Exact renormalization group exhibiting tricritical fixed point for a spin-one Ising model in one dimension

S. Krinsky*
Department of Physics, Brookhaven National Laboratory, Upton, New York 11973

D. Furman
Department of Physics, State University of New York at Stony Brook, Stony Brook, New York 11794
(Received 5 September 1974)

Exact renormalization-group recursion relations are derived in closed form for a one-dimensional spin-1 Ising model. The recursion relations possess *tricritical* and critical fixed points. We have studied the flow diagram corresponding to the renormalization-group transformations, and we have linearized about the fixed points. The predicted asymptotic homogeneity is shown to be satisfied by the true free energy, which has been computed using the transfer matrix. The pseudocritical singularities, existing in the zero-temperature limit, are marked by a double degeneracy of the largest eigenvalue of the transfer matrix, and the pseudotricritical point is marked by a triple degeneracy. The increased instability of the tricritical fixed point, relative to the critical fixed points, is shown to be directly related to the eigenvalue degeneracy of the transfer matrix being greater at a tricritical point than at a critical point.

I. INTRODUCTION

Insight into Wilson's renormalization-group ideas[1] has been obtained through the work of Fisher and Nelson[2] and Kadanoff[3] on exactly soluble systems. They constructed exact renormalization-group recursion relations exhibiting critical fixed points for the spin-$\frac{1}{2}$ Ising model in one dimension. Here, we consider a one-dimensional spin-1 Ising model[4] and find exact recursion relations exhibiting *tricritical* and critical fixed points. We have studied the flow diagrams corresponding to these recursion relations. We have linearized about the fixed points and shown that the predicted asymptotic homogeneity is observed in the true free energy, which has been computed by using the transfer matrix. In this model, the pseudocritical lines, existing in the zero-temperature limit, are marked by a double degeneracy of the largest eigenvalue of the transfer matrix. The pseudotricritical point is marked by a triple degeneracy.

We consider the one-dimensional model defined by the Hamiltonian

$$\mathcal{H} = -\sum_{n=1}^{N} [JS_n S_{n+1} + KS_n^2 S_{n+1}^2 + \tfrac{1}{2} L S_n S_{n+1} (S_n + S_{n+1}) - \Delta S_n^2 + HS_n] , \qquad (1.1)$$

where N is the number of spins and $S_n = 1, 0, -1$. The corresponding model in three dimensions, with $L = 0$, was studied within the mean-field approximation by Blume, Emery, and Griffiths.[5] The case of $L \neq 0$ was considered by Mukamel and Blume.[6] In Fig. 1, we qualitatively sketch the phase diagram found within mean-field theory for $K = L = 0$. The tricritical point C is the terminus of three critical lines α, β, and γ bounding the three coexistence surfaces Σ_1, Σ_2, and Σ_3, respectively. The three coexistence surfaces intersect in a line of triple points τ. Consider a limit in which the three-dimensional system becomes one dimensional, e.g., by allowing appropriate interaction constants to vanish. As the one-dimensional system is approached, the tricritical point will lie at lower and lower temperatures until, in the limit, it becomes a pseudotricritical point at zero temperature. We can think of the phase diagram in Fig. 1 as flattening out. Finally, the first-order surfaces become the coexistence lines (a) and (f) in the $T = 0$ plane shown in Fig. 2. When (a) and (f) are crossed in the $T = 0$ plane, one finds discontinuous changes in the magnetization $M = \langle S_n \rangle$ and/or the quadrupolar average $Q = \langle S_n^2 \rangle$. On approaching the lines (a) and (f) from $T > 0$, one observes pseudocritical behavior. When the triple point (c) is approached from $T > 0$, one finds pseudotricritical behavior in the appropriate limit.

Our motivation for considering all of the interactions included in (1.1) is that we wish to construct the renormalization-group equations in closed form. We shall see that if we begin with only an exchange interaction J and a crystal field Δ, then application of the renormalization group generates biquadratic exchange K. If the initial Hamiltonian contains a magnetic field H, then the renormalization group generates nonsymmetric exchange L. In order to gain some familiarity with the Hamiltonian (1.1), we plot in Figs. 2–5 some of the possible phase diagrams at zero temperature. In Fig. 3 we consider $J = 0$, $K > 0$, $L = 0$. When the two-phase coexistence lines (f) are approached from $T > 0$, pseudocritical behavior is observed. On approach to the line (d), corresponding to the coexistence of an infinite number (when

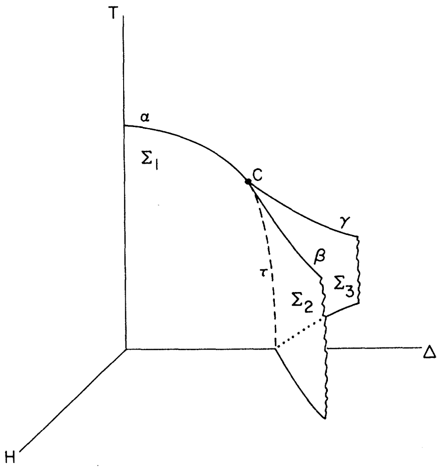

FIG. 1. Qualitative sketch of the phase diagram in the T, Δ, H space for the spin-1 model in three dimensions, with $K=L=0$. The tricritical point C is the terminus of the three critical lines α, β, γ, which bound the coexistence surfaces Σ_1, Σ_2, Σ_3, respectively. The three coexistence surfaces meet in a line of triple points τ.

$N\to\infty$) of phases, no pseudocritical behavior is observed. Physically, this is because the system responds to a magnetic field like a collection of free spins. In Fig. 4, we consider $J>0$, $|L|\le J+K$. The phase diagram is a simple deformation of that presented in Fig. 2 for $L=0$. However, for $J>0$, $J+K<|L|$, there exists staggered quadrupolar order, as shown in Fig. 5.

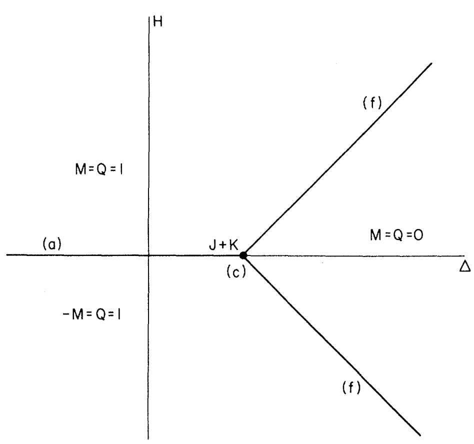

FIG. 2. Plot of the phase diagram at $T=0$, in the Δ, H plane, for the one-dimensional spin-1 model, with $J>0$, $J+K>0$, $L=0$. Two phases coexist along the lines (a) and (f), and three phases coexist at the point (c). When (a) and (f) are traversed in the $T=0$ plane, one finds discontinuous changes in the magnetization M and the quadrupolar average Q. When the lines (a) and (f) are approached from $T>0$, one finds pseudocritical behavior, and when the triple point (c) is approached from $T>0$, one finds pseudotricritical behavior in the appropriate limit.

