# Josephson tunneling current in the presence of a time-dependent voltage

Richard E. Harris
*United Aircraft Research Laboratories, East Hartford, Connecticut 06108*
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The expression for the current through a small Josephson tunnel junction in the presence of a time-dependent voltage is presented. Four terms appear: the usual sine, cosine, and quasiparticle terms, and a reactive part of the quasiparticle current. The latter is displayed graphically as a function of both energy and temperature. It is shown that in the limit of zero dc voltage and small ac voltage, the Josephson device behaves linearly. Interpretation of the in- and out-of-phase components of the current in this linear limit is given to provide physical insight into some of the details of the general expression. Finally, the tunneling current in the linear limit is shown for thin tunneling barriers to be proportional to the current in a single superconductor in the presence of an electromagnetic field.

## I. INTRODUCTION

The theoretical expression for the current through a small Josephson tunnel junction[1] is quite complicated and seemingly defiant to intuitive physical interpretation. In this paper we will introduce the general expression for the current and then consider a special case in which physical insight is possible. One term in the general expression, a reactive part of the usual quasiparticle current, has not previously been discussed in detail. It is given graphically as a function of energy and temperature. Then it is shown that when the general expression for the tunneling current is specialized to the case of a small sinusoidal voltage the junction behaves linearly and contains terms easily identifiable as components in and out of phase with the applied voltage. The interpretation of all of the terms then becomes more clear. Finally it is shown that the expression for the tunneling current in the limit of low junction resistance is proportional to that for the current in a single superconductor exposed to an electromagnetic field.

## II. GENERAL EXPRESSION FOR THE TUNNELING CURRENT

We have generalized an unpublished calculation by Schrieffer[2] to include voltages having arbitrary time dependence. The resulting, rather general, form of the tunneling current follows:

$$\mathcal{I}(t) = \mathrm{Im} \int\int d\omega\, d\omega' [\, U^*(\omega) U(\omega')\, e^{i(\omega-\omega')t} I_{qp}(\hbar\omega') + U(\omega) U(\omega')\, e^{-i(\omega+\omega')t} I_J(\hbar\omega')] \tag{1}$$

where

$$\exp\left(-i\frac{e}{\hbar}\int_0^t V(t')\,dt' - \tfrac{1}{2} i\varphi_0\right) = \int U(\omega)\, e^{-i\omega t}\, d\omega \ .$$

In the above, $V(t)$ is the voltage appearing across the junction, $t$ is the time, $\hbar$ is Planck's constant divided by $2\pi$, $\varphi_0$ is a constant phase determined by the circuit,[3] and $e$ is the absolute value of the charge of the electron. The complex-valued function $I_{qp} = I_{qp1} + iI_{qp2}$ describes the flow of quasiparticles. $I_{qp2}$ is real and the usual quasiparticle current.[4] (In Ref. 4, $I_{qp2}$ is labeled $I_{qp}$.) $I_{qp1}$ is real and is a reactive part of the quasiparticle current. It will be discussed further below. The complex-valued function $I_J = I_{J1} + iI_{J2}$ describes the Josephson terms.[4] $I_{J1}$ is the amplitude of the usual Josephson current, sometimes called the sine term. $I_{J2}$ is the amplitude of the cosine term.

It is interesting to note that all of the details of the superconductivity enter Eq. (1) primarily through the functions $I_{qp}$ and $I_J$. In addition, of course, the general form of the equation is mandated by the existence of superconducting electrodes. The driving voltage, on the other hand, enters solely through the function $U(\omega)$.

In the derivation of this equation, it is assumed that at all times the superconductors on each side of the insulating barrier are described as equilibrium superconductors of negligible spatial extent. In a real junction, of course, equilibrium may be disturbed for large currents. Furthermore, the theory does not include losses occurring in the superconductors themselves due to the high frequency currents which commonly flow in these devices. Finally, because tunneling is introduced using a perturbation approach, the result is strictly valid only in the limit of low coupling between the two superconducting electrodes.

Equation (1) agrees with the general result previously obtained by Werthamer,[1] but not with Werthamer's result at $T=0$. Details are discussed in the reference to Werthamer.

