Dispersion relations for nonlinear systems of arbitrary degree

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A time-independent causal system is considered, in which the effect depends nonlinearly in an nth-degree way on the cause. It is shown that there are Kramers-Kronig-type dispersion relations for the transform of the response function.

I. INTRODUCTION

Recently dispersion relations for a one-dimensional third-degree nonlinear system were developed.[1] The purpose of the present paper is to show that the arguments used in the third-degree problem can be extended to the nth-degree nonlinear problem and to derive the Kramers-Kronig-type dispersion relations for that general system.

This paper rests on the previous one (Ref. 1) and we refer to it for introductory and background material.

II. DERIVATION

Consider a time-independent nth-degree nonlinear system where the cause $C(t)$ and the effect $E(t)$ are related by

$$E(t)=\int dt_1\int dt_2\cdots\int dt_n G(t-t_1,\,t-t_2,\ldots,t-t_n)\times C(t_1)C(t_2)\cdots C(t_n)\,. \tag{1}$$

The integrals extend from minus to plus infinity and all these quantities are real. The response function is symmetric with respect to interchange of any two variables,

$$G(\tau_1,\tau_2,\tau_3,\ldots)=G(\tau_2,\tau_1,\tau_3,\ldots),\ \text{etc.} \tag{2}$$

Suppose also that the system is causal so that the effect at any time depends only on the cause at earlier times,

$$G(\tau_1,\tau_2,\ldots)=0\ \ \text{if}\ \tau_1<0\,. \tag{3}$$

The transform of the response function is defined by

$$g(\omega_1,\omega_2,\ldots,\omega_n)=\int_0^\infty d\tau_1\int_0^\infty d\tau_2\cdots\int_0^\infty d\tau_n \times G(\tau_1,\tau_2,\ldots,\tau_n)\times\exp i(\omega_1\tau_1+\omega_2\tau_2+\cdots\omega_n\tau_n)\,. \tag{4}$$

The symmetry of G implies the symmetry of g,

$$g(\omega_1,\omega_2,\omega_3,\ldots)=g(\omega_2,\omega_1,\omega_3,\ldots),\ \text{etc.}, \tag{5}$$

and the reality of G implies a type of crossing symmetry of g,

$$g^*(\omega_1,\omega_2,\ldots,\omega_n)=g(-\omega_1,-\omega_2,\ldots,-\omega_n)\,. \tag{6}$$

Because of these symmetries, the function g is completely determined if it is specified in the fundamental region

$$\omega_1>\omega_2>\cdots>\omega_n\,,$$
$$\omega_1+\omega_2+\cdots+\omega>0\,.$$

The conditions that describe the fundamental region suggest an appropriate set of frequency variables:

$$\Omega_0=\omega_1+\omega_2+\cdots+\omega_n\,,$$
$$\Omega_1=\omega_1-\omega_2\,,$$
$$\Omega_2=\omega_2-\omega_3\,,$$
$$\vdots$$
$$\Omega_{n-1}=\omega_{n-1}-\omega_n\,.$$

These are positive throughout the fundamental region and form a nonorthogonal set of coordinates for points inside.

A new function may be defined by

$$\bar g(\Omega_0,\Omega_1,\ldots,\Omega_{n-1})=g(\omega_1,\omega_2,\ldots,\omega_n)\,. \tag{7}$$

The variable Ω_0 has special significance, as will be seen below; the dispersion relations involve integrations over Ω_0 and the response of a system at frequency ω_{out} to a set of applied frequencies is governed by $\bar g(\Omega_0,\Omega_1,\ldots,\Omega_{n-1})$ evaluated at $\Omega_0=\omega_{out}$.

