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Extended universality of the Ising model*

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An extension of the critical-phenomena universality hypothesis assumed previously to be valid for less singular terms is shown to be invalid. A reformulated extension of the universality hypothesis is found to be valid exactly for the spin-1/2 Ising model on the triangular, square, and honeycomb lattices, and, further support is supplied by high- and low-temperature expansions for the susceptibility. The extension fails however for the Ising model on the kagomé lattice and for the spherical model. At present there are insufficient data to test the extended principle for other models or for experimental systems.

I. INTRODUCTION

During the development of the theory of critical phenomena[1] evidence accumulated that within a few broad classes, different systems behave in a very similar manner in the critical region.[2] This evidence has led to Kadanoff's principle of universality[3] according to which all phase-transition problems can be divided into a small number of classes, depending only on the dimensionality of the system and the symmetries of the order parameter(s), within which the systems have identical behavior in the critical region. The universality hypothesis for thermodynamic properties has been stated independently with varying degrees of generality and precision by a number of authors.[4–6] It has also been generalized to correlations by Stauffer, Ferer and Wortis.[7] Tested predictions of universality include the invariance within a universality class of critical exponents[2,8] and certain products of powers of critical amplitudes.[5–7,9]

In previous statements of the universality hypothesis it has usually been assumed that scaling[10] is also valid and that only the *most singular part* of the free energy and its derivatives are universal, less singular parts, being corrections to scaling presumably also being corrections to universality. Guttmann[11] has recently assumed the principle of universality to be valid for the *second most singular part* of the Ising-model susceptibility in order to "predict" the corresponding amplitudes on the triangular, honeycomb and kagomé lattices from the exact results[12] for the square lattice. However, universality in the form assumed by Guttmann is not valid for the two-dimensional Ising model.

In Sec. II we reformulate the principle of universality so that it has extended validity at least for the Ising model on the triangular, square, and honeycomb lattices. Section III contains tests of the extended hypothesis using exact solutions for the two-dimensional Ising model. Section IV contains further tests using series expansions of the susceptibility of the two-dimensional Ising model. Section V contains discussion of validity of extended universality for the spherical model and other models. Summary and discussion are in Sec. VI.

II. EXTENDED UNIVERSALITY HYPOTHESIS

In magnetic language let us employ the reduced field variable $h = mH/kT$ and the reduced energy variable $j = 1 - K/K_c = 1 - T_c/T$. As the above variables differ from the more usual variables $h' = mH/kT_c$ and $t = 1 - T/T_c$ only in second order it is immaterial which set are used in discussing the most singular part of the free energy. However, the difference is crucial for the less singular parts of the free energy of interest in the extension of the universality hypothesis, which we now state.

The singular parts of the dimensionless free energy per spin $f(j, h)$ of two systems X and Y of the same universality class satisfy the relation

$$n_X f_X(j_X, h_X) = n_Y f_Y(j_Y, h_Y) , \qquad (1)$$

where the reduced energy and field variables are related by

$$g_X j_X = g_Y j_Y , \qquad (2a)$$

$$n_X h_X = n_Y h_Y . \qquad (2b)$$

The critical parameters g and n of (2) are those previously introduced[6] for the more restricted universality. For the two-spin correlation function the extended universality relation is

$$\Gamma_X(\vec{r}_X, j_X, h_X) = \Gamma_Y(\vec{r}_Y, j_Y, h_Y) , \qquad (3)$$

where

$$r_X/l_X = r_Y/l_Y , \qquad (4)$$

as before[7]

The extended universality hypothesis can most readily be tested by comparing the behavior of the second most singular term of different models of the same universality class. The critical exponent of the second most singular term then should be the same for all members of the same universality class. This aspect of extended universality has recently been tested by Camp and Van Dyke[13] for the three-dimensional Ising model of general spin.

TABLE I. Values of the critical inverse temperature, critical amplitudes of the free energy and spontaneous magnetization and universality scale factors for the two-dimensional Ising model.

