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Kapitza conductance, temperature gradients, and solutions to the Boltzmann equation

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In the belief that the study of heat transport requires the study of the transport equation, we present an approach to the problem of the Kapitza conductance h_K between two materials which involves the solutions of the Boltzmann equation. One of our purposes is to investigate the origin of the apparent temperature discontinuity ΔT that is associated with this phenomenon. The hydrodynamic solutions of the Boltzmann equation, which (by definition) are describable in terms of local thermohydrodynamic variables, can transfer heat but are not at all responsible for ΔT; whereas the nonhydrodynamic solutions are completely responsible for ΔT but do not transfer heat. An effective temperature $\tilde{T}$ is defined which approaches the thermodynamic temperature T far from the interface, and which is assumed to be continuous across the interface. With this assumption, formal expressions for ΔT and h_K are derived. In the limit as the properties of the two materials become identical, R_K $(=h_K^{-1})$ approaches zero, as should be the case. Further, this approach has a natural generalization to finite frequencies and includes lifetime effects. It is pointed out that thermometers do not measure $\tilde{T}$ but rather T_R which reflects, in a complicated fashion, the presence of the nonhydrodynamic modes, whose amplitudes fall off exponentially as one moves from the interface. In He II, determination of the exponential damping lengths (as a function of temperature and pressure) would provide information about phonon dispersion and phonon interactions which is at least as detailed as could be obtained by other means.

I. INTRODUCTION

This paper is concerned with two phenomena—the Kapitza conductance h_K between two materials, and the temperature gradients in the vicinity of a heated surface—which are usually treated as distinct and indirectly related. [Other names for the first phenomenon are Kapitza resistance R_K $(=h_K^{-1})$ and thermal boundary resistance.] We have found that there exists a direct and intimate relationship between these phenomena, since they must be described by using the same solutions of the Boltzmann equation. The implication for h_K calculations is that the traditional "independent-phonon" approach to energy transfer (to be discussed in Sec. II) must be abandoned in favor of a more rigorous approach involving the solutions to the Boltzmann equation. Unfortunately, the difficulties with applying this approach may not easily be overcome, so for the time being it will primarily be of conceptual, rather than practical, value.

Our research into these questions began with a study of temperature gradients in He II,[1] using the formulation of linear-response theory. It was found that they fall off exponentially in space, with a characteristic distance

$$\delta=(\kappa\Lambda/\rho^2S^2T)^{1/2}, \qquad (1)$$

where κ is the thermal conduction coefficient, ρ is the mass per unit volume, S is the entropy per unit mass, and T is the temperature. Here

$$\Lambda=\tfrac{4}{3}\mu_n+\zeta_2+\rho^2\zeta_3-2\rho\zeta_1, \qquad (2)$$

where μ_n is the viscosity of the normal fluid, and the ζ's are the coefficients of second viscosity.[2] It should be noted that Kronig and Thellung derived a similar result much earlier.[3]

The difficulty in applying this formula is that the transport coefficients κ and Λ are dependent on the wave vector $\vec{k}$. In fact, they are functions of $|\vec{k}|=\delta^{-1}$, and the problem must be solved self-consistently. It turns out that use of $|\vec{k}|\approx 0$ for κ and Λ (the hydrodynamic approximation) yields values for δ which are smaller than a characteristic mean free path. Therefore, the use of hydrodynamics is not consistent.[4] Further, the self-consistent solution of Eq. (1) does not appear to be feasible. It is difficult enough to solve for hydrodynamic transport coefficients, let alone nonhydrodynamic ones (as a function of wave vector).

