Magnetic resonance and spin waves in the A phase of superfluid ^{3}He

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Extending an idea due to Leggett, that the longitudinal resonance in the A phase of superfluid ^{3}He is considered as an internal Josephson effect, we discuss the nuclear magnetic resonance as well as spin waves in the A phase of superfluid ^{3}He. The spin-wave dispersion in the hydrodynamic regime is determined explicitly.

I. INTRODUCTION

Recent longitudinal resonance experiments[1,2] in the A phase of liquid ^{3}He confirmed a theoretical prediction of Leggett.[3,4] Prior to these experiments, exploiting Leggett's[4] idea that the longitudinal resonance is considered as an internal Josephson effect, we had shown the existence of the longitudinal resonance in a simple manner and predicted the transient behavior of the magnetization after sudden application of magnetic field.[5] The latter prediction is borne out in a recent experiment by Webb *et al.*[6] However, the above work is limited to a spatially homogeneous situation. In this paper we will study more general situations, where the spin current is nonvanishing in general.

As in the previous work, we imagine that the A phase of superfluid ^{3}He in the presence of a magnetic field along the z axis consists of two superfluidities associated with the up-spin atoms and the down-spin atoms. The corresponding order parameters are given by $\Delta_\uparrow$ and $\Delta_\downarrow$. Furthermore, these two superfluidities are completely independent of each other but for a small dipole interaction which transfers one superfluid to the other.[5] In the absence of the dipole interaction and in the low-frequency limit (i.e., in the hydrodynamic limit) we have two conserved currents: the up-spin supercurrent and the down-spin supercurrent. However, it is more convenient to write these two supercurrents as the ordinary supercurrent associated with the mass flow and the new supercurrent associated with the (z component of) spin flow. Introduction of the dipole interaction breaks the conservation of the latter current weakly, while the former current is still conserved. We are not interested here in the former current, since the behavior of the mass current in the superfluid ^{3}He is already well understood.

The hydrodynamic equation for the spin current yields a differential equation for the relative phase $\phi(\equiv\phi_\uparrow-\phi_\downarrow)$, where $\phi_\uparrow$ and $\phi_\downarrow$ are phases of $\Delta_\uparrow$ and $\Delta_\downarrow$. In the limit of small amplitude oscillation, the oscillation of ϕ is described in terms of spin waves, while in a more general situation, the oscillation of ϕ is described in terms of solitons. Therefore, the present theory predicts existence of both spin waves and solitons in the hydrodynamic regime of the A phase. In Appendix C we will describe briefly how the present approach can be extended for the transverse resonance.

II. FORMULATION

As in our previous work,[5] we imagine that the A phase of the superfluid ^{3}He consists of two interpenetrating superfluids associated with the up-spin atoms and the down-spin atoms. The total Hamiltonian is decomposed as

$$H=E_\uparrow+E_\downarrow+H_I+E_d\ , \tag{1}$$

where $E_\uparrow$ and $E_\downarrow$ describe the free motion of the up-spin quasiparticles and the down-spin quasiparticles, H_I is the spin exchange interaction given by

$$H_I=I\int \delta n_\uparrow\, \delta n_\downarrow \tag{2}$$

(in a more general situation H_I describes the effective interaction term between quasiparticles as envisioned in the Landau theory of Fermi liquid), and finally E_d is the dipole interaction energy of the system. In particular, in terms of $\phi_\uparrow$ and $\phi_\downarrow$, E_d is calculated for the P-wave pairs as (see Appendix A)

$$E_d=\langle H_d\rangle=-(\pi\gamma^2/20g^2)\Delta^2(T)[1+3\cos(\phi_\uparrow-\phi_\downarrow)], \tag{3}$$

where $\phi_\uparrow$ and $\phi_\downarrow$ are the phases of the order parameters $\Delta_\uparrow$ and $\Delta_\downarrow$, respectively, γ is the gyromagnetic ratio of the ^{3}He nucleus, and g is the pairing interaction constant.