Property	Triangular (T)	Square (Q)	Lattice (X) Honeycomb (H)	Kagomê (K)
K_c^X	$\frac{1}{4}\ln 3$	$-\frac{1}{2}\ln(\sqrt{2}-1)$	$-(\frac{1}{2})\ln(2-\sqrt{3})$	$-(\frac{1}{4})\ln[(2/\sqrt{3})-1]$
$F_{1+}^X = F_{1-}^X$	$-6\sqrt{3}\,(K_c^T)^2/\pi$	$-4\,(K_c^Q)^2/\pi$	$-\sqrt{3}\,(K_C^H)^2/\pi$	$-2\sqrt{3}(K_C^K)^2/\pi$
B_1^X	$(16K_c^T)^{1/8}$	$(2^{7/2}K_c^Q)^{1/8}$	$(16K_C^H/\sqrt{3})^{1/8}$	$[144(\sqrt{3}-1)^8 K_C^K]^{1/8}$
$g_X K_c^T/K_c^X$	1	$1/\sqrt{2}$	$1/\sqrt{3}$	$9(\sqrt{3}-1)^8$
n_X	1	$3\sqrt{3}/4$	2	$3^5(\sqrt{3}-1)^{16}$
$F_{2+}^X = -F_{2-}^X$	$-12\sqrt{3}\,(K_c^T)^3/\pi$	$-4\sqrt{2}\,(K_c^Q)^3/\pi$	$-2\,(K_C^H)^3/\pi$	$-4\sqrt{3}\,(\sqrt{3}-1)\,(K_C^K)^3/\pi = -0.163\,965\,041$
Universal expressions	...	$-4\sqrt{2}\,(K_c^Q)^3/\pi$	$-2\,(K_C^H)^3/\pi$	$-36\sqrt{3}(\sqrt{3}-1)^8(K_C^K)^3/\pi = -0.166\,257\,920$
B_2^X	$-9\sqrt{2}\,(K_c^T)^{9/8}/8$	$-9\,(K_c^Q)^{9/8}/8.\,2^{1/16}$	$-3^{23/16}\sqrt{2}\,(K_C^H)^{9/8}/8$	$-9.\,12^{1/4}(\sqrt{3}-1)^2(K_C^K)^{9/8}/8 = -0.475\,945\,821$
Universal expressions	...	$-9\,(K_c^Q)^{9/8}/8.\,2^{1/16}$	$-3^{23/16}\sqrt{2}\,(K_C^H)^{9/8}/8$	$-9^{17/8}\sqrt{2}\,(\sqrt{3}-1)^9(K_C^K)^{9/8}/8 = -0.482\,601\,422$

They find the critical exponent of the second most singular part of the susceptibility for $H=0$, $T>T_c$ to be $\gamma_2 \approx 0.75$, independent of spin in agreement with extended universality.

III. TESTS OF THE EXTENDED UNIVERSALITY HYPOTHESIS USING EXACT RESULTS FOR THE TWO-DIMENSIONAL ISING MODEL

From Onsager's celebrated exact solution[14] and subsequent developments[15] we now have a number of exact results for the two-dimensional Ising model, which have already proved of great value in testing various theoretical developments. In particular for the triangular, square, honeycomb and kagomê lattices, we know exactly the dimensionless inverse critical temperature K_c^X, the amplitudes of the dimensionless free energy per spin in

$$f_X(j_X, h_X) = F_{1\pm}^X j_X^2 \ln|j_X| + F_{2\pm}^X j_X^3 \ln|j_X| + \cdots \quad T \gtrless T_c\,, \tag{5}$$

and the amplitudes in the dimensionless spontaneous magnetization,

$$m_X \equiv \frac{\partial f_X}{\partial h_X}\bigg|_{h_X \to 0}\,,$$

$$= B_1^X(-j_X)^{1/8} + B_2^X(-j_X)^{9/8} + \cdots, \quad T < T_c\,. \tag{6}$$

The exact values of these critical parameters are displayed in Table I.

From the amplitudes, $F_{1\pm}^X$ and B_1^X, of the most singular terms we form the critical scale factors n_X and g_X as also listed in Table I.

According to extended universality the amplitudes of the second most singular terms for the spin-$\frac{1}{2}$ Ising model are related by

$$F_{2+}^X/F_{2+}^T = F_{2-}^X/F_{2-}^T = g_X^3/n_X\,, \tag{7}$$

and

$$B_2^X/B_2^T = g_X^{9/8}\,. \tag{8}$$

From Table I we see that (7) and (8) are satisfied exactly for the square and honeycomb lattices with respect to the triangular lattice as standard. However both relations fail by about 1% for the kagomê lattice.