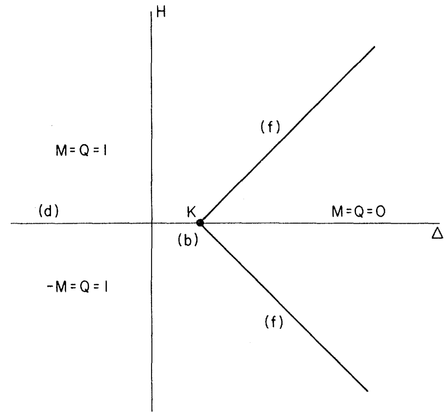

FIG. 3. Plot of the phase diagram at $T=0$ in the Δ, H plane for the one-dimensional spin-1 model, with $J=0$, $K>0$, $L=0$. Two phases coexist along the lines marked (f); three phases coexist at the point (b), and an infinite number of phases coexist along the line (d). Pseudocritical behavior is observed upon approaching the lines (f) from $T>0$. No pseudocritical behavior is observed upon approaching the line (d) from $T>0$. Physically, the reason for the absence of pseudocritical behavior is that the system responds to a magnetic field like a collection of free spins.

We construct the renormalization-group recursion relations by relating the transfer matrix raised to a power b to a constant times the transfer matrix corresponding to new interaction parameters [see Eq. (3.7)]. For integer b, this is equivalent to the spin-decoration transformation considered by Fisher and Nelson[2] and Kadanoff[3] for the spin-$\frac{1}{2}$ Ising chain. However, the renormalization group can be defined for noninteger b. Choosing $b=e^l$ and letting $l\to 0$, one can derive differential renormalization-group equations.

To study nonstaggered ordering, one may choose

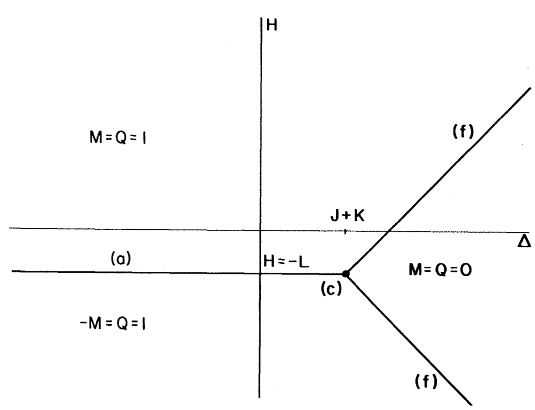

FIG. 4. Plot of the phase diagram at $T=0$ in the Δ, H plane for the one-dimensional spin-1 model, with $J>0$, $|L|\le J+K$. The lines (a) and (f) and the point (c) are characterized in the same manner as in Fig. 2, for the special case $L=0$.

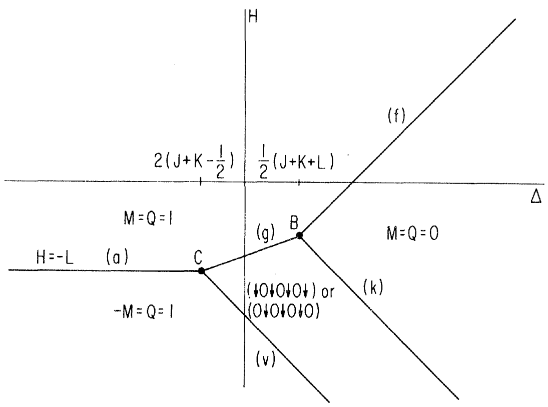

FIG. 5. Plot of the phase diagram at $T=0$ in the Δ, H plane for the one-dimensional spin-1 model, with $J>0$, $J+K<|L|$. Note the existence of staggered quadrupolar order.

$b=2$. To study staggered order, one can take $b=3$. The renormalization-group transformations corresponding to $b=2$ are found to possess three classes of fixed points. *Tricritical* fixed points correspond to transfer matrices with three positive equal eigenvalues. *Critical* fixed points correspond to transfer matrices with the largest eigenvalue positive and doubly degenerate and the third eigenvalue zero. *Infinite-temperature* fixed points correspond to transfer matrices with at least two zero eigenvalues. If we let $b=3$, then, in addition to the fixed points discussed above, the renormalization-group transformations possess fixed points related to staggered ordering. There exists a tricritical fixed point corresponding to transfer matrices with eigenvalues 1, 1, −1 and critical fixed points corresponding to transfer matrices with eigenvalues λ, $-\lambda$, 0.

Consider the case of nonstaggered ordering. At criticality, the transfer matrix has a doubly degenerate largest eigenvalue and a smaller nonzero eigenvalue. Repeated application of the renormalization group results in the largest eigenvalues becoming more and more dominant over the third eigenvalue. This causes movement along a critical surface toward a critical fixed point. If the initial transfer matrix has a nondegenerate largest eigenvalue, repeated application of the renormalization group causes convergence to an infinite-temperature fixed point. There is no choice of initial transfer matrix such that repeated application of the renormalization group causes convergence to the tricritical fixed point. The increased instability of the tricritical fixed point relative to the critical fixed point was noted by Riedel and Wegner.[7] From our work, we see that this greater instability is directly related to the fact that the eigenvalue degeneracy of the transfer matrix is larger at a tricritical point than at a critical point.

For $H=L=0$, we find there exist a tricritical fixed point and two distinct critical fixed points (see Fig. 6). Let us define ξ_{I} to be the correlation length of $\langle S_i S_j\rangle$ and ξ_{II} to be the correlation length of $\langle (S_i^2-Q)(S_j^2-Q)\rangle$. The tricritical fixed point corresponds to $\xi_{\text{I}}=\xi_{\text{II}}=\infty$. The first critical fixed point ($\xi_{\text{I}}=\infty$, $\xi_{\text{II}}=0$) is connected to the tricritical fixed point by a critical line ($\xi_{\text{I}}=\infty$, ξ_{II} finite). The second critical fixed point ($\xi_{\text{I}}=0$, $\xi_{\text{II}}=\infty$) is connected to the tricritical fixed point by a critical line (ξ_{I} finite, $\xi_{\text{II}}=\infty$). At the second critical fixed point the exchange $J=0$. Let us note that the quantity

$$C\equiv \xi_{\text{I}}/\xi_{\text{II}}$$

is an invariant of the renormalization group and therefore is a convenient label for trajectories in a flow diagram. In the immediate neighborhood of the tricritical fixed point, the invariant C can take on any value from 0 to ∞. Since C is a measure of the strength of fluctuations in M relative to those in Q, it is a useful measure of the crossover observed upon approaching the tricritical point along different paths. Heuristically, the crossover behavior can be thought of as resulting from the competition between the two critical fixed points. For $H\neq 0$, we find critical fixed points characteristic of the nonsymmetric "wing" critical points.

