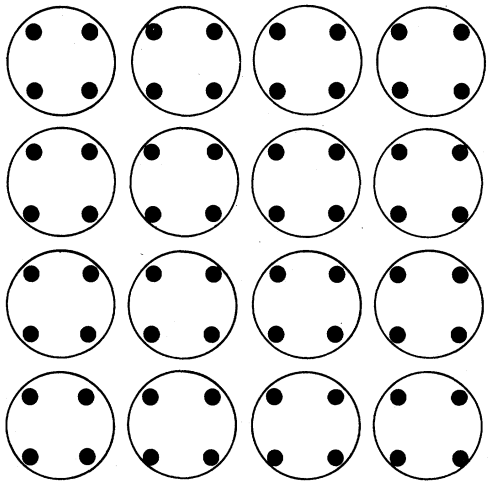

FIG. 1. Original lattice, illustrated here by a two-dimensional array of dots, is divided up into blocks with two sites on a side, 2^d sites altogether.

$$\begin{aligned} Z' &\equiv \int_S e^{\mathcal{H}'[S]} \\ &= \int_S \int_\sigma e^{\mathcal{T}_{a,b}[S,\sigma]+\mathcal{H}[\sigma]} \\ &= \int_\sigma \int_S e^{\mathcal{T}_{a,b}[S,\sigma]+\mathcal{H}[\sigma]} \,, \\ Z' &= \text{const}\times Z \,, \end{aligned} \tag{2.4}$$

where the constant is independent of the Hamiltonian $\mathcal{H}$ and might have been absorbed in the definition of $T_{a,b}$ as an uninteresting normalization factor. These factors will be ignored in all of the equations that follow, since they do not affect any of the results. We shall also omit all spin-independent constants in the Hamiltonians $\mathcal{H}'$ generated from $\mathcal{H}$, since these constants should not contain any information about the singular behavior of the system near its critical point. (It is by keeping track of these terms that Nauenberg and Nienhuis[7] are able to obtain the complete thermodynamic functions in their calculations.)

A second property of the transformation (2.3) is that the parameter a in the transformation is nearly superfluous, since one may easily relate results obtained with one value of a to those obtained with another value. In fact, by changing integration variables in Eq. (2.3b) to $\sigma_{\vec{n}} = \lambda\tilde{\sigma}_{\vec{n}}$, one readily shows that, if the Hamiltonians $\mathcal{H}[\sigma]$ and $\mathcal{H}'[S]$ are related as in Eq. (2.3a), then one also has

$$e^{\mathcal{H}'[\lambda S]} = T_{\lambda^2 a,b}[S]\, e^{\mathcal{H}[\lambda\sigma]} \,. \tag{2.5}$$

Thus, if one knows the outcome of the transformation $T_{a,b}$ for all Hamiltonians $\mathcal{H}[\sigma]$ differing only in

the over-all normalization of the spin variables σ, then one also knows the outcome of the transformations $T_{\lambda^2 a,b}$ for any λ.

In the limit $a \to \infty$ the Gaussian functions of $S_{\vec{n}'} - b\sum_{\vec{j}\in\vec{n}'}\sigma_{\vec{j}}$ in Eq. (2.3b) approach δ functions (when the transformation is appropriately normalized), and one has

$$\lim_{a\to\infty} T_{a,b}[S]\, e^{\mathcal{K}} = \int_\sigma \prod_{\vec{n}'} \delta\left(S_{\vec{n}'} - b\sum_{\vec{j}\in\vec{n}'}\sigma_{\vec{j}}\right) e^{\mathcal{K}} . \quad (2.6)$$

Such a transformation has been considered in detail by Høye[16] and Jona-Lasinio.[17] The physical interpretation of the transformation for $a \to \infty$ is clear: The new block-spin variable is just proportional to the average spin in the block. However, we shall be more interested here in the transformation with finite a.

A more important property of the transformation involves the parameter b. The transformation is "linear," in the sense defined in the Introduction, and this leads to a condition which must be satisfied by the parameter b if the transformation $T_{a,b}$ is to carry the critical Hamiltonian to a useful fixed point. This condition is easily derived by considering the spin-spin correlation functions determined by $\mathcal{K}[\sigma]$ and $\mathcal{K}'[S]$, which are defined by

$$\Gamma_{\vec{n}\vec{0}}[\mathcal{K}] \equiv Z^{-1} \int_\sigma \sigma_{\vec{n}}\sigma_{\vec{0}} e^{\mathcal{K}[\sigma]} . \quad (2.7)$$

By proceeding in a manner very similar to that used in Ref. 14, one may show that a useful fixed point can be reached only if b satisfies the condition

$$b = 2^{-(d+2-\eta)/2} \equiv b^* . \quad (2.8)$$

A discussion of this argument can also be found in Ref. 18. Since the critical exponent η is also related to the magnetic eigenvalue Λ_h by Eq. (1.4b), the condition (2.8) establishes a relation between Λ_h and b:

$$\Lambda_h = b^{-1} . \quad (2.9)$$

Finally, because of the form of the transformation $T_{a,b}$ defined in Eq. (2.3), one can show that if $T_{a,b}{}^*$ has one fixed point $\mathcal{K}^*$ satisfying the equation

$$T_{a,b}{}^*[S]\, e^{\mathcal{K}^*[\sigma]} = e^{\mathcal{K}^*[S]} , \quad (2.10)$$

then there must in fact exist a line of fixed points of $T_{a,b}{}^*$.[19] This is readily demonstrated if a second transformation U can be found that commutes with $T_{a,b}{}^*$ in the sense of

$$T_{a,b}{}^* U e^{\mathcal{K}} = U T_{a,b}{}^* e^{\mathcal{K}} .$$

For if such a transformation exists, then one has immediately

$$T_{a,b}{}^*[U e^{\mathcal{K}^*}] = U e^{\mathcal{K}^*} ,$$

and the Hamiltonian generated by U is also a fixed point of $T_{a,b}{}^*$.

A transformation that serves this purpose is

$$U_{\alpha,\beta} e^{\mathcal{K}} \equiv \int_{\tilde{S}} \exp\left(-\tfrac{1}{2}\alpha \sum_{\vec{n}} (S_{\vec{n}} - \beta\tilde{S}_{\vec{n}})^2 + \mathcal{K}[\tilde{S}]\right) . \quad (2.11)$$

Note that *every* old spin variable $\tilde{S}_{\vec{n}}$ is associated with a new spin variable $S_{\vec{n}}$, in contrast with the reduction in the number of variables effected by $T_{a,b}$. The commutation of $U_{\alpha,\beta}$ with $T_{a,b}{}^*$ is most easily shown by first verifying the identities

$$U_{\alpha,\beta} T_{a,b} = T_{\alpha a/(a+\alpha\beta^2),\beta b} \quad (2.12a)$$

and

$$T_{a,b} U_{\alpha,\beta} = T_{\alpha a/(\alpha+2^d b^2 a),\beta b} , \quad (2.12b)$$

which are proved by carrying out the Gaussian integrals over the intermediate spin variables. (Note that we continue to ignore normalization factors.) For given a and b^*, then, $U_{\alpha,\beta}$ commutes with $T_{a,b}{}^*$ for

$$a + \alpha\beta^2 = \alpha + 2^d (b^*)^2 a$$

or

$$\alpha = a[1 - 2^d (b^*)^2]/(1-\beta^2) . \quad (2.13)$$

Thus, the set of Hamiltonians

$$e^{\mathcal{K}^*_\beta} \equiv U_{\alpha,\beta} e^{\mathcal{K}^*} , \quad (2.14)$$

with α given by Eq. (2.13) and $\mathcal{K}^*$ a solution of Eq. (2.10), are all fixed points of $T_{a,b}{}^*$. Note that α becomes infinite for $\beta \to 1$, and that the form of the transformation $U_{\alpha,\beta}$ is such that it becomes just an identity transformation in this limit, and $\mathcal{K}^*_{\beta=1} = \mathcal{K}^*$.