In terms of the original response function G, $\bar g$ is given by

$$\bar g(\Omega_0,\Omega_1,\ldots,\Omega_{n-1})=\int_0^\infty d\tau_1\int_0^\infty d\tau_2\cdots\int_0^\infty d\tau_n G(\tau_1,\tau_2,\ldots,\tau_n)\exp((i/n)\{\Omega_0(\tau_1+\tau_2+\cdots+\tau_n) - [\Omega_1+2\Omega_2+\cdots(n-1)\Omega_{n-1}](\tau_1+\tau_2+\cdots+\tau_n)+n[\Omega_1\tau_1+\Omega_2(\tau_1+\tau_2) +\cdots\Omega_{n-1}(\tau_1+\tau_2+\cdots+\tau_{n-1})]\})\,. \tag{8}$$

This result is found by solving for the ω's in terms of the Ω's and substituting into Eq. (4). The interesting feature of this result is that the coefficient of $i\Omega_0$ in the exponent is $n^{-1}(\tau_1+\tau_2+\cdots+\tau_n)$, which is always positive. Consequently, for fixed $\Omega_1, \Omega_2, \ldots, \Omega_{n-1}$, Eq. (8) serves to define $\bar{g}(\Omega_0, \Omega_1, \ldots, \Omega_{n-1})$ as an analytic function of a complex variable Ω_0 in the upper half-plane.

The function $\bar{g}(\Omega_0, \Omega_1, \ldots, \Omega_{n-1})$ does not have crossing symmetry in the Ω_0 alone. However, one can find functions that do have the required property. Equations (5) and (6) imply

$$g(\omega_1, \omega_2, \ldots, \omega_{n-1}, \omega_n) = g(\omega_n, \omega_{n-1}, \ldots, \omega_2, \omega_1) = g^*(-\omega_n, -\omega_{n-1}, \ldots, -\omega_2, -\omega_1) \quad (9)$$

which, upon translating into terms of $\bar{g}$ and the Ω's, becomes

$$\bar{g}(\Omega_0, \Omega_1, \Omega_2, \ldots, \Omega_{n-2}, \Omega_{n-1}) = \bar{g}^*(-\Omega_0, \Omega_{n-1}, \Omega_{n-2}, \ldots, \Omega_2, \Omega_1) . \quad (10)$$

Consequently one defines two new functions by

$$\bar{g}_1(\Omega_0, \Omega_1, \Omega_2, \ldots, \Omega_{n-2}, \Omega_{n-1}) = \tfrac{1}{2}\bar{g}(\Omega_0, \Omega_1, \Omega_2, \ldots \Omega_{n-2}, \Omega_{n-1}) + \tfrac{1}{2}\bar{g}(\Omega_0, \Omega_{n-1}, \Omega_{n-2}, \ldots, \Omega_2, \Omega_1) , \quad (11a)$$

$$\bar{g}_2(\Omega_0, \Omega_1, \Omega_2, \ldots, \Omega_{n-2}, \Omega_{n-1}) = \tfrac{1}{2}i\bar{g}(\Omega_0, \Omega_1, \Omega_2, \ldots, \Omega_{n-2}, \Omega_{n-1}) - \tfrac{1}{2}i\bar{g}(\Omega_0, \Omega_{n-1}, \Omega_{n-2}, \ldots, \Omega_2, \Omega_1) , \quad (11b)$$

since they have the standard crossing symmetry

$$\bar{g}_i^*(\Omega_0, \Omega_1, \Omega_2, \ldots, \Omega_{n-2}, \Omega_{n-1}) = \bar{g}_i(-\Omega_0, \Omega_1, \Omega_2, \ldots, \Omega_{n-2}, \Omega_{n-1}) . \quad (12)$$