The local and the global properties of the renormalization group play an important role in Wilson's theory of critical phenomena. Local questions to be answered are "What fixed points exist?" and "What scaling properties are predicted upon linearizing about the fixed points?" Global ques-

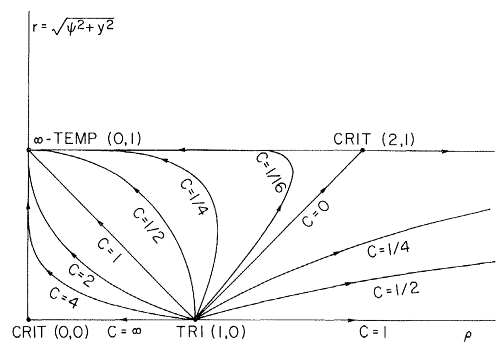

FIG. 6. Plot of the flow diagram in the invariant plane $x=0$, $\phi=$const, $\psi/y=$const. Fixed points are marked by dark circles, and their character and coordinates are labeled. For $b=2$, we need not consider $r>1$, since $r\leq 1$ after one iteration. The trajectories are given by Eq. (4.17) and are labeled by the invariant $C=\xi_{\text{I}}/\xi_{\text{II}}$. The two lines of critical Hamiltonians correspond to $C=0$ and $C=\infty$.

tions are "What Hamiltonians converge to a given fixed point?" and "What is the flow diagram?" For the spin-1 Ising chain, these local and global questions can be answered exactly. It would be of interest if future work could determine exactly the nonlinear scaling fields introduced by Wegner[1] in his study of corrections to scaling.

This paper is organized as follows: In Sec. II, the transfer matrix is used to compute the free energy and correlation functions corresponding to the Hamiltonian (1.1). In Sec. III, exact recursion relations are derived for the spin-1 Ising model (1.1). The fixed points are located, and the flows are described. Linearizing about the tricritical fixed point, we find five relevant eigenfields. Linearizing about the critical fixed points, we find two relevant eigenfields, two marginal eigenfields, and one nonlinearizable recursion relation. In Sec. IV, we present our concluding remarks.

II. CALCULATION OF FREE ENERGY AND CORRELATION FUNCTIONS

A. Transfer matrix

The free energy and correlation functions of the system described by the Hamiltonian (1.1) are easily computed by using the transfer matrix. For a Hamiltonian of the form

$$\mathcal{H}=\sum_{n=1}^{N} V(S_n,\, S_{n+1}), \tag{2.1}$$

the transfer matrix is defined by

$$T_{SS'}=e^{-\beta V(S,\,S')} . \tag{2.2}$$

The transfer matrix corresponding to the Hamiltonian (1.1) is

$$\underline{T}=\begin{bmatrix} e^{\beta(J+K-\Delta+L+H)} & e^{\beta(-\Delta+H)/2} & e^{\beta(-J+K-\Delta)} \\ e^{\beta(-\Delta+H)/2} & 1 & e^{-\beta(\Delta+H)/2} \\ e^{\beta(-J+K-\Delta)} & e^{-\beta(\Delta+H)/2} & e^{\beta(J+K-\Delta-L-H)} \end{bmatrix} . \tag{2.3}$$

The partition function Z_N is expressed in terms of the transfer matrix by

$$Z_N=\operatorname{Tr}\,\underline{T}^N . \tag{2.4}$$

For nonzero temperature, the transfer matrix has positive elements so the Frobenius theorem implies that the largest eigenvalue is positive and nondegenerate. The free energy per spin f_N is defined by $-\beta f_N=N^{-1}\ln Z_N$. In the thermodynamic limit $N\to\infty$, the free energy per spin f is determined by the largest eigenvalue λ_1 of the transfer matrix,

$$-\beta f=\ln\lambda_1 . \tag{2.5}$$

The correlation functions are expressed in terms of the transfer matrix by

$$\langle S_1^{\alpha} S_{1+R}^{\beta}\rangle_N=Z_N^{-1}\operatorname{Tr}\underline{S}^{\alpha}\,\underline{T}^{R}\,\underline{S}^{\beta}\,\underline{T}^{N-R} , \tag{2.6}$$

where the exponents $\alpha,\beta=1$ or 2, and

$$\underline{S}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} .$$

The transfer matrix has three eigenvalues λ_1, λ_2, and λ_3, with corresponding eigenvectors Ψ_1, Ψ_2, and Ψ_3. In the thermodynamic limit $N\to\infty$,

$$\langle S_1^{\alpha} S_{1+R}^{\beta}\rangle=\sum_{K=1}^{3}\langle\Psi_1|\underline{S}^{\alpha}|\Psi_K\rangle\langle\Psi_K|\underline{S}^{\beta}|\Psi_1\rangle\left(\frac{\lambda_K}{\lambda_1}\right)^{R} . \tag{2.7}$$

We see that the exponential falloff of the correlations at large separations is determined by the ratio of the next largest to the largest eigenvalue. If the largest eigenvalue becomes degenerate in the zero-temperature limit, this will correspond to an infinite correlation length, giving rise to what is known as pseudocritical behavior.

The magnetization is related to the transfer matrix by

$$M_N=Z_N^{-1}\operatorname{Tr}\underline{S}\,\underline{T}^N , \tag{2.8}$$

and the quadrupolar average by

$$Q_N=Z_N^{-1}\operatorname{Tr}\underline{S}^2\,\underline{T}^N . \tag{2.9}$$

In the thermodynamic limit $N\to\infty$, the magnetization and quadrupolar average per spin are determined by the eigenvector corresponding to the largest eigenvalue by

$$M=\langle\Psi_1|\underline{S}|\Psi_1\rangle , \tag{2.10}$$

$$Q=\langle\Psi_1|\underline{S}^2|\Psi_1\rangle . \tag{2.11}$$

B. $H=L=0$

In the case of vanishing magnetic field H and nonsymmetric exchange L, the Hamiltonian (1.1) is invariant under the reversal of all spins, $S_n\to -S_n$. This leads to the symmetry property of the transfer matrix, $T_{SS'}=T_{-S,-S'}$. Upon introducing the parameterization

$$\rho=1+z=e^{-\beta(J+K-\Delta)},\quad x=e^{-\beta(J+K-\Delta/2)},\quad y=e^{-2\beta J},$$

the eigenvalues are given by

$$\lambda_1=\rho^{-1}\{1+\tfrac{1}{2}(y+z)+\tfrac{1}{2}[(y-z)^2+8x^2]^{1/2}\} , \tag{2.12a}$$

$$\lambda_2=\rho^{-1}(1-y) , \tag{2.12b}$$

$$\lambda_3=\rho^{-1}\{1+\tfrac{1}{2}(y+z)-\tfrac{1}{2}[(y-z)^2+8x^2]^{1/2}\} , \tag{2.12c}$$

and the corresponding eigenvectors are

$$\Psi_1=(1/\sqrt{2})(\sqrt{Q},\ \sqrt{2(1-Q)},\ \sqrt{Q}) , \tag{2.13a}$$

$$\Psi_2=(1/\sqrt{2})(1,\ 0,\ -1) , \tag{2.13b}$$

$$\Psi_3=(1/\sqrt{2})(\sqrt{1-Q},\ -\sqrt{2Q},\ \sqrt{1-Q}) , \tag{2.13c}$$

where the quadrupolar average Q is

$$Q = \tfrac{1}{2}\{1 + (y-z)/[(y-z)^2 + 8x^2]^{1/2}\} \ . \quad (2.14)$$

With the positive choice of square root, λ_1 is the largest eigenvalue, and the free energy per spin is determined by (2.5). From (2.7), we find that the correlation functions can be expressed by

$$\langle S_1 S_{1+R}\rangle = Q(\lambda_2/\lambda_1)^R \ , \quad (2.15a)$$

$$\langle S_1^2 S_{1+R}^2\rangle - Q^2 = Q(1-Q)(\lambda_3/\lambda_1)^R \ . \quad (2.15b)$$