## III. REACTIVE PART OF THE QUASIPARTICLE CURRENT

Of the four functions $I_{qp1}$, $I_{qp2}$, $I_{J1}$, and $I_{J2}$ containing the details of the superconductivity in Eq. (1), the last three have been discussed in Ref. 4. These three functions are the only ones which en-

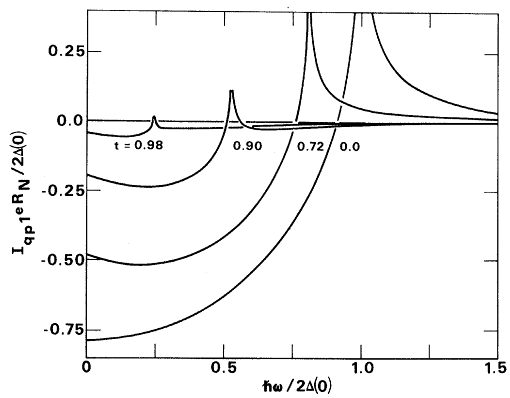


FIG. 1. Reactive part $I_{qp1}$ of the quasiparticle current for identical superconductors.

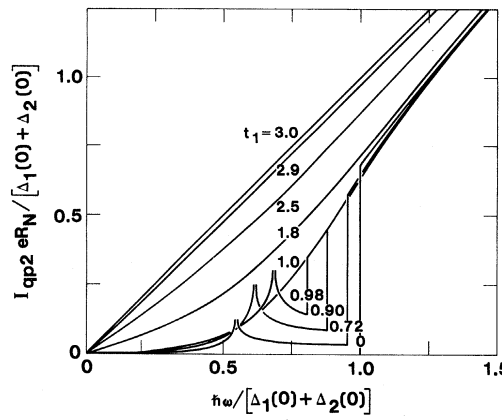


FIG. 3. Usual quasiparticle current $I_{qp2}$ for different superconductors.

ter expressions for the time-dependent current through a Josephson tunnel junction when the voltage is constant, and the only ones which enter the expression for the time-averaged current when a more general voltage appears across the junction. These are the only cases considered in Ref. 4.

However, when a time-dependent voltage appears across the junction, the time-dependent part of the current also depends on the additional function $I_{qp1}$, which we call the reactive part of the quasiparticle current.

Larkin and Ovchinnikov[5] have given a formula which we have used to numerically evaluate $I_{qp1}$ as a function of energy $\hbar\omega$ and temperature $T$. We restate this formula here for completeness:

$$I_{qp1}(\omega)=\frac{1}{2eR_N}\int_{-\infty}^{\infty}\left(\frac{(\omega'-\omega)\theta(\Delta_1-|\omega'-\omega|)\theta(|\omega'|-\Delta_2)}{[\Delta_1^2-(\omega'-\omega)^2]^{1/2}(\omega'^2-\Delta_2^2)^{1/2}}+\frac{(\omega'+\omega)\theta(|\omega'|-\Delta_1)\theta(\Delta_2-|\omega'-\omega|)}{(\omega'^2-\Delta_1^2)^{1/2}[\Delta_2^2-(\omega'+\omega)^2]^{1/2}}\right)\times\omega'[1-2f(|\omega'|)]\,d\omega\,. \tag{2}$$

Definitions of the quantities in Eq. (2) are given in Ref. 4. Note that we have used $\hbar=1$ in this equation. The function is plotted in Fig. 1 for identical superconductors on each side of the insulating barrier.

It can be shown algebraically, and more physically in Sec. IV, that $I_{qp1}$ is the Kramers-Kronig transform of the usual quasiparticle current $I_{qp2}$. $I_{qp1}$ is plotted as a function of energy $\hbar\omega$ in Fig. 1 for several reduced temperatures. The logarithmic singularity in $I_{qp1}$ at $2\Delta$ seen in Fig. 1 thus results from the Kramers-Kronig transform of the corresponding abrupt jump in the quasiparticle current at $2\Delta$.

When the superconductors are different, the logarithmic singularity in $I_{qp1}$ occurs at $\Delta_1+\Delta_2$ and again corresponds to the onset of the usual quasiparticle current at that energy. Here, in addition, a step is seen in $I_{qp1}$ at the difference in the energy gaps $\Delta_2-\Delta_1$. This step arises from the Kramers-Kronig transform of the logarithmic singularity in $I_{qp2}$ at the difference in the energy gaps. In Fig. 2 we show $I_{qp1}$ evaluated for a hypothetical case in which $\Delta_2(0)=3\Delta_1(0)$, the same case considered for the other three functions in Ref. 4. The reduced temperature $t_1=T/T_{c1}$ is that for the superconductor having the smaller energy gap. For reference we have plotted in Fig. 3 the usual quasiparticle current $I_{qp2}$ for the same cases for which $I_{qp1}$ is plotted in Fig. 2.

