High-temperature series for the susceptibility of the spin-s Ising model: Analysis of confluent singularities*

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(Received 10 October 1974)

We have extended the series for the zero-field susceptibility of the spin-s Ising model through tenth order in the reduced inverse temperature K on the square, triangular, simple-cubic, body-centered-cubic, and face-centered-cubic lattices. The series coefficients $h_n(s)$ are expressed as simple polynomials in $X = s(s+1)$. Using extended methods of analysis we have estimated the nature of the leading singularities on the face-centered lattice and conclude with good confidence that the susceptibility exponent γ_1 equals 1.25, independent of s. The exponent of the leading correction term is estimated to be $\gamma_2 \simeq 0.75 \pm 0.08$ in good agreement with renormalization-group theory. For $s = 1/2$ only, the amplitude of the confluent correction apparently vanishes. We have also studied the leading singularities on the triangular lattice and conclude that $\gamma_1 = 7/4$, independent of s, with, however, much stronger corrections than in three dimensions. These results provide a very strong corroboration of the universality hypothesis.

I. INTRODUCTION

In a previous paper[1] (hereafter referred to as I) we presented high-temperature series for the zero-field susceptibility of the spin-s Ising model on the triangular (TRI), simple cubic (sc), body-centered-cubic (bcc), and face-centered-cubic (fcc) lattices through eighth order. Domb and Sykes[2] had previously reported results through sixth order on the fcc lattice. In this work we extend the series through order 10 on those lattices and include the 10th order series on the square (SQ) net. As in I the notation of Domb and Sykes[2] will be followed throughout.

In I we relied on rather straightforward ratio and Padé methods to study the apparent spin dependence of the susceptibility. We found two principal results: (i) the apparent critical point varies smoothly with s as $s^2 K_c(s)^{-1} = s(s+1)\, K_c(\infty)^{-1} + K_0 + K_1/s$, which—since K_0 and K_1 are much smaller than $K_c(\infty)^{-1}$—is very close to the prediction of molecular-field theory.[3] Namely, $K_c(s)^{-1} \propto (s+1)s$. We also found (ii) that the apparent critical exponent $\gamma(s)$ obtained from ratio analysis remains constant at $\gamma \simeq 1.23$ for all $s \geq 3$. As s is reduced from 3 to $\frac{1}{2}$ the apparent exponent increases to $\gamma \simeq 1.25$.

Repeating the end-shifted ratio and Padé analysis of I on the longer series produces no consequential changes from the results reported therein. Even for small s the estimates for $K_c(s)$ change only in the fifth place, while those for γ change in the fourth place. In particular, we find $\gamma(\infty) \simeq 1.232$ (unchanged) and $\gamma(\frac{1}{2}) \simeq 1.248$ (as opposed to 1.246 using eighth-order series). The latter agrees very well with that found by analysis of the twelfth-order $s = \frac{1}{2}$ series constructed by Moore, Jasnow, and Wortis[4] and by Sykes, Gaunt, Roberts, and Wyles.[5] These results are an apparent contradiction of the universality hypothesis,[6] which states that γ should depend upon dimension but not spin.

Wortis, in unpublished work,[7] has suggested that γ is indeed independent of s, and that the apparent spin dependence of γ is due to a weaker, confluent singularity in the susceptibility. Similarly, Wegner[8] has used renormalization-group theory (RGT) to find the effect of irrelevant variables on the leading power-law divergences at the critical point. Within the context of RGT, Wegner[8] finds that all Hamiltonians in the same "universality class" have equal critical exponents, and that there are weaker confluent corrections to the leading power law. These corrections are due to "irrelevant" operators (such as that equivalent to a spin shift in the Ising model). The exponent of the confluent correction term, $\gamma_2 = \gamma - \delta$, is also a universal quantity. However, the relative amplitudes of the principal and secondary singularities vary from Hamiltonian to Hamiltonian in a given class. Wegner has estimated that $\delta \sim 0.5$.[8]

Following these suggestions we have carried out ratio analysis—appropriately extended to allow for the existence of confluent singularities—on the fcc series. In addition, we have recast the series in the manner proposed for the study of confluent singularities by Baker and Hunter.[9] The results of ratio analysis are certainly consistent with $\gamma = 1.25$, a confluent singularity with exponent $\gamma_2 = \gamma - \delta$, and spin-dependent relative amplitudes for the principal and secondary singularities. The exponent δ is estimated to be $\delta = 0.50 \pm 0.08$. The Baker-Hunter analysis for $s = \infty$ (where straightforward analysis of the bare series shows high apparent convergence to $\gamma = 1.232$) shows clear evidence of confluent singularities and yields $\gamma = 1.249$–1.253 for the dominant singularity. The apparent exponent of the weaker singularity is $\gamma_2 \simeq 0.68$–

0.73, yielding $\delta \simeq 0.52-0.57$. The analysis of the series for finite s is somewhat less clean, but tends to agree with the $s=\infty$ results, except for spin-$\frac{1}{2}$ which stands out as a special case. In this case the analysis shows no evidence of the weaker singularity. (This has already been noted by Baker and Hunter,[9] and is implicit in the results of Sykes *et al.*[5]) Accurate two-parameter fits to the critical points $K_c(s)$ are presented in Sec. III for both the confluent singularity analysis and the direct end-shifted ratio analysis.

In an accompanying paper,[10] Saul, Wortis, and Jasnow report independent studies of the fcc series which—although quite different in the details of analysis—are in excellent agreement with our conclusions regarding the nature of the two leading confluent singularities. These authors have actually obtained the eleventh- and twelfth-order contributions for several values ($\frac{1}{2}$, 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, 3, $\frac{7}{2}$, 4, $\frac{9}{2}$, 5, $\frac{11}{2}$, and ∞) of the spin s.

We have also studied the initial susceptibility on the sc lattice. The series here exhibits much poorer convergence than that on the fcc net. The principal cause of this degraded convergence is the loose-packed structure of the sc lattice which allows the existence of a weaker singularity in the susceptibility at the antiferromagnetic Curie point $K=-K_c$.[11] Although the series oscillations caused by the antiferromagnetic singularity can be partially removed by extrapolation of alternate or square-root alternate ratios,[11] the most effective way to deal with the series is to introduce the Euler transformed variable $W\equiv 2K^*K/(K+K^*)$, where K^* is a (hopefully fairly accurate) estimate for K_c. If $K^*\equiv K_c$, the ferromagnetic critical point is invariant under the transformation while the antiferromagnetic critical point is transformed to $W=-\infty$. For $K^*\simeq K_c$, the ferromagnetic critical point is only slightly changed by the transformation, while the antiferromagnetic critical point is shifted far off the circle of convergence.[9,11] Padé analysis of the Euler transformed sc series yields results very similar to those found for the fcc net. That is, the apparent exponent γ is best converged for large spin s, for which $\gamma \simeq 1.23$. The convergence is poorer at small s but there is a distinct trend to $\gamma \simeq 1.25$ at $s=\frac{1}{2}$.

Unfortunately, the Baker-Hunter analysis[9] which was quite successful on the fcc lattice does very poorly in the sc case. The point is that on a loose-packed lattice the second most significant singularity is not the confluent singularity, but rather the antiferromagnetic singularity.[12] On the other hand, the similarity between the apparent spin dependences of γ on the two lattices leaves us little doubt that confluent singularities are also important on the sc lattice and that γ is independent of lattice structure (for given dimensionality) as well as spin value.[6]

Finally, we have also studied the susceptibility of the spin-s model on the (close-packed) TRI lattice. In this case, both the critical point V_c [$\equiv \tanh(K_c)$] and critical exponent γ are known exactly for the spin-$\frac{1}{2}$ model. The exponent has the value $\gamma(\frac{1}{2})=\frac{7}{4}$, and the critical point is located at $V_c=2-\sqrt{3}$.[5,11] In keeping with the universality hypothesis,[3,6,11] we assume that $\gamma\equiv\frac{7}{4}$—independent of s. The apparent exponent $\gamma(s)$ depends strongly on s, however, and is not settled down through tenth order *for any value of s*. For large s $\gamma(s) \simeq 1.9$, while for $s=\frac{1}{2}$, $\gamma(\frac{1}{2})\simeq 1.75\pm 0.02$. Thus, in contrast to the $d=3$, fcc case, the apparent large-s exponent is larger than the $s=\frac{1}{2}$ exponent.

We are able to reconcile the apparent spin dependence of γ with a universal exponent $\gamma=\frac{7}{4}$ by allowing for rather large correction terms with exponent $\gamma_2=\frac{5}{4}$ and $\gamma_3=\frac{3}{4}$. $\gamma_2=\frac{5}{4}$ means $\delta=\frac{1}{2}$, as in three dimensions. We also fit the series with $\delta=\frac{1}{4}$, $\frac{3}{4}$, and 1, but found most reasonable results for $\delta=\frac{1}{2}$. In any case, the amplitudes of the confluent singularities are much larger relative to the principal amplitude than they are for the fcc case.

In Sec. II we present the series for the susceptibility. The remaining sections are then devoted to the various analyses of series described above: Sec. III presents the fcc analysis, both direct and extended. The analysis of the sc series is given in Sec. IV, while the analysis of the two-dimensional series is discussed in Sec. V. Finally, we summarize our results and conclusions in Sec. VI.

II. SERIES EXPANSIONS

The series presented herein have been derived by generalization of the recursive method of Stanley and Kaplan.[13] We have considered the general class of models with Hamiltonian of the form

$$-\beta\mathcal{H}=\sum_{\vec{r}} W[Q(\vec{r})]+\tfrac{1}{2}K\sum_{\vec{r}}\sum_{\vec{\delta}} Q(\vec{r})\cdot Q(\vec{r}+\vec{\delta}), \tag{2.1}$$

where $\beta\,(=1/kT)$ is the inverse temperature, $Q(\vec{r})$ is a *classical* tensor variable with arbitrary domain, W is an even function of Q, $Q(\vec{r})\cdot Q(\vec{r}+\vec{\delta})$ is the (in general, weighted) inner product of $Q(\vec{r})$ and $Q(\vec{r}+\vec{\delta})$ with $\vec{r}$ and $\vec{r}+\vec{\delta}$ nearest-neighbor sites. Series expansions for the susceptibility have been derived through tenth order for arbitrary models of the above type. Details of the method are described elsewhere.[14]

The spin-s Ising model belongs to the scalar Q subclass of the models described by Eq. (2.1). For such models the coefficients of the high-temperature series are sums of products of bare vertex weights I_{2l} defined by

$$I_{2l}=\mathrm{Tr}(Q^{2l}e^{W[Q]})/\mathrm{Tr}(e^{W[Q]}). \tag{2.2}$$

In particular, the Nth coefficient of the susceptibility has the form

$$\sum_{p_1}\cdots\sum_{p_M} A(p_1,\ldots,p_M)\prod_{n=1}^{M}(I_{2n})^{p_n}, \tag{2.3}$$

where the sums over $p_1,\ldots,p_M$ run over all values of $\{p_1,\ldots,p_M\}$ such that $p_l \geq 0$ for all l and that

$$\sum_{n=1}^{M}(2n)p_n = 2N+2.$$

The integer M has value $\frac{1}{2}(N+2)$ for even N and $\frac{1}{2}\times(N+1)$ for odd N. The coefficients $A(p_1,\ldots,p_M)$ are sums and difference of high temperature lattice constants[14] and thus vary from lattice to lattice.

For the spin-s Ising model in zero field $W=0$, and Q is the z component of an angular momentum operator of magnitude s. Thus, Q takes on values $-s, -s+1, \ldots, s$ and the bare vertex weights are given by

$$I_{2l}=\sum_{m=-s}^{+s}\frac{m^{2l}}{2s+1} \tag{2.4}$$

Furthermore, we replace K in Eq. (2.1) by $K(s)=K/s^2$ in conformity with the normalization of Domb and Sykes.[2] In Appendix A we show that the vertex weights I_{2l} entering the series expansions can all be written as polynomials in $X\equiv s(s+1)$ of degree l. Therefore, if we write the susceptibility χ (as in I) as

$$\chi=\frac{m^2K(s)\,s(s+1)}{3J}\sum_{n=0}^{\infty}h_n(s)K(s)^n, \tag{2.5}$$

we may write the nth coefficient $h_n(s)$ as a polynomial of degree n in $X=s(s+1)$. Namely,

$$h_n(s)=\sum_{l=1}^{n}\frac{C_l^{(n)}X^l}{D_n}, \tag{2.6}$$

where we have explicitly included a common denominator D_n in each polynomial. (Note that the coefficient of X^0 is zero for all n.)