In the absence of the dipole interaction energy, the superfluids associated with the up-spin atoms and the down-spin atoms behave independent of each other. Making use of the conservation law in the absence of the dipole interaction, we obtain the following relations for the densities of the up-spin atoms $n_\uparrow$ and the down-spin atoms $n_\downarrow$:

$$\dot{n}_\uparrow + \nabla\vec{j}_\uparrow = \frac{\delta E_d}{\delta\phi_\uparrow} = \frac{3\pi\gamma^2}{20g^2}\Delta^2(T)\sin(\phi_\uparrow - \phi_\downarrow),$$
$$\dot{n}_\downarrow + \nabla\vec{j}_\downarrow = \frac{\delta E_d}{\delta\phi_\downarrow} = -\frac{3\pi\gamma^2}{20g^2}\Delta^2(T)\sin(\phi_\uparrow - \phi_\downarrow), \tag{4}$$

where a dot implies the time derivative, and $\vec{j}_\uparrow$ and $\vec{j}_\downarrow$ are the currents associated with the up-spin atoms and the down-spin atoms, respectively. The right-hand side of Eq. (4) describes the transfer of the up-spin superfluid into the down-spin superfluid and vice versa due to the dipole interaction.[5] We can rewrite Eq. (4) as

$$\dot{n} + \vec{\nabla}\vec{j} = 0, \tag{5a}$$

$$\dot{S}_z + \vec{\nabla}\vec{j}_{S_z} = -(3\pi\gamma^2/20g^2)\Delta^2(T)\sin\phi, \tag{5b}$$

where

$$n = n_\uparrow + n_\downarrow, \qquad \vec{j} = \vec{j}_\uparrow + \vec{j}_\downarrow,$$
$$S_z = \tfrac{1}{2}(n_\uparrow - n_\downarrow), \quad \vec{j}_{S_z} = \tfrac{1}{2}(\vec{j}_\uparrow - \vec{j}_\downarrow),$$

and

$$\phi = \phi_\uparrow - \phi_\downarrow. \tag{6}$$

Equation (5a) implies the conservation of the total atoms, while Eq. (5b) expresses the conservation of the total spin weakly broken by the dipole interaction.

In the low-frequency limit (i.e., in the hydrodynamic limit), $\vec{j}_\uparrow$ and $\vec{j}_\downarrow$ are related to the superfluid velocity fields $\vec{\nabla}\phi_\uparrow$ and $\vec{\nabla}\phi_\downarrow$ by

$$\vec{j}_{\uparrow(\downarrow)} = (N/4m)(\overleftrightarrow{\rho}_s/\rho)(\vec{\nabla}\phi_{\uparrow(\downarrow)}) . \tag{7}$$

As in an ordinary superconductor, where the superfluid density tensor $\overleftrightarrow{\rho}_s$ is defined in terms of the retarded product of $\vec{j}_\uparrow$ and $\vec{j}_\downarrow$,

$$\overleftrightarrow{\rho}_s = \lim_{\omega\to 0} 2\langle(\vec{j}_\uparrow, \vec{j}_\uparrow)\rangle(0,\omega)$$
$$= \lim_{\omega\to 0} 2\langle(\vec{j}_\downarrow, \vec{j}_\downarrow)\rangle(0,\omega); \tag{8}$$

we note also

$$\langle(\vec{j}_\uparrow, \vec{j}_\downarrow)\rangle(0,\omega) = 0 \tag{9}$$

in the A phase. In the weak-coupling limit Eq. (8) is explicitly calculated as (see Appendix B)

$$\frac{(\rho_s)_{ij}}{\rho} = 3\left\langle k_i k_j 2\pi T \sum_{n=0}^{\infty} \frac{\Delta^2(\Omega)}{[\omega_n^2 + \Delta^2(\Omega)]^{3/2}} \right\rangle , \tag{10}$$

where $\vec{k}$ is a unit vector in the directions designated by Ω and $\langle A\rangle$ implies the angular average of A. In particular, for the case of the P-wave pairs, $\overleftrightarrow{\rho}_s/\rho$ has the following asymptotic forms:

$$\frac{(\rho_s)_\parallel}{\rho} = \frac{1-T}{T_c}, \quad \frac{(\rho_s)_\perp}{\rho} = 2\left(1 - \frac{T}{T_c}\right) \text{ for } T \cong T_c \;;$$
$$\frac{(\rho_s)_\parallel}{\rho} = 1 - 6\left(\frac{\pi T}{3\Delta}\right)^2, \quad \frac{(\rho_s)_\perp}{\rho} = 1 - \frac{84}{5}\left(\frac{\pi T}{3\Delta}\right)^4 \text{ for } T \cong 0, \tag{11}$$

where the subscripts $\parallel$ and $\perp$ mean the components parallel and perpendicular to the symmetry axis of the orbital wave function of the condensed pairs.