For this reason, it became clear that another approach was necessary; specifically a study of the solutions to the Boltzmann (or transport) equation. For most systems this would be as difficult a task as the original one but, fortunately, solutions to the Boltzmann equation for a relevant model recently became available. Ma has obtained a class of eigenvalues (and their eigenfunctions) of the Boltzmann equation for the weakly interacting Bose gas at low temperatures.[5] This class corresponds to the longest-lived temporal transients. Using these solutions as basis functions, we were able to obtain the spatially most extended solutions of the eigenvalue equation for the modes relevant to steady-state heat transport. (The details of this work will be published elsewhere.[6]) With these solutions at hand, it was then possible to consider the problem of heat transport in some detail We concluded that these solutions transport no heat but do simulate a rapid spatial variation of temperature. Those properties are the central themes of this paper.

In Sec. II we present a critique of previous theories of h_K and of temperature gradients in He II.[7] In Sec. III we discuss the relationship between temperature gradients near an interface between two media, and the solutions of the Boltzmann equation. An effective temperature $\tilde{T}$ is defined which we suggest is continuous across the interface. In Sec. IV these ideas are applied to the h_K problem, and the resulting approach is found to eliminate a number of difficulties that exist for the "independent-phonon" approach. Section V contains a summary, some discussion, and our conclusions. The Appendix contains a discussion of the properties of the Boltzmann equation and of its solutions under various circumstances.

II. CRITIQUE OF EARLIER WORK

In order to discuss the difficulties of the independent-phonon approach,[8,9] it is appropriate to define the theoretical problem. We consider steady-state heat flow between two materials (each treated as a continuous medium) separated by a perfect planar surface. There are no imperfections or impurities and there is no roughness: the surfaces are considered to be perfectly smooth. This neglects the atomicity of matter. For long wavelength phonons, it should be reasonable to do this. The independent phonon approach assumes that heat is transferred by phonons of momentum $\vec{p}$ (with infinite lifetime) whose transmission coefficient w_p is obtained by matching the acoustically or microscopically determined stresses and velocities across the interface. By giving the phonon $\vec{p}$ the (Boson) thermal weighting n_p, and accounting for the component of velocity normally incident on the interface, the energy of the phonon $\hbar\omega_p$, and the transmission coefficient w_p, one can determine the heat flux from one material to a second, in thermal equilibrium. By detailed balance, an equal flux should be going from the second to the first. Considering the two materials to have a small temperature difference ΔT, one can then determine the net heat flow from the hotter to the cooler material. Most theoretical work has concerned itself with the determination of values for w_p, assuming the correctness of the approach (just described) which employs w_p.

The first difficulty with this approach was pointed out by Little,[10] who showed that the existence of a finite temperature discontinuity ΔT across the infinitesimal barrier between the two materials implies that the acoustic mismatch value for the Kapitza resistance R_K does not go to zero as the properties of the two materials become identical. Hence, the independent phonon value of R_K does not have the correct behavior in the limit of zero mismatch. In order to resolve this difficulty, Little pointed out that in the limit of identical materials, the two points between which ΔT occurs must be separated by a finite distance on the order of a phonon mean free path. However, in estimating this separation distance he employs the long-wavelength thermal conductivity, which is inappropriate because of the short wavelength involved. As with the work of Ref. 1, which was discussed in the Introduction, the separation distance must be determined self-consistently. Therefore, we consider Ref. 10 to have presented a qualitatively correct resolution of the problem, but we feel there is a need to find a quantitative one as well. Although this paper does not give such a quantitative resolution, it does indicate the direction such work should take.

A second difficulty with the independent phonon approach is that there is no natural way of formulating the h_K problem at finite frequencies. However, experiments have been performed at finite frequencies (i.e., second-sound transmission through metal foils[11]) and the results differ from those obtained by the usual steady-state measurements. (Note that different sample preparation probably explains this discrepancy in measurements.) A complete theory must account for finite-frequency effects.

A third difficulty is that it is unclear how to incorporate phonon lifetime effects: the work of Peterson and Anderson[12] and of Haug and Weiss,[13] attempts to do this in an *ad hoc* fashion. Vuorio has discussed some difficulties with this approach.[14] Our particular concern is with the fact that this approach does not obviously satisfy reversibility. In Sec. III it will be shown that the transport-theory approach is manifestly reversible.