The coefficients $C_l^{(n)}$ and common denominators D_n for $n=1,2,\ldots,10$ on each lattice are listed in Table I. In each order D_n is listed first followed by $C_1^{(n)},\ldots,C_n^{(n)}$. For all lattices the leading term h_0 is unity.

III. ANALYSIS OF SERIES ON THE fcc LATTICE

As noted in Sec. I we have performed our most detailed and extensive analyses on the fcc series, primarily because apparent convergence is drastically improved by the absence of antiferromagnetic singularities on the circle of convergence. In this section we describe the various analyses performed and the results to be drawn from them. In Sec. IIIA we discuss the results of repeating the end-shifted ratio analysis of I (Ref. 1) on the tenth-order series. In Sec. IIIB the asymptotic behavior of ratios is discussed for series representing two and three confluent singularities. We then describe our (quite successful) efforts to fit the spin-s ratios to such a confluent singularities form. Section IIIC presents the Baker-Hunter[9] series transformation which allows one to find the critical exponents $\gamma_1,\ldots,\gamma_n$ of N confluent singularities from the poles of the $[N-1/N]$ Padé approximant to the transformed series.

A. End-shifted ratio analysis

The ferromagnetic susceptibility is normally expected to exhibit a "power law" or branch cut singularity at the critical point K_c. Thus we expect that near $K=K_c$,[3,11]

$$\chi\approx A(K)(1-K/K_c)^{-\gamma}, \tag{3.1}$$

although generally there will be additive corrections which may be singular (but more weakly so) at K_c. The critical exponent γ describes the strength of the singularity in χ and thus is a key quantity in characterizing critical point phenomena.

For a singularity of the form (3.1), with $A(K)$ analytic at K_c, the ratio, $R_n\equiv h_n/h_{n-1}$ of successive K series coefficients is expected[1,3,9,11] to behave as

$$R_n\approx K_c^{-1}[1+(\gamma-1)/n] \tag{3.2}$$

for large n. If A is not strictly constant, there will be correction terms in higher integer powers of $1/n$. In the case that $A(K)$ is nonanalytic at K_c, Eq. (3.2) still holds; but the higher-order corrections are, in general, noninteger powers of $1/n$. We defer further discussion of the nonanalytic case to Secs. IIIB and IIIC. For now, we assume that (3.1) holds with $A(K)$ analytic at K_c. Then, as discussed in I,[1] the method of endshifted ratios[1] should yield accurate estimates for K_c and γ. That is, we assume that

$$R_n\approx K_c^{-1}[1+(\gamma-1)/(n+\Delta)], \tag{3.3}$$

and obtain a sequence of estimates for K_c^{-k}, γ, and the end shift Δ. As in I,[1] we have used (3.3) to estimate the best apparent critical parameters $K_c(s)^{-1}$, $\gamma(s)$, and $\Delta(s)$ for a large number of spin values logarithmically distributed between $s=\frac{1}{2}$ and $s=\infty$. The addition of two further terms in the series produces no consequential changes in the results of I.[1] Table II is a list of the "best" tenth-order estimates for the spin values studied. The differences between this table and Table II of I (Ref. 1) are unimportant. The best value of $\gamma(s)$ is still $\gamma(s)=1.23$ for all $s\geq\frac{5}{2}$. Indeed, the apparent convergence to $\gamma=1.232$ at large s is spectacular. For example, with $s=\infty$ we find, using sixth- through tenth-order series, the successive estimates 1.2316, 1.2333, 1.2318, 1.2318, and 1.2318 for γ. For $s\leq 2$, γ increases with decreasing s to 1.25 at $s=\frac{1}{2}$. The most significant difference from the order eight results of I (Ref. 1) is the change in the estimates for

TABLE I. Susceptibility series through tenth-order for the spin-s Ising model. For each order the expansion coefficient $h_n(s)$ is given by

$$D_n h_n(s) = \sum_{l=1}^{n} C_l^{(n)} X^l ,$$

where $X = s(s+1)$. The first number listed in a given order n is D_n. This is then followed by $C_1^{(n)}, \ldots, C_n^{(n)}$ in ascending order.

N	SQ	TRI	sc	bcc	fcc
1	3	1	1	3	1
	4	2	2	8	4
2	45	5	5	45	5
	−6	−1	−1	−12	−2
	68	18	18	296	76
3	675	75	75	675	75
	6	1	1	12	2
	−216	−66	−56	−912	−272
	1 144	464	484	10 928	4 248
4	28 350	6 300	6 300	14 175	3 150
	−45	−15	−15	−45	−15
	1 806	1 116	948	3 882	2 322
	−16 236	−15 956	−13 268	−75 972	−70 772
	50 744	64 904	70 952	551 368	656 648
5	1 190 700	661 500	26 460	297 675	330 750
	270	225	9	135	225
	−12 960	−23 652	−684	−14 040	−49 104
	190 152	549 228	15 612	444 348	2 440 236
	−1 024 032	−4 010 864	−134 688	−5 361 168	−39 096 208
	2 235 808	11 092 944	519 376	27 795 632	251 682 608
6	35 721 000	3 969 000	3 969 000	8 930 250	1 984 500
	−1 890	−315	−315	−945	−315
	100 260	38 070	26 640	109 710	79 290
	−1 630 728	−1 024 404	−675 348	−3 893 436	−4 607 196
	11 497 632	10 828 976	7 445 912	60 404 784	105 206 144
	−41 433 696	−51 683 088	−45 048 576	−499 442 352	−1 125 263 472
	68 310 016	106 529 088	134 113 696	1 979 241 472	5 480 403 392
7	13 395 375 000	59 535 000	19 845 000	133 953 750	29 767 500
	141 750	945	315	2 835	945
	−8 505 000	−155 790	−30 240	−374 220	−322 920
	163 501 200	5 059 764	911 376	15 821 136	22 986 144
	−1 483 268 400	−67 444 248	−12 891 672	−315 219 672	−664 684 728
	7 190 638 800	442 284 696	100 431 384	3 428 921 064	9 548 691 096
	−19 711 689 600	−1 550 331 552	−476 289 088	−21 984 134 208	−76 329 628 032
	25 954 467 200	2 524 174 144	1 153 557 056	70 437 239 296	297 051 037 504
8	70 727 580 000	110 020 680 000	3 143 448 000	35 363 790 000	55 010 340 000
	−212 625	−496 125	−14 175	−212 625	−496 125
	13 908 510	95 829 480	1 490 076	30 793 770	198 604 710
	−286 090 380	−3 418 716 780	−48 243 924	−1 408 662 900	−15 730 522 380
	2 834 637 552	51 936 829 488	743 902 992	30 878 814 384	522 161 817 264
	−16 115 073 984	−420 350 361 696	−6 684 759 648	−390 360 493 728	−9 196 404 968 448
	57 157 269 408	1 974 825 335 232	39 578 788 800	3 164 326 789 536	96 001 685 872 416
	−125 021 664 576	−5 447 227 764 544	−153 502 202 048	−16 391 099 923 392	−612 403 917 558 592
	136 519 395 712	7 291 822 764 928	312 149 311 616	43 833 285 137 024	1 976 994 515 599 744
9	2 334 010 140 000	18 153 412 200 000	1 037 337 840 000	1 167 005 070 000	9 076 706 100 000
	1 771 875	20 671 875	1 181 250	1 771 875	20 671 875
	−127 575 000	−5455 107 000	−137 025 000	−283 500 000	−11 240 964 000
	2 915 701 380	219 572 014 140	4 951 515 960	14 513 848 380	1 022 154 935 940
	−32 655 191 040	−3 736 467 377 712	−86 683 603 680	−362 671 231 680	−38 426 692 327 488
	216 201 641 904	34 847 273 833 200	900 597 250 656	5 338 970 710 992	781 384 549 051 440
	−923 056 564 608	−198 561 705 025 920	−6 228 641 221 632	−51 359 920 308 864	−9 756 313 531 512 960
	2 629 938 997 056	724 887 263 981 120	30 410 648 276 864	339 474 962 774 208	80 185 615 539 775 040
	−4 805 264 923 392	−1 645 784 670 954 240	−100 410 894 389 248	−1 486 064 221 437 696	−425 829 648 399 806 720
	4 468 880 440 064	1 864 771 814 420 736	175 890 198 647 296	3 408 695 720 774 912	1 171 663 580 597 467 904
10	6 371 847 682 200 000	283 193 230 320 000	471 988 717 200 000	41 417 009 934 300 000	707 983 075 800 000
	−1 616 162 625	−107 744 175	−179 573 625	−21 010 114 125	−538 720 875
	125 415 722 250	34 064 787 330	22 518 783 000	3 638 990 832 750	350 344 849 050
	−3 025 742 906 700	−1 471 018 205 196	−862 337 502 900	−198 557 342 018 100	−34 744 487 761 620
	35 695 584 786 600	27 028 694 292 768	15 933 946 020 480	5 280 876 942 929 640	1 424 537 364 737 880
	−253 070 383 970 592	−277 966 027 290 816	−176 673 144 849 984	−83 688 772 541 984 064	−32 123 083 739 615 136
	1 198 433 342 376 000	1 810 359 274 718 304	1 336 387 037 162 880	888 381 687 543 517 440	456 441 911 630 847 360
	−3 998 236 247 726 400	−7 925 971 293 760 704	−7 443 030 356 201 920	−6 766 760 745 776 191 680	−4 418 075 580 182 500 800
	9 444 732 079 338 240	23 678 823 685 360 128	30 843 987 031 111 040	37 492 343 506 036 323 840	30 057 929 530 585 443 840
	−14 794 167 627 828 480	−45 666 137 074 603 008	−88 558 940 112 284 160	−141 691 361 091 197 548 800	−136 839 339 534 638 795 520
	11 974 432 007 975 936	44 769 163 917 919 744	136 050 699 748 572 672	283 891 578 221 538 506 752	327 541 325 856 325 700 608

K_c^{-1} for $s \leqslant 2$ (0.01% for $s = \frac{1}{2}$, an order of magnitude less for $s = 1, \frac{3}{2}$, and 2). It is interesting that spin-$\frac{1}{2}$ has the worst apparent convergence, yet—as we discuss in Secs. IIIB and IIIC—is the only case not marked by a confluent singularity.

As noted in I,[1] the critical estimates vary smoothly with s and (especially for large s) have a variation close to that predicted by molecular-field theory,[3] namely,

$$s^2 K_c(s)^{-1} \propto s(s+1).$$

The order-ten end-shift results for the critical point fit the following formula to within 0.001%:

$$s^2K_c(s)^{-1}=s\,(s+1)K_c(\infty)^{-1}+K_0+K_1/s\,, \qquad (3.4)$$

with $K_0=-0.208716$ and $K_1=0.013146$. Note that (3.4) does not have a fitting parameter multiplying a term linear in s. A three-parameter fit finds such a parameter to be $\sim 10^{-3}$ and does not improve the quality of the fit. This indicates that the molecular field result may be exact to lowest order [i.e., that the coefficient of s is *exactly* $K_c(\infty)^{-1}$]. It is tempting to assume that $s^2/K_c(s)$ depends on $s(s+1)$ only. However the replacement of K_1/s in Eq. (3.4) by $K_1/[s(s+1)]$ leads to a small but definite deterioration of the fit. This is also the case with the confluent-singularity analysis discussed below. The values for K_0 and K_1 quoted for tenth-order series are within 0.3 and 4.2% of the respective results quoted in I (Ref. 1) for eighth-order series.

B. Ratio analysis for confluent singularities

In this section we analyze the fcc series for χ using the extension of the ratio method appropriate to an assumed singularity of the form

$$\chi(K)\approx A_1(1-K/K_c)^{-\gamma}+A_2(1-K/K_c)^{-(\gamma-1)} + B(1-K/K_c)^{-(\gamma-\delta)}. \qquad (3.5)$$

Here the first term represents the dominant singularity in χ; the second term arises by expanding the amplitude $A(K)$ of the dominant term about $K=K_c$ and keeping terms to order $K-K_c$. Finally, the third term is added to allow for the confluent singularity proposed by Wortis[7] and by Wegner.[8] In Appendix B we obtain the form of the ratios $R_n \equiv h_n/h_{n-1}$ for a function of the form (3.5). The appropriate form as n tends to infinity is given by

$$K_cR_n=\left(1+\frac{\gamma-1}{n}\right)\left(1-\frac{B'\delta}{n^{1+\delta}}-\frac{A_2'}{n^2}+\frac{(B')^2}{n^{1-2\delta}}+\frac{B'\delta(\delta+1)(2\gamma-\delta-2)+2(1+\delta)A_2'B'}{2n^{2+\delta}}-\frac{(B')^3\delta}{n^{1+3\delta}}+\cdots\right), \qquad (3.6)$$

where the coefficients A_2' and B' are given by

$$A_2'=\Gamma(\gamma)A_2/\Gamma(\gamma-1)A_1 \qquad (3.7a)$$

and

$$B'=\Gamma(\gamma)B/\Gamma(\gamma-\delta)A_1, \qquad (3.7b)$$

respectively. The most significant effect of the confluent singularity is the replacement of $A_2'n^{-2}$ by $B'\delta n^{-(1+\delta)}$ as the leading correction to $(\gamma-1)/n$.