The spin current is given from Eq. (7) as

$$\vec{j}_{S_z} = (N/8m)(\overleftrightarrow{\rho}_s/\rho)(\vec{\nabla}\phi). \tag{12}$$

In order to complete the above equation we have to relate the time derivative of ϕ to n which is easily formed from the Josephsen relation[4,5]

$$\dot{\phi} = -\delta H/\delta(\tfrac{1}{2}S_z) = -2(\mu_\uparrow - \mu_\downarrow) + 4IS_z. \tag{13}$$

(Note that the conjugate variable to ϕ is $\frac{1}{2}S_z$.) In deriving Eq. (13) we have assumed that the local equilibrium is attained both among the up-spin atoms and among the down-spin atoms separately.

Finally, eliminating $\vec{S}_z$ and $\vec{j}_{S_z}$ from Eq. (5), we obtain

$$\ddot{\phi} - \tfrac{1}{3}(1-\bar{I})v_F^2(\vec{\nabla}\overleftrightarrow{\rho}_s\vec{\nabla}\phi)/\rho = -\Omega_l^2\sin\phi, \tag{14}$$

where $\Omega_l = [(1-\bar{I})6\pi\gamma^2/5g^2N(0)]^{1/2}\Delta(T)$ is the longitudinal resonance frequency.[4,7] For small ϕ (i.e., $|\phi| \ll 1$), Eq. (14) predicts the existence of the spin wave with the dispersion

$$\omega^2 = \Omega_l^2 + \tfrac{1}{3}(1-\bar{I})v_F^2\vec{q}\overleftrightarrow{\rho}_s\vec{q}/\rho, \tag{15}$$

where v_F is the Fermi velocity.

At $T=0$ K, Eq. (15) agrees with the spin wave dispersion determined previously within the random-phase approximation[8,9] (RPA). However, at $T \neq 0$ K, Eq. (15) differs significantly from the RPA calculation. This is because in the present derivation we assumed that the thermal equilibrium is attained in the up-spin states and the down-spin states separately, which seems more physical for low-frequency phenomena. On the other hand, the RPA calculation[8,9] was limited to the collisionless limit, as the effects associated with the quasiparticle lifetime are completely neglected.

So far we have neglected any dissipation term which may appear in Eq. (14) for ϕ. It is not difficult to introduce the dissipation terms phenomenologically. If we include the contribution from the normal spin current as well as the spin lifetime due to the spin-nonconserving scattering, Eq. (10) would have to be generalized as

$$\ddot{\phi} - \tfrac{1}{3}(1-\bar{I})v_F^2\vec{\nabla}\overleftrightarrow{\rho}_s\vec{\nabla}\phi/\rho - \vec{\nabla}\overleftrightarrow{D}\vec{\nabla}\phi - T_1^{-1}\dot{\phi} = -\Omega_l^2\sin\phi, \tag{16}$$

where $\overleftrightarrow{D}$ is the spin diffusion tensor and T_1 is the intrinsic-spin lifetime due to the scattering.

III. SOLITONS

In the following we will neglect dissipation terms in Eq. (16) for simplicity. Furthermore, we limit ourselves to the one-dimensional situation, where the spatial variation in ϕ takes place only in one direction (say the x direction). Then Eq. (16) reduces to

$$\ddot{\phi} - C^2\frac{\partial^2}{\partial x^2} = -\Omega_l^2\sin\phi,$$

where

$$C^2 = \tfrac{1}{3}(1-\bar{I})v^2(\rho_s)_{xx}/\rho \ . \tag{17}$$

An equation similar to Eq. (17) has been encountered in the literature on the problem of the vortex structure in the Josephsen junction.[11,12] In particular, it is known that Eq. (17) allows moving solutions (or solitons) in the infinite system. A class of periodic solutions can be obtained by assuming that