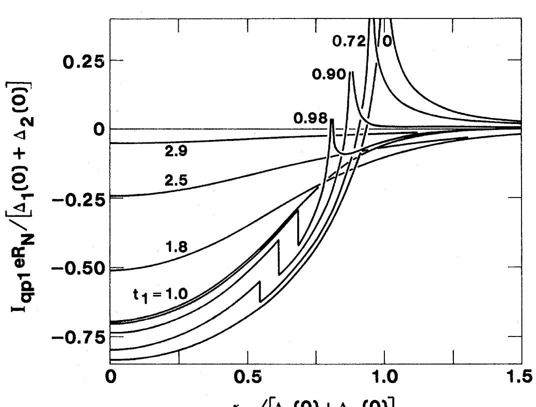


FIG. 2. Reactive part $I_{qp1}$ of the quasiparticle current for different superconductors.

As can be seen also in Fig. 2, structure still remains in $I_{qp1}$ when only one of the electrodes is superconducting. This is a necessary consequence of the structure remaining in $I_{qp2}$ under the same circumstances. In contrast, both $I_{J1}$ and $I_{J2}$ vanish whenever either electrode is normal, since these functions involve the tunneling of pairs which is not possible when one electrode is normal.

One additional point may be useful. It can be shown that adding a constant to $I_{qp1}$ does not change the value given by Eq. (1) for the current. Thus we have chosen this arbitrary constant so that $I_{qp1}$ vanishes for large argument. We prefer this because one expects the superconducting device it describes to behave like a normal device at high frequencies.[6]

## IV. LINEAR LIMIT OF A JOSEPHSON JUNCTION

Since the quantities $I_{qp}$ and $I_J$ have real and imaginary parts which are the Kramers-Kronig transforms of each other, it is natural and will turn out to be very helpful to further specialize Eq. (1) to the case for which the Kramers-Kronig relations are derived.[7] There are three usual general conditions assumed in this derivation. First is that the process described is casual, an assumption not likely to be violated in a Josephson tunnel junction. Second, the response of the device to a disturbance must be bounded, again an expected result in a real physical system.[8] Finally, the response must be linearly related to the disturbance. That is, in this case, that the current must be linearly related to the voltage. This last assumption is usually violated in a nonlinear device like the Josephson junction.

The tunneling current is, however, linear in the special case of Eq. (1) in which the voltage is purely sinusoidal [$V(t) = v\cos\omega t$] and is small ($eV/\hbar\omega \ll 1$). In this case, the Kramers-Kronig assumptions are satisfied. The linear relation between current and voltage has the following form:

$$\begin{aligned}\mathcal{I}(t) = &-I_{J1}(0)\sin\varphi_0 \\ &+ v\frac{e}{\hbar\omega}\left\{-[I_{qp1}(\hbar\omega) - I_{qp1}(0)]\sin\omega t + [I_{qp2}(\hbar\omega)]\cos\omega t\right\} \\ &+ v\cos\varphi_0\left(\frac{e}{\hbar\omega}\right)\left\{-[I_{J1}(\hbar\omega) + I_{J1}(0)]\sin\omega t + [I_{J2}(\hbar\omega)]\cos\omega t\right\}.\end{aligned} \quad (3)$$

The physical significance of each of the terms in this equation can readily be seen. The first term is the usual dc Josephson supercurrent. The minus sign results because $I_{J1}$ is a negative number using the sign convention of Ref. 4.

The second and third terms of Eq. (3) (in large curly brackets) give the time-dependent parts of the quasiparticle and Josephson currents, each of which has a part out of phase and a part in phase with the applied voltage. These are the terms proportional to $\sin\omega t$ and $\cos\omega t$, respectively. In discussing the amplitudes of these terms below, we have divided by the amplitude $v$ of the time-dependent voltage, obtaining quantities having the units of conductance.

The second term of Eq. (2) gives the out-of-phase and in-phase parts of the quasiparticle current. The amplitude of the former is described by $-(e/\hbar\omega)[I_{qp1}(\hbar\omega) - I_{qp1}(0)]$ and that of the latter is $(e/\hbar\omega)I_{qp2}(\hbar\omega)$. It might appear that an extra minus sign has arisen. It was stated earlier that $I_{qp1}$ and $I_{qp2}$ are the Kramers-Kronig transforms of each other. By simple manipulations of the Kramers-Kronig relations it follows that $(e/\hbar\omega)I_{qp2}(\hbar\omega)$ and $-(e/\hbar\omega)[I_{qp1}(\hbar\omega) - I_{qp1}(0)]$ are the real and imaginary parts, respectively, of another complex-valued function whose real and imaginary parts are the Kramers-Kronig transforms of each other. Thus the minus sign accompanies the $\hbar\omega$ in the denominator of the amplitudes.