To study possible confluent singularities we have attempted to fit the ratio sequences to the form

$$R_n=K_c^{-1}\,[1+(\gamma-1)/n+a/n^{1+\delta}+b/n^2], \qquad (3.8)$$

with K_c, γ, δ, a, and b parameters of the fit. Any attempt to derive a fit with all five parameters free is an exercise in futility. No well-behaved fits can be obtained with b and either or both of γ and δ simultaneously fit. Thus, in fits with γ and/or δ free we have set b equal to zero. Table III lists the results for K_c^{-1}, γ, δ, and a when all four are free (with $b=0$) for several values of s. The listed numbers result from fitting R_7 through R_{10}. We have also fit Eq. (3.8) using R_4 through R_7, R_5 through R_8 and R_6 through R_9. There is a great deal of scatter in the results from order to order; and, as may be seen from Table III, the scatter (in δ especially) within a given order is very great. Since we seek universal values for both γ and δ, this fitting procedure is not considered meaningful.

Thus, given that we cannot treat the problem in its full generality, i.e., with γ as a free parameter, two natural choices for γ arise from the endshift analysis of Sec. IIIA, $\gamma=1.232$ and $\gamma=1.25$. The former value arises because it is the apparent exponent favored by endshifted ratio analysis for all spin values greater than $s=5$ (in fact, $\gamma\simeq1.23$ is favored for all $s>2$). On the other hand, the longer $s=\frac{1}{2}$ series favors $\gamma=1.25$, with a nearly zero end shift. Additional evidence for $\gamma=1.25$ is provided by the analysis of the spin-$\frac{1}{2}$ series by Moore, Jasnow, and Wortis,[4] by Sykes *et al.*,[5] and by Baker and Hunter,[9] all of whom have strongly concluded that $\gamma(\frac{1}{2})=1.25$.

Consider first our fit of Eq. (3.8) with b set equal to zero and γ forced to be 1.232, independent of s. The results of this fit (using R_7 through R_{10}) are

TABLE II. Best end-shift results (order 10) for γ and K_c^{-1} on the fcc lattice for various spin values.

s	K_c^{-1}	γ	Δ
0.5	9.79474	1.2482	0.029
1.0	6.82074	1.2401	0.311
1.5	5.75798	1.2371	0.477
2.0	5.21166	1.2355	0.562
2.5	4.87886	1.2344	0.609
3.0	4.65485	1.2338	0.637
3.5	4.49378	1.2333	0.656
4.0	4.37238	1.2330	0.669
4.5	4.27761	1.2328	0.678
5.0	4.20157	1.2326	0.685
5.5	4.13920	1.2325	0.690
6.0	4.08712	1.2324	0.695
8.0	3.94344	1.2322	0.705
10.0	3.85687	1.2320	0.709
15.0	3.74110	1.2319	0.714
20.0	3.68304	1.2318	0.715
30.0	3.62485	1.2318	0.717
50.0	3.57822	1.2318	0.717
50.5	3.57753	1.2318	0.717
51.0	3.57685	1.2318	0.717
100.0	3.54321	1.2318	0.717
999.0	3.51166	1.2318	0.718
9999.0	3.50850	1.2318	0.718
∞	3.50814	1.2318	0.718

shown in Table IV. Requiring γ to be 1.232 produces no single reasonable value of δ, which varies from about 0.9 at $s=\infty$ to -0.1 at $s=\frac{1}{2}$. Furthermore the scatter in δ from order to order is extremely large. But, universality dictates that δ should be independent of s.[6–8] The lack of a single reasonable estimate for δ when $\gamma=1.232$ forces us to conclude that this cannot be the universal value of γ for the Ising model.

In Table V we list the results of fitting Eq. (3.8) with $b=0$ and γ forced to be 1.250, for several values of s. Again, the results displayed are obtained by fitting R_7 through R_{10}. In contrast to the cases with γ free and $\gamma=1.232$, examination of the fits to R_4 through R_7, R_5 through R_8 and R_6 through R_9 indicates that the results are converging with increasing order fit. Within a given order the results (for $s\neq\frac{1}{2}$) are practically independent of spin. For example, from Table V we find $\delta\ (10)\simeq 0.59 \pm 0.01$. The value of δ estimated decreases with increasing order and extrapolates to $\delta\simeq 0.49\pm 0.05$ at infinite order. For $s=\frac{1}{2}$, the value of δ estimated is very scattered as a function of order. More importantly the amplitude a is small for all orders. In the light of these results we conclude that $\delta\simeq 0.50\pm 0.06$ for all $s>\frac{1}{2}$, and that for $s=\frac{1}{2}$ this leading confluent correction probably disappears.

If we are willing to dictate *both* γ and δ, we should be able to obtain more detailed information concerning the spin variation of $K_c(s)$ and of the relative amplitudes of the singularities at K_c. We have therefore fit the series to Eq. (3.8), with γ and δ forced to equal 1.25 and 0.50, respectively. The results (as a function of s) of such a fit are shown in Table VI. Note from the table that the relative amplitude, $a/(\gamma-1)$, of the $1/n^{1+\delta}$ term to the $1/n$ term in Eq. (3.8) is an order of magnitude smaller for spin-$\frac{1}{2}$ than for $s=\infty$. Exactly the same statement holds for the relative amplitude $b/(\gamma-1)$ of the $1/n^2$ term to the $1/n$ term. This aspect of Table VI should not be surprising; Table II shows that $\frac{1}{4}n^{-1}$ comes very close to fitting the spin-$\frac{1}{2}$ ratios. The drop in a and b as s decreases from $s=1$ to $s=\frac{1}{2}$ appears abrupt in Table VI. However, extrapolation of a and b versus $1/s$ (using $s=1$ through $s=\infty$) shows them to be decreasing linearly with s^{-1} for $s<3$ and to apparently pass through zero at $s^{-1}\simeq 2$ ($s\simeq\frac{1}{2}$). To follow this apparent decrease with more detail we have analytically continued the susceptibility series [Eqs. (2.5) and (2.6)] to the interval $\frac{3}{4}\leq X\leq 2(\frac{1}{2}\leq s\leq 1)$. Estimates of K_c, γ, and Δ from bare end shifts together with estimates for K_c a and b obtained by fitting Eq. (3.8) with $\gamma=\frac{5}{4}$ and $\delta=\frac{1}{2}$ forced are given for several values of s in $[\frac{1}{2}, 1]$ in Table VII. The end-shift analysis is quite smooth. The apparent exponent γ increases rapidly from 1.240 at $s=1$ to 1.248 at $s=\frac{1}{2}$; at the same time the end shift decreases rapidly. The near linearity of the decrease of a and b is confirmed by the analytically continued series. In fact, to the accuracy of the fit a and b are *zero* at $s=\frac{1}{2}$. This is clearly seen by comparing order nine estimates to the order ten estimates for $s=\frac{1}{2}$ of Table VII. In ninth order of magnitudes of a and b are unchanged from tenth order. However, the sign of both quantities changes on going from ninth to tenth order, an indication that a and b are both zero for $s=\frac{1}{2}$.

We have fit $K_c(s)$, as estimated from Eq. (3.8) with $\gamma=\frac{5}{4}$ and $\delta=\frac{1}{2}$, to the form, Eq. (3.4), suggested by molecular-field theory. The parameters K_0 and K_1 of the resulting fit are -0.207681 and 0.012949, respectively, which fit all estimates for K_c^{-1} listed in Table VI to better than 0.001%. Again, as with end shifts, we find no evidence for a term linear in s. Indeed, the end-shift and "confluent" estimates for K_0 are within $\frac{1}{2}$% of one another, as are those for K_1.

As a check on the validity of the use of the asymptotic formula, Eq. (3.8), to extrapolate tenth-order series we have also fit the ratios to the form

TABLE III. Parameters of tenth-order fit to Eq. (3.8) with b set equal to zero. For $s=\frac{1}{2}$, no solution was found with δ in the range $-0.5\leq\delta\leq 4.0$.

s	K_c^{-1}	γ	a	δ
$\frac{1}{2}$	...	...	...	...
1	6.82040	1.242	−0.061	0.829
$\frac{3}{2}$	5.75781	1.239	−0.089	0.854
3	4.65454	1.238	−0.108	0.791
8	3.94304	1.238	−0.115	0.753
∞	3.50775	1.238	−0.116	0.744

TABLE IV. Parameters of the tenth-order fit to Eq. (3.8) with b set equal to zero, and γ set equal to 1.232.

s	K_c^{-1}	a	δ
$\frac{1}{2}$	9.79401	0.014	−0.076
1	6.82274	−0.181	1.922
$\frac{3}{2}$	5.75893	−0.111	1.179
3	4.65520	−0.118	0.957
8	3.94359	−0.122	0.896
∞	3.50824	−0.123	0.885

TABLE V. Parameters of the tenth-order fit to Eq. (3.8) with b set equal to zero, and γ equal to 1.250.

s	K_c^{-1}	a	δ
$\frac{1}{2}$	9.79446	−0.005	0.385
1	6.81944	−0.059	0.586
$\frac{3}{2}$	5.75661	−0.084	0.596
3	4.65352	−0.105	0.586
8	3.94223	−0.113	0.578
∞	3.50704	−0.115	0.576

TABLE VI. Coefficients of three-parameter fit to Eq. (3.8) with $\gamma=1.25$ and $\delta=0.50$. R_8 through R_{10} are fit.

s	K_c^{-1}	a	b
0.5	9.79441	−0.007	0.003
1.0	6.81959	−0.043	−0.020
1.5	5.75670	−0.058	−0.031
2.0	5.21049	−0.067	−0.034
2.5	4.87770	−0.073	−0.035
3.0	4.65370	−0.076	−0.035
3.5	4.49264	−0.079	−0.035
4.0	4.37125	−0.080	−0.035
4.5	4.27649	−0.082	−0.035
5.0	4.20046	−0.082	−0.035
5.5	4.13810	−0.083	−0.035
6.0	4.08604	−0.084	−0.035
8.0	3.94238	−0.085	−0.035
10.0	3.85585	−0.086	−0.035
15.0	3.74008	−0.086	−0.035
20.0	3.68203	−0.087	−0.035
30.0	3.62386	−0.087	−0.035
50.0	3.57724	−0.087	−0.035
50.5	3.57655	−0.087	−0.035
51.0	3.57587	−0.087	−0.035
100.0	3.54223	−0.087	−0.035
999.0	3.51069	−0.087	−0.035
9999.0	3.50753	−0.087	−0.035
∞	3.50718	−0.087	−0.035

exactly appropriate to a susceptibility of the form in Eq. (3.5). That is, we have employed Eq. (B3) of Appendix B in forming the function to which we fit the ratios. As in Tables VI and VII, γ and δ are, respectively, set to 1.250 and 0.50. The parameters of the fit are then K_c^{-1}, B/A_1, and A_2/A_1, where A_1, B, and A_2 are the amplitudes of the singularities in Eq. (3.5). The results of such a fit using R_8 through R_{10} in the fit are displayed in Table VIII for several values of spin s. In comparing Tables VI and VIII, the first thing to note is that the largest relative difference in the estimates for $K_c(s)$ is $\simeq 0.003\%$ for $s=\infty$; the estimates for $K_c(s)$ obtained from the two methods being in even closer agreement for all other values of spin. We can use Eqs. (3.7) and (3.8) to obtain estimates for A_2/A_1 and B/A_1 from b and a, respectively. Using Table VI, we find estimates for A_2/A_1 and B/A_1 which differ from the estimates in Table VIII by about 8% in all cases. Considering the large-n character of Eq. (3.8) the extremely

TABLE VII. Ratio analysis of the fcc series for χ (s) analytically continued to $\frac{1}{2}\le s\le 1$.

s	K_c^{-1}	γ	Δ	K_c^{-1}	a	b
0.50	9.79474	1.248	0.021	9.79441	−0.007	0.003
0.55	9.27602	1.247	0.051	9.27546	−0.013	0.002
0.60	8.83664	1.246	0.082	8.83592	−0.018	0.001
0.75	7.84610	1.243	0.177	7.84509	−0.030	−0.007
1.00	6.82074	1.240	0.311	6.81959	−0.043	−0.020