A fourth difficulty is that the role of surface (Rayleigh) modes is unclear. If the modes are short lived (long lived) they are expected to contribute (not contribute) to the heat transfer.[15] It should be noted that this is, in some sense, a lifetime effect.

A fifth difficulty is that the independent-phonon approach does not cope with the question of how much of ΔT is due to a rapid variation in temperature near the interface, and how much is a discontinuity (except when viewed on an atomic scale). In this sense, it does not explain the origin of ΔT.

In short, we have reason to believe that the approach of summing the contributions of independent and infinitely long-lived phonons is fundamentally inadequate. One of the purposes of this paper is to outline, using the solutions to the Boltzmann equation, what we consider to be a correct approach.

We note that our original work[1] on temperature gradients in He II was motivated by an interest in studying a mechanism,[16] within the framework of the independent phonon approach, which would increase the theoretical value of h_K (bringing it more in line with experiment). We concluded that the rapid change in temperature of the He II near the surface of a heater would not require one to make significant corrections in estimating the discontinuity in T. We now believe this conclusion to have been erroneous. Indeed, it will be argued that the rapid change in temperature (in both substances) near the interface between He II and a solid is an inseparable part of the apparent temperature discontinuity ΔT. Further, these rapid changes in temperature occur at the interface when heat flows between any two materials, even for two solids placed in contact. As noted in Ref. 1, such spatially rapid changes in temperature have not been observed in He II (and, *a fortiori*, in other condensed materials) because the scale over which they occur (on the order of an appropriate mean free path) is usually too small to measure with a macroscopic thermometer. However, for He II (at $T \lesssim 0.5$ K) and for rarified gases, the effect should be observable. The primary difference between our earlier paper and this one (in their conclusions about temperature gradients) is that we now believe that temperature falls off, in principle, as the sum of an infinite number of exponentials. This point will be discussed in Sec. III. In terms of the self-consistent solution of Eq. (1), there is no unique solution δ but rather an infinite number of solutions δ.

III. HEAT FLOW AND TEMPERATURE VIA THE BOLTZMANN EQUATION

We restrict our considerations to materials with only one significant type of excitation: phonons. We do not consider metals, although we believe that this general approach will apply to all materials. The Boltzmann equation that we discuss is, therefore, the Boltzmann equation for phonons.

The Boltzmann equation has as many solutions, for a disturbance of given frequency, as there are modes of motion in the absence of collisions.[17] Almost all of these are nonhydrodynamic in nature (i.e., the distribution function cannot be described as arising from deviations in local temperature, pressure, and velocity of the material in question.) Such modes either grow or decay spatially in a distance on the order of a relevant mean free path, with an exponential law for a uniform medium (see Appendix). They can be generated by the flow of heat, so that within a mean free path of a heater it is not correct to speak of a local temperature, etc. Nevertheless, at zero frequency, despite the fact that these nonhydrodynamic modes are present, they do not contribute to heat flow, as will now be shown.

At zero frequency $\partial\epsilon/\partial t = 0$, where ϵ is the energy density. Therefore, the divergence of the energy current density $\vec{q}$ must be zero, by

$$\vec{\nabla} \cdot \vec{q} = -\frac{\partial \epsilon}{\partial t} = 0 \ . \tag{3}$$

Now for heat flow generated by a perfect infinite planar heater, the energy current density must always point normal to the heater surface. Hence

$$\vec{\nabla} \cdot \vec{q} = \frac{\partial q_z}{\partial z} = 0 \ . \tag{4}$$

Since all quantities associated with nonhydrodynamic modes have a finite spatial derivative (see Appendix), q_z must be zero for all nonhydrodynamic modes (otherwise $\partial q_z/\partial z \neq 0$.) By using the appropriate conservation laws, one can similarly show that the nonhydrodynamic modes do not transport either mass or momentum. On the other hand, they do contribute to quantities like ϵ.