The relation between $\mathcal{K}^*_\beta$ and the original fixed point $\mathcal{K}^*$ can be understood by considering the correlation functions for the two Hamiltonians. The generating functional for correlation functions is convenient for this purpose:

$$\begin{aligned}\mathcal{F}[J] &\equiv \int_S \exp\left(\sum_{\vec{n}} J_{\vec{n}} S_{\vec{n}} + \mathcal{K}_\beta{}^*[S]\right) \\ &= \int_S \exp\left(\sum_{\vec{n}} J_{\vec{n}} S_{\vec{n}}\right) U_{\alpha,\beta} e^{\mathcal{K}^*} \\ &= \int_{\tilde{S}} \exp\left(\sum_{\vec{n}} (\beta J_{\vec{n}} \tilde{S}_{\vec{n}} + \tfrac{1}{2}\alpha^{-1} J_{\vec{n}}^2) + \mathcal{K}^*[\tilde{S}]\right) .\end{aligned}$$

This result and the definition of the correlation function in (2.7) imply that

$$\Gamma_{\vec{n}\vec{0}}[\mathcal{K}^*_\beta] = \beta^2 \Gamma_{\vec{n}\vec{0}}[\mathcal{K}^*] + \alpha^{-1}\delta_{\vec{n},\vec{0}} .$$

One may thus interpret the action of the transformation $U_{\alpha,\beta}$ to be simply a change in the normalization of the spins by a factor β, as well as a change in the Hamiltonian that affects only the self-correlation of the spins (such as the $\alpha^{-1}\delta_{\vec{n},\vec{0}}$ term), not the long-range correlations of the system.

Many properties of the renormalization-group transformation $T_{a,b}$ can be derived by means of the transformation $U_{\alpha,\beta}$. For example, it can be shown

that if $T_{a,b}*$ carries the Hamiltonian $\mathcal{H}_c$ to a fixed point for a given value of the parameter a, then it will carry $\mathcal{H}_c$ to fixed points for any value of a, and the eigenvalues obtained will not depend on a (though the eigenoperators will, of course). Nor will the eigenvalues depend on which Hamiltonian on the line of fixed points one chooses to work with. The transformation $U_{\alpha,\beta}$ can also be used to generate several redundant eigenoperators of $T_{a,b}*$. However, we shall content ourselves with deriving these results explicitly for the Gaussian model, to which we now proceed in Sec. III.

III. RENORMALIZATION GROUP FOR GAUSSIAN MODEL

A. The iterated transformation

The results of Sec. II did not require knowledge of the specific form of the Hamiltonian $\mathcal{H}[\sigma]$. In this and the following sections we shall limit ourselves to Hamiltonians of the form (2.1). In order to find fixed points of the transformation $T_{a,b}$, we shall want to know the effect of iterated transformations of the original Hamiltonian, given by

$$e^{\mathcal{H}^{(k)}[S]} = [T_{a,b}]^k e^{\mathcal{H}[\sigma]} . \tag{3.1}$$

Because the transformation $T_{a,b}$ involves only Gaussian functions of the new and old spins, the iterated transformation $[T_{a,b}]^k$ also has such a form. It is shown in the Appendix that it is given by

$$T^k_{a,b} e^{\mathcal{H}[\sigma]} \equiv [T_{a,b}]^k e^{\mathcal{H}[\sigma]} \tag{3.2a}$$

$$= \int_\sigma e^{\mathcal{T}^k_{a,b}[S,\sigma] + \mathcal{H}[\sigma]} , \tag{3.2b}$$

where we find

$$\mathcal{T}^k_{a,b}[S,\sigma] = -\tfrac{1}{2} a_k \sum_{\vec{n}'} \left(S_{\vec{n}'} - b_k \sum_{\vec{j} \in \vec{n}'} \sigma_{\vec{j}} \right)^2 \tag{3.2c}$$

with

$$a_k = a(1 - 2^d b^2)/[1 - (2^d b^2)^k] \tag{3.3a}$$

and

$$b_k = b^k . \tag{3.3b}$$

The sum $\sum_{\vec{j} \in \vec{n}'}$ includes all L^d spins $\sigma_{\vec{j}}$ now represented by the block spin $S_{\vec{n}'}$ with

$$L = 2^k , \tag{3.4}$$

where L is the length of the side of one of the blocks of old spins which is associated with a single new block-spin variable after k transformations.

Although the exact renormalization group is defined for an infinite lattice, we shall later wish to know the results of the transformation for a finite lattice. In order to evaluate $\mathcal{H}^{(k)}$ for a finite lattice, we shall require a more detailed characterization of the quantities above in terms of the lattice parameters.

The original lattice will be assumed to be a cube with N spins on a side, or N^d spins altogether. We shall eventually obtain the exact renormalization-group equations by letting N tend to infinity. Since the lattice will be subdivided k times into a final lattice with

$$N' = N/L \tag{3.5}$$

block spins on a side, we must assume that N is an integral multiple of L. Sums over all lattice sites may be written as

$$\sum_{\vec{n}} = \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} \cdots \sum_{n_d=0}^{N-1} \equiv \sum_{\vec{n}=0}^{N-1} . \tag{3.6}$$

The transformation kernel $\mathcal{T}^k_{a,b}[S,\sigma]$ defined in Eq. (3.2c) may now be written as

$$\mathcal{T}^k_{a,b}[S,\sigma] = -\tfrac{1}{2} a_k \sum_{\vec{n}'=0}^{N'-1} \left(S_{\vec{n}'} - b_k \sum_{\vec{m}=0}^{L-1} \sigma_{L\vec{n}+\vec{m}} \right)^2 , \tag{3.7}$$

where the summations $\sum_{\vec{n}'=0}^{N'-1}$ and $\sum_{\vec{m}=0}^{L-1}$ are defined as in Eq. (3.6).

Because we shall assume periodic boundary conditions for the lattice, the quantity $\rho(\vec{r})$ and the spin variables $\sigma_{\vec{n}}$ satisfy

$$\rho(\vec{r} + N\hat{e}_m) = \rho(\vec{r}) , \tag{3.8a}$$

$$\sigma_{\vec{n}+N\hat{e}_m} = \sigma_{\vec{n}} , \quad m = 1, \ldots, d , \tag{3.8b}$$

where $\hat{e}_m$ is a lattice vector in the mth direction. The periodicity makes it convenient to introduce Fourier components defined by the equations

$$\sigma_{\vec{n}} = N^{-d} \sum_{\vec{q}} e^{i\vec{q}\cdot\vec{n}} \sigma_{\vec{q}} , \tag{3.9a}$$

$$\rho(\vec{r}) = N^{-d} \sum_{\vec{q}} e^{i\vec{q}\cdot\vec{r}} \rho(\vec{q}) , \tag{3.9b}$$

$$S_{\vec{n}'} = (N')^{-d} \sum_{\vec{q}'} e^{i\vec{q}'\cdot\vec{n}'} S_{\vec{q}'} , \tag{3.9c}$$

where the vector $\vec{q}$ may only take the values $\vec{q} = (2\pi/N)\vec{n}$ and the summation $\sum_{\vec{q}}$ means

$$\sum_{\vec{q}} f(\vec{q}) \equiv \sum_{\vec{n}=0}^{N-1} f\left(\frac{2\pi\vec{n}}{N}\right), \tag{3.10}$$

and where the vector $\vec{q}'$ is similarly defined, with N replaced by N'. One may now proceed to carry out the transformation in Eq. (3.2) using the simplifications that appear when all quantities are expressed in terms of their Fourier components. Details of this calculation are given in the Appendix. One finds

$$\mathcal{H}^{(k)}[S] = -\tfrac{1}{2}(N')^{-d} \sum_{\vec{q}'} \rho^{(k)}(\vec{q}') S_{\vec{q}'} S_{-\vec{q}'} + h^{(k)} \sum_{\vec{n}'=0}^{N'-1} S_{\vec{n}'} , \tag{3.11}$$

where the new interactions $\rho^{(k)}$ and $h^{(k)}$ are given in terms of the old ones ρ and h by

$$\rho^{(k)}(\vec{q}') = a_k/[1 + a_k b_k^2 \omega^{(k)}(\vec{q}')] \tag{3.12}$$

and

$$h^{(k)} = (2^d b)^k [\rho^{(k)}(\vec{q}' = 0)/\rho(\vec{q} = 0)] h \,, \tag{3.13}$$

with

$$\omega^{(k)}(\vec{q}') = L^{-d} \sum_{\vec{l}=0}^{L-1} \frac{|u_L[(\vec{q}' + 2\pi\vec{l})/L]|^2}{\rho[(\vec{q}' + 2\pi\vec{l})/L]} \,, \tag{3.14}$$

and

$$|u_L(\vec{x})|^2 = \prod_{i=1}^{d} \frac{[\sin(Lx_i/2)]^2}{[\sin(x_i/2)]^2} \,. \tag{3.15}$$