Hence the correlation lengths ξ_{I} for fluctuations in the magnetization and ξ_{II} for fluctuations in the quadrupolar average are

$$\xi_{\mathrm{I}}^{-1} = -\ln|\lambda_2/\lambda_1| \ , \quad (2.16a)$$

$$\xi_{\mathrm{II}}^{-1} = -\ln|\lambda_3/\lambda_1| \ . \quad (2.16b)$$

The susceptibilities can be obtained from (2.15) by using the fluctuation theorem, and we find

$$\frac{\partial M}{\partial(\beta H)} = \frac{Q}{1-\lambda_2/\lambda_1} \ , \quad (2.17a)$$

$$\frac{\partial Q}{\partial(\beta\Delta)} = \frac{Q(1-Q)}{1-\lambda_3/\lambda_1} \ . \quad (2.17b)$$

C. Nonzero H and L

Let us compute the free energy corresponding to the Hamiltonian (1.1) with nonzero magnetic field H and nonsymmetric exchange L. We consider the case of nonstaggered order; so we restrict our attention to $J \geq 0$, $|L| \leq J+K$. The algebra involved in solving the cubic characteristic equation for the eigenvalues of the transfer matrix (2.3) is reduced by introducing the parameterization

$$\rho = 1 + z = e^{-\beta(J+K-\Delta)}/\cosh\beta(L+H) \ ,$$

$$x = \rho e^{-\beta\Delta/2}, \quad y = \rho e^{\beta(-J+K-\Delta)} \ ,$$

$$\phi = e^{\beta H/2}, \quad \Psi = \tanh\beta(L+H) \ .$$

The transfer matrix (2.3) can be written

$$\underline{T} = \rho^{-1}\begin{bmatrix} 1+\Psi & x\phi & y \\ x\phi & 1+z & x\phi^{-1} \\ y & x\phi^{-1} & 1-\Psi \end{bmatrix} . \quad (2.18)$$

For nonzero temperatures, we denote the eigenvalues by $\Lambda_1 > \Lambda_2 \geq \Lambda_3$. Let us write $\Lambda_m = \rho^{-1}(1+\overline{\Lambda}_m)$. Letting an asterisk denote complex conjugation and $\omega = e^{2\pi i/3}$, the solution to the cubic equation has the form

$$3\overline{\Lambda}_1 = z + \Gamma + \Gamma^* \ , \quad (2.19a)$$

$$3\overline{\Lambda}_2 = z + \omega^*\Gamma + \omega\Gamma^* \ , \quad (2.19b)$$

$$3\overline{\Lambda}_3 = z + \omega\Gamma + \omega^*\Gamma^* \ , \quad (2.19c)$$

where

$$\Gamma = (B + i\sqrt{27D})^{1/3} \ .$$

To ensure the ordering $\Lambda_1 > \Lambda_2 \geq \Lambda_3$, the cube root is cut along the negative real axis and is real analytic. The square root is cut along the negative real axis and is positive on the positive real axis. In the limit $H \to 0$, $L \to 0$, one finds $\Lambda_1 \to \lambda_1$, $\Lambda_2 \to \max(\lambda_2, \lambda_3)$, $\Lambda_3 \to \min(\lambda_2, \lambda_3)$, where λ_m are defined in (2.12). Introducing

$$r^2 = y^2 + \Psi^2, \quad 2c = \phi^2 + \phi^{-2}, \quad 2s = \phi^2 - \phi^{-2} \ ,$$

it is straightforward to show

$$B = z^3 - 9zr^2 + 9(cz + 3y + 3s\Psi)x^2 \ , \quad (2.20a)$$

$$\begin{aligned} D = {} & r^2(z^2 - r^2)^2 \\ & + [(18zr^2 - 2z^3)(y + s\Psi) + cr^2(10z^2 + 6r^2)]x^2 \\ & + [c^2(12r^2 + z^2) - 18cz(y + s\Psi) - 27(y + s\Psi)^2]x^4 \\ & + 8c^3x^6 \ . \end{aligned} \quad (2.20b)$$

In the thermodynamic limit $N \to \infty$, the free energy per spin f is given by

$$-\beta f = -\ln\rho + \ln(1 + \overline{\Lambda}_1) \ . \quad (2.21)$$

The *critical* singularities corresponding to nonstaggered ordering occur when $B < 0$ and $D \to 0$. In this limit,

$$\overline{\Lambda}_1 \approx \tfrac{1}{3}z + (-B)^{1/3}(\tfrac{1}{3} - \sqrt{D}/B) \ . \quad (2.22)$$

The first class of critical singularities corresponds to

$$z < 0, \quad x\phi \to 0, \quad x\phi^{-1} \to 0, \quad y \to 0, \quad \Psi \to 0 \ , \quad (2.23a)$$

and is realized upon approach to the first-order lines labeled (a) in Figs. 2 and 4. One finds

$$B = z^3 + O(r^2, cx^2) \ , \quad (2.23b)$$

$$\begin{aligned} D = {} & z^2(x^2 - zy)^2 + z^2(sx^2 - z\Psi)^2 \\ & + O(cr^2x^2, crx^4, c^3x^6) \ . \end{aligned} \quad (2.23c)$$

The second class of critical singularities, "wing critical points," corresponds to

$$z > 0, \quad x\phi \to 0, \quad x\phi^{-1} \to 0, \quad r^2 - z^2 \to 0 \ , \quad (2.24a)$$

and is realized upon approach to the first-order lines marked (f) in Figs. 2–4, where in addition $y \to 0$. The cases with $0 < y \leq 1$ correspond to different paths of approach to the triple points in Figs. 2–4 when one considers $J \to 0+$. One finds,

$$B = -8z^3 + O(z^2 - r^2, cx^2) \quad (2.24b)$$

$$\begin{aligned} D = {} & z^2(z^2 - r^2)^2 + 16[z^3(y + s\Psi) + cz^4]x^2 \\ & + O(cx^4, (z^2 - r^2)x^2, c^2x^4) \ . \end{aligned} \quad (2.24c)$$

The third class of critical singularities corresponds to

$$\Psi < 0, \quad z < -\Psi, \quad y \to 0, \quad x\phi \to \sqrt{2\Psi(\Psi + z)}, \quad x\phi^{-1} \to 0 \ . \quad (2.25a)$$

These singularities are realized upon approach to the triple point (c) in Fig. 4, with $-L=J+K>0$. Defining

$$\zeta=(x\phi)^2-2\Psi(\Psi+z) \ ,$$

we find

$$B=(z+3\Psi)^3+O(\zeta,\ y^2,\ x^2y,\ x^4) \tag{2.25b}$$

$$D=(3\Psi+z)^2\zeta^2+(\Psi+z)^{-1}(3\Psi+z)^3[x^2-(\Psi+z)y]^2 + O(\zeta^3,\ x^6,\ x^4\zeta,\ x^2y\zeta,\ \zeta y^2,\ x^4y^2,\ y^4) \ . \tag{2.25c}$$

The final class of critical singularities corresponds to

$$\Psi>0,\quad z<\Psi,\quad y\to 0,\quad x\phi\to 0,\quad x\phi^{-1}\to\sqrt{2\Psi(\Psi-z)} \ , \tag{2.26}$$

and is realized upon approach to the triple point (c) in Fig. 4, with $L=J+K>0$. The expressions for B and D follow from (2.25c) and (2.25d) upon the replacement $\phi\to\phi^{-1}$, $\Psi\to-\Psi$.