Now consider heat to be flowing through an interface with an apparent temperature discontinuity ΔT. Further, note that there are three scales of spatial variation in this problem: (a) macroscopic (i.e., thermohydrodynamic description of the phonon distribution function), (b) collisional (i.e., nonthermohydrodynamic description of the phonon distribution function), and (c) quantum mechanical (i.e., associated with the interface on an atomic scale). Since it is experimentally well known that ΔT does not occur over a macroscopic scale, we eliminate that possibility. Next, we note that there is some overlap of the collisional and the atomic scales, since many of the nonhydrodynamic modes have characteristic damping lengths on the order of or smaller than the atomic scale. A proper description of such modes would require a detailed knowledge of atomic scattering processes at the interface. We cannot prove that ΔT does not occur over this scale, but for the purpose of treating the interface like a mathematical boundary (so that boundary conditions may be employed), we assume that ΔT occurs over a scale on the order of a characteristic damping length which is much larger than an atomic scale. The question is how.

To answer this, we consider the problem of defining an effective temperature which: (a) includes the effect of the nonhydrodynamic modes, (b) approaches the hydrodynamic temperature far from the interface, and (c) is continuous at the interface. First, note that far from the interface, the nonhydrodynamic modes may be neglected, so that changes in the energy per unit mass $E(=\epsilon/\rho)$ are given by

$$dE = T dS + (P/\rho^2)\, d\rho \ , \tag{5}$$

where S is the entropy per unit mass, P is the pressure, and ρ is the mass density. For heat flow without mass flow, P is constant, so

$$dE\Big|_P = \frac{\partial E}{\partial T}\Big|_P dT \equiv \bar{C}_P\, dT \ , \tag{6}$$

thus defining $\bar{C}_P$. When the nonhydrodynamic-mode contribution $\tilde{E}$ to the energy per unit mass is included, one has

$$dE = T\,dS + (P/\rho^2)\,d\rho + d\tilde{E} \ . \tag{7}$$

Now, for heat flow without mass flow, the stress σ_{zz} ($= P$ far from the interface) is constant, so

$$dE\,|\,\sigma_{zz} = [T\,dS + (P/\rho^2)\,d\rho]_{\sigma_{zz}} \ . \tag{8}$$

We therefore believe it natural to define $\tilde{T}$ by

$$dE\,|_{\sigma_{zz}} = \bar{C}_P\, d\tilde{T} \ . \tag{9}$$

We use $\bar{C}_P$ because $\sigma_{zz} = P + O(q_z)$, and the energy flux density q_z is considered to be a small quantity. Comparing with Eq. (6), we find that

$$d\tilde{T} = dT + d\tilde{E}/\bar{C}_P \tag{10}$$

Since $\bar{C}_P$ is nearly constant over the (small) scale for which $d\tilde{E}$ is non-zero,

$$\tilde{T}(z) = T(z) + \tilde{E}(z)/\bar{C}_P \ . \tag{11}$$

Since $\tilde{E}(z) \to 0$ far from the interface, $\tilde{T}(z) \to T(z)$ far from the interface. However, it is not clear that T is continuous at the interface. Nevertheless, because it reflects the presence of the nonhydrodynamic modes (which help smooth the transition at the interface), $\tilde{T}$ should be more continuous than T. We assume that $\tilde{T}$ is continuous at the interface. Note that if $\tilde{T}$ turns out to vary on an atomic scale (due to the presence of nonhydrodynamic modes whose characteristic damping lengths are on such a scale), then it would not be appropriate to employ this boundary-condition approach. Therefore, a more precise approach, such as that outlined in Sec. V, would be necessary. With this proviso, we continue to develop the boundary-condition approach, with $\tilde{T}$ continuous at the interface. Then, if the subscripts 1 and 2 apply to the regions to the left and right of the interface, we have

$$\tilde{T}_1(0) = \tilde{T}_2(0) \ ,$$

or

$$T_1 - T_2 = (\tilde{E}/\bar{C}_P)_2 - (\tilde{E}/\bar{C}_P)_1 \ .$$

Here $T_{1,2}$ and $\tilde{E}_{1,2}$ are evaluated at $z = 0$. For heat flow from left to right we expect that $T_1 > T_2$, $\tilde{E}_1 < 0$, and $\tilde{E}_2 > 0$, so

$$T_1 - T_2 = |(\tilde{E}/\bar{C}_P)_1| + |(\tilde{E}/\bar{C}_P)_2| \ . \tag{12}$$