TABLE VIII. Estimates for K_c and the relative amplitudes B/A_1 and A_2/A_1 obtained by fitting the spin-s ratios to the exact form appropriate to Eq. (3.5). As in Tables VI and VII, $\gamma=1.25$ and $\delta=0.50$. R_8 through R_{10} are fit, although no significant changes occur by fitting R_7, R_8 and R_9.

s	K_c^{-1}	B/A_1	A_2/A_1
$\frac{1}{2}$	9.79440	0.0196	−0.0128
1	6.81968	0.1242	0.0432
$\frac{3}{2}$	5.75691	0.1709	0.0825
2	5.21061	0.1968	0.0970
$\frac{5}{2}$	4.87782	0.2123	0.1031
3	4.65382	0.2222	0.1061
$\frac{7}{2}$	4.49275	0.2288	0.1077
4	4.37136	0.2334	0.1086
$\frac{9}{2}$	4.27660	0.2368	0.1093
5	4.20057	0.2393	0.1097
$\frac{11}{2}$	4.13821	0.2412	0.1100
6	4.08614	0.2428	0.1102
8	3.94249	0.2464	0.1106
10	3.85595	0.2482	0.1108
15	3.74018	0.2501	0.1110
20	3.68212	0.2507	0.1110
30	3.62395	0.2513	0.1111
50	3.57733	0.2515	0.1111
$50\frac{1}{2}$	3.57664	0.2515	0.1111
51	3.57596	0.2515	0.1111
100	3.54232	0.2516	0.1111
999	3.51078	0.2516	0.1111
9999	3.50762	0.2516	0.1111
99999	3.50731	0.2516	0.1111
∞	3.50727	0.2516	0.1111

close agreement between the two sets of estimates for $K_c(s)$ is very gratifying. We have further made similar checks on the results in Table V and find that order-by-order the estimates for K_c *and* δ obtained by the two methods agree extremely closely with each other. We take this close agreement as a strong verification of the validity of Eq. (3.8) in extrapolating finite series of the form in Eq. (3.5). Finally we note that we have fit $K_c(s)$, as listed in Table VIII, to the molecular-field formula (3.4). The resulting parameters of the fit are $K_0 = -0.203140$ and $K_1 = 0.016080$. An added term $\tilde{K}s$ in the fit finds $\tilde{K}\sim 0.0007$, which, as above, is taken to mean there is no term linear in s in Eq. (3.4).

In summary, we find that our generalized ratio analysis is consistent with universal critical behavior in the spin-s Ising model. The universal exponent is found to be $\gamma=1.25$, and the exponent of the leading correction is found to be $\gamma-\delta$ with $\delta=0.50\pm 0.08$. The relative amplitudes of the singularities vary smoothly with s and evidently the leading correction vanishes at $s=\frac{1}{2}$.

C. Baker-Hunter transformation

To further explore the existence of confluent singularities, we recast the series in a form suggested by Baker and Hunter.[9] This procedure requires only an accurate estimate y for K_c^{-1}. Generalizing Eq. (3.5), we consider a function $F(K)$ with N confluent singularities at $K_c \equiv y^{-1}$, namely,

$$F(K)=\sum_{l=1}^{N} \frac{A_l}{(1-yK)^{\gamma_l}} \ . \tag{3.9}$$

Introduction of the variable ξ via the transformation

$$\xi = \ln(1-yK)$$

followed by reexpression of F as a function ξ leads to

$$F(K(\xi)) \equiv f(\xi) = \sum_{l=1}^{N} A_l e^{\xi\gamma_l}$$

$$= \sum_{l=1}^{N} A_l \sum_{n=0}^{\infty} \frac{(\gamma_l \xi)^n}{n!} \ . \tag{3.10}$$

Having formed the ξ series for $f(\xi)$, we multiply the nth coefficient of $f(\xi)$ by $n!$ to obtain the auxiliary function

$$\mathfrak{F}(\xi)=\sum_{l=1}^{N} \frac{A_l}{1-\gamma_l \xi} \ . \tag{3.11}$$

The auxiliary function has simple poles at $\xi_l=\gamma_l^{-1}$ with residues $-A_l/\gamma_l$. We note that the transformation from K to ξ has the property that the coefficient of K^n affects the coefficient of ξ^k only for $k \geq n$. Thus given an M term series for $\mathfrak{F}(K)$ and a satisfactory estimate for y, we can construct an M term series for $F(\xi)$. Further, if $F(K)$ is of the form described by Eq. (3.9), then $\mathfrak{F}(\xi)$ is very suitable for analysis by Padé approximants, in particular using the $[N-1/N]$ Padé approximants.[9] In examining the sequence of $[k-1/k]$ Padé approximants, it is found that the sensitivity to y (which is an input parameter to this process) increases rapidly with k. Baker and Hunter[9] considered the effect of deviations from the assumed form (3.9) on the approximants to $\mathfrak{F}(\xi)$. However, they did not discuss in detail the effect of small errors in y coupled with such deviations. Even for test series with strictly constant amplitudes, we find the higher order estimates to be extremely sensitive to y.

One possible criterion for fine tuning y is to attempt to make the $[k-1/k]$ sequence of approximants yield a single value for γ_1. For clean test series this is a very successful method. However, for real series we should keep in mind that the $[k-1/k]$ Padé approximant uses only the first $2k$ terms in the series for $\mathfrak{F}(\xi)$ and thus makes *direct* use of only the first $2k$ terms in $\chi(K)$. Hence, the $[1/2]$ approximant is determined from information we would regard as far short of the asymptotic character of the series.[15]

We first apply the Baker-Hunter transformation to the spin infinity Ising model. The y dependence of the estimates is seen clearly in the results depicted in Table IX. In this table we list the estimates for γ_1, γ_2, γ_3, A_1, A_2, and A_3 using the $[k-1/k]$ Padé approximants with $k=2$, 3, 4, 5, and 6 for five values of y. (We have used the eleventh-order coefficient provided to us by Saul, Wortis, and Jasnow.[10]) The third column corresponds to $y=K_c^{-1}$ as obtained from the confluent singularity ratio analysis, and the fourth column corresponds to $y=K_c^{-1}$ as obtained from "bare" end shifts. As we vary y outside the range shown, apparent convergence progressively worsens. On the other hand, small variations ($|\delta y/y| \lesssim 10^{-5}$) in y produce little variation in apparent convergence, or in ab-

TABLE IX. γ_i and amplitude A_i from Padé-approximant table of Baker-Hunter series for the spin-∞ model on the fcc lattice. The parameter y is the value of K_c^{-1} assumed in the series transformation.

y	3.50650	3.50690	3.50718	3.50814	3.50850
			γ_1		
[1/2]	1.756	1.753	1.751	1.746	1.743
[2/3]	1.256	1.253	1.251	1.244	1.241
[3/4]	1.258	1.254	1.251	1.244	1.241
[4/5]	1.258	1.253	1.249	1.227	1.213
[5/6]	1.255	1.252	1.248	1.225	1.203
			γ_2		
[1/2]	1.091	1.091	1.091	1.091	1.091
[2/3]	0.727	0.712	0.702	0.662	0.645
[3/4]	0.740	0.721	0.708	0.662	0.645
[4/5]	0.739	0.713	0.681	...	...
[5/6]	0.720	0.698	0.671	...	...
			γ_3		
[1/2]	...	...	...	...	...
[2/3]	−0.378	−0.356	−0.340	−0.281	−0.258
[3/4]	−0.420	−0.381	−0.357	−0.283	−0.258
[4/5]	−0.419	−0.347	−0.242	−0.822	−0.732
[5/6]	−0.337	−0.275	−0.188	−0.764	−0.657
			A_1		
[1/2]	0.025	0.025	0.025	0.025	0.025
[2/3]	0.254	0.257	0.260	0.267	0.269
[3/4]	0.251	0.256	0.259	0.267	0.269
[4/5]	0.252	0.257	0.262	0.292	0.316
[5/6]	0.255	0.259	0.263	0.295	0.337
			A_2		
[1/2]	0.309	0.308	0.308	0.308	0.308
[2/3]	0.082	0.079	0.077	0.071	0.069
[3/4]	0.085	0.081	0.078	0.071	0.069
[4/5]	0.084	0.080	0.076	...	...
[5/6]	0.082	0.078	0.076	...	...
			A_3		
[1/2]	...	...	...	...	...
[2/3]	−0.003	−0.003	−0.004	−0.005	−0.005
[3/4]	−0.003	−0.003	−0.004	−0.005	−0.005
[4/5]	−0.003	−0.004	−0.005	−0.001	−0.001
[5/6]	−0.004	−0.004	−0.005	−0.001	−0.001

solute estimates. For example, we studied the effect of varying y in the range $3.5070 \le y \le 3.5075$ about the value obtained from confluent ratio analysis and found only smooth and minor variation of estimated parameters. Before discussing these results in further detail we remark that in all cases studied the [3/4] and [4/5] approximants contain nearly cancelling pole zero pairs that in some cases are closer to the origin than the poles of interest.

The best choice of y according to the "smoothness" criterion is given by $y \simeq 3.5069$ ($K_c \simeq 0.28515$), which differs from K_c^{-1} estimated using the confluent singularity ratio analysis by 0.007%. However, there is very little to choose between the above value and the confluent singularity ratio value ($y \simeq 3.50718$, $K_c \simeq 0.28513$) as is seen by comparing estimates for γ in Table IX. In contrast, the estimates for γ obtained using $y = 3.50814$ ($K_c = 0.28505$), so as to agree with "bare" end shifts, are much more scattered than those obtained from either of the above values of y.

Thus, our smoothness criterion yields $\gamma_1 \simeq 1.250 \pm 0.003$ and $\gamma_2 \simeq 0.68 \pm 0.07$ ($\delta \simeq 0.57 \pm 0.08$). This result is the best evidence to date for universality in the spin-s Ising model. In obtaining it we assumed only that there exists a sequency of confluent singularities. The location of the singularities was chosen by a smoothness criterion which we consider relatively unbiased. The amplitudes A_1 and A_2 are relatively well converged and given respectively by $A_1 = 0.257 \pm 0.005$ and $A_2 = 0.080 \pm 0.004$. Note that all cases considered yield $A_1\gamma_2 \sim 2A_2\gamma_1$ or $A_2 \sim 0.3A_1$. The series is not long enough to estimate γ_3 accurately, although its presence is probably real. In this regard, note that for *all* values of y listed in Table VIII, the [1/2] approximant without the possibility of a third pole yields values for γ_1 completely inconsistent with the remaining estimates. The value $\gamma_3 \simeq -0.35$ is consistent, to the accuracy we would claim, with $\gamma_3 = -\frac{1}{4}$. As we have noted above,[15] higher-order corrections are expected in χ which means weaker poles are expected in $\mathcal{F}(\xi)$. The third pole detected ($\gamma_3 \simeq -0.35$) evidently corresponds to the first correction to $A_2/(1-yK)^{\gamma_2}$. In fact, we find no evidence for a $\gamma \simeq +0.25$ which would correspond to the first (analytic) correction to $A_1(1-yK)^{\gamma_1}$. (We shall find that for $s = \frac{1}{2}$ this term will be detectable.)

As we have already seen in the endshift analysis (Table II), large spin behavior, i.e., behavior similar to $s = \infty$, prevails roughly for $s > 3$. The $s = \infty$ results just discussed are much cleaner than the analysis for $s < 3$. For small s the effects of a singularity or singularities not described by Eq. (3.9) are strongly apparent.

The results of the Baker-Hunter analysis for various spin values are summarized in Table X. (For $s = \infty$ we have chosen $y = 3.50718$ to be consistent with confluent singularity ratio analysis.) The $s = \infty$ analysis is typical of that for all $s > 3$ and is in good agreement with spin infinity ratio results namely, $\gamma_1 \simeq 1.25$ and $\gamma_2 \simeq 0.7$ ($\delta \simeq 0.55$). The first thing to note about smaller spin results is that internal consistency down through the $[k-1/k]$ list is poorer than for $s = \infty$. Nonetheless, they are consistent with $\gamma_1 \simeq 1.25 \pm 0.01$ for all s, and with $\gamma_2 \sim 0.60 - 0.70$ independent of s (neglecting spin-$\frac{1}{2}$ for the moment). Note that the ratio A_2/A_1 of the secondary to the primary singularity decreases from 0.29 at $s = \infty$ to 0.15 at $s = 1$. We found evidence from "confluent" ratio analysis that A_2 vanishes as s tends to $\frac{1}{2}$. Below we shall examine how this is manifested in the Baker-Hunter analysis.