$$\phi(x,t) = \phi(u), \tag{18}$$

with

$$u = (x - vt)/\lambda$$

and

$$\lambda = \Omega_l^{-1}(C^2 - v^2)^{1/2}, \tag{19}$$

where we have assumed $v < C$. Then Eq. (17) becomes

$$\frac{\partial^2}{\partial u^2}\phi = \sin\phi. \tag{20}$$

The first integral of Eq. (20) is given as

$$\left(\frac{\partial\phi}{\partial u}\right)^2 = C - 2\cos\phi, \tag{21}$$

where C is a constant. We have different classes of solutions depending on C:

$$\begin{aligned} &\sin\{\tfrac{1}{2}[\phi(x,t)-\pi]\} = \mathrm{sn}(u/k\,|\,k^2) && \text{for } C>2,\\ &\tan[\tfrac{1}{4}\phi(x,t)] = \exp(-u) && \text{for } C=2,\\ &\sin\{\tfrac{1}{2}[\phi(x,t)-\pi]\} = k^{-1}\,\mathrm{sn}(u\,|\,k^{-2}) && \text{for } C<2, \end{aligned} \tag{22}$$

where $k = 2(C+2)^{-1/2}$ and sn is the Jacobian elliptic integral. Furthermore, $\phi(u)$ satisfies

$$\phi(u+2\tau) = \phi(u) + 2\pi, \tag{23}$$

where

$$\begin{aligned} \tau &= kK(k) \quad \text{for } C>2\\ &= K(k^{-1}) \quad \text{for } C<2. \end{aligned} \tag{24}$$

The local magnetizations associated with the above solutions are then given by

$$S_z(x,t) = -\tfrac{1}{2}X_0\,\frac{\partial\phi}{\partial t}\ , \tag{25}$$

which yields

$$\begin{aligned} S_z(x,t) &= X_0\left(\frac{v}{\lambda}\right)k^{-1}\mathrm{dn}[(u/k)\,|\,k^2] && \text{for } C>2\\ &= X_0(v/\lambda)\,\mathrm{sech}(u) && \text{for } C=2\\ &= X_0(v/\lambda)k^{-1}\,\mathrm{cn}(u\,|\,k^{-2}) && \text{for } C<2, \end{aligned} \tag{26}$$

respectively, where X_0 is the susceptibility in the normal state, and dn and cn are the Jacobian elliptic functions. These solutions represent a series of identical peaks, a single peak and a series of alternating peaks in sign moving in the x direction. In the case $v > C$, on the other hand, the corresponding solutions are easily constructed from those for $v < C$ by the following replacement as noted already by Kulik[11]:

$$\begin{aligned} &\Phi(x,t) \to \phi(x,t) + \pi,\\ &\lambda \to \lambda' = \Omega_l^{-1}(v^2 - c^2)^{1/2}. \end{aligned} \tag{27}$$

IV. DETECTION OF SPIN WAVES

In this section we consider a possibility of detecting the spin waves predicted in the present calculation. For this purpose we will confine ourselves to the case where the amplitude ϕ is small [i.e., we will use the linear version of Eq. (14)]. Furthermore, we consider one dimensional problems, where everything varies only along the x axis. When a nonuniform magnetic field $H(x)$ is applied suddenly, the subsequent oscillation of ϕ is determined by Eq. (17) with the initial condition

$$\phi = 2\Delta\omega_L(x) = 2|\gamma|H(x) \quad \text{at } t = 0. \tag{28}$$

Assuming that the liquid is confined in a box with the length b along the x axis, $\phi(x,t)$ is given as

$$\phi(x,t) = \sum_{n=0}^{\infty} a_n(t)\cos\frac{n\pi x}{b}\ , \tag{29}$$

where $a_n(t)$ is determined as

$$\begin{aligned} a_n(t) &= a_n\sin(\lambda_n t),\\ \lambda_n &= [\Omega_l^2 + C^2(n\pi/b)^2]^{1/2},\\ a_0 &= \frac{2}{b\lambda_0}\int_0^b dx\,\Delta\omega_L(x),\\ a_n &= \frac{4}{b\lambda_n}\int_0^b dx\,\Delta\omega_L(x)\cos\frac{n\pi x}{b}\ . \end{aligned} \tag{30}$$

In expansion (29), it is assumed that no spin current flows in or out at the boundary. (This assumption may be modified if the container is made of magnetic material.) Since $C = 10^2 \sim 10^3$ cm/sec in the A phase of ^{3}He, the difference between λ_n and λ_0 becomes appreciable if b is smaller than, say, 1 mm. Therefore, by choosing an appropriate $\Delta\omega_L(x)$, it is possible to excite a standing spin wave. Making use of Eq. (12), we can treat the decay of the standing spin wave as well as the nonlinear interaction between spin waves.