Certainly, Eq. (11) requires microscopic justification, but is is not clear how one should proceed to obtain this. Note that if the two materials are identical (e.g., two identical pieces of electropolished copper) the discontinuous transition at the interface will cause a thermal boundary resistance. This is because the fit will not be perfect, and nonhydrodynamic modes can be expected to be generated. On the other hand, if the fit is perfect (i.e., the boundary is purely a mathematical device), then no nonhydrodynamic modes are generated, there is no temperature discontinuity, and $R_K = 0$. This is in agreement with the discussion given in Sec. II.

We note that a thermometer is unlikely to read $\tilde{T}$. In the case of a resistance thermometer, a resistance (with which is associated the effective temperature T_R) is measured. Certainly the resistance as a function of the amplitudes of the nonhydrodynamic modes is not proportional to the ΔE coming from the nonhydrodynamic modes. Nevertheless the resistance R will reflect the presence of the nonhydrodynamic modes.[18] By plotting $R(z)$, and trying to fit it to a sum of exponentials, one may find the damping lengths of the most important nonhydrodynamic modes:

$$R(z) = R_0 + \sum_i A_i\, e^{-z/\delta_i} \ . \tag{13}$$

In the above, the A_i's may vary with the measuring device, but the δ_i's are a property of the medium. Figure 1 contains a schematic diagram of T, $\tilde{T}$, and T_R as functions of z.

It is appropriate to make two additional comments. First, note that if there is any scattering at all (which is certainly the case at finite temperatures), then we expect nonhydrodynamic modes with characteristic lengths which are infinitesimally small. This is implicit in the work of Ref. 6, where only a few of the longest characteristic lengths are exhibited. Decreasing the scattering

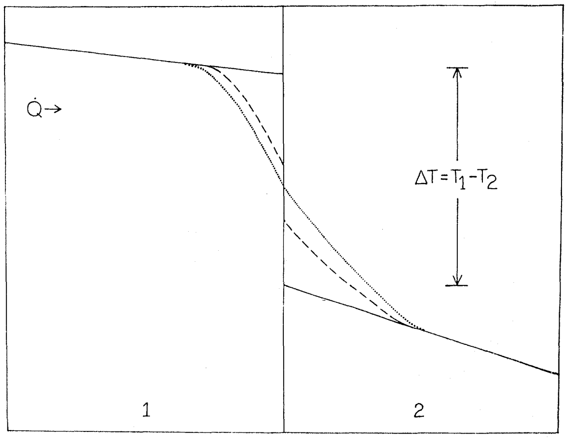

FIG. 1. Solid line: thermodynamic temperature T; dotted line: effective temperature $\tilde{T}$; and dashed line: thermometer temperature T_R, all as functions of z. We have drawn the diagram under the assumption that $T_{R1} > T_{R2}$ at the interface.

cross section changes the scale of the characteristic lengths, but it does not exhaust the infinite number of characteristic lengths which can "peel out" of the zero limit point. Therefore, we believe the transport-theory approach is appropriate even to systems whose phonon mean free paths are limited by surface scattering. Second, note that the transport-theory approach is manifestly reversible: a change in the direction of heat flow simply causes the amplitudes of all modes (hydrodynamic and nonhydrodynamic) to change sign.