The *tricritical* singularities corresponding to nonstaggered order occur when $B\to 0$ and $D\to 0$, that is,

$$z\to 0,\quad x\phi\to 0,\quad x\phi^{-1}\to 0,\quad y\to 0,\quad \Psi\to 0 \ , \tag{2.27a}$$

and are realized upon approach to the triple points marked (c) in Figs. 2 and 4. In this limit,

$$-\beta f\approx -z+\overline{\Lambda}_1 \ . \tag{2.27b}$$

Since B is homogeneous in $x\phi^{-1}$, $x\phi$, y, z, Ψ of order 3, and D of order 6, the free energy is asymptotically homogeneous of order 1. Finally, we note there is *no singularity* when $B>0$ and $D\to 0$.

III. RENORMALIZATION GROUP FOR SPIN-1 ISING CHAIN

A. Derivation of equations

It is instructive to perform a renormalization-group analysis upon the spin-one Ising chain. The simplicity of the one-dimensional chain allows the construction of exact renormalization-group equations in closed form. The model exhibits interesting and varied pseudocritical behavior in the zero-temperature limit, which is reflected in the structure of the renormalization-group transformations. These equations possess fixed points characteristic of both critical and tricritical phenomena. We shall construct exact renormalization-group equations for the spin-one Ising chain described by the Hamiltonian (1.1).

In Sec. II, we computed the free energy and correlation functions for the model specified by (1.1) by solving for the eigenvalues λ_i and the eigenvectors Ψ_i ($i=1, 2, 3$) of the transfer matrix $\underline{T}(J,K,L,\Delta,H)$. The renormalization-group equations relating J, K, L, Δ, H to new parameters J', K', L', Δ', H' will be defined by

$$\underline{T}^b(J,K,L,\Delta,H)=A(J,K,L,\Delta,H)\times\underline{U}\underline{T}(J',K',L',\Delta',H')\underline{U}^{-1} \ . \tag{3.1}$$

Equation (3.1) specifies an infinity of different renormalization groups, each labeled by a 3×3 nonsingular matrix $\underline{U}$. The partition functions corresponding to primed and unprimed coupling parameters are related by

$$Z_{Nb}(J,K,L,\Delta,H)=A^N(J,K,L,\Delta,H)\times Z_N(J',K',L',\Delta',H') \ , \tag{3.2}$$

which follows from (2.4) and (3.1). The correlation lengths[8]

$$\xi_{\rm I}^{-1}=-\ln|\lambda_2/\lambda_1| \tag{3.3}$$

and

$$\xi_{\rm II}^{-1}=-\ln|\lambda_3/\lambda_1| \tag{3.4}$$

are transformed under (3.1) according to

$$\xi(J',K',L',\Delta',H')=\xi(J,K,L,\Delta,H)/b \ . \tag{3.5}$$

For any choice of nonsingular $\underline{U}$, the quantity

$$C\equiv\xi_{\rm I}/\xi_{\rm II} \tag{3.6}$$

is left invariant under the transformation (3.1). In the following, we shall make the special choice $\underline{U}=\underline{1}$. Then (3.1) becomes

$$\underline{T}^b(J,K,L,\Delta,H)=A(J,K,L,\Delta,H)\underline{T}(J',K',L',\Delta',H') . \tag{3.7}$$

Since $\underline{T}^b$ and $\underline{T}$ commute, they both have the same eigenvectors. As a result, the magnetization $M=\langle S_n\rangle$ and the quadrupolar average $Q=\langle S_n^2\rangle$ are invariant under (3.7), because they can be expressed as matrix elements between eigenvectors of the transfer matrix. However, M and Q are not invariant under (3.1) for an arbitrary choice of $\underline{U}$.

The explicit fixed-point structure of the transformations (3.7) depends on the choice of b. Let us first consider the choice $b=2$, which is appropriate for the study of nonstaggered ordering. Then, the transformations (3.7) possess three classes of fixed points:

(a) Tricritical fixed points, which correspond to a transfer matrix with three positive equal eigenvalues. (b) Critical fixed points, which correspond to a transfer matrix with the largest eigenvalue positive and doubly degenerate and the third eigenvalue zero. (c) Infinite-temperature fixed points, which correspond to a transfer matrix with at least two zero eigenvalues.

At criticality, the transfer matrix possesses doubly degenerate largest eigenvalues and a smaller eigenvalue, not in general zero. Repeated application of (3.7) will result in the largest eigenvalues becoming increasingly dominant over the third eigenvalue. Therefore (3.7) will cause movement along a critical surface toward a critical fixed point. On the other hand, if the initial transfer matrix has a nondegenerate largest eigenvalue, then repeated application of (3.7) will cause movement toward an infinite-temperature fixed point. It is significant to note that there is no choice of initial transfer matrix such that repeated application of (3.7) will cause convergence to the tricritical fixed point. The increased instability of the tricritical fixed point relative to the critical fixed point was emphasized by Riedel and Wegner.[7] We see that this greater instability is directly related to the fact that the eigenvalue degeneracy of the transfer matrix is larger at a tricritical point than at a critical point.

If we let $b=3$, then, in addition to the fixed points discussed above, the transformations (3.7) possess fixed points related to *staggered* ordering. There exist tricritical fixed points corresponding to transfer matrices with eigenvalues 1, 1, -1 and critical fixed points corresponding to transfer matrices with eigenvalues λ, $-\lambda$, 0.

B. Invariant subspace $H = L = 0$

For $H=L=0$, the renormalization-group equations close within the subspace spanned by J, K, Δ. In this region, the susceptibilities $\partial M/\partial H$ and $\partial Q/\partial\Delta$ are explicitly given by (2.16a) and (2.16b), respectively. Under (3.7), the susceptibilities transform according to

$$\left(\frac{\partial M}{\partial(\beta H)}\right)_b=\frac{Q}{1-(\lambda_2/\lambda_1)^b}\quad, \tag{3.8a}$$

$$\left(\frac{\partial Q}{\partial(\beta\Delta)}\right)_b=\frac{Q(1-Q)}{1-(\lambda_3/\lambda_1)^b}\quad. \tag{3.8b}$$

Here, we have used the invariance of Q.