B. Fixed points of the transformation

With these results in hand it is relatively straightforward to determine the fixed points of $T_{a,b}$ for an infinite lattice, by setting the parameters in $\mathcal{H}[\sigma]$ to their critical values and finding the limiting form of $\mathcal{H}^{(k)}[S]$ for $k \to \infty$. We first note that, in the infinite-lattice limit ($N \to \infty$), the Fourier-transform variable $\vec{q}$ becomes continuous and sums over $\vec{q}$ become integrals, as $N^{-d} \sum_{\vec{q}} \to (2\pi)^{-d} \int d^d q$. If the interaction strengths $\rho(\vec{r})$ in $\mathcal{H}[\sigma]$ are short range, then $\rho(\vec{q})$ for small $\vec{q}$ must have the form

$$\rho(\vec{q}) \sim r_0 + zq^2 + \cdots \,. \tag{3.16}$$

The Hamiltonian becomes critical for $r_0 = h = 0$. Because we shall later be interested in considering Hamiltonians with long-range interactions, we shall admit the possibility of more general behavior of $\rho(\vec{q})$ for small $\vec{q}$,

$$\rho(\vec{q}) \sim r_0 + z|\vec{q}|^{2+\epsilon} + \cdots \,. \tag{3.17}$$

The parameter ϵ characterizes how $\rho(\vec{r})$ decreases with distance for large r, as $\rho(\vec{r}) \sim 1/r^{d+2+\epsilon}$ ($\epsilon \neq 0$). Note, however, that interactions with small-$\vec{q}$ behavior like (3.17) for positive ϵ will be rather artificial, since any short-range interaction is likely to dominate with a term proportional to q^2.

With the choice (3.17) for $\rho(\vec{q})$, and $r_0 = h = 0$, it is not difficult to find the limiting form of $\rho^{(k)}(\vec{q})$. One finds that a nontrivial fixed point is reached only if one chooses

$$b = 2^{-(d+2+\epsilon)/2} \,, \tag{3.18}$$

and that with this choice $\rho^{(k)}(\vec{q})$ approaches the fixed point

$$\rho^*(\vec{q}) = a^*/[1 + (a^*/z)\Omega_\epsilon^*(\vec{q})] \,, \tag{3.19}$$

with

$$a^* = a(1 - 2^d b^2) \tag{3.20}$$

and

$$\Omega_\epsilon^*(\vec{q}) = \sum_{\vec{l}=-\infty}^{\infty} \frac{\prod_{i=1}^{d} [\sin(q_i/2)]^2/(q_i/2 + \pi l_i)^2}{|\vec{q} + 2\pi\vec{l}|^{2+\epsilon}} \,. \tag{3.21}$$

Many of the properties of the fixed points $\rho^*(\vec{q})$ given in Eq. (3.19) are readily discovered. We note first of all that the fixed point reached depends on the original Hamiltonian only through the parameters z and ϵ, and that for small $\vec{q}$, $\rho^*(\vec{q}) \sim z|\vec{q}|^{2+\epsilon}$ behaves just as $\rho(\vec{q})$ did. The renormalization-group transformation therefore has nonlocal ($\epsilon \neq 0$) fixed points; but these fixed points are inaccessible to the transformation if the initial Hamiltonian $\mathcal{H}[\sigma]$ is a local one ($\epsilon = 0$). In fact, as we have just mentioned in the derivation of Eq. (3.19), if the initial Hamiltonian is local ($\epsilon = 0$), then a nontrivial fixed point is reached for only one choice of the parameter b,

$$b = b^* \equiv 2^{-(d+2)/2} \,; \tag{3.22}$$

for other values of b the iterated transformation $T_{a,b}$ carries the original Hamiltonian either to zero or to a fixed point describing a completely noninteracting system of block spins, and not to any of those given in Eq. (3.19) with $\epsilon \neq 0$.

Let us therefore ignore the nonlocal fixed points for the moment. We note next that there is a line of fixed points of T_{a,b^*}, as was predicted in Sec. II; these fixed points are parametrized by the parameter z. The range of the interactions described by $\rho^*(\vec{r})$, the Fourier inverse of $\rho^*(\vec{q})$, varies along this fixed line. A rough estimate of the range may be obtained by locating the pole in $\rho^*(\vec{q})$ nearest the real-$\vec{q}$ axis. Using $\Omega_0^*(\vec{q}) \sim 1/q^2$ for small $\vec{q}$, one quickly ascertains that there is certainly a pole at $q^2 \approx -a^*/z$ if a^*/z is small enough, and so the range of interactions described by $\rho^*(\vec{r})$ is roughly $(z/a)^{1/2}$. The fixed points become less and less local as $a/z \to 0$. This estimate fails for large a/z, but numerical inversion of $\rho^*(\vec{q})$ indicates that the range of $\rho^*(\vec{r})$ continues to decrease until $a/z \approx 8$, and then increases to a finite limit for $a/z \to \infty$. Note that in the limit $a \to \infty$, when the transformation $T_{a,b}$ approaches the form given in Eq. (2.6), all fixed points described by $\rho^*(\vec{q})$ in Eq. (3.19) (still with $\epsilon = 0$) have the same range, and differ from one another only in the over-all normalization factor z. As we shall see later, it is this fact which makes the transformation $T_{a,b}$ with finite a more interesting to us.

Finally, we note that the fixed points $\rho^*(\vec{r})$ of $T_{a,b}$ for small ϵ cannot be qualitatively distinguished from the local fixed points for $\vec{r}$ smaller than the range of interactions of the local fixed point; in fact, the distinction does not become obvious until the local fixed-point interactions, which decline exponentially with $\vec{r}$, have dropped sufficiently in size to become comparable with the strength of the nonlocal fixed-point interactions, which behave as $\epsilon/r^{d+2+\epsilon}$ for large distances.

Let us summarize what one finds for the behavior of the renormalization-group transformation $T_{a,b}$

for an infinite lattice: There is a line of local fixed points for $b=b^*$, but with varying range of interactions (unless $a\to\infty$). For $b\neq b^*$, there are nonlocal fixed points, which cannot be distinguished from local ones unless one looks at interactions involving sufficiently large separations, but which are inaccessible to the transformation when the original Hamiltonian is a local one.

Finally, we must discuss the eigenvalues one finds for perturbations about the fixed point. These eigenvalues are easily obtained by introducing perturbations of the original Hamiltonian $\mathcal{H}_c$ and looking for the behavior indicated in Eq. (1.2) when the transformation is iterated many times. When the Hamiltonians are local and $b=b^*$, one finds the eigenvalues

$$\Lambda_t=4 \tag{3.23}$$

and

$$\Lambda_1=1\ . \tag{3.24}$$

The eigenvalue $\Lambda_t=4$ determines the critical exponent $\nu=\frac{1}{2}$ from Eq. (1.3a), as expected. The eigenvalue $\Lambda_1=1$ corresponds to a "marginal" eigenoperator. It is associated with perturbations along the direction of the line of fixed points (i.e., changes in z), which are, of course, stationary under the renormalization-group transformation. Eigenvalues $\Lambda_m=4^{1-m}$, $m=2,3,\ldots$, are found which are smaller than one and therefore irrelevant. The eigenvalue corresponding to the addition of a magnetic field can be calculated from Eq. (3.13), and it is found to be

$$\Lambda_h=2^{(d+2)/2}\ , \tag{3.25}$$

as we could have predicted from Eq. (2.9). This implies $\eta=0$ from Eq. (1.3b), as expected.