The functions $\bar{g}_i$ have crossing symmetry in Ω_0 and are also analytic functions of complex Ω_0 in the upper half-plane, for fixed $\Omega_1, \Omega_2, \ldots, \Omega_{n-1}$. Consequently, assuming no subtractions, one can write dispersion relations for these functions,

$$\mathrm{Re}\,\bar{g}_i(\Omega_0, \Omega_1, \ldots, \Omega_{n-1}) = \frac{2}{\pi} P \int_0^\infty \frac{d\Omega_0' \Omega_0' \mathrm{Im}\,\bar{g}_i(\Omega_0', \Omega_1, \ldots, \Omega_{n-1})}{\Omega_0'^2 - \Omega_0^2} , \quad (13a)$$

$$\mathrm{Im}\,\bar{g}_i(\Omega_0, \Omega_1, \ldots, \Omega_{n-1}) = -\frac{2}{\pi} \Omega_0 P \int_0^\infty \frac{d\Omega_0' \mathrm{Re}\,\bar{g}_i(\Omega_0', \Omega_1, \ldots, \Omega_{n-1})}{\Omega_0'^2 - \Omega_0^2} . \quad (13b)$$

These are the standard Kramers-Kronig type of dispersion relation on the Ω_0 dependence of the functions g_i, the frequency variables $\Omega_1, \ldots, \Omega_{n-1}$ being held constant.

III. DISCUSSION

In cases $n=2$ and $n=3$, Eqs. (13) specialize to the results obtained earlier.[1]

The interpretation of g in terms of the response of the system to a superposition of applied frequencies was developed before for the cases $n=2$ and $n=3$.[1] For higher n the same type of interpretation applies. For example, a single input frequency ω_a with amplitude A_a leads to output frequencies 0, $2\omega_a$, $4\omega_a, \ldots, n\omega_a$ when n is even and to frequencies ω_a, $3\omega_a$, $5\omega_a, \ldots, n\omega_a$ when n is odd, each with amplitude proportional to A_a^n. The amplitude and phase of the responses determine the transform function. The complete result, in this special case, is that the cause

$$C(t) = A_a \cos(\omega_a t - \eta_a)$$

implies the effect

$$E(t) = \frac{n!}{2^n (n/2)!\,(n/2)!} A_a^n g(\omega_a, \omega_a, \ldots, -\omega_a, -\omega_a) + \frac{A_a^n}{2^{n-1}} \sum_{\kappa=0}^{n/2-1} \frac{n!}{(n-\kappa)!\,\kappa!} |g(\omega_a, \omega_a, \ldots, -\omega_a, -\omega_a)| \times \cos[(n-2\kappa)\omega_a t - (n-2\kappa)\eta_a - \theta(\omega_a, \omega_a, \ldots, -\omega_a, -\omega_a)]$$

for n even, and the effect

$$E(t) = \frac{A_a^n}{2^{n-1}} \sum_{\kappa=0}^{(n-1)/2} \frac{n!}{(n-\kappa)!\,\kappa!} |g(\omega_a, \omega_a, \ldots, -\omega_a, -\omega_a)| \cos[(n-2\kappa)\omega_a t - (n-2\kappa)\eta_a - \theta(\omega_a, \omega_a, \ldots, -\omega_a, -\omega_a)]$$

for n odd. Here the notation is that $g = |g|\, e^{i\theta}$. In the time-independent term half the n arguments of g are ω_a and the other half are $-\omega_a$; in this case g itself is real. In the terms in the sums on κ, the n arguments of $|g|$ and θ are always $+\omega_a$ or $-\omega_a$, the $-\omega_a$ occurring κ times. When two frequencies ω_a and ω_b are applied with amplitudes A_a and A_b, all possible sums and differences of n frequencies chosen from ω_a and ω_b are found in the output and the amplitude is proportional to $A_b^k A_b^{n-k}$,

k being the number of ω_a's contributing. The situation is progressively more involved as more frequencies are used in the input, but no qualitatively new features appear. A general feature, illustrated in the above equation, is that the response at an output frequency ω_{out} always involves g or $\bar{g}$ evaluated at $\Omega_0 = \omega_{out}$.

In this general nth-degree problem, the values of Reg and Img can be determined separately by making time averages of the appropriate responses with the nth power of the cause, as illustrated for the $n = 2$ and $n = 3$ cases formerly.

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[1]F. L. Ridener, Jr. and R. H. Good, Jr., Phys. Rev. B 10, 4980 (1974).