A further exact confirmation of extended universality is provided for the honeycomb-triangular lattice pair by star-triangle relation for their susceptibilities.[16] If the dimensionless initial susceptibility per site,

$$\chi_X \equiv \frac{\partial m_X}{\partial h_X}\bigg|_{h_X \to 0}\,,$$

$$= C_{1\pm}^X |j_X|^{-\gamma} + C_{2\pm}^X |j_X|^{-\gamma+1} + \cdots\,, \quad T \gtrless T_c\,, \tag{9}$$

then the star-triangle relation

$$2\chi_T(v_T) = \chi_H(v_H) + \chi_H(-v_H)\,, \tag{10}$$

where

$$v_H^2 = v_T/(1 - v_T + v_T^2) \tag{11}$$

yields not only the original universality relation

$$C_{1\pm}^H = (n_H/g_H^\gamma)C_{1\pm}^T\,, \quad \gamma = \tfrac{7}{4}\,, \tag{12}$$

but also the extended universality relation

$$C_{2\pm}^H = (n_H/g_H^{\gamma-1})C_{2\pm}^T\,. \tag{13}$$

IV. COMPARISON OF SERIES ESTIMATES AND UNIVERSALITY PREDICTIONS FOR SUSCEPTIBILITY AMPLITUDES

The four susceptibility amplitudes in (9) for the spin-$\frac{1}{2}$ Ising model have recently been evaluated exactly by Barouch *et al.*[12] for the square lattice. Universality then permits the prediction of the amplitudes $C_{1\pm}^X$ for the triangular, honeycomb, and kagomê lattices as already noted by Guttmann.[11]

TABLE II. Critical amplitudes of the second most singular part of the susceptibility of the two-dimensional $S=\frac{1}{2}$ Ising model as determined by exact solution, extended universality, direct series analysis, and analysis of difference series.

Amplitude	Method	Lattice (X) Triangular (T)	Square (Q)	Honeycomb (H)
C^X_{2+}	Exact solution[a]	⋯	0.074 988 154	⋯
	Extended universality	0.063 459 070	⋯	0.099 454 879
	Original series[b]	0.0633	0.0742	⋯
	Difference series	0.0635 ± 0.0001	0.0750 ± 0.0003	0.0972 ± 0.0006
C^X_{2-}	Exact solution[a]	⋯	−0.001 989 411	⋯
	Extended universality	−0.001 683 548	⋯	−0.002 638 505
	Original series	⋯	⋯	⋯
	Difference series	−0.00175 ± 0.000 30	−0.0021 ± 0.0002	−0.0026 ± 0.0006

[a]See Ref. 12. [b]See Refs. 18 and 19.

Extended universality, in the form postulated above rather than the form assumed by Guttmann, may be valid for the further prediction of the amplitudes $C^X_{2\pm}$ for the triangular and honeycomb lattices only. The latter amplitudes are listed in Table II.

Independent estimates of C^X_{2+} for the square and triangular lattices have been made by Sykes *et al.*[18] by analysis of the high-temperature series expansions of the susceptibility on the respective lattices. These estimates are also given in Table II. It does not seem possible to estimate C^X_{2-} directly from existing low-temperature series using presently available methods.

However, accepting the universality of the most singular part of the susceptibility, we can obtain estimates of C^X_{2-} and more precise estimates of C^X_{2+}. As not only K^X_c and γ but also $C^X_{1\pm}$ are then known, the series expansion of the leading singularity of the susceptibility can be subtracted from the original susceptibility series to give a difference series. The resulting difference series can then be analysed by standard ratio and Padé approximant techniques to yield estimates of exponents $\gamma_2 = \gamma_2'$ (known exactly) and amplitudes $C^X_{2\pm}$ of the second most singular part of the susceptibility.

From the low-temperature difference series,

$$\chi(z) - C^z_{1-}(1 - z/z_C)^{-7/4} = \sum_n a_n z^n \,, \tag{14}$$

where $z = e^{-2K}$ and C^z_{1-} is the appropriate critical amplitude, ratio estimates, $\gamma_{2,n}$ of the exponent γ_2 can be formed using the relation

$$\gamma'_{2,n} = n(a_n/a_{n-1}K_c - 1) + 1 \,. \tag{15}$$

A plot of these estimates versus $1/n$ is illustrated in Fig. 1. For the square lattice the independent variable $u = z^2$ has been used as usual while for the triangular lattice the variable $x = u/(1+2u)$ has been used to decrease the effect of a pole on the negative u axis.

Good convergence to the known value, $\gamma' = \frac{3}{4}$, is observed for the low-temperature ratio plots of Fig. 1. The high-temperature difference series show similarly good behavior. The difference series are also quite amenable to Padé approximant analysis. The overall best estimates of $C^X_{2\pm}$ as determined from our analysis of the difference series are also listed in Table II.