At the tricritical fixed point both $\partial M/\partial H$ and $\partial Q/\partial\Delta$ are divergent, since $\lambda_1=\lambda_2=\lambda_3$. A critical surface $\lambda_1=\lambda_2>\lambda_3$ connects the tricritical fixed point to the critical fixed point $\lambda_1=\lambda_2$, $\lambda_3=0$. Along this critical surface, $\partial M/\partial H$ is divergent, while $\partial Q/\partial\Delta$ is finite. Another critical surface $\lambda_1=\lambda_3>\lambda_2$ connects the tricritical fixed point to the critical fixed point $\lambda_1=\lambda_3$, $\lambda_2=0$. On this surface, $\partial Q/\partial\Delta$ is divergent, while $\partial M/\partial H$ is finite. The second critical fixed point, at which $\partial Q/\partial\Delta$ is divergent, corresponds to a Hamiltonian with vanishing exchange $J=0$, but with nonzero biquadratic exchange K and crystal field Δ. The above considerations are illustrated in Fig. 6.

To proceed, it is useful to introduce the parameterization discussed in Sec. II B. There, we defined $\rho=e^{-\beta(J+K-\Delta)}$, $x=e^{-\beta(J+K-\Delta/2)}$, and $y=e^{-2\beta J}$. The transfer matrix was written

$$\underline{T}=\rho^{-1}\begin{bmatrix}1 & x & y\\ x & \rho & x\\ y & x & 1\end{bmatrix} \tag{3.9}$$

It is easy to see that the transformation (3.7) possesses the invariant plane

$$x=0\quad. \tag{3.10}$$

Also, since $\underline{T}^b$ commutes with $\underline{T}$, (3.7) possesses the invariant planes, labeled by the constant E,

$$(1+y-\rho)/x=E\quad. \tag{3.11}$$

The invariance of the planes (3.11) is equivalent to the invariance of $Q=\frac{1}{2}[1+E/(E^2+8)^{1/2}]$.

The transformations (3.7) induce flows within the invariant planes (3.10) and (3.11). The trajectories describing these flows can be determined using the invariance of $C=\xi_{\rm I}/\xi_{\rm II}$, which was noted in (3.6). Let us consider only the ferromagnetic case $J\geq 0$; so $0\leq y\leq 1$. Then, within the $x=0$ plane, the trajectories parameterized by C are

$$\rho=(1-y)^C(1+y)^{1-C}\quad(0\leq C\leq\infty) \tag{3.12a}$$

and

$$\rho^{1-C}=(1+y)(1-y)^{-C}\quad(0\leq C\leq 1)\quad. \tag{3.12b}$$

These trajectories are plotted in Fig. 6. Note that the tricritical fixed point is ($x=0$, $y=0$, $\rho=1$), the critical fixed point at which $\xi_{\rm I}=\infty$ is ($x=0$, $y=0$, $\rho=0$), and the critical fixed point at which $\xi_{\rm II}=\infty$ is ($x=0$, $y=1$, $\rho=2$). There is an infinite-temperature fixed point ($x=0$, $y=1$, $\rho=0$).

The invariant C provides a natural measure of the relative strength of fluctuations in M compared to fluctuations in Q. One can think of the crossover behavior of the susceptibilities on approaching the tricritical point as resulting from the competing influence of the two critical fixed points (corresponding to $C=0$ and $C=\infty$) and as being measured by C. (See Fig. 6.)

C. Linearization about fixed points for $b = 2$

We will now consider nonvanishing magnetic field and nonsymmetric exchange. Staggered order will not be considered in this subsection. A renormalization group appropriate in the absence of staggered ordering corresponds to choosing $b=2$ in (3.7). To proceed, we use the parameterization introduced in Sec. II C

$$\rho=1+z=e^{-\beta(J+K-\Delta)}/\cosh\beta(L+H)\quad,$$

$$x=\rho e^{-\beta\Delta/2},\quad y=\rho e^{\beta(-J+K-\Delta)},$$

$$\phi=e^{\beta H/2},\quad \Psi=\tanh\beta(L+H)\quad.$$

The transfer matrix can be written

$$\underline{T}=\rho^{-1}\begin{bmatrix} 1+\Psi & x\phi & y \\ x\phi & 1+z & x\phi^{-1} \\ y & x\phi^{-1} & 1-\Psi \end{bmatrix} .$$

We obtain the recursion relations

$$\underline{T}^2(\rho,x,y,\phi,\Psi)=\rho^{-2}A\rho'\,\underline{T}(\rho',x',y',\phi',\Psi') \ . \quad (3.13)$$

Explicitly,

$$A=1+\Psi^2+y^2+cx^2 \ ,$$
$$A\Psi'=2\Psi+sx^2 \ ,$$
$$Ay'=2y+x^2 \ ,$$
$$A\rho'=\rho^2+2cx^2 \ ,$$
$$Ax'\phi'=(1+\Psi+\rho)x\phi+yx\phi^{-1} \ ,$$
$$Ax'(\phi')^{-1}=(1-\Psi+\rho)x\phi^{-1}+yx\phi \ .$$

Here, we have again defined $2c=\phi^2+\phi^{-2}$ and $2s=\phi^2-\phi^{-2}$.

The fixed points of these transformations are easily located, since they have a simple characterization in terms of the eigenvalue structure of the transfer matrix. The tricritical, critical, and infinite-temperature fixed points correspond, respectively, to transfer matrices with the largest eigenvalue triply, doubly, and singly degenerate, and the other eigenvalues zero. The fixed points are tricritical,

$$x\phi=x\phi^{-1}=y=z=\Psi=0 \ , \quad (3.14)$$

critical,

$$x\phi=x\phi^{-1}=y=\rho=\Psi=0 \ , \quad (3.15a)$$

$$-1\le\Psi\le1, \quad x\phi=0, \quad x\phi^{-1}=0, \quad y=(1-\Psi^2)^{1/2}, \quad \rho=2 \ , \quad (3.15b)$$

$$-1<\Psi<0, \quad x\phi=[-2\Psi(1+\Psi)]^{1/2}, \quad x\phi^{-1}=0, \quad y=0, \quad \rho=-2\Psi \ , \quad (3.15c)$$

$$0<\Psi<1, \quad x\phi=0, \quad x\phi^{-1}=[2\Psi(1-\Psi)]^{1/2} \ , \quad y=0, \quad \rho=2\Psi \ , \quad (3.15d)$$

infinite temperature,

$$-1<\Psi<1, \quad \phi^4=(1+\Psi)/(1-\Psi), \quad y=(1-\Psi^2)^{1/2}, \quad \rho=x^2/(1-\Psi^2)^{1/2} \ , \quad (3.16a)$$

$$\Psi=\pm1, \quad x\phi=x\phi^{-1}=y=\rho=0 \ , \quad (3.16b)$$

$$\rho^{-1}=x\phi\rho^{-1}=x\phi^{-1}\rho^{-1}=y\rho^{-1}=\Psi\rho^{-1}=0 \ . \quad (3.16c)$$

In the last section, we considered $H=L=0$, i.e., $\Psi=0$, $\phi=1$. We have noted that this region is composed of the invariant planes $x=0$ and $(1+y-\rho)/x=E$. For all E, the second set of planes contain the tricritical fixed point ($x=0$, $y=0$, $\rho=1$), the critical fixed point ($x=0$, $y=1$, $\rho=2$), and the infinite-temperature fixed point ($x=x_0(E)$, $y=1$, $\rho=x_0^2(E)$), where $x_0(E)=-\frac{1}{2}E+\frac{1}{2}(E^2+8)^{1/2}$. Since

$$\frac{x'}{\rho'}=\frac{2x/\rho+Ex^2/\rho^2}{1+2x^2/\rho^2} \ ,$$

it follows that $x'/\rho'\le1/x_0(E)$, so we need consider only $x/\rho\le1/x_0(E)$. For $0<x/\rho<1/x_0(E)$, we see that $x'/\rho'>x/\rho$. In addition,

$$1-y'=(1-y)^2/(1+x^2+y^2) \ ,$$

which implies $1-y'<1-y$ for $x>0$. Hence all flows starting with $x>0$ approach the infinite-temperature fixed point $x=x_0(E)$, $y=1$, $\rho=x_0^2(E)$.