TABLE X. Baker–Hunter analysis of the fcc series for several values of s.

s	$\frac{1}{2}$	1	$\frac{3}{2}$	$\frac{5}{2}$	8	∞
y	9.7912	6.8190	5.7570	4.8770	3.9425	3.50718
			γ_1			
[1/2]	1.253	1.509	1.633	1.709	1.745	1.751
[2/3]	1.220	1.230	1.234	1.246	1.249	1.251
[3/4]	1.251	1.249	1.245	1.253	1.250	1.251
[4/5]	1.251	1.249	1.243	1.251	1.247	1.249
			γ_2			
[1/2]	0.284	1.080	1.093	1.093	1.091	1.091
[2/3]	···	···	···	0.011	0.686	0.702
[3/4]	0.213	0.615	0.598	0.689	0.695	0.708
[4/5]	0.212	0.606	0.566	0.675	0.664	0.681
			A_1			
[1/2]	0.971	0.144	0.067	0.042	0.029	0.025
[2/3]	1.140	0.627	0.494	0.383	0.295	0.260
[3/4]	0.975	0.582	0.471	0.312	0.294	0.259
[4/5]	0.975	0.583	0.476	0.374	0.298	0.262
			A_2			
[1/2]	0.029	0.522	0.489	0.425	0.346	0.031
[2/3]	···	···	···	0.092	0.084	0.077
[3/4]	0.025	0.087	0.090	0.099	0.085	0.078
[4/5]	0.025	0.085	0.088	0.097	0.083	0.076

For spin-$\frac{1}{2}$ the leading singularity is found to have exponent $\gamma_1 \simeq 1.251$. There is clear evidence for a secondary pole with $\gamma_2 \simeq 0.21$. The value of γ_2 is extremely sensitive to y although the value of γ_1 is not, within the range of y considered. If instead of $y = 9.7912$ we choose 9.7895 (a 0.02% shift), the [3/4] and [4/5] approximants yield $\gamma_s = 0.250 \pm 0.001$. If one looks for maximum agreement between [3/4] and [4/5] estimates, $y = 9.7914$ is perhaps favored, with γ_1 and γ_2 then estimated as 1.251 ± 0.001 and 0.206 ± 0.002, respectively. It is tempting to speculate that $\gamma_2(\frac{1}{2}) = \frac{1}{4}$, which is exactly right to be the leading correction term in the (analytic) amplitude function multiplying $(1-yK)^{-5/4}$. However, analysis of the V (=tanh$K$) series for $\chi(\frac{1}{2})$ leads, in agreement with Baker and Hunter,[9] to no evidence for *any* confluent singularity. This would then mean that the amplitude of any such singularity estimated from the K series should be very small ($|V_c - K_c|/V_c$ is of order 10^{-3}, which would imply an A_2 for the K series an order of magnitude less than in Table IX). In addition, we discuss below the presence of complex poles in $\mathfrak{F}(\xi)$, perhaps indicative of nonconfluent singularities.

At this point it is worth emphasizing that the results of Tables IX and X constitute *very* strong evidence for a universal value of $\gamma(s)$, namely $\gamma(s) = 1.250 \pm 0.005$, for all s. Since the value of γ estimated from the Baker-Hunter transformation depends somewhat on the value y for K_c^{-1} employed in the transformation, one could argue that we have biased the results by choosing a value of y which confluent-singularity ratio analysis yielded when γ and δ were respectively set to 1.250 and 0.50. However this is not at all the case. In fact comparison of the values listed for $K_c(s)^{-1}$ in Tables IX and X with those listed in Table VI show small but significant differences between the estimates for given s. What we have actually done is perform the Baker-Hunter transformation *as a function* of y for a rather large range of y centered on the estimate for K_c^{-1} obtained from Table VI. As noted above the result $\gamma = 1.25$ is rather insensitive to small changes in y. On the other hand, changes in y large enough to significantly alter the estimate for γ introduce a great deal of scatter in the estimates from various Padé approximants. (See, for example, columns 4 and, especially, 5 of Table IX.) Thus, in reality, using a smoothness-of-convergence criterion we obtain unbiased estimates for $\gamma(s)$ which overwhelmingly favor $\gamma = 1.25$ for all s.

To follow the behavior of the assumed confluent singularities near spin-$\frac{1}{2}$, we have analytically continued the series to continuous spin values in $0.50 \leq s \leq 1.0$, as we did with ratio analysis above. The results are summarized in Table XI. In this table we see the crossover to $s = \frac{1}{2}$ behavior in the behavior of γ_2 in particular. That is, for s as small as 0.75 the results look very much like the small s results in Table X. For $s = 0.60$, and particular for $s = 0.55$, one sees the $s = \frac{1}{2}$ behavior manifesting itself. The change is not found principally in A_2, but rather in γ_2 which is decreasing rapidly from its large-s value ($\simeq 0.7$) to its spin-$\frac{1}{2}$ value ($\lesssim 0.3$). In fact, $\gamma_2(0.60) \lesssim 0.5$ and $\gamma_2(0.55) \lesssim 0.33$. Clearly, this apparent smooth variation of γ_2 near $s = \frac{1}{2}$ is not real. Actually, from studies of crossover behavior,[16] we know that near a point ($s = \frac{1}{2}$) where the character of a divergence changes discontinuously (i.e., γ_2 changes discontinuously from $\gamma_2 \simeq 0.75$ for $s > \frac{1}{2}$ to $\gamma_2 \simeq 0.25$ for $s = \frac{1}{2}$), analysis of the critical exponent (γ_2) based on finite series produces estimates for the exponent which (i) are poorly converged near the crossover point ($s = \frac{1}{2}$), and (ii) change rapidly (from the large-s to the $s = \frac{1}{2}$ value) near the crossover point. On this point there is a discrepancy between Baker-Hunter analysis and confluent singularity ratio analysis. Using ratios we found evidence that the amplitudes of *both* $(1 - yK)^{-3/4}$ and $(1 - yK)^{-1/4}$ become very small (and perhaps zero) at $s = \frac{1}{2}$. Here we interpret the Baker-Hunter results as implying that the amplitude of $(1 - yK)^{-0.7}$ tends to zero at $s = \frac{1}{2}$. However, that of $(1 - yK)^{-0.2}$ is found to be ~ 0.02 at $s = \frac{1}{2}$, an order of magnitude too large to agree with ratios. As noted above, in our analysis of the V series we found, in agreement with Baker and Hunter,[9] *no* evidence for *any* confluent singularity at $s = \frac{1}{2}$. We pointed out above that this result implies that the amplitude of $(1 - yK)^{-\gamma_2}$ is zero and that of $(1 - yK)^{-\gamma_3}$ is $\sim 10^{-3}$ ($\gamma_2 \simeq 0.75 \pm 0.08$, $\gamma_3 = 0.25$) in agreement with confluent singularity ratio analysis.

We cannot, without much longer series, reconcile this discrepancy. Basically, there are other nonconfluent singularities which interfere with both kinds of analysis. However, this discrepancy not withstanding, we find from every kind of analysis performed strong evidence for a confluent singularity $A_2(s)(1 - yK)^{-\gamma_2}$ (with $\gamma_2 \simeq 0.75 \pm 0.08$) whose amplitude $A_2(s)$ vanishes at $s = \frac{1}{2}$.

Caveats—Nonconfluent singularities. The Padé-approximant analysis of the Baker-Hunter series $\mathfrak{F}(\xi)$ finds pairs of complex poles entering for all s. Typically, these poles are found in [3/4] and [4/5] approximants to be closer to the origin than is γ_1^{-1}. Now Baker and Hunter[9] point out that should y be greater than K_c^{-1}, instead of having a single pole at γ^{-1}, $\mathfrak{F}(\xi)$ will have a pair of complex poles. One might thus surmise that these complex poles found in our analysis are due to an error in our location in y. However, this cannot be the case. The decrease in y required to make the complex poles go away is simply inconsistent with how well we know y. For spin-$\frac{1}{2}$ the change $\delta y/y_0$ (where y_0 is our best estimate for K_c^{-1}) is about 10%, whereas we know K_c^{-1} to within about 0.01%. The locations of the complex poles are distinctly a function of y for variations of 10% in y. When the complex poles

TABLE XI. Baker-Hunter analysis of the fcc series for $\chi(s)$ analytically continued to $0.5 \leq s \leq 1$. ($y = K_c^{-1}$)

s	0.50	0.55	0.60	0.75	1.0
y	9.7912	9.2721	8.8300	7.8444	6.8190
			γ_1		
[3/4]	1.251	1.252	1.259	1.250	1.249
[4/5]	1.251	1.252	1.259	1.250	1.249
			γ_2		
[3/4]	0.213	0.447	0.623	0.588	0.615
[4/5]	0.212	0.446	0.622	0.577	0.606
			A_1		
[3/4]	0.975	0.897	0.818	0.705	0.582
[4/5]	0.975	0.897	0.818	0.707	0.583
			A_2		
[3/4]	0.025	0.043	0.071	0.087	0.087
[4/5]	0.025	0.042	0.071	0.088	0.085

TABLE XII. Series for $\mathcal{F}(\xi)$ [see Eq. (3.11)] for the spin-$\frac{1}{2}$ Ising model with $y = 9.7912$. The thirteen-term series of Ref. 18 has been used to obtain the last two terms.

N	Coeff. of ξ^N
0	1.00000
1	1.22559
2	1.52821
3	1.91315
4	2.38932
5	2.97003
6	3.81252
7	4.86377
8	4.46108
9	15.72241
10	−30.35825
11	−52.81200
12	−495.00264

finally disappear (at systematically decreasing y for increasing order k of the $[k-1/k]$ approximant), the resulting simple pole is at $\gamma^{-1} \approx 1.4$ ($\gamma \approx 0.7$). Although we were initially inclined to ignore these as "defects" in the approximants, at this point it seems likely that they are a manifestation of nonconfluent singular behavior in χ which cannot be described by Eq. (3.8). It is tempting to speculate that the extra singularity has precisely the form $\dot{A}(1-K/\dot{K})^{-0.75}$. In Table XII we give the coefficients of $\mathcal{F}(\xi)$ (using $y = 9.7912$) for the $s = \frac{1}{2}$ model. Note, in particular, that the last three terms listed are negative. Since the leading behavior is supposed to be

$$A_i/(1-\gamma_i\xi) \approx A_i[1+\gamma_i\xi+(\gamma_i\xi)^2+\cdots],$$

where γ_i^{-1} is the closest pole to the origin; the dominant singularity in $\mathcal{F}(\xi)$ cannot be located at $\gamma_1^{-1} = \frac{4}{5}$. Now $\gamma_1^{-1} = \frac{4}{5}$ is the closest pole to the origin on the positive real ξ axis, and poles with $|\gamma_i^{-1}| < \gamma_1^{-1}$ on the negative real axis would make an oscillatory contribution to the series coefficients. Thus, the cause of the sign change would seem to be associated with the complex poles appearing in $\mathcal{F}(\xi)$. For larger values of s (≥ 1) no sign change is found in the first 11 coefficients of $\mathcal{F}(\xi)$. However, the two fewer terms for $s \geq 1$ could be significant in this regard. In Sec. III D we construct model series which lend further support to the idea that these complex poles could be due to nonconfluent singularities. We can summarize the results of our analysis using the Baker-Hunter transformation as follows. We find strong evidence for universality of the critical exponents. The analysis produces a rather unbiased estimate that $\gamma_1 = 1.250 \pm 0.005$ for *all* s. Furthermore, we would estimate $\gamma_2 = 0.67 \pm 0.09$ independent of s although evidence of crossover to $\gamma_2 = 0.25$ at $s = \frac{1}{2}$ is evident. This latter result is consistent with our ratio analysis in which we found no evidence of nonanalytic corrections to the dominant singularity.

D. Model series

We have constructed two model series χ_1 and χ_2 whose parameters were obtained by directly fitting the series coefficients h_n for the spin-infinity model. The parameters of these model series were obtained by least-square fitting the coefficients h_n to the coefficients of the assumed model series. The coefficients were not equally weighted in the fitting procedure. Rather we proceeded as follows. Estimates of the parameters of the model series were obtained using the first N coefficients h_0, $h_1 \cdots h_{N-1}$, with h_k being weighted in direct proportion to its magnitude. Since the magnitude of h_k increases exponentially with k this procedure very heavily weights the higher-order coefficients.

In the first case we assumed a very general confluent singularity form

$$\chi_1 = A_1(1-yK)^{-5/4} + A_2(1-yK)^{-1/4} + B_1(1-yK)^{-3/4} + B_2(1-yK)^{1/4}, \quad (3.12)$$

and in the second case we explicitly assumed an additive nonconfluent singularity. That is, we fit the series to the form

$$\chi_2 = A_1(1-yK)^{-5/4} + A_2(1-yK)^{-1/4} + B_1(1-yK)^{-3/4} + \tilde{B}_2(1-\tilde{y}K)^{-3/4}. \quad (3.13)$$

Consider first the model series χ_1 with only confluent singularities. In Table XIII, Part a we list the parameters y, A_1, A_2, B_1, and B_2 obtained using the first $N+1$ coefficients as a function of N. (We have used h_{11} and h_{12} provided to us by Saul, Wortis and Jasnow.[10]) The apparent convergence of these results is spectacular. On the basis of these results alone, we would quote $K_c^{-1} = 3.50733 \pm 0.00001$, $A_1 = 0.25894 \pm 0.00001$, and $B_1 = 0.06748$

TABLE XIII. Parameters of the least-squares fit to Eq. (3.12) obtained using $h_0, h_1, \ldots, h_{N-1}$.

a. Spin-∞ Ising model on the fcc net.