IV. APPLICATION OF KAPITZA CONDUCTANCE

Let us now apply these ideas to the problem of the Kapitza conductance h_K. It is usually defined as

$$h_K = \dot{Q}/A\Delta T \,, \tag{14}$$

where $\dot{Q} = (q_z A)$ is the magnitude of total heat flow across the interface, A is the area of the interface, and ΔT is the apparent temperature discontinuity between the two materials. (Note that when calculating h_K by the independent-phonon approach it is most convenient to assume ΔT known, and to compute $\dot{Q}$. By the transport-theory approach it appears most convenient to assume $\dot{Q}$ known, and to compute ΔT.) In light of the previous section an equivalent definition would be

$$h_K = \frac{\dot{Q}}{A[\,|(\Delta E/\overline{C}_P)_1| + |(\Delta E/\overline{C}_P)_2|\,]} \;. \tag{15}$$

This can also be written as

$$h_K = (h_{K1}^{-1} + h_{K2}^{-1})^{-1} \,, \tag{16}$$

where

$$h_{K1,2} = \frac{\dot{Q}}{A(\Delta E/\overline{C}_P)_{1,2}} \;. \tag{17}$$

Since $\overline{C}_P \propto T^3$ for phonons at low temperatures, then if $\dot{Q}/\Delta E$ is only weakly temperature dependent, one can explain the observed approximate temperature dependence of h_K.[7] The key to solving this problem (besides determining the properties of the nonhydrodynamic modes) lies in a determination of the amplitudes of the nonhydrodynamic modes generated by the heat current. This is a difficult boundary-value problem, and we have no particular insights into its solution. Some aspects of the problem will be discussed in Sec. V.

In regard to the second difficulty mentioned earlier, we note that, unlike the independent-phonon approach, the Boltzmann-equation approach has a natural generalization to finite frequencies. It would, in principle, enable one to compute h_K's appropriate to those determined in second-sound measurements. However, for finite frequency the nonhydrodynamic modes contribute to the heat current $\vec{q}$, so the problem becomes more complicated.

V. SUMMARY, DISCUSSION, AND CONCLUSIONS

We have studied the problem of Kapitza conductance h_K between two uniform materials with a perfect planar interface, using the properties of the relevant solutions of the Boltzmann equation. We found it essential to define an effective temperature $\tilde{T}$ which was assumed to be continuous across an interface, despite the fact that the thermodynamic temperature T is not. From that definition, an expression for h_K was derived. This approach has the advantages that: (a) $h_K^{-1} = 0$ if the properties of the two materials become identical; (b) it has a clear generalization to finite frequency; (c) it includes lifetime effects for both bulk and surface modes (the surface modes will be considered shortly); (d) it is manifestly reversible; and (e) it elucidates the origin of the apparent temperature discontinuity.

On the other hand, this approach has the disadvantage that it requires a knowledge of the appropriate solutions of the Boltzmann equation and at the boundary conditions with which to match the solutions in both materials. This latter problem is particularly difficult, for a complete theory would determine the boundary conditions from a knowledge of the microscopic coupling between the two materials. Only recently has progress been made on the microscopic theory of coupling between phonons in liquid He^4 and a solid.[19] This work is still at an early stage of development but in the future it should yield answers to the problem of boundary conditions for the h_K problem. Clearly, it is desirable to discuss what information should be contained in more advanced work of the type initiated by Ref. 19.

Let us therefore consider the coupling between an insulating solid and liquid helium. Near the surface of the solid, the atomic sites will have a spacing which differs slightly from the spacing of the bulk. This effect, known as relaxation, will depend on the interactions of the solid atoms with one another and with the atoms of the liquid. Most likely it will be sufficient to treat the solid as consisting of classical particles. On the other hand, liquid He^4 is distinctly quantum mechanical, and this must be accounted for at the start. The (time-averaged) density profile of the liquid will conform to the shape of the solid at the surface (i.e., it will be periodic parallel to the surface, with the periodicity of the solid substrate), approaching the bulk liquid density a few atomic distances from the interface. The relaxation of the solid and the density profile of the liquid He^4 will have to be determined simultaneously and self-consistently. This point was not treated in Ref. 19. Of course, the effect of temperature complicates this problem considerably, so it is implicit that this work be done at $T = 0$.