IV. THE FINITE-LATTICE APPROXIMATION

A. Justification for approximation

Now that we know the behavior of the exact renormalization group, we are ready to investigate the finite-lattice approximation for it. As we mentioned in the Introduction, a successful renormalization-group transformation generates recursion relations for the interaction strengths in the new Hamiltonian in terms of analytic functions of the old interaction strengths, even though the thermodynamic functions may be singular; the recursion relations are analytic because the interactions among block spins in one region are scarcely affected by regions of the lattice far distant from the one of interest, in spite of the possibly infinite correlation lengths of the system. This same phenomenon—effective isolation of distant regions of the lattice from one another—provides the justification for the finite-lattice approximation.[6]

The renormalization-group transformation discussed here provides a particularly good example of how this decoupling of distant regions of the lattice from each other occurs. Let us rewrite the transformation in Eq. (3.2) in the form

$$T_{a,b}e^{\mathcal{H}[\sigma]}=\exp\left(-\tfrac{1}{2}a\sum_{\vec{n}'}S_{\vec{n}'}^2\right)\times\int_\sigma\exp\left(ab\sum_{\vec{n}'}S_{\vec{n}'}\sum_{\vec{j}\in\vec{n}'}\sigma_{\vec{j}}+\mathcal{H}_{\rm eff}[\sigma]\right)\ , \tag{4.1}$$

with

$$\mathcal{H}_{\rm eff}[\sigma]\equiv\mathcal{H}[\sigma]-\tfrac{1}{2}ab^2\sum_{\vec{n}'}\left(\sum_{\vec{j}\in\vec{n}'}\sigma_{\vec{j}}\right)^2\ , \tag{4.2}$$

and define the bracket operation

$$\langle\mathcal{O}[\sigma]\rangle\equiv\int_\sigma\mathcal{O}[\sigma]e^{\mathcal{H}_{\rm eff}[\sigma]}/Z_{\rm eff}\ ,$$

$$Z_{\rm eff}\equiv\int_\sigma e^{\mathcal{H}_{\rm eff}[\sigma]}\ .$$

Then Eq. (4.1) becomes

$$e^{\mathcal{H}'[S]}=\exp\left(-\tfrac{1}{2}a\sum_{\vec{n}'}S_{\vec{n}'}^2+\ln Z_{\rm eff}\right)\times\left\langle\exp\left(ab\sum_{\vec{n}'}S_{\vec{n}}\sum_{\vec{j}\in\vec{n}'}\sigma_{\vec{j}}\right)\right\rangle\ . \tag{4.3}$$

The cumulant expansion of the bracketed exponential above gives

$$\mathcal{H}'[S]=\text{const}-\tfrac{1}{2}a\sum_{\vec{n}'}S_{\vec{n}'}^2+\sum_{p=1}^{\infty}\frac{1}{p!}\left\langle\left(ab\sum_{\vec{n}'}S_{\vec{n}'}\sum_{\vec{j}\in\vec{n}'}\sigma_{\vec{j}}\right)^p\right\rangle_c\ , \tag{4.4}$$

where the subscript c of the bracket indicates that only the "connected part" of the expression is to be kept. If the original Hamiltonian $\mathcal{H}[\sigma]$ is quadratic in the spin variables $\sigma_{\vec{n}}$, then the only nonvanishing term in the sum in Eq. (4.4) is the $p=2$ term, so that we have

$$\mathcal{H}'[S]=\text{const}-\tfrac{1}{2}a\sum_{\vec{n}'}S_{\vec{n}'}^2+\tfrac{1}{2}a^2b^2\times\sum_{\vec{n}_1'}\sum_{\vec{n}_2'}S_{\vec{n}_1'}S_{\vec{n}_2'}\left\langle\left(\sum_{\vec{i}\in\vec{n}_1'}\sigma_{\vec{i}}\right)\left(\sum_{\vec{j}\in\vec{n}_2'}\sigma_{\vec{j}}\right)\right\rangle\ ; \tag{4.5}$$

the subscript c is no longer necessary. One can now see that the range of interactions between block spins is determined by the degree of correlation of the spins $\sigma_{\vec{n}}$ dictated by $\mathcal{H}_{\rm eff}[\sigma]$ (not $\mathcal{H}[\sigma]$). But $\mathcal{H}_{\rm eff}[\sigma]$ is considerably different from $\mathcal{H}[\sigma]$ because of the additional local interactions between spins introduced by the $(\sum_{\vec{j}}\sigma_{\vec{j}})^2$ terms in it, and the long-range correlations characteristic of the critical point are unlikely after such a modification of the Hamiltonian. This means that the interaction between block spins is largely determined by the behavior of the original Hamiltonian in the vicinity of these spins.

If the Hamiltonians involved in the calculation are sufficiently local, and if the interactions among block spins are determined only by their immediate environment, then one is quickly led to approximate the renormalization-group equations by keeping only a confined set of interactions and ignoring contributions to them from distant spins. We have already indicated the variety of methods that have been introduced to take advantage of the local determinism of the renormalization group.[4–10]

B. Exact results for finite-lattice approximation

In our calculation a lattice of a certain size N is used. Although the periodic boundary conditions used here in the realization of the finite-lattice approximation may appear artificial, from a graphical point of view they can be seen to introduce errors of the same order as would be introduced by other forms of truncation at the boundaries of the lattice. However, the particular errors introduced by the periodic boundary conditions tend to cooperate to produce results that are especially simple.

Only the interactions $\rho_N(\vec{r})$ that fit within this lattice of size N are kept. The renormalization-group transformation $T_{a,b}$ generates a new set of interactions $\rho'_{N/2}(\vec{r}')$ defined on a lattice of size $\frac{1}{2}N$. The recursion relations for the Fourier components of these interactions can be obtained directly from Eqs. (3.12) and (3.14) by setting $k=1$:

$$\rho'_{N/2}(\vec{q}') = a/[1+ab^2\omega(\vec{q}')] , \tag{4.6}$$

with

$$\omega(\vec{q}') = 2^{-d}\sum_{\vec{l}=0}^{1} \frac{|u_2[(\vec{q}'+2\pi\vec{l})/2]|^2}{\rho_N[(\vec{q}'+2\pi\vec{l})/2]} . \tag{4.7}$$

Since it is assumed that $\rho'_{N/2}(\vec{r}')$ is a good approximation to the interactions that would have been generated by an exact calculation, we set

$$\rho'_N(\vec{r}=\vec{r}') = \rho'_{N/2}(\vec{r}') \tag{4.8}$$

in order to iterate the transformation.[20] Those interactions which are possible in the larger lattice but cannot fit in the smaller lattice are assumed to vanish, which would be a good approximation if the interactions are sufficiently local. Note that we use the prime now in two ways: The prime on ρ' distinguishes it from the interactions ρ from which it was generated by the transformation $T_{a,b}$; the prime on $\vec{r}'$ and $\vec{q}'$ distinguishes vectors defined on the smaller block-spin lattice of size $\frac{1}{2}N$ from those of the larger lattice.

Some features of the fixed point and associated eigenvalues within this approximation can be determined exactly, without resorting to numerical methods.

First, if the finite-lattice equations have a fixed point, then $\rho^*_N(\vec{q}=0)$ is determined—in fact, we shall see that it must vanish if the fixed point is nontrivial. We first note, from Eqs. (3.15) and (4.7), that

$$\omega(\vec{q}'=0) = 2^d/\rho_N(\vec{q}=0) , \tag{4.9}$$

and, from the definition of the Fourier components in Eq. (3.9b), that if the nonvanishing interactions satisfy Eq. (4.8), then their Fourier components must satisfy

$$\rho'_N(\vec{q}=0) = \rho'_{N/2}(\vec{q}'=0) . \tag{4.10}$$

At a fixed point, where one has $\rho'_N(\vec{q}) = \rho_N(\vec{q}) = \rho^*_N(\vec{q})$, Eqs. (4.6), (4.9), and (4.10) require

$$\rho^*_N(\vec{q}=0) = 0 \quad \text{or} \quad a(1-2^d b^2) . \tag{4.11}$$

The second, nonzero possibility can be shown to correspond to a trivial fixed point with $\rho^*_N(\vec{q}\neq 0)=0$, representing a completely noninteracting system with Hamiltonian

$$\mathcal{H}^*[S] = -\tfrac{1}{2}a(1-2^d b^2)\sum_{\vec{n}} S_{\vec{n}}^2 , \tag{4.12}$$

and so we shall not consider it further.