Examination of Table II shows that for the square lattice the difference series, requiring an assumption of only the original universality hypothesis, yields an estimate of C^Q_{2+} in more precise agreement with the exact result than the estimate obtained by direct analysis of the original series. The difference series estimate of C^Q_{2-} is less precise than that for C^Q_{2+} but is still in agreement with the exact result to within 5%. Thus we can have some confidence in the procedure of analysing the difference series. This confidence is increased by the agreement between the estimates of C^T_{2+} from the

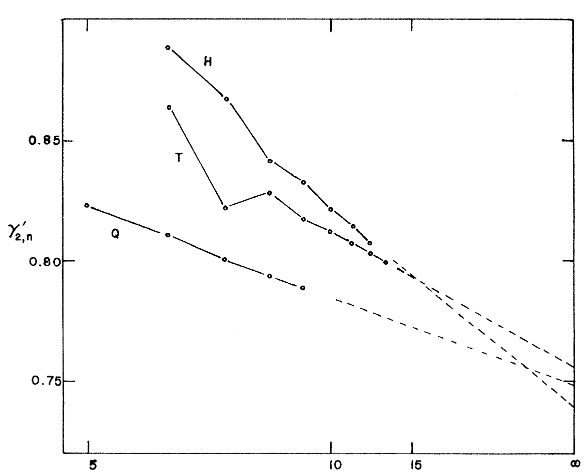

FIG. 1. Plot on a $1/n$ scale of the successive ratio estimates $\gamma'_{2,n}$ of the exponent $\gamma'_2 = \frac{3}{4}$ for the spin-$\frac{1}{2}$ Ising model on the honeycomb (*H*), triangular (*T*) and square (*Q*) lattices.

original and the difference series.

Accepting the validity of the estimates from the difference series we see satisfactory support of extended universality from the estimates of $C^T_{2\pm}$ and $C^H_{2\pm}$. As noted in Sec. III extended universality fails for the kagomé lattice.

V. VALIDITY OF EXTENDED UNIVERSALITY FOR OTHER MODELS

Mean-field theory and the renormalization-group ϵ-expansion expressions are trivially universal in the sense that there is only one system of each universality class, that is , $n_X = g_X = 1$. A test of extended universality can however be made for the spherical model for which $g_X = 1$ for all lattices but $n_X \neq 1$ in general. Joyce[17] has given an extensive treatment of the spherical model. Extended universality applied to the spherical model predicts

$$C^X_{2+}/C^Y_{2+} = n_X/n_Y \; . \tag{16}$$

For the simple-cubic lattice (S) we have[6] $n_B = 1.065\,566\,95$ and for the bcc lattice we have $n_S = 1.394\,566\,57$. On the other hand the amplitudes[17] $C^S_{2+} = 0.773\,173\,68$ and $C^B_{2+} = 0.867\,640\,51$. So we see that extended universality fails for the spherical model for the only pair of lattices for which the data is available.

To test the extended universality for other classes of models we have to rely on estimations of amplitudes of second most singular terms of thermodynamic functions based on series expansions. This requires, for at least two members of the universality class, sufficiently long series to determine simultaneously with precision five critical properties—the critical temperature, two critical exponents, and two critical amplitudes. We have analysed existing series expansions using ratio and Padé approximant techniques for a variety of models such as the $d = 3$, $S = \frac{1}{2}$ Ising model, the $d = 3$, $S = \infty$ Ising model, etc. to obtain estimates of the amplitude of the second most singular term. In no case have we found existing series to be of great enough length to determine the second amplitude with adequate precision. With the completion of work presently in progress on series expansions for $S = \infty$ models,[20] it may soon be possible to test further the extended universality hypothesis.

VI. DISCUSSION AND SUMMARY

The universality hypothesis for the free energy has been expressed in terms of the independent variables $j = 1 - K/K_c$ and $h = mH/kT$. For the spin-$\frac{1}{2}$ Ising model for the set consisting of the triangular, square and honeycomb lattices the universality hypothesis has been confirmed exactly for the second most singular terms of the zero-field free energy both above and below T_c and of the magnetization. The star-triangle transformation for the susceptibility also is in agreement with the extended universality hypothesis. Somewhat surprisingly, the extended hypothesis fails in second order by a numerically small amount for the kagomé lattice.

The extended universality hypothesis definitely fails for the spherical model. It is still an open question whether it may hold for any other nontrivial examples. As mentioned above, Camp and Van Dyke[13] have found some evidence for $\gamma_2(S) = \gamma_1 - \frac{1}{2} \approx 0.75$ for the three-dimensional Ising model. Saul *et al.*[21] also find $\gamma_2 \approx 0.75$ for the spin ∞ Ising model on the fcc lattice, but neither group has been able to obtain precise amplitude estimates. For the two-dimensional Ising model $\gamma_2 = \gamma_1 - 1$ exactly,[12] which may indicate that this model is rather special with regard to extended universality.

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