Another invariant subspace is defined by $x\phi=x\phi^{-1}=0$. This region is composed of the invariant planes $\Psi'/y'=\Psi/y=\text{const}$. Because of the manifest cylindrical symmetry, we define $r^2=\Psi^2+y^2$. Then,

$$r'=2r/(r^2+1) \ ,$$

which implies $r'\le1$, and also $r'>r$ for $0<r<1$. Within the plane $\Psi/y=\text{const}$ the flows follow the trajectories

$$\rho=(1-r)^C(1+r)^{1-C} \quad (0\le C\le\infty) \ , \quad (3.17a)$$

$$\rho^{1-C}=(1+r)(1-r)^{-C} \quad (0\le C\le1) \ . \quad (3.17b)$$

These trajectories are derived exactly as were those of Eq. (3.12).

Let us recall[1] that if μ_K are fields measuring deviation from a fixed point, all $\mu_K=0$ at the fixed point, then μ_K are called eigenfields with eigenvalues y_K if $\mu_K'=b^{y_K}\mu_K+O(\mu_m\mu_n)$. The field μ_K is relevant, marginal, or irrelevant if $y_K>0$, $y_K=0$, or $y_K<0$, respectively. The scaling piece of the free energy is predicted to asymptotically satisfy $f_s(b^{y_1}\mu_1, b^{y_2}\mu_2, \cdots)=bf_s(\mu_1,\mu_2,\cdots)$, in the limit $\mu_K\to0$.

Linearizing the recursion relations (3.13) about the tricritical fixed point (3.14), we find five relevant fields $x\phi$, $x\phi^{-1}$, y, z, and Ψ, all corresponding to eigenvalue 1. This implies the free energy is asymptotically homogeneous of order 1 in these variables, which is confirmed by (2.20) and (2.27).

The critical fixed point (3.15a) is characteristic of the critical surface (2.23). Linearizing about this fixed point, we find two relevant eigenfields corresponding to eigenvalue 1,

$$y'+x'^2=2(y+x^2)+\cdots \ , \quad (3.18a)$$

$$\Psi'+sx'^2=2(\Psi+sx^2)+\cdots \ , \quad (3.18b)$$

and two marginal eigenfields,

$$x'\phi'=x\phi+\cdots \ , \quad (3.18c)$$

$$x'\phi'^{-1}=x\phi^{-1}+\cdots \ . \quad (3.18d)$$

The recursion relation

$$\rho'=\rho^2+2cx^2+\cdots \quad (3.18e)$$

cannot be linearized. This may be termed marginal behavior, since it represents a slow variation in the neighborhood of the fixed point. Equations (2.22) and (2.23) show that the singular part of the true free energy is asymptotically homogeneous of order 1 in $-zy+x^2$ and $-z\Psi+sx^2$ near the critical surface (2.23). This agrees with the prediction of the linearized renormalization group, since terms of order $(z+1)\Psi$ and $(z+1)y$ have been dropped in (3.18).

The critical fixed points (3.15b) are characteristic of the critical surface (2.24). Consider linearizing about the fixed point with $\Psi=\Psi_0$, $y=y_0 \equiv (1-\Psi_0^2)^{1/2}$, $\rho=2$, where $-1<\Psi_0<1$. Define $\overline{\Psi}=\Psi-\Psi_0$, $\overline{y}=y-y_0$, and $\overline{\rho}=\rho-2$. We find the two relevant eigenfields with eigenvalue 1,

$$R_1=2(\overline{\rho}-\Psi_0\overline{\Psi}-y_0\overline{y}) \ , \tag{3.19a}$$

$$R_2=x\phi\sqrt{(1+\Psi_0)/2}+x\phi^{-1}\sqrt{(1-\Psi_0)/2} \ , \tag{3.19b}$$

and the two marginal eigenfields,

$$M_1=x\phi\sqrt{(1-\Psi_0)/2}-x\phi^{-1}\sqrt{(1+\Psi_0)/2} \ , \tag{3.19c}$$

$$M_2=y_0\overline{\Psi}-\Psi_0\overline{y} \ . \tag{3.19d}$$

The recursion relation for

$$\Psi_0\overline{\Psi}+y_0\overline{y} \tag{3.19e}$$

cannot be linearized. To check the prediction that the singular part of the free energy is asymptotically homogeneous of order 1 in R_1 and R_2 near the fixed point, note that

$$R_1^2\approx(z^2-r^2)^2 \ \text{for} \ z\approx 1 \ ,$$

and

$$R_2^2=(c+y_0+s\Psi_0)x^2 \ .$$

Near the fixed point ($z\approx 1$), Eq. (2.23c) becomes $D\approx R_1^2+16R_2^2$. Using (2.21) and (2.22), this establishes the required homogeneity property.

The critical fixed points (3.15c) are characteristic of the critical surface (2.25). Consider linearizing about the fixed point with $\Psi=\Psi_0$, $\rho=-2\Psi_0$, $x\phi=\sqrt{-2\Psi_0(1+\Psi_0)}$, where $-1<\Psi_0<0$. Let us define $\overline{\Psi}=\Psi-\Psi_0$ $\overline{\rho}=\rho+2\Psi_0$, and $W=x\phi-\sqrt{-2\Psi_0(1+\Psi_0)}$. The linearized recursion relations are

$$(1-\Psi_0)\begin{bmatrix} x'\phi'^{-1} \\ y' \end{bmatrix}=\begin{bmatrix} 1-3\Psi_0 & \sqrt{-2\Psi_0(1+\Psi_0)} \\ \sqrt{-2\Psi_0(1+\Psi_0)} & 2 \end{bmatrix}\begin{bmatrix} x\phi^{-1} \\ y \end{bmatrix} \tag{3.20a}$$

and

$$(1-\Psi_0)\begin{bmatrix} \overline{\Psi}' \\ W' \\ \overline{\rho}' \end{bmatrix}=\begin{bmatrix} 2(1-\Psi_0^2) & (1-\Psi_0)\sqrt{-2\Psi_0(1+\Psi_0)} & 0 \\ (1-2\Psi_0)\sqrt{-2\Psi_0(1+\Psi_0)} & 1+\Psi_0+2\Psi_0^2 & \sqrt{-2\Psi_0(1+\Psi_0)} \\ 4\Psi_0^2 & 2(1+\Psi_0)\sqrt{-2\Psi_0(1+\Psi_0)} & -4\Psi_0 \end{bmatrix}\begin{bmatrix} \overline{\Psi} \\ W \\ \overline{\rho} \end{bmatrix} \ . \tag{3.20b}$$