The third term in Eq. (3) gives the Josephson contribution to the tunneling current. The out-of-phase part of the Josephson current is $-\cos\varphi_0(e/\hbar\omega)[I_{J1}(\hbar\omega) + I_{J1}(0)]$. Thus the out-of-phase part arises from the usual Josephson, or sine, term. The in-phase part has amplitude $\cos\varphi_0(e/\hbar\omega) \times I_{J2}(\hbar\omega)$ and thus arises from the cosine term $I_{J2}$. Both the out-of-phase and in-phase Josephson terms are modulated by $\cos\varphi_0$, where $\varphi_0$ is the usual constant phase difference.

Of course the out-of-phase parts of the current are lossless and the in-phase parts lossy. The dissipation is simply calculated from

$$P = \lim_{T\to 0} \frac{1}{T}\int_0^T \mathcal{I}(t)V(t)\,dt$$

giving

$$P = \tfrac{1}{2}v^2\left(\frac{e}{\hbar\omega}\right)[I_{qp2}(\hbar\omega) + \cos\varphi_0 I_{J2}(\hbar\omega)] .$$

Thus it is clear that the quasiparticle current is not purely lossy because $I_{qp1}$ does not enter the expression for the dissipation. We can therefore denote $I_{qp1}$ as a reactive part of the usual quasiparticle current $I_{qp2}$. On the other hand, one sees that the Josephson current is not purely lossless because $I_{J2}$ does enter the expression for the dissipation.

It should be noted that in Eq. (3) the quasiparticle and Josephson parts of the tunneling current are separated only because they are conventionally considered separately. From an experimental point of view what is observed is the sum of these two parts of the current. While the sums of the quasiparticle and Josephson contributions to the in- and out-of-phase parts of the tunneling current can be measured using well-known techniques, more effort is required to separate the quasiparticle and Josephson contributions. Another parameter must be varied, such as the phase $\varphi_0$ or the temperature.

Although the preceding applies rigorously only to tunnel junctions, it suggests that theories of other kinds of Josephson devices may have the same sort of linear limit in which the Kramers-Kronig relations apply. For any kind of device there may then be quantities corresponding to $I_{qp}$ and $I_J$ which can be measured and which may be a significant experimental test of any theory to be developed.

## V. RELATION OF TUNNELING TO CONDUCTIVITY OF SINGLE SUPERCONDUCTOR

As in Ref. 4, Josephson tunneling can be looked at in terms of the coherence effects which dominate many different kinds of experiments on superconductors. In Ref. 4, it is shown that the Josephson effect exhibits the same type of coherence effects as the response of a single superconductor to electromagnetic radiation. This is intuitively reasonable because in the limit in which the tunneling barrier has zero thickness, one expects Eq. (3) to describe a single superconductor[9] except for a constant factor. Normalizing Eq. (3) to the normal state current removes the constant factor. We now note that because no constant phase difference can develop in a single superconductor, setting $\varphi_0 = 0$ in Eq. (3) should give the normalized expression for the current in a single superconductor. Aside from the supercurrent, we can express the resulting relation between current and voltage as a complex conductivity $\sigma = \sigma_1 + i\sigma_2$ (Ref. 10) where

$$\frac{\sigma_1}{\sigma_N} = \frac{eR_N}{\hbar\omega}[I_{qp2}(\hbar\omega) + I_{J2}(\hbar\omega)] ,$$
$$\frac{\sigma_2}{\sigma_N} = -\frac{eR_N}{\hbar\omega}[I_{qp1}(\hbar\omega) - I_{qp1}(0) + I_{J1}(\hbar\omega) + I_{J1}(0)] . \quad (4)$$

Indeed, as suggested by the notation, this simply reproduces the Mattis and Bardeen expression[11] for the complex conductivity of a single superconductor in the local limit. Equation (4) relates tunneling and the response of a single superconductor to electromagnetic radiation. This relation demonstrates that tunneling is yet another example of the many experiments on superconductors which, although seeming to differ widely, can be accurately described in terms of the coherence effects originally discussed by BCS.[12]

In the preceding we have given the general expression for the Josephson tunneling current in the presence of a time-dependent voltage. We have shown that it can be applied to a simple case in which physical interpretation of the components of the current is possible because the junction behaves linearly in that case. Finally a relation is given between the components of the tunneling current and the conductivity of a single superconductor in the local limit.