The next step is to find the normal modes of the system, in the absence of collisions. Clearly, they will be coupled modes of both the liquid and the solid. Besides bulk modes, there will also be modes localized near the interface (surface modes).

Both of the above steps would be taken by advocates of the independent-phonon approach. It is in what follows that the Boltzmann-equation approach differs from the independent-phonon approach.

The third step is to find the appropriate transport equation for the coupled modes, in order to include the effect of collisions. Separate Boltzmann equations for each system will not do. The inhomogeneous nature of the problem will prevent Fourier analysis perpendicular to the surface. Most likely a transport equation resembling that for the Wigner distribution will be found, complicated by the effects of the inhomogeneity of the system. However, the equation, if soluble, will yield the modes describing transport processes in the system. All the boundary conditions will be built into this equation describing the real system, so only one particular hydrodynamic mode will be generated by heat flow in the absence of matter flow. Clearly, lifetime effects for both bulk and surface modes, and effects associated with the interface on an atomic scale, will be accounted for.

It should be noted that, for the problem with ideal surfaces and nonmicroscopically determined boundary conditions, an enormously large number of combinations of nonhydrodynamic modes could be responsible for a given static value of h_K. If h_K were known as a function of frequency, the number of possibilities would be vastly decreased. As a practical matter, one would hope that only a few nonhydrodynamic modes are generated to any significant extent. This is probably the only practical working hypothesis—otherwise, analysis of any experiment using the Boltzmann-equation approach would be virtually impossible.

We now consider the experimental consequences of the nonhydrodynamic modes. First, they produce a rapidly varying "temperature" in the vicinity of any heat source, as qualitatively indicated in Fig. 1. Such an effect can be seen in the work of Ref. 20. This paper discusses a computer experiment on heat transfer in anharmonic linear chains. At the edges, where the chain interacts with thermal reservoirs, Fig. 3 of that reference shows a rapidly varying "temperature," just as one would expect there were nonhydrodynamic modes being generated. In real systems, a rapidly varying "temperature" can probably be observed in He II for $T \lesssim 0.5$ K and perhaps in rarified gases. (The thermometry problem in this latter case is a formidable obstacle. However, attempts to measure the temperature profile in such systems have been made.[21]) For He II, experimental determination of the damping lengths [via Eq. (13)] as a function of temperature and pressure for the first few important nonhydrodynamic modes would provide information about phonon dispersion and phonon interactions which is at least as detailed as could be obtained by other means (e.g., sound attenuation). The problem of phonon dispersion is of much current interest.[22] Second, nonhydrodynamic modes are also generated in the vicinity of any pressure source—indeed, they are associated with any boundary. Perhaps generation of finite-frequency nonhydrodynamic modes is responsible for recent surprising observations on the generation and detection of first and second sound.[23] This possibility deserves further consideration.

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APPENDIX

In this Appendix we discuss some properties of the nonhydrodynamic modes appropriate to heat flow. The Boltzmann equation for a uniform medium is taken to be

$$\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial \vec{x}} \cdot \frac{d\vec{x}}{dt} + \frac{\partial \psi}{\partial \vec{p}} \cdot \frac{d\vec{p}}{dt} = J(\psi) + S . \qquad \text{(A1)}$$

Here $\psi = \psi(\vec{x}, \vec{p}, t)$ is the particle distribution function, which depends on position ($\vec{x}$), momentum ($\vec{p}$), and time (t). $J(\psi)$ is a linear functional (for small amplitude disturbances) representing the effect of collisions upon ψ, and S is a source term which describes how an external agent can modify ψ (e.g., introduction of heat). The $d\vec{p}/dt$ term can also serve as a source (e.g., an external gravitational or electric field). Here it will be dropped as being inessential to our argument, but we note that as a practical matter it is quite important in plasma physics and for describing the excitations in condensed matter.