Second, if a fixed point exists, then the magnetic eigenvalue can be determined immediately by using Eqs. (3.12) and (3.13). We find

$$\Lambda_h = a2^d b/[\rho^*_N(\vec{q}=0) + a2^d b^2] , \tag{4.13}$$

and if the fixed point is nontrivial, then we must have

$$\Lambda_h = b^{-1} . \tag{4.14}$$

Note that the significance of this equation is entirely different from that of the identical relation given in Eq. (2.9). Here it is a consequence of the finite-lattice approximation for the recursion relations of the Gaussian model. There it was a consequence of the exact renormalization group.

Finally, we can determine the thermal eigenvalue Λ_t in the finite-lattice approximation from the relation

$$\rho'_N(\vec{q}=0) = a/[1+2^d b^2 a/\rho_N(\vec{q}=0)] , \tag{4.15}$$

which follows from Eqs. (4.6), (4.9), and (4.10). The deviation of $\rho_N(\vec{q}=0)$ from zero is proportional to the deviation of the temperature from its critical value; when $\rho_N(\vec{q}=0)$ is small but nonzero, Eq. (4.14) shows that it is increased with each iteration of the transformation by a factor

$$\Lambda_t = 2^{-d}b^{-2} . \tag{4.16}$$

V. CRITERIA FOR SELECTING A FIXED POINT

In Sec. IV we obtained the relevant eigenvalues of the finite-lattice approximation to the recursion relations for the Gaussian model, in terms of the transformation parameter b. But we have neither determined this parameter nor demonstrated the existence of a nontrivial fixed point of the approxi-

mate renormalization-group equations. We learned in Secs. II and III that for an infinite lattice, if the renormalization-group transformation has a fixed point related to the original system, then it will only be found for one value of the parameter $b = b^*$, and that there will in fact be a line of fixed points, with the same eigenvalues but with different ranges of interaction. How much of this will survive in the approximation scheme?

Since the approximation does not permit interactions beyond a certain range, it can accurately represent the behavior of the transformation for only the most localized of fixed points. It is therefore unlikely that the approximate renormalization-group equations will have a line of fixed points, since the approximation will break down where the range of interactions for the Hamiltonians on the fixed line exceeds the limit imposed by the approximation. Furthermore, there is no reason to suppose that the approximate renormalization-group equations will not have fixed points for $b \neq b^*$. Fixed points for $b \neq b^*$ exist for the infinite-lattice transformation as well, but they are nonlocal and do not interfere with the determination of critical exponents of a system governed by a local Hamiltonian, because they cannot be reached by renormalization-group transformations of a local Hamiltonian. The parameter b can therefore be fixed in an exact calculation by requiring that the iterated renormalization-group transformation take the original critical Hamiltonian to a nontrivial fixed point. But in the finite-lattice approximation, fixed points for $b \neq b^*$ will be accessible, starting from a local, critical Hamiltonian, and so some other means for choosing the parameter b must be found.

Criteria for choosing b that immediately suggest themselves all require that some property expected of the infinite-lattice calculation be searched for in the finite-lattice approximation. For instance, one might hope that the approximate renormalization-group equations have fixed points only for $b = b^*$, but we have already suggested that this is unlikely; in actual calculations it is not observed.

A possible criterion might be the locality of the fixed-point Hamiltonians. Of course, in the finite-lattice approximation all fixed points are local, but one would choose the fixed point for which the interactions seem to fall off most quickly. This can provide an excellent clue to the proper value of b, but appears rather arbitrary when formulated quantitatively, for the Gaussian model it is found to be not so accurate, in fact, as another criterion to be discussed later. As we have noted in Sec. III, the nonlocal fixed points cannot be qualitatively distinguished from the local ones unless the long-range tail of the interaction can be examined, and when b is in the neighborhood of b^* this is impossible in any practical approximation.

Another such criterion might be provided by a relation between the eigenvalue Λ_h and the parameter b in the transformation, which is given in Eq. (2.9) and is a consequence of the linearity of the renormalization-group transformation. One would require that this relation, known to be true for the exact renormalization group, be satisfied as nearly as possible by the eigenvalue Λ_h obtained from the finite-lattice calculation. Since Λ_h will depend on the value of b used in the calculation, one would hope that the relation $\Lambda_h = b^{-1}$ might work well for $b \approx b^*$, and not so well elsewhere. Such a criterion was introduced by Subbarao, [9] without much success, though that might be attributable to the size of the lattice used in the calculation. In our model, as we have seen in Eq. (4.14), the relation $\Lambda_h = b^{-1}$ is in fact satisfied for every value of b for which a nontrivial fixed point of the finite-lattice equations exists, and therefore completely fails to distinguish one value of b from another. That the relation is satisfied in this approximation for every value of b is, to be sure, a consequence of the periodic boundary conditions used in the calculation, but it seems likely that, with other boundary conditions, the value of b selected by this criterion would prove to be more dependent on details of the approximation than on the correct value of the parameter. We may conjecture that this criterion fails because the infinite-lattice relation $\Lambda_h = b^{-1}$ is derived without assuming the locality of the fixed-point Hamiltonian, and so will be true for nonlocal as well as local fixed points of the transformation. Since it does not distinguish the "correct" value of b from others for the infinite-lattice equations, there is no guarantee that the relation will work better for $b = b^*$ than for other values in the finite-lattice approximation.

Although the criteria suggested above fail to distinguish the proper value of b from other values, there does seem to be a criterion that is eminently successful for the Gaussian model. We have already mentioned that the line of fixed points expected for the infinite-lattice equations cannot be seen in the finite-lattice approximation for them, because only the most localized of local fixed-point Hamiltonians will be adequately represented in the finite-lattice approximation. But if the neighborhood of this most localized of fixed points is treated accurately enough by the approximation, then, when the eigenvalues and eigenoperators for perturbations about this fixed point are obtained, a nearly marginal eigenoperator (with eigenvalue $\Lambda_1 \approx 1$) should be found among them—a vestige of the line of fixed points of the infinite-lattice equations.

At first sight it might appear that this criterion should be found as inadequate as the one using $\Lambda_h = b^{-1}$ discussed in the previous paragraph, and for the same reason, for the exact renormalization-

group equations should have lines of nonlocal fixed points as well as lines of local fixed points, just as the exact equations had nonlocal fixed points satisfying $\Lambda_h = b^{-1}$ as well as local ones satisfying the relation. The crucial difference between the two cases seems to reside in the local or nonlocal nature of the eigenoperators. The eigenoperator associated with the magnetic field eigenvalue Λ_h is highly local (for the Gaussian model it is just $\sum_{\vec{n}} S_{\vec{n}}$) for both the local and nonlocal fixed points; but the eigenoperator with eigenvalue $\Lambda_1 = 1$ is local only for the local fixed points. It would appear from our calculations that the nonlocal marginal eigenoperators are not at all well represented in the finite-lattice approximation, and that as a consequence the fixed point with $b = b^*$ is distinguished in the approximation from the other fixed points by being the only one associated with a marginal eigenoperator.

VI. NUMERICAL RESULTS

Fixed points of the finite-lattice equations (4.7) and (4.8) were searched for numerically for a three-dimensional lattice, with $N = 6$ in the first case (that is, a $6\times6\times6$ lattice) and $N = 10$ in the second. All spin-spin interactions $\rho'_{N/2}(\vec{r}')$ generated by the renormalization-group transformation were kept. Because of the periodic boundary conditions assumed for the lattice, only four independent interactions $\rho(\vec{r})$ can be accommodated in the three-dimensional $\frac{1}{2}N = 3$ block-spin lattice, and only ten independent interactions $\rho(\vec{r})$ in the $\frac{1}{2}N = 5$ block-spin lattice.

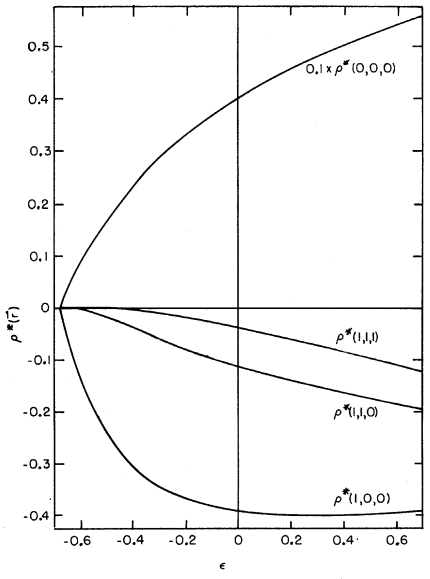

FIG. 2. Fixed-point interaction strengths $\rho^*(\vec{r})$ in the finite-lattice approximation using a $6\times6\times6$ lattice, as a function of ϵ ($b = 2^{-(d+2+\epsilon)/2}$). Note that the interaction strength $\rho^*(\vec{0})$ has been plotted reduced by a factor of 10.