N	$y=K_c^{-1}$	A_1	B_1	A_2	B_2
5	3.50748	0.25852	0.06914	0.01514	−0.006
6	3.50748	0.25876	0.06746	0.02232	−0.003
7	3.50734	0.25894	0.06747	0.01753	−0.009
8	3.50734	0.25894	0.06747	0.01742	−0.009
9	3.50733	0.25894	0.06748	0.01744	−0.009
10	3.50733	0.25894	0.06748	0.01747	−0.009
11	3.50733	0.25894	0.06748	0.01748	−0.008
12	3.50733	0.25894	0.06748	0.01744	−0.009

b. B_2 set identically to zero.

N	$y=K_c^{-1}$	A_1	B_1	A_2
8	3.50741	0.25883	0.06731	0.02328
9	3.50738	0.25886	0.06735	0.02265
10	3.50737	0.25888	0.06739	0.02211
11	3.50736	0.25889	0.06740	0.02180
12	3.50736	0.25889	0.06740	0.02173

TABLE XIV. Model series results.

a. Parameters of the fit to Eq. (3.13) using h_0 through h_N for N=8, 9, and 10.

N	$y=K_c^{-1}$	A_1	B_1	A_2	$\tilde{B}_2$	$\tilde{y}$
8	3.5070	0.25899	0.06821	0.0290	−0.003	3.0975
9	3.5072	0.25896	0.06855	0.0288	−0.003	3.1682
10	3.5072	0.25895	0.06796	0.0275	−0.003	3.1693

b. End-shifted ratio analysis of model series χ_1 and χ_2, and of the $s=\infty$ susceptibility $\chi(s=\infty)$. χ_1 was determined from the tenth-order fit to Eq. (3.12) with $B_2 \neq 0$, and χ_2 from the tenth-order fit to Eq. (3.13).

	χ_1			χ_2			$\chi(s=\infty)$		
N	$y(1)$	$\gamma(1)$	$\Delta(1)$	$y(2)$	$\gamma(2)$	$\Delta(2)$	$y(\infty)$	$\gamma(\infty)$	$\Delta(\infty)$
4	3.4550	1.391	1.99	3.4942	1.268	1.01	3.4831	1.300	1.30
5	3.5017	1.253	0.96	3.5060	1.237	0.75	3.5053	1.241	0.82
6	3.5066	1.238	0.80	3.5080	1.232	0.69	3.5082	1.232	0.72
7	3.5078	1.233	0.74	3.5084	1.230	0.67	3.5079	1.233	0.74
8	3.5081	1.232	0.72	3.5084	1.230	0.67	3.5081	1.232	0.72
9	3.5082	1.232	0.71	3.5083	1.230	0.67	3.5081	1.232	0.72
10	3.5082	1.232	0.71	3.5083	1.231	0.68	3.5081	1.232	0.72

± 0.00002. However, even though they appear well converged the parameters A_2 and B_2 are quite "soft." To show this we have set B_2 equal to zero and fit the series to Eq. (3.12) just as above. In Table XIII, Part b we showed the results obtained by fitting the first $N+1$ coefficients with $N=8$, 9, 10, 11, and 12. The apparent convergence is still quite good, but the estimates for K_c^{-1}, A_1, and B_1 are slightly changed, while that for A_2 is changed by 25%. With $B_2=0$, $K_c^{-1}=3.50736$, $A_1=0.25889$, and $B_1=0.6740$ are the best estimates.

One might feel that we are bound to get good apparent convergence because in each order we change the number of parameters fit by only 1. Thus for $N-1=11$ this is only one new parameter out of 12. However, the Nth coefficient is assigned approximately 3.5 times the weight of the $(N-1)$th coefficient, 12.3 times the weight of the $(N-2)$th coefficient, etc., in the fitting procedure. Nevertheless to check our procedure, we have also fit Eq. (3.12) with B_2 set equal to zero by exactly solving for the estimates of K_c^{-1}, A_1, B_1, and A_2 obtained using h_N, h_{N-1}, h_{N-2}, and h_{N-3} for $N=7$ through 12. The value of K_c^{-1} estimated using this procedure is 3.5073 ± 0.0001 independent of N for $N=7$ through 12. The estimates for A_1 and B_1 are somewhat more scattered, but yield $A_1 \simeq 0.2589 \pm 0.0003$ and $B_1=0.067\pm 0.001$ in excellent agreement with the above fitting procedure. A_2 is found to be 0.023 ± 0.005 which is consistent with the results of Table XIII. In summary, the inclusion of A_2 and/or B_2 seems to be important, although their values are not well determined by fitting to order-twelve series.

The results of fitting the nonconfluent form, Eq. (3.13) are presented in Table XIV, Part a. The results for K_c^{-1} are apparently well converged to $K_c^{-1} \simeq 3.5072\pm 0.0001$; and A_1 is estimated to be $A_1 \simeq 0.2590\pm 0.0001$, while B_1 is found to be 0.068 ± 0.001. Although these results are not so well converged, they are fully consistent with the confluent-singularity results above. The nonconfluent singularity is located about 10% further from the origin than the dominant singularity. Both its position and amplitude are quite "soft." Different "starting" values for the parameters to be fit lead to different results for $\tilde{y}$ and $\tilde{B}_2$, the range being about 10% in both. However the parameters y, A_1, and B_1 are "hard," i.e., insensitive to change of starting values.

To compare the two types of model functions we first compare how well they fit the series coefficients h_n. Both χ_1 and χ_2 fit the higher-order terms in $\chi(s=\infty)$ with negligible relative error, but χ_1 produces a significantly better fit to the lower-order series coefficients. In Table XIV, Part b we compare the endshifted ratio analysis of $\chi(s=\infty)$ with that of the model functions χ_1 and χ_2 obtained from the tenth-order parameters of Tables XIII, Part a and XIV, Part a, respectively. The extreme similarities among the three sets of results is striking. The fact that χ_1 is in somewhat closer agreement with $\chi(s=\infty)$ than is χ_2 should probably not be taken seriously. These fits were obtained by fitting the series coefficients directly. Thus the presence of such innocuous terms as De^{-K/K_c} (which would have little effect on higher order ratios) could easily change the relative quality of the fits of the two model series. Baker-Hunter analysis of the two model series also produces results very similar to the analysis of $\chi(s=\infty)$. However, there are no complex poles in $\mathfrak{F}[\chi_1]$, whereas the complex poles first appear in the [4/5] approximant to $\mathfrak{F}[\chi_2]$ {recall that $\mathfrak{F}[\chi(\infty)]$ had complex poles in

TABLE XV. Padé analysis for $D\ln\chi(W)$ for various spin values on the sc lattice $[W=2KK_c/(K+K_c)]$. The results in row N reflect all $[L/M]$ approximants with $L+M-1=N$. Within an N class the results are quite independent of L (or M).

	$s=\infty$		$s=3$		$s=2$		$s=1.5$		$s=1$		$s=\frac{1}{2}$	
N	K_c^{-1}	γ	K_c^{-1}	γ	K_c^{-1}	γ	K_c^{-1}	γ	K_c^{-1}	γ	K_c^{-1}	γ
7	1.6644	1.224	2.2035	1.224	2.4615	1.234	2.7129	1.239	3.1951	1.248	...	...
8	1.6636	$1.23\frac{1}{2}$	2.2027	1.235	2.4609	1.239	2.7123	1.242	3.1949	1.249	4.4992	1.273
9	1.6636	1.231	2.2029	1.234	2.4610	1.237	2.7126	1.249	3.1956	1.247	4.5055	1.255
10	1.6639	1.230	2.2030	1.234	2.4612	1.236	2.7127	1.239	3.1957	1.244	4.5060	$1.25\frac{5}{8}$

both the [3/4] and [4/5] approximants}. We feel that the lack of complex poles in the analysis of χ_1 and their presence in χ_2 lends strength to the hypothesis that they arise from nonconfluent singularities.

This completes our discussion of the fcc series. Using both ratio and Baker-Hunter analysis we have found conclusive evidence that $\gamma=1.25$ independent of s, the apparent spin effect being due to a weaker confluent singularity with exponent $\gamma_2\simeq 0.75\pm 0.08$. There remain subtle problems with the analysis as discussed above. However, we feel that resolution of these relatively minor difficulties would require significantly longer series and/or new methods of analysis. Based on all methods of analysis we would claim that the leading singularities in $\chi(s)$ are of the form given in Eq. (3.13). For $s=\infty$, we estimate conservatively that $K_c^{-1}=3.5073\pm 0.0001$, $A_1=0.2589\pm 0.0002$, and $B_2=0.067\pm 0.001$.

IV. ANALYSIS OF SERIES ON THE SIMPLE CUBIC NET

The simple cubic lattice is typical of loose-packed lattices in that it may be divided into two identical interpenetrating sublattices. For the Ising magnet this has the effect that the antiferromagnetic singularity is on the radius of convergence at $K=-K_c$, where K_c is the ferromagnetic critical point. The location of a competing singularity at $-K_c$ leads to characteristic oscillatory behavior in the ratios. This oscillation considerably degrades the apparent convergence of the various ratio analyses. It may be taken into account and partially removed by extrapolating sequences of alternate ratios R_n, R_{n-2}, ..., R_{n-4}, or of square-root alternate ratios.[9] Nevertheless, apparent convergence is never as good as on close packed lattices, such as the fcc and TRI nets.

Convergence of estimates based on Padé approximants to $d\ln\chi(K)/dK$ are likewise more poorly converged on the sc net than on the fcc net. However, the scatter and number of defects[9] can be significantly reduced by an Euler transformation to the variable $W\equiv 2K_cK/(K+K_c)$ which leaves K_c unchanged ($W_c=K_c$) but shifts $-K_c$ to $W=-\infty$. An error δK_c in the estimate for K_c used in the Euler transformation leads to a critical point $W_c\approx K_c-\frac{1}{2}\delta K_c+O(\delta K_c^2)$, i.e., with the same magnitude shift from K_c as in the estimate used. More important, the antiferromagnetic singularity is shifted to $W^*\approx K_c^2/\delta K_c$, which is far from the radius of convergence. Thus, our method consisted of estimating K_c by analysis of $d\ln\chi(K)/dK$ and using this estimate in an Euler transformation to a series in W. We then used Padé analysis of $d\ln\chi(W)/dK$ to estimate W_c and γ. The transformation was iterated until W_c obtained from Padé analysis agreed with K_c. The final estimates are listed in Table XV for a number of spin values. These estimates are the "best" estimates obtained from $[l/m]$ approximants with $N=l+m-1$ for $N=7$, 8, 9, and 10.

The general trend of the estimates for $\gamma(s)$ is the same as that described above on the basis of end-shifted ratio analysis of the fcc series. Spin infinity corresponds to the best apparent convergence, as above to $\gamma(\infty)\simeq 1.23$. Spin-$\frac{1}{2}$ is not well behaved: on the basis of the last two orders (we did not use the Sykes *et al.* eighteen-term series[5] but rather to maintain consistency used order-ten series) one would conclude $\gamma(\frac{1}{2})\simeq 1.25-1.26$. This also follows the trend found in the fcc series. The [5/3] approximant is defective or nearly defective[17] in all cases studied. The reason for this is not understood. In all cases except spin-$\frac{1}{2}$, the estimates $K_c[l/m]$ obtained for the class $\{l+m-1=10, l, m\geq 3\}$ agree to the accuracy shown.

We attempted to use the Baker-Hunter confluent singularity analysis on the sc series. In contrast to our experience with the fcc analysis, the sc analysis failed completely. The estimate for γ_1 is about 1.37 which is clearly unreasonable. The obvious point is that the second most significant singularity on a loose packed lattice is not confluent with the dominant singularity, but is rather the antiferromagnetic singularity[12] [which behaves as $\tilde{A}\cdot(K+K_c)^{1-\alpha}$ with $\alpha\simeq\frac{1}{8}$[11]]. This of course means that a sophisticated form of analysis suitable to a particular class of *confluent* singularities will not do very well for this case. So, in retrospect, the failure of Baker-Hunter analysis is unsurprising.