We consider the situation near thermal equilibrium for which the source term drives the system so that the distribution function has a part ψ

$\propto \exp[i(\vec{k}\cdot\vec{x}-\nu t)]$. The Boltzmann equation then becomes

$$-i(\nu-\vec{k}\cdot\vec{v}_p)\psi_p = J(\psi)+S\ . \tag{A2}$$

Here we use $\vec{v}_p = d\vec{x}/dt$, where $\vec{v}_p$ is taken to be the group velocity of an excitation of momentum $\vec{p}$, and ψ_p is the deviation from thermal equilibrium of the distribution function for such an excitation. For static Kapitza-conductance (h_K) measurements $\nu=0$; for second sound measurements of h_K, ν is finite. Since the heater surface is taken to be perfectly smooth, by symmetry $\vec{k}$ must point along the normal to the heater. Therefore we take $\vec{k}\cdot\vec{v}_p = k\,v_p\cos\theta_p$, with θ_p measured with respect to the normal.

The modes generated by the heater may be determined by solving (A2) with the source term set to zero. First we write $J(\psi)$ explicitly as $J_{pp'}\psi_{p'}$, where summation on p' is implied. We then have

$$-i(\nu-k\,v_p\cos\theta_p)\psi_p = J_{pp'}\psi_{p'}\ . \tag{A3}$$

Since p takes on a continuum of values, this can be considered to be a matrix equation. If ν is fixed, then k can be determined by solving

$$\det[i(\nu-k\,v_p\cos\theta_p)\delta_{pp'}+J_{pp'}]=0\ . \tag{A4}$$

This is a determinant of infinite rank, so one can anticipate an infinite number of solutions k. We will discuss two of the properties of these solutions.

First, the collision integral $J_{pp'}$ has complete rotational symmetry, so that the solutions to (A3) can be labeled by the azimuthal "quantum number" m. The term in $v_p\cos\theta$ destroys the "quantum number" l, but m is not affected. We expect that a heater can only generate the $m=0$ solutions.

Let us now restrict ourselves to zero frequency and rewrite (A3) as

$$J_{pp'}\psi_{p'} = \lambda v_p\cos\theta_p\,\psi_p\ , \tag{A5}$$

where $\lambda = ik$ is to be determined. [$\lambda=\delta^{-1}$ of Eq. (1).] This is a "weighted" eigenvalue problem, with the weighting factor $v_p\cos\theta_p$. With a suitably defined inner product, $J_{pp'}$ can be made a self-adjoint operator. [This statement is true, to the best of my knowledge, for all collision integrals that have so far been studied. Specifically, this includes the gas of Maxwell molecules (intermolecular potential varying as r^{-4})[17] and the weakly interacting bose gas at low temperatures.[5]] It is straightforward to show that the eigenvalues of a weighted eigenvalue problem are real, provided that the operator is self-adjoint and that the weighting function is real. From this, we conclude that at $\nu=0$ the allowed values of k are imaginary. In addition, under a reflection through the k axis $\cos\theta_p$ changes sign, but $J_{pp'}$ is unchanged. This implies that a relabeling of the coordinate axes produces a new eigenstate with eigenvalue of opposite sign. Hence the solutions λ (and k) come in pairs of opposite sign. The physical origin of this symmetry is simple: An external source of planar geometry can generate a mode from either of its sides.

Those modes with $\lambda\neq 0$ either damp or grow exponentially in space. They are what we have called nonhydrodynamic modes, and constitute the vast majority of the solutions. Since

$$J_{pp'}\psi_{p'} = \lambda\psi_p \tag{A6}$$

shares its $\lambda=0$ solutions with (A5), and vice versa, a knowledge of the $\lambda=0$ solutions of (A5) characterizes the corresponding solutions of (A6). In fact, if $\psi_{p'}$ corresponds to a deviation from local equilibrium which can be characterized by T, P, and the local velocity $\vec{v}$, it is a solution of (A6) and (A5) with $\lambda=0$. This is a consequence of energy and momentum conservation (for conserved particles, number conservation generates an additional solution with $\lambda=0$). See Ref. 17.

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