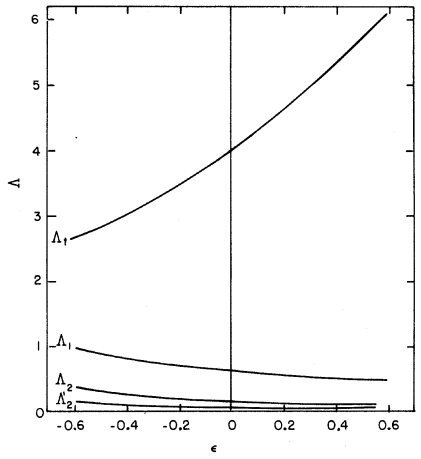

FIG. 3. Eigenvalues associated with the fixed points given in Fig. 2 for the $6\times6\times6$ lattice.

The transformation parameter a in $T_{a,b}$ may be chosen arbitrarily, since the relation (2.5) holds as well for the finite-lattice approximation as for the exact equations. In our calculations the parameter was assigned the value $a = 8$. In the first case, with $N = 6$, fixed points exist for a considerable range of values of b around b^*. Coupling-constant strengths are plotted in Fig. 2 as a function of the parameter ϵ defined in Eq. (3.18) (that is, $b = 2^{-(d+2+\epsilon)/2}$). The value $b = b^*$ corresponds to $\epsilon = 0$. The eigenvalues associated with these fixed points are plotted as a function of ϵ in Fig. 3. It is apparent that nothing significant happens near $\epsilon = 0$; in particular, no marginal eigenoperator appears (except possibly where the fixed point vanishes trivially). Evidently the number of interactions allowed by the smaller lattice is not large enough.

When the lattice is increased in size to $N = 10$, no nontrivial fixed points are found for $\epsilon < 0$ (more precisely, no fixed point was found for $-0.5 \leq \epsilon \leq -1.0\times10^{-10}$). At $\epsilon = 0$ a fixed point appears, with the interaction strengths $\rho^*(\vec{r})$ given in Table I. For $\epsilon > 0$, two fixed points are found. Some representative interaction strengths are plotted as a function of ϵ in Figs. 4 and 5.

At $\epsilon = 0$ we see the first sign of a line of fixed points. The point $\epsilon = 0$ where the slope is infinite is a signal that, for $b = 2^{-5/2}$, a sufficiently local

TABLE I. Fixed-point interaction strengths $\rho^*(\vec{r})$ for $\epsilon = 0$, $N = 10$.

$\rho^*(0,0,0) = 4.0276$	$\rho^*(2,1,0) = -0.000\,88$
$\rho^*(1,0,0) = -0.3922$	$\rho^*(2,1,1) = -0.000\,29$
$\rho^*(1,1,0) = -0.1120$	$\rho^*(2,2,0) = 0.000\,62$
$\rho^*(1,1,1) = -0.0389$	$\rho^*(2,2,1) = 0.000\,39$
$\rho^*(2,0,0) = -0.001\,51$	$\rho^*(2,2,2) = 0.000\,15$

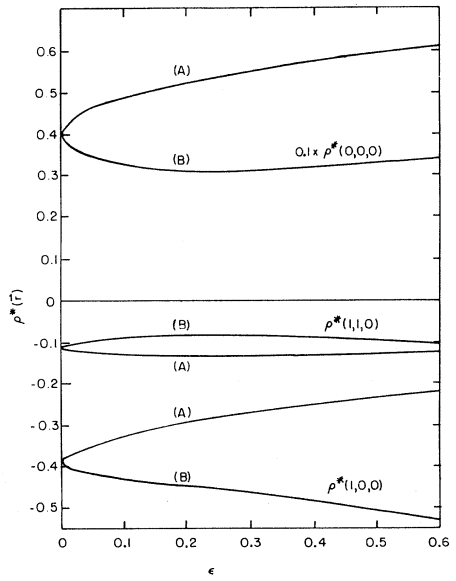

FIG. 4. Some fixed-point interaction strengths $\rho^*(\vec{r})$ obtained with a $10\times10\times10$ lattice. In order to distinguish the two fixed points observed for $\epsilon > 0$, the two branches of fixed points have been labeled A and B, respectively. More interaction strengths are plotted in Fig. 5.

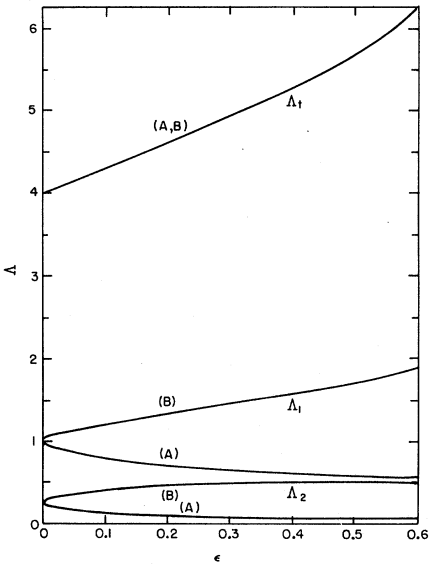

FIG. 6. Eigenvalues associated with the fixed points given in Figs. 4 and 5 for the $10\times10\times10$ lattice. The labels A and B are explained in the caption to Fig. 4. The eigenvalues Λ_2' were nearly degenerate with the eigenvalues Λ_2 on the scale of this plot and are not shown. See the text for more precise eigenvalues at $\epsilon = 0$.

fixed point and associated marginal eigenoperator are adequately represented in the approximation scheme. The appearance of the marginal eigenoperator selects b, and therewith the eigenvalues Λ_t and Λ_h via Eqs. (4.14) and (4.16). Some numerical results for the eigenvalues associated with the fixed points as a function of ϵ are plotted in Fig. 6. At $\epsilon = 0$ the numerical results for the eigenvalues are

$$\Lambda_t = 4.000\,00\,, \quad \Lambda_1 = 1.000\,00\,,$$

$$\Lambda_2 = 0.250\,00\,, \quad \Lambda_2' = 0.241\,21\,.$$

Both the eigenvalues Λ_2 and Λ_2' are expected from the infinite-lattice results of Sec. III to equal $\frac{1}{4}$. Two of the six smaller eigenvalues not listed above were complex.

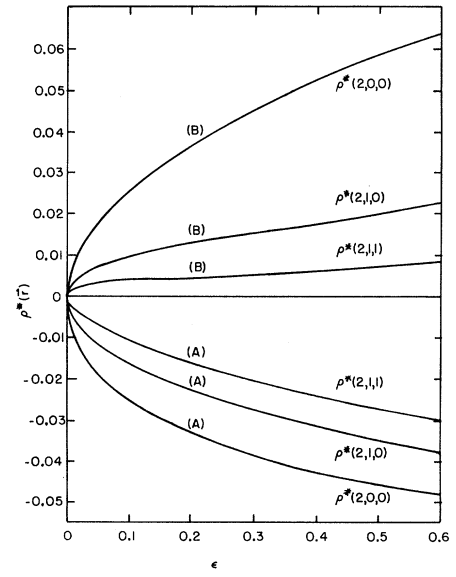

FIG. 5. More of the fixed-point interaction strengths $\rho^*(\vec{r})$ obtained with the $10\times10\times10$ lattice. The labels A and B are explained in the caption to Fig. 4. All ten interaction strengths for $\epsilon = 0$ are given in Table I.

VII. DISCUSSION

The simplicity of the Gaussian model, and the periodic boundary conditions for the lattice, are responsible for the ideal behavior of this example of a finite-lattice approximation to renormalization-group equations and of the determination of the parameter b within the approximation scheme. A tractable finite-lattice approximation for the linear renormalization group of a more complicated system such as the Ising model is less likely to yield an exactly marginal eigenoperator signaling the correct value of the transformation parameter; but if the number of interactions included in the approximation is large enough, the transformation parameter should be unambiguously selected by requiring the transformation to have a fixed point with an eigenoperator as nearly marginal as possible.