Although we could not analyze the confluent correction terms, we have no doubt of their existence; nor do we doubt their essential similarity to those found in the fcc case. Indeed, we can infer their existence from the marked similarity of the apparent spin dependence of $\gamma(s)$ on the two lattices.

V. ANALYSIS OF TRIANGULAR-NET SUSCEPTIBILITY

The spin-$\frac{1}{2}$ Ising model has been quite thoroughly studied in two dimensions using exact analytical techniques.[18] In particular γ is known exactly to be $\gamma=\frac{7}{4}$,[18] and $V_c[=\tanh(K_c)]$ on the TRI lattice is known to be $V_c=2-\sqrt{3}$.[18] Further Barouch *et al.*[19] have recently obtained the leading terms in the exact expansion of $\chi(V)$ *about the critical point* on the SQ net. They find that

$$\chi(V)\approx\chi_a\epsilon^{-7/4}+\chi_b\epsilon^{-3/4}+\chi_c\epsilon^{1/4}+\cdots\,, \qquad (5.1)$$

where $\epsilon=(V_c-V)/V_c$, and the coefficients χ_a, χ_b, and χ_c confirm closely the previous numerical results of Sykes *et al.*[18] Of course, since Eq. (5.1) is an expansion about V_c, it says nothing about nonconfluent singularities. However, it does say that for $s=\frac{1}{2}$ there are no nonanalytic confluent singularities appearing, at least through order $\epsilon^{1/4}$. This result is in agreement with our conclusions about the fcc susceptibility, for which we decided that the term in $\epsilon^{-\gamma+1/2}$ was absent for $s=\frac{1}{2}$. Although Eq. (5.1) was derived for the SQ lattice, previous experience with two-dimensional $s=\frac{1}{2}$ Ising models indicates that it will also hold on other two-dimensional lattices. Indeed, Sykes *et al.*[18] using seventeen-term series for $\chi(V)$ on the TRI net found

$$\chi(V)\approx A\epsilon^{-7/4}+B\epsilon^{-3/4}+C\epsilon^{1/4}+\cdots\,, \qquad (5.2)$$

with $A=0.847086\pm0.00001$, $B=0.1756\pm0.001$, and $C=0.0287\pm0.01$. We therefore expect to find no evidence for non-analytic confluent correction terms for $s=\frac{1}{2}$.

We have studied the spin-s susceptibility on the triangular net. The series does not exhibit good apparent convergence for any value of s. Sykes *et al.*[18] note that on the TRI net $s=\frac{1}{2}$ extrapolation becomes constant only beyond eleventh order. As with the three-dimensional results, there is a marked spin dependence in the apparent critical exponent γ obtained from endshifted ratio analysis. In contrast with the three-dimensional spin effect, the apparent value of γ *increases* with increasing s. That is, using tenth order estimates γ varies from $\gamma\approx1.75$ at $s=\frac{1}{2}$ to $\gamma\simeq1.89$ at $s=\infty$, the latter value prevailing for $s\gtrsim3$.

By studying the Baker-Hunter series $\mathfrak{F}(\xi)$ we see that there are strong *nonconfluent* singularities masking the dominant behavior in χ. In fact, these nonconfluent singularities are significant enough that they strongly affect, through the appearance of complex poles, all but the dominant ($\gamma\simeq1.75$) singularity in $\mathfrak{F}(\xi)$. We have made no effort to identify and remove the nonconfluent terms, but only note that we must keep them in mind when interpreting confluent singularity analyses which ignore them.

TABLE XVI. Parameters of model series, $\chi_1=\mathring{A}_1/(1-yK)^{7/4}+B/(1-yK)^{5/4}+A_2/(1-yK)^{3/4}$ and $\chi_2=\mathring{A}_1/(1-yK)^{7/4}+\mathring{B}\ln(1-yK)/(1-yK)^{5/4}+\mathring{A}_2/(1-yK)^{3/4}$, obtained by fitting the first N coefficients of the K-series for the $s=\infty$ susceptibility on the triangular lattice.

		χ_1		
N	$y=K_c^{-1}$	A_1	B	A_2
5	1.4363	0.221	−0.063	0.210
6	>1.437	...	...	...
7	1.4319	0.244	−0.136	0.292
8	1.4327	0.240	−0.122	0.275
9	1.4311	0.251	−0.161	0.326
10	1.4311	0.251	−0.161	0.326
		χ_2		
N	$y=K_c^{-1}$	$\mathring{A}_1$	$\mathring{B}$	$\mathring{A}_2$
5	1.4279	0.308	0.102	0.042
6	1.4354	0.248	0.046	0.101
7	1.4274	0.321	0.117	0.035
8	1.4298	0.296	0.092	0.055
9	1.4286	0.310	0.107	0.045
10	1.4292	0.302	0.098	0.050

A. Simulation by model series

We have attempted to fit the TRI susceptibility series to two general kinds of confluent singularities. To motivate the choices we first recall the apparent spin dependence of $\gamma(s)$. Namely, $\gamma(s)$ increases with increasing s. Now we assume that the amplitude of any *nonanalytic* confluent correction term is zero for $s=\frac{1}{2}$ [in agreement with Eqs. (5.1) and (5.2)]. We have seen above that a confluent correction $B(1-K/K_c)^{-\gamma+1/2}$, with a positive amplitude B, leads to an apparent exponent γ_{eff} less than γ. Thus, we expect that for an apparent exponent greater than γ, B will have to be negative. We do not present the argument here, but rather only note that $\gamma_{\text{eff}}>\gamma$ could also be due to a confluent correction of the form $\mathring{B}\ln(1-K/K_c)(1-K/K_c)^{-\gamma+1/2}$, with $\mathring{B}$ *positive.*

With these points in mind we have chosen to fit the coefficients $h_n(s)$ of the spin-s series to two model forms[20]

$$\chi_1=\frac{A_1}{(1-yK)^{7/4}}+\frac{B}{(1-yK)^{5/4}}+\frac{A_2}{(1-yK)^{3/4}} \qquad (5.3)$$

and

$$\chi_2=\frac{\mathring{A}_1}{(1-yK)^{7/4}}+\frac{\mathring{B}\ln(1-yK)}{(1-yK)^{5/4}}+\frac{\mathring{A}_2}{(1-yK)^{3/4}}\,. \qquad (5.4)$$

The parameters (A_1, A_2, B, and y for χ_1; $\mathring{A}_1$, $\mathring{A}_2$, $\mathring{B}$, and y for χ_2) of the test functions are then chosen, as above, to give the best fit to the first N terms of the series for $\chi(s)$. In Table XVI we list the fitting parameters obtained using orders $N=5$ through 10 for the $s=\infty$ series. There is little to choose between χ_1 and χ_2, as far as smoothness of the se-

TABLE XVII. End-shifted ratio analysis of the $s=\infty$, triangular-net susceptibility series, together with end-shift analysis of the model series χ_1 and χ_2 obtained from the $N=10$ results of Table XIII.

		χ_1			$\chi(s=\infty)$			χ_2	
N	K_c^{-1}	γ	Δ	K_c^{-1}	γ	Δ	K_c^{-1}	γ	Δ
4	⋯	⋯	⋯	1.411	2.04	1.78	1.390	2.21	2.17
5	1.350	2.78	3.80	1.412	2.03	1.74	1.413	2.03	1.78
6	1.402	2.19	2.41	1.430	1.87	1.28	1.420	1.97	1.60
7	1.416	2.03	1.90	1.420	1.98	1.65	1.423	1.94	1.49
8	1.421	1.96	1.62	1.426	1.91	1.37	1.425	1.92	1.42
9	1.424	1.92	1.44	1.424	1.92	1.44	1.426	1.90	1.36
10	1.426	1.89	1.31	1.426	1.89	1.31	1.426	1.89	1.31

quence of fitting parameters is concerned. (Although, it is somewhat surprising to find $A_2 > A_1$ in the case of χ_1.) Note that as predicted in the preceding discussion the estimates for B are all negative while those for $\mathring{B}$ are positive. To see how well χ_1 and χ_2 simulate $\chi(s=\infty)$ we list in Table 15 the endshifted ratio analysis for the three series. Again, there is little to choose between the model functions. Both simulate the $s=\infty$ results very well. However, based on over-all fit (including lower-order behavior) we would choose the second form χ_2. On the other hand, there is validity in the argument that one should not introduce more complicated functional forms than needed to fit the series unless there is outside supporting evidence for the more complicated function. In this case we have no *a priori* arguments that either function, χ_1 or χ_2, is more appropriate than the other.

We have also used χ_1 and χ_2 to simulate the susceptibility for $s=1$, 2, and 3. In these cases the sequence of fits to the parameters of χ_1 is much smoother than that to the parameters of χ_2. Whereas for $s=\infty$ the last two ($N=9, 10$) parameter sets were slightly closer to one another for χ_1 than for χ_2, here the differences is considerable. For example, with $s=3$, the estimates for A_1 obtained using ninth- and tenth-order series are within 0.3% of one another while the ninth- and tenth-order estimates for $\mathring{A}_1$ differ by more than 3% from one another. The ninth- and tenth-order parameters are summarized in Table XVIII for $s=1$, 2, and 3. On the basis of apparent convergence for these values of s we definitely would choose χ_1 as providing a better simulation of $\chi(s)$ than χ_2, although the caveats concerning omission of corrections discussed in Sec. III D would apply here also. As with $s=\infty$, the endshifted ratio analyses of χ_1 and χ_2 are very similar to that of $\chi(s)$.

In the case of $s=\frac{1}{2}$, no successful fit could be obtained using either χ_1 or χ_2. This was already expected in the light of the exact results on the SQ net[19] and of the Sykes *et al.* numerical results[18] on the TRI net. We saw no reason to redo the excellent $s=\frac{1}{2}$ analysis of Sykes *et al.*[18]

B. Baker-Hunter analysis

For $s>1$ the Baker-Hunter series $\mathfrak{F}(\xi)$ is replete with complex poles and does not really do an adequate job of estimating the exponent γ of the dominant singularity. There are two factors involved here: (i) we do not know K_c at all well for $s>\frac{1}{2}$, and (ii) there are evidently very strong nonconfluent correction terms (which have much less effect on the ratio and test series analysis than on Baker-Hunter analysis). For $s=\frac{1}{2}$ we know $V_c[=\tanh(K_c)]$ exactly and also have six further terms in the V

TABLE XVIII. Parameters of the model series χ_1 and χ_2 defined in Table XIII, for various values of spin.

		χ_1					χ_2		
N	$y=K_c^{-1}$	A_1	B	A_2	N	$y=K_c^{-1}$	$\mathring{A}_1$	$\mathring{B}$	$\mathring{A}_2$
$s=1$					$s=1$				
9	2.675	0.580	−0.246	0.387	9	2.672	0.672	0.165	−0.042
10	2.676	0.569	−0.204	0.329	10	2.674	0.635	0.126	−0.022
$s=2$					$s=2$				
9	2.096	0.405	−0.247	0.445	9	2.093	0.496	0.164	0.015
10	2.097	0.401	−0.233	0.424	10	2.094	0.475	0.142	0.026
$s=3$					$s=3$				
9	1.885	0.348	−0.221	0.419	9	1.882	0.430	0.147	0.034
10	1.886	0.347	−0.214	0.409	10	1.883	0.415	0.131	0.042

TABLE XIX. Baker-Hunter results for γ_1, γ_2, A_1 and A_2 using the V series for χ ($s=\frac{1}{2}$). The exact critical point $V_c = 2-\sqrt{3}$ has been used in the transformation.

	γ_1	γ_2	A_1	A_2
[1/2]	1.726	0.633	0.892	0.108
[2/3]	1.726	0.631	0.892	0.108
[3/4]	1.740	0.749	0.866	0.134
[4/5]	1.756	0.904	0.829	0.170
[5/6]	1.752	0.846	0.841	0.160
[6/7]	1.752	0.850	0.840	0.161
[7/8]	1.750	0.789	0.846	0.162

series. Thus we would expect Baker-Hunter analysis to provide more information for $s=\frac{1}{2}$ than for $s>\frac{1}{2}$. Table XIX lists the Baker-Hunter estimates γ_1, γ_2, A_1, and A_2 appropriate to an assumed form

$$\chi = A_1/(1-V/V_c)^{\gamma_1} + A_2/(1-V/V_c)^{\gamma_2} + \cdots \ .$$

These results were obtained using the exact value $2-\sqrt{3}$ for V_c.[18] The convergence of γ_1 to 1.750 is rather good, and the estimate $A_1 \simeq 0.846 \pm 0.005$ is in reasonable agreement with the accurate value $A_1 \simeq 0.84701$ given by Sykes *et al.*[18] On the other hand, the rather poor convergence of γ_2 to 0.75 is almost certainly due to the presence of nonconfluent singularities [the approximants to $\mathcal{F}(\xi)$ are replete with complex poles]. Nevertheless, $A_2 \simeq 0.16$ is within 10% of the accurate result $A_2 \simeq 0.176$ given by Sykes *et al.*[18] These results are interesting not so much because they lead to the estimate $\gamma = 1.75$ in agreement with exact results,[19] but rather because they show how Baker-Hunter analysis may break down (presumably because of the presence of nonconfluent correction terms). They perhaps serve to put an upper band on our confidence in Baker-Hunter analysis, when the analysis is marred by complex poles.