We find[9] two relevant eigenfields corresponding to eigenvalue 1,

$$R_1=\sqrt{-2\Psi_0(1+\Psi_0)}\ x\phi^{-1}+(1+\Psi_0)y \ , \tag{3.21a}$$

$$R_2=\overline{\Psi}+\sqrt{-2\Psi_0(1+\Psi_0)}\ W-\Psi_0\overline{\rho} \ , \tag{3.21b}$$

and two marginal eigenfields,

$$M_1=\sqrt{-2\Psi_0(1+\Psi_0)}\ x\phi^{-1}+2\Psi_0 y \ , \tag{3.21c}$$

$$M_2=-\sqrt{-2\Psi_0(1+\Psi_0)}\ \overline{\Psi}+(1+3\Psi_0)W+\sqrt{-2\Psi_0(1+\Psi_0)}\ \overline{\rho} \ . \tag{3.21d}$$

The recursion relation for

$$2\Psi_0\overline{\Psi}+2\sqrt{-2\Psi_0(1+\Psi_0)}\ W-(1+\Psi_0)\overline{\rho} \tag{3.21e}$$

cannot be linearized. Near the fixed point

$$R_1\approx(x\phi)^2-2\Psi(\Psi+z), \quad R_2\approx x^2-(\Psi+z)y \ ,$$

so Eq. (2.25c) shows

$$D\approx(1-\Psi_0)^2R_1^2+[(1-\Psi_0)^3/(1+\Psi_0)]R_2^2 \ ,$$

which establishes the asymptotic homogeneity of the singular part of the free energy. Treatment of the fixed points (3.15d) follows that of (3.15c) with the replacement $\phi\leftrightarrow\phi^{-1}$ and $\Psi\rightarrow-\Psi$.

D. Recursion relations for $b=3$

Let us briefly consider the renormalization group (3.7) corresponding to $b=3$. The fixed points existing for $b=2$ are also fixed points when $b=3$. However, there exist fixed points of the recursion relations for $b=3$, related to staggered order, which are not fixed points of the renormalization group for $b=2$. For example, in addition to the tricritical fixed point (3.14), corresponding to a transfer matrix with eigenvalues (1, 1, 1) for $b=3$, there exist tricritical fixed points corresponding to a transfer matrix with eigenvalues proportional to (1, 1, −1). This illustrates the important point that one must choose a renormalization group appropriate for the problem of interest. As noted by Fisher and Nelson,[2] if one uses an inappropriate renormalization group, there might be no fixed points characteristic of the phenomenon being studied.

Recall that an eigenfield μ of the linearized renormalization group is supposed to satisfy[1] (for arbitrary b)

$$R_b u = b^y u + \cdots \quad . \tag{3.22}$$

While it is clear that the eigenfields for $b = 2$ will be eigenfields for $b = 4$, a nontrivial check on (3.22) is to linearize the $b = 3$ recursion relations around the $b = 2$ fixed points (3.14), and (3.15). It is found that the eigenfields computed for $b = 2$ are also the eigenfields for $b = 3$.

IV. CONCLUDING REMARKS

Renormalization-group recursion relations have been obtained for continuous-spin models by varying the momentum cutoff.[1] Under these renormalization groups, the magnetic field transforms in a very simple manner,[10]

$$H' = b^{(d+2-\eta)/2} H \quad . \tag{4.1}$$

The magnetic field also transforms according to (4.1) under the exact recursion relations derived by Baker[11] for the hierarchical model. For these renormalization groups, all fixed points lie at $H = 0$ or $H = \infty$. On the other hand, the magnetic field does not transform so simply under the recursion relations (3.7) derived for the discrete spin model. And for this renormalization group, we have found critical fixed points at finite $H \neq 0$, corresponding to the "wing" critical points.

We have also found that the recursion relations (3.7) do not close unless one allows for the existence of biquadratic exchange. As a result of including biquadratic exchange, we find there exist two distinct critical fixed points at $H = L = 0$. Letting $\xi_{\rm I}$ denote the correlation length of $\langle S_i S_j \rangle$ and $\xi_{\rm II}$ the correlation length of $\langle (S_i^2 - Q)(S_j^2 - Q) \rangle$, one critical fixed point corresponds to $\xi_{\rm I} = \infty$, $\xi_{\rm II} = 0$, and the other critical fixed point to $\xi_{\rm I} = 0$, $\xi_{\rm II} = \infty$ (see Fig. 6). To our knowledge neither the analogue of biquadratic exchange $(\nabla\phi^2)^2$ nor the analogue of nonsymmetric exchange $(\nabla\phi)(\nabla\phi^2)$ has been considered in the continuous-spin models. Consequently, the fixed point with $\partial M/\partial H$ strongly divergent and $\partial Q/\partial\Delta$ weakly divergent has been found, but the fixed point with $\partial M/\partial H$ weakly divergent and $\partial Q/\partial\Delta$ strongly divergent has not been observed. As emphasized by Fisher and Nelson,[2] work on the one-dimensional models provides the warning that not all renormalization groups possess the same fixed points. Care must be taken to choose a renormalization group appropriate to the phenomenon being studied.

---

*Work performed under the auspices of the U.S. Atomic Energy Commission.

[1]K. G. Wilson, Phys. Rev. B 4, 3174 (1971); 4, 3184 (1971); K. G. Wilson and J. Kogut, Phys. Rep. 12, 75 (1974); see also F. J. Wegner, Phys. Rev. B 5, 4529 (1972).

[2]D. R. Nelson and M. E. Fisher, AIP Conf. Proc. 18, 888 (1973).

[3]L. P. Kadanoff, in Proceedings of the Conference on the Renormalization Group, Temple University, Philadelphia, Pennsylvania, 1973 (unpublished).

[4]Our results were reported in S. Krinsky and D. Furman, Phys. Rev. Lett. 32, 731 (1974).

[5]M. Blume, V. J. Emery, and R. B. Griffiths, Phys. Rev. A 4, 1071 (1971). See also, J. Bernasconi and F. Rys, Phys. Rev. B 9, 3045 (1971).

[6]D. Mukamel and M. Blume, Phys. Rev. A 10, 610 (1974).

[7]E. K. Riedel and F. J. Wegner, Phys. Rev. Lett. 29, 349 (1972); Phys. Rev. B 7, 248 (1973).

[8]The largest eigenvalue is denoted λ_1, and we consider the branch cuts chosen so that λ_1, λ_2, λ_3 reduce to the values given in Eqs. (2.12a)–(2.12c), respectively, when $H \to 0$ and $L \to 0$.

[9]The eigenfields are determined by finding the row eigenvectors of the matrices in (3.20).

[10]J. Hubbard, Phys. Lett. A 40, 111 (1972).

[11]G. A. Baker, Phys. Rev. B 5, 2622 (1972).