Not all renormalization groups are linear, how-

ever. Many of the renormalization groups introduced for dealing directly with the Ising model are nonlinear, and all possess at least implicitly one or more parameters. Recent work[13,14] suggests that critical exponents calculated from these nonlinear renormalization groups will be independent of the parameters in the group, within certain limits, but this independence will be lost in a finite-lattice approximation. The reason for this loss is basically the same as for the disappearance of the line of fixed points in the finite-lattice approximation to a linear renormalization group. Although the nonlinear renormalization group will not have a line of fixed points, it will have different fixed-point Hamiltonians for different values of the parameters of the transformation, and the range of interactions in these Hamiltonians will vary with the transformation parameters. The finite-lattice approximation will therefore represent the renormalization-group equations best when the transformation parameters lead to the most local of fixed points. This suggests one means of choosing the parameters in the transformation: One would choose the transformation parameters to minimize the dependence of the thermal eigenvalue on the parameters. In order to smooth the transition from linear to nonlinear renormalization groups, a better form of the criterion might be to minimize the dependence of the thermal eigenvalue on the fixed point. But such a criterion would only begin to function adequately for lattice sizes of the same order as is necessary in order that a nearly marginal eigenoperator appear for a linear renormalization group, unless the nonlinear renormalization group has much more localized fixed-point Hamiltonians than has the linear renormalization group.

Another criterion has been proposed by Kadanoff and Houghton.[10] They obtain a relation between the magnetic eigenvalue Λ_h and the parameter of their renormalization group (a generalization of relations of the sort $\Lambda_h = b^{-1}$ which hold for linear renormalization groups) and choose the renormalization-group parameter so that the eigenvalue Λ_h, which in the approximation depends on the parameter, satisfies the relation as well as possible. However, we have seen above that, for a linear renormalization group, such a relation completely fails to distinguish from other fixed points the fixed point that gives the best values for critical indices, and therefore cannot be used indiscriminately as a criterion for choosing transformation parameters.

With the criteria so far available for choosing the parameters in the renormalization group in order to obtain the best values for critical indices, calculations with the finite-lattice approximation must usually be large enough to require the aid of a computer, but are well worth the effort, since the method offers quantitative results from the renormalization group where no other method is available.

APPENDIX: DETAILS OF CALCULATION

In verifying Eqs. (3.2), (3.3), and (3.11)–(3.15), one must evaluate numerous integrals of Gaussian functions of the form

$$I = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \cdots \int_{-\infty}^{\infty} dx_N \exp(-\tfrac{1}{2} M_{ij} x_i x_j + h_j x_j) \ . \tag{A1}$$

It is easy to show the value of such an integral to be

$$I = (2\pi)^{N/2} \, | \, | M \, | \, |^{-1/2} \, \mathrm{Max}_x [\exp(-\tfrac{1}{2} M_{ij} x_i x_j + h_j x_j)] \ , \tag{A2}$$

where $| \, | M \, | \, |$ is the determinant of the matrix M_{ij} and $\mathrm{Max}_x[\ ]$ means the maximum value of the quantity in brackets with respect to the variables x_i. The factors multiplying $\mathrm{Max}_x[\ ]$ in (A2) will always prove to be irrelevant normalization factors or constant spin-independent contributions to a Hamiltonian, and will in the future be omitted.

In order to verify Eqs. (3.2) and (3.3), consider the equation

$$T^{k+1}_{a,b} e^{\mathcal{H}} = T_{a,b} T^{k}_{a,b} e^{\mathcal{H}} \ , \tag{A3}$$

and assume that the form of $T^{k}_{a,b}$ is

$$T^{k}_{a,b}[S] \, e^{\mathcal{H}[\sigma]} = \int_{\sigma} \exp\left[-\tfrac{1}{2} a_k \sum_{\vec{n}'} \left(S_{\vec{n}'} - b_k \sum_{\vec{j} \epsilon \vec{n}'} \sigma_{\vec{j}} \right)^2 + \mathcal{H}[\sigma] \right] . \tag{A4}$$

Equation (A3) may then be written

$$T^{k+1}_{a,b}[S'] \, e^{\mathcal{H}[\sigma]} = \int_{S} \exp\left[-\tfrac{1}{2} a \sum_{\vec{n}''} \left(S'_{\vec{n}''} - b \sum_{\vec{j}' \epsilon \vec{n}''} S_{\vec{j}'} \right)^2 \right] \times \int_{\sigma} \exp\left[-\tfrac{1}{2} a_k \sum_{\vec{n}'} \left(S_{\vec{n}'} - b_k \sum_{\vec{j} \epsilon \vec{n}'} \sigma_{\vec{j}} \right)^2 + \mathcal{H}[\sigma] \right] .$$

If each integration variable $S_{\vec{n}'}$ is shifted by $b_k \sum_{\vec{j} \epsilon \vec{n}'} \sigma_{\vec{j}}$, the equation above becomes

$$T^{k+1}_{a,b}[S'] \, e^{\mathcal{H}} = \int_{\sigma} \int_{S} \exp\left[-\tfrac{1}{2} a \sum_{\vec{n}''} \left(s_{\vec{n}''} - b \sum_{\vec{j}' \epsilon \vec{n}''} S_{\vec{j}'} \right)^2 - \tfrac{1}{2} a_k \sum_{\vec{n}'} S^2_{\vec{n}'} + \mathcal{H}[\sigma] \right] , \tag{A5}$$

with

$$s_{\vec{n}''} \equiv S'_{\vec{n}''} - b b_k \sum_{\vec{j} \epsilon \vec{n}''} \sigma_{\vec{j}} \ . \tag{A6}$$

Now one makes use of Eq. (A2) to carry out the integrations over the spin variables $S_{\vec{n}'}$ in Eq. (A5). The maximum of the Gaussian occurs for $S_{\vec{j}'}$ satisfying the equations

$$ab \left(s_{\vec{n}''} - b \sum_{\vec{j}' \epsilon \vec{n}''} S_{\vec{j}'} \right) - a_k S_{\vec{n}'} = 0 \ , \quad \vec{n}' \epsilon \vec{n}'' . \tag{A7}$$

The set of equations implied by (A7) may be solved for $S_{\vec{n}'}$ by first summing the equation over all $\vec{n}' \in \vec{n}''$ to get an equation for the quantity $\sum_{\vec{j}', \vec{n}'} S_{\vec{j}'}$ in terms of $s_{\vec{n}''}$, and then substituting this result back into Eq. (A7) to obtain the solutions $S_{\vec{n}'}$ that maximize the Gaussian in Eq. (A5). After some algebraic simplification, one obtains

$$T^{k+1}_{a,b}[S']\, e^{\mathcal{H}} = \int_{\sigma} \exp\left(-\tfrac{1}{2} a_{k+1} \sum_{\vec{n}''} s^2_{\vec{n}''} + \mathcal{H}[\sigma] \right), \tag{A8}$$

with

$$a_{k+1} = a a_k / (a_k + 2^d b^2 a) . \tag{A9}$$

Because the form of Eq. (A8) for $T^{k+1}_{a,b}$ agrees with the assumed form of $T^{k}_{a,b}$ in Eq. (A4), the identification of a_{k+1} in Eq. (A9) is a legitimate one, and, from Eq. (A6), we may further identify

$$b_{k+1} = b b_k . \tag{A10}$$

The recursion relations (A9) and (A10), with the initial conditions $a_1 = a$ and $b_1 = b$, may readily be solved for a_k and b_k; the solution obtained has been given in Eqs. (3.3).