VI. SUMMARY AND CONCLUSIONS

We have presented a rather thorough discussion of the spin-s Ising ferromagnet in two and three dimensions. Our results are consistent with the conclusion that the dominant behavior of the susceptibility is given by ($\epsilon = 1 - K/K_c$).

$$\chi \approx \chi_a \epsilon^{-\gamma} + \chi_b \epsilon^{-\gamma+1/2} + \chi_c \epsilon^{-\gamma+1} + \cdots \ , \qquad (6.1)$$

with $\gamma = \frac{7}{4}$ and $\frac{5}{4}$ in two and three dimensions, respectively. In all cases χ_b is found to be zero, within the accuracy of our analyses, for $s=\frac{1}{2}$. In the case of the fcc lattice especially we found strong unbiased evidence for $\gamma = 1.25$ independent of s using the Baker-Hunter confluent singularity analysis. We found good evidence both from ratio and Baker-Hunter analysis of the fcc series that the leading correction term has exponent $\gamma_2 = \gamma - \delta$ with $\delta \simeq 0.50 \pm 0.08$, independent of s. We also conclude that in *all* cases studied there were nonconfluent singularities marring the convergence of the confluent singularity analysis. These nonconfluent terms are evidently specially important in two dimensions.

The conclusions drawn for the TRI lattice ($d=2$) are much less certain than for the fcc case. Here we had to resort almost exclusively to fitting model series. That is, we assume a form like Eq. (6.1) and fitted the first N terms of the series to obtain a model series. The parameters of the test series were found to exhibit rather good convergence with increasing N, and they behaved smoothly as a function of s. From such analysis we can draw no stronger conclusion than that the TRI series is consistent with Eq. (6.1). To do better would require longer series (say 13 or more terms), and we have no plans to generate such series.

For the loose packed lattices the series convergence is degraded by characteristic oscillatory behavior due to the existence of an antiferromagnetic singularity at $-K_c$ on the radius of convergence. We were unable to carry out detailed confluent singularity analyses on loose-packed lattices, although we found indirect evidence for the importance of confluent corrections in the case of the sc lattice.

To extend the verification of the universality hypothesis, we are studying other Ising-like models (scalar order parameter models) such as the continuum models introduced by Wilson and others.[21] Further we shall apply the confluent singularity analysis to the X-Y, planar Heisenberg and isotropic Heisenberg models. This work will be reported on elsewhere.

ACKNOWLEDGMENTS

We would like to thank Dr. D. Jasnow for a discussion concerning the apparent spin dependence of the Ising model, and Dr. M. Wortis for several discussions of this work. The two extra terms provided by Dr. D. M. Saul, Dr. M. Wortis, and Dr. Jasnow for $s=\infty$ were very valuable in confirming the convergence of our analysis.

APPENDIX A

The bare vertex weights entering the high-temperature expansion for the spin-s Ising susceptibility have the form

$$I_{2l} = (2s+1)^{-1} \sum_{m=-s}^{s} m^{2l} \ . \qquad (A1)$$

This form can be expressed as a polynomial of degree $2l$ in s. However, it has an even simpler expression in terms of $X = s(s+1)$

$$I_{2l} = \sum_{n=1}^{l} a_n X^n \ . \qquad (A2)$$

The coefficients a_n are easily obtained by comparison of the equations for s and $s-1$. It is first noted that $a_l = 1/(2l+1)$. Then a_k is found recursively for $k = l-1,\ l-2, \ldots$ as

$$a_k = \sum_{j=k+1}^{l'} \binom{2j-2k}{j} \left(\frac{a_j}{2j-2k+1}\right), \tag{A3}$$

where l' is the smaller of l or $2k$. The results needed to find the series coefficients through order 10 are listed here:

$$I_2 = \tfrac{1}{3} X, \tag{A4a}$$

$$I_4 = \tfrac{1}{15}(3X^2 - X), \tag{A4b}$$

$$I_6 = \tfrac{1}{21}(3X^3 - 3X^2 + X), \tag{A4c}$$

$$I_8 = \tfrac{1}{45}(5X^4 - 10X^3 + 9X^2 - 3X), \tag{A4d}$$

$$I_{10} = \tfrac{1}{33}(3X^5 - 10X^4 + 17X^3 - 15X^2 + 5X), \tag{A4e}$$

$$I_{12} = \tfrac{1}{1365}(105X^6 - 525X^5 + 1435X^4 - 2360X^3 + 2073X^2 - 641X). \tag{A4f}$$

APPENDIX B

Consider the function $f(K)$ given by [with $\gamma > \max(1, \delta)$]

$$f(K) = \frac{A_1}{(1-K/K_c)^{\gamma}} + \frac{A_2}{(1-K/K_c)^{\gamma-1}} + \frac{B}{(1-K/K_c)^{\gamma-\delta}}. \tag{B1}$$

Here A_1 and A_2 are the leading terms in the Taylor series expansion of the amplitude of the dominant singularity about K_c, and B is the amplitude of the weaker confluent singularity. Let $f(K)$ have a Taylor series

$$f(K) = \sum_{n=0}^{\infty} f_n K^n. \tag{B2}$$

Then the coefficients of the series are given by

$$\Gamma(n+1) f_n = K_c^{-n} A_1 \left(\frac{\Gamma(n+\gamma)}{\Gamma(\gamma)}\right)\left(1 + \frac{A_2 \Gamma(n+\gamma-1)\Gamma(\gamma)}{A_1 \Gamma(n+\gamma)\Gamma(\gamma-1)} + \frac{B\Gamma(n+\gamma-\delta)\Gamma(\gamma)}{A_1\Gamma(n+\gamma)\Gamma(\gamma-\delta)}\right), \tag{B3}$$

where $\Gamma(x)$ is the standard Γ function $(x-1)!$. The large-n asymptotic expression[22]

$$\frac{\Gamma(n+\gamma-\delta)}{\Gamma(n+\gamma)} \approx n^{-\delta}\left(1 - \frac{\delta(2\gamma-\delta-1)}{2n} + \frac{\delta(\delta+1)}{24n^2}[3(2\gamma-\delta-1)^2 + (\delta-1)] + O(n^{-3})\right) \tag{B4}$$

will form the basis of our analysis of the ratios $R_n \equiv f_n/f_{n-1}$. Using (B4) we write f_n for large n as

$$K_c^n \Gamma(n+1) f_n \approx A_1\left[1 + \frac{A_2\Gamma(\gamma)}{A_1\Gamma(\gamma-1)}[n^{-1} - (\gamma-1)n^{-2} + O(n^{-3})] + \frac{B\Gamma(\gamma)}{A_1\Gamma(\gamma-\delta)} n^{-\delta} \times \left(1 - \frac{\delta(2\gamma-\delta-1)}{2n} + \frac{\delta(\delta+1)}{24n^2}[3(2\gamma-\delta-1)^2 + (\delta-1)] + O(n^{-3})\right)\right]. \tag{B5}$$

To facilitate the calculation of R_n we rewrite f_n and f_{n-1} in a reduced form, namely,

$$f_n = [K_c^{-n} A_1 \Gamma(n+\gamma)/\Gamma(\gamma)\Gamma(n+1)](1+y) \quad \text{and} \quad f_{n-1} = [K_c^{-n+1} A_1 \Gamma(n+\gamma-1)/\Gamma(\gamma)\Gamma(n)](1+y+\epsilon). \tag{B6}$$

Here y and ϵ are expressed as

$$y = \mathring{A}_2 n^{-1} + \mathring{B}[n^{-\delta} - \tfrac{1}{2}\delta(2\gamma-\delta-1)n^{-1-\delta}] + O(n^{-2}) \tag{B7}$$

and

$$\epsilon = \mathring{A}_2 n^{-2} + \mathring{B}\delta n^{-1-\delta} - \tfrac{1}{2}\mathring{B}\delta(\delta+1)(2\gamma-\delta-2)n^{-2-\delta} + O(n^{-3}). \tag{B8}$$

We have introduced reduced amplitudes $\mathring{A}_2$ and $\mathring{B}$ given by $\mathring{A}_2 = A_2\Gamma(\gamma)/A_1\Gamma(\gamma-1)$; $\mathring{B} = B\Gamma(\gamma)/A_1\Gamma(\gamma-\delta)$. At this point one may worry about our neglect of terms of order n^{-2} in y, since we certainly will want to keep all terms of this order (and smaller) in R_n. However, this is no worry because of the manner of which y enters R_n. That is, $(1+y)/(1+y+\epsilon)$ may be written as

$$\sum_{n=0}^{\infty} \frac{(-1)^n \epsilon^n}{(1+y)^n},$$

which means that the lowest-order contribution involving y^m enters as ϵy^m. The first neglected term in this product is of order $n^{-(2m+1+\delta)}$, which means that we may neglect terms of order n^{-2} in y since this corresponds to a neglect of terms of order $n^{-3-\delta}$ in R_n. We thus write R_n as

$$R_n \approx K_c^{-1}[1 + (\gamma-1)/n][1 - \epsilon + y\epsilon - y^2\epsilon + O(\epsilon^2, y^3\epsilon)],$$

which, using (B7) and (B8), becomes

$$R_n \approx K_c^{-1}\,[1+(\gamma-1)/n]\{1-\mathring{A}_2 n^{-2}-\mathring{B}\delta n^{-(1+\delta)}+\mathring{B}^2\delta n^{-(1+2\delta)}\,[\mathring{B}\delta(\delta+1)\,(2\gamma-\delta-2)+2(1+\delta)\mathring{A}_2\mathring{B}]n^{-(2+\delta)} - \mathring{B}^3\delta n^{-(1+3\delta)}+\cdots\}\,, \tag{B9}$$

in which we have neglected terms of order $n^{-(2+2\delta)}$, $n^{-(1+4\delta)}$, n^{-3}, or smaller.

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*Work supported by the U. S. Atomic Energy Commission.

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[11] On this point consult the general review of theoretical studies of critical phenomena prior to 1967; M. E. Fisher, Rept. Prog. Phys. 30, 615 (1967). In the case of the two-dimensional spin-$\frac{1}{2}$ model the antiferromagnetic singularity is known to behave similar to the internal energy, i.e., as $(K+K_c)\ln(K+K_c)$.

[12] This is not strictly true since the confluent singularity is probably more singular at $|K|=K_c$. However, it is most likely that the antiferromagnetic amplitude is much greater than the confluent singularity amplitude, so that the antiferromagnetic singularity has a greater effect on finite series extrapolations.

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[15] In order to weight higher order terms more heavily in selecting y, one might think of using ratios on $\mathfrak{F}(\xi)$ to evaluate the choice of y, for example, one might vary y so as to force the ratios to approach γ_l with zero slope (assuming γ_l^{-1} to have the smallest absolute value among the set $\gamma_1,\ldots,\gamma_N$). If one chooses a simple test series, this method and Padé analysis of the logarithmic derivative of $\mathfrak{F}(\xi)$ produce well behaved measures of the deviation of y from K_c^{-1}. However, if the amplitudes of the singularities are made even weakly K dependent, these criteria fail. An analytic amplitude $A_l(K)=A_l^{(0)}+A_l^{(1)}(K_c-K)+\cdots+A_l^{(n)}(K_c-K)^n+\cdots$ leads to an infinite sequence of progressively weaker confluent singularities. For n such that $|n-\gamma|>\gamma$ there will be weak poles in $\mathfrak{F}(\xi)$ which must dominate convergence of $\mathfrak{F}(\xi)$. Indeed there will be an infinite sequence of weak poles with accumulation point at $\xi=0$ [the $n\to\infty$ limit of $1/(n-\gamma)$ is zero].

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[17] By "nearly defective" we mean that there is a pole zero pair just beyond the pole of interest on the positive real K axis. By "defective" we mean that the pair lies between the origin and the pole of interest.

[18] Consult M. E. Fisher, Ref. 11, for a review of results known exactly prior to 1967. The spin-$\frac{1}{2}$ susceptibility on two-dimensional lattices has been very thoroughly studied using very long series by M. F. Sykes, D. S. Gaunt, P. D. Roberts, and J. A. Wyles, J. Phys. A 5, 624 (1972). These authors present an extensive bibliography of previous analytical and numerical results.

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