It is a please for the author to acknowledge a stimulating note by M. R. Beasley and M. Tinkham.[13] Several conversations with J. R. Schrieffer and the use of his unpublished notes on tunneling are also greatly appreciated. We are indebted to D. N. Langenberg for providing his unpublished study of Schrieffer's calculation; we used his study extensively. Discussions with D. G. McDonald, S. Shapiro, and E. B. Treacy have also been helpful.

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[1]N. R. Werthamer, Phys. Rev. 147, 255 (1966). Our Eq. (1) is algebraically equivalent to the result of Werthamer as expressed in his Eqs. (11) and (12). However, the $I_{qp}$ used here is the negative of the complex conjugate of Werthamer's $j_1$ ($I_{qp} = -j_1^*$). Unfortunately, Werthamer's Eq. (13) and his Fig. 2 present $I_{qp}$ even though labeled $j_1$. Furthermore, although $j_2$ as defined by Werthamer in Eq. (12) is identical with our $I_J$ ($I_J = j_2$), his Eq. (13) and Fig. 2 give $-I_J$ even though labelled $j_2$. To simplify Werthamer's Eq. (11), we have introduced $U(\omega)$ which includes both the constant and time-dependent parts of the voltage across the junction; he introduces the constant part explicitly in a phase $\varphi(t)$. Finally, Werthamer includes the possibility of a spatial variation in the phase which we neglect for a small junction.

[2]J. R. Schrieffer, private communication. D. N. Langenberg, D. J. Scalapino, and J. R. Schrieffer (unpublished).

[3]Another way of thinking about $\varphi_0$ demonstrates that it is determined by the history of the voltage applied to the junction. The expression defining $U(\omega)$ in Eq. (1) arises formally in the theory as

$$\exp\left(-i\frac{e}{\hbar}\int_{-\infty}^{t} V(t')\,dt'\right) = \int U(\omega)e^{-i\omega t}\,d\omega.$$

Comparing this with the form used in Eq. (1) shows that $\varphi_0 = 2(e/\hbar)\int_{-\infty}^{0} V(t')dt'$.

[4]Richard E. Harris, Phys. Rev. B 10, 84 (1974).

[5]A. I. Larkin and Yu. N. Ovchinnikov, Zh. Eksp. Teor. Fiz. 51, 1535 (1966) [Sov. Phys.—JETP 24, 1035 (1967)].

[6]In Ref. 1, Werthamer chooses this constant so that $I_{qp1}$ (his Re$j_1$) vanishes at zero argument.

[7]See, for example, F. Stern, in *Solid State Physics*, edited by F. Seitz and D. Turnbull (Academic, New York, 1963), Vol. 15.

[8]The singularities found in $I_{qp}$ and $I_J$ seem to indicate that the response of the Josephson tunnel junction is not bounded. In a real device the singularities are broadened into peaks and this difficulty is not present. In the theory the singularities are logarithmic and therefore have finite integrals suggesting that they represent the proper limit of very narrow peaks to be found in a nearly ideal device.

[9]This intuitive argument was pointed out to the author by M. Tinkham. Although the result of this argument agrees with that of the theory, rigorous justification of the argument may be rather involved.

[10]It is imperative to note here that we have used the time dependence conventionally used by physicists: $e^{-i\omega t}$. The time dependence conventionally used by electrical engineers and physicists studying the far-infrared properties of superconductors is $e^{j\omega t}$. Often one can convert from one approach to the other by substituting $j$ for $-i$. The effect of these differing conventions upon the complex conductivity of a single superconductor is discussed briefly in a footnote by D. M. Ginsberg and L. C. Hebel in ***Superconductivity***, edited by R. D. Parks (Dekker, New York, 1969), p. 206. Changing

the convention also changes the sign of the Kramers-Kronig relations. For example, Ref. 4 uses the physicists' convention while the engineers' convention is used by M. J. Stephen, Phys. Lett. A 46, 289 (1973).

[11]D. C. Mattis and J. Bardeen, Phys. Rev. 111, 412 (1958).

[12]J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. 106, 162 (1957); 108, 1175 (1957).

[13]The substance of this note will appear in M. Tinkham, *Introduction to Superconductivity* (McGraw-Hill, New York, to be published).