Verification of Eqs. (3.11)–(3.15) proceeds in a similar fashion. If the Fourier expansions of $\sigma_{\vec{n}}$ and $\rho(\vec{r})$ given in Eqs. (3.9) are substituted into Eq. (2.1) for $\mathcal{H}[\sigma]$, one finds

$$\mathcal{H}[\sigma] = -\tfrac{1}{2} N^{-d} \sum_{\vec{q}} \rho(\vec{q}) \sigma_{\vec{q}} \sigma_{-\vec{q}} + h \sigma_{\vec{q}=0} , \tag{A11}$$

where the summation over $\vec{q}$ is defined in Eq. (3.10). We require the Fourier expansion of Eq. (3.7) for $\mathcal{T}^{k}_{a,b}[S, \sigma]$ as well. One finds

$$\sum_{\vec{n}'=0}^{N'-1} S^2_{\vec{n}'} = (N')^{-d} \sum_{\vec{q}'} S_{\vec{q}'} S_{-\vec{q}'} , \tag{A12}$$

where N' is defined in Eq. (3.5); and

$$\sum_{\vec{n}'=0}^{N'-1} S_{\vec{n}'} \sum_{\vec{m}=0}^{L-1} \sigma_{L\vec{n}'+\vec{m}} = N^{-d} \sum_{\vec{q}} u_L(-\vec{q}) S_{L\vec{q}} \sigma_{-\vec{q}} , \tag{A13}$$

with

$$u_L(\vec{q}) \equiv \sum_{\vec{m}=0}^{L-1} e^{i\vec{q}\cdot\vec{m}} = \prod_{j=1}^{d} \frac{1 - e^{iq_j L}}{1 - e^{iq_j}} ; \tag{A14}$$

the identity $S_{\vec{q}'+2\pi\vec{n}'} = S_{\vec{q}'}$ for any lattice vector $\vec{n}'$ has been used in obtaining (A13).

After the terms involving σ^2 in Eq. (3.7) are likewise expanded in terms of their Fourier components, and the integrations over $\sigma_{\vec{q}}$ performed using Eq. (A2), one has

$$\mathcal{H}^{(k)}[S] = \mathrm{Max}_{\sigma} \Big[-\tfrac{1}{2} a_k (N')^{-d} \sum_{\vec{q}'} S_{\vec{q}'} S_{-\vec{q}'} + a_k b_k N^{-d} \sum_{\vec{q}} u_L(-\vec{q}) S_{L\vec{q}} \sigma_{-\vec{q}}$$
$$- \tfrac{1}{2} a_k b_k^2 N^{-d} \sum_{\vec{q}} u_L(-\vec{q}) L^{-d} \sum_{\vec{l}=0}^{L-1} u_L\left(\vec{q} + \frac{2\pi\vec{l}}{L}\right) \sigma_{-\vec{q}} \sigma_{\vec{q}+2\pi\vec{l}/L} - \tfrac{1}{2} N^{-d} \sum_{\vec{q}} \rho(\vec{q}) \sigma_{\vec{q}} \sigma_{-\vec{q}} + h \sigma_{\vec{q}=0} \Big] . \tag{A15}$$

The maximum occurs when the spin variables $\sigma_{\vec{q}}$ satisfy the equations

$$a_k b_k N^{-d} u_L(-\vec{q}) S_{L\vec{q}} - a_k b_k^2 N^{-d} u_L(-\vec{q}) L^{-d} \sum_{\vec{l}=0}^{L-1} u_L\left(\vec{q} + \frac{2\pi\vec{l}}{L}\right)$$
$$\times \sigma_{\vec{q}+2\pi\vec{l}/L} - N^{-d} \rho(\vec{q}) \sigma_{\vec{q}} + h \delta_{\vec{q},0} = 0 . \tag{A16}$$

In order to solve these equations for $\sigma_{\vec{q}}$, one uses the same device employed in the solution of Eqs. (A7). Let us define

$$r_{\vec{q}} \equiv L^{-d} \sum_{\vec{l}=0}^{L-1} u_L\left(\vec{q} + \frac{2\pi\vec{l}}{L}\right) \sigma_{\vec{q}+2\pi\vec{l}/L} . \tag{A17}$$

Then multiply Eq. (A16) by $u_L(\vec{q})/\rho(\vec{q})$ and sum the equation over the set of vectors $\vec{q} + 2\pi\vec{l}/L$ to get an equation for $r_{\vec{q}}$:

$$a_k b_k (N')^{-d} \varphi_{\vec{q}} S_{L\vec{q}} - a_k b_k^2 (N')^{-d} \varphi_{\vec{q}} r_{\vec{q}} - (N')^{-d} r_{\vec{q}}$$
$$+ h \frac{u_L(0)}{\rho(\vec{q}=0)} \sum_{\vec{l}} \delta_{\vec{q}, 2\pi\vec{l}/L} = 0 , \tag{A18}$$

with

$$\varphi_{\vec{q}} = L^{-d} \sum_{\vec{l}=0}^{L-1} \frac{|u_L(\vec{q} + 2\pi\vec{l}/L)|^2}{\rho(\vec{q} + 2\pi\vec{l}/L)} . \tag{A19}$$

Once one has obtained $r_{\vec{q}}$ from Eq. (A18) and used this to find the solutions of Eq. (A16) for $\sigma_{\vec{q}}$, one can obtain the maximum required in Eq. (A15). The algebraic simplification of the result is facilitated by use of the periodicity of $\varphi_{\vec{q}} = \varphi_{\vec{q}+2\pi\vec{l}/L}$ and of $S_{\vec{q}'} = S_{\vec{q}'+2\pi\vec{l}}$, as well as the identity

$$N^{-d} \sum_{\vec{q}} f(\vec{q}) = (N')^{-d} \sum_{\vec{q}'} L^{-d} \sum_{\vec{l}=0}^{L-1} f\left(\frac{\vec{q}'}{L} + \frac{2\pi\vec{l}}{L}\right) . \tag{A20}$$

Equations (3.11)–(3.15) are obtained by setting

$$\omega^{(k)}(\vec{q}') = \varphi_{\vec{q}'/L} . \tag{A21}$$

The term of order h^2 in $\mathcal{H}^{(k)}[S]$ has not been included since it is a constant independent of the spin variables $S_{\vec{n}'}$.

*Supported in part by the National Science Foundation.

[1]K. G. Wilson, Phys. Rev. B 4, 3174 (1971); 4, 3184 (1971).

[2]K. G. Wilson and M. Fisher, Phys. Rev. Lett. 28, 240 (1972).

[3]A thorough discussion of the renormalization group may be found in K. G. Wilson and J. Kogut, Phys. Rep. C 12, 76 (1974).

[4]Th. Niemeijer and J. M. J. van Leeuwen, Phys. Rev. Lett. 31, 1411 (1973); Physica 71, 17 (1974).

[5]L. P. Kadanoff in *Renormalization Group in Critical Phenomena and Quantum Field Theory: Proceedings of a Conference*, edited by J. D. Gunton and M. S. Green (Temple University, Philadelphia, 1974), p. 21.

[6]K. G. Wilson, in Cargèse Lectures in Theoretical Physics (1973) (unpublished).

[7]M. Nauenberg and B. Nienhuis, Institute for Theoretical Physics, University of Utrecht, Holland, report, 1974 (unpublished); Phys. Rev. Lett. 33, 944 (1974).

[8]S. C. Hsu, Th. Niemeijer, and J. D. Gunton, Phys. Rev. B 11, 2699 (1975).

[9]K. Subbarao, Phys. Rev. B 11, 1165 (1975).

[10]L. P. Kadanoff and A. Houghton, Phys. Rev. B 11, 377 (1975).

[11]L. P. Kadanoff, Physics 2, 263 (1966).

[12]L. P. Kadanoff *et al.*, Rev. Mod. Phys. 39, 395 (1967).

[13]F. J. Wegner, J. Phys. C 7, 2098 (1974).

[14]T. L. Bell and K. G. Wilson, Phys. Rev. B 10, 3935 (1974).

[15]T. H. Berlin and M. Kac, Phys. Rev. 86, 821 (1952).

[16]J. S. Høye, SUNY report, 1974 (unpublished).

[17]G. Jona-Lasinio, Istituto di Fisica dell'Universitàli Padova report, 1974 (unpublished).

[18]K. Subbarao, Laboratory of Nuclear Studies, Cornell University report, 1974 (unpublished).

[19]The argument given below is similar to one given in Ref. 6 for a linear-renormalization-group transformation for Ising models.

[20]Because of the periodic boundary conditions, one must be a bit more careful when $\frac{1}{2}N$ is an even integer. In fact, it can be seen by comparison of the lattice calculation with a graphical expansion that if any component of $\vec{r}'$ is of magnitude $\frac{1}{4}N$, then one should use $\rho'_N(\vec{r}=\vec{r}') = \frac{1}{2}\rho'_{N/2}(\vec{r}')$ instead of Eq. (4.8). Since we shall only use lattices with $\frac{1}{2}N$ odd, this exception need not concern us.