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APPENDIX A: CALCULATION OF THE DIPOLE INTERACTION ENERGY

The dipole interaction energy is expressed as the expectation value of H_d:

$$E_d=\langle H_d\rangle$$
$$=-\frac{\gamma^2}{8}\left\langle\int d^3r\int d^3r'\psi^+(\vec{r})\sigma_i\psi(\vec{r})^+(\vec{r}')\sigma_j\psi(\vec{r}')\right\rangle\frac{\delta_{ij}-3e_ie_j}{|\vec{r}-\vec{r}'|^3}\,, \tag{A1}$$

where $\vec{e}=(\vec{r}-\vec{r}')/|\vec{r}-\vec{r}'|$ and γ is the gyromagnetic ratio of the ^{3}He nucleus. The above expectation value can be expressed in terms of the Green's function as

$$E_d=-\frac{\pi\gamma^2}{6}T^2\sum_{nm}\int\frac{d^3p}{(2\pi)^3}$$
$$\times\int\frac{d^3p'}{(2\pi)^3}T_r\{G(\vec{p},\omega_n)\alpha_iG(\vec{p}',\omega_m)\alpha_j\}\Lambda_{ij}(\Omega,\Omega'), \tag{A2}$$

where $\Lambda_{ij}(\Omega,\Omega')=3n_in_j-\delta_{ij}$, $\vec{n}=(\vec{p}-\vec{p}')/|\vec{p}-\vec{p}'|$, α_i are Pauli spin operators[9] in the four-dimensional representation, and $G(\vec{p},\omega_n)$ is the Green's function[9] given as

$$G^{-1}(\vec{p},\omega_n)=i\omega_n-[\xi+(\Omega_0/2)\sigma_3]\rho_3-\tfrac{1}{2}\{\rho_+[\tfrac{1}{2}(1+\sigma_3)\Delta_\uparrow^*$$
$$+\tfrac{1}{2}(1-\sigma_3)\Delta_\downarrow^*+\rho_-[\tfrac{1}{2}(1+\sigma_3)\Delta_\uparrow+\tfrac{1}{2}(1-\sigma_3)\Delta_\downarrow]\} \tag{A3}$$

and

$$\rho\pm=\rho_1\pm i\rho_2,$$
$$\xi=(1/2m)p^2-\mu,\ \Omega_0=\omega_0/(1-\bar{I})=-\gamma H/(1-\bar{I}), \tag{A4}$$

where ω_n and ω_m are the Matsubara frequencies, Ω_0 is the renormalized Larmor frequency and ρ_i, σ_i are the Pauli spin matrices operating in the particle-hole space and in the ordinary spin space, respectively. The integrals over d^3p and d^3p' are easily carried out by replacing them by $N(0)(d\Omega/4\pi)d\xi$ and $N(0)(d\Omega'/4\pi)d\xi'$, respectively, as

$$E_d=(-\tfrac{2}{3}\pi\gamma^2)[\pi TN(0)]^2\sum_{n,m}\sum_{i,j}\int\frac{d\Omega d\Omega'}{(4\pi)^2}\frac{\Delta_i^*(\Omega)\Lambda_{ij}(\Omega,\Omega')\Delta_j(\Omega')}{(\omega_n^2+\Delta^2(\Omega))^{1/2}(\omega_m^2+\Delta^2(\Omega))^{1/2}}=-\frac{2\pi\gamma^2}{3g^2}\Delta^2\sum_{ij}\langle f_i|\Lambda_{ij}|f_j\rangle, \tag{A5}$$

where

$$\langle f_i|\Lambda_{ij}|f_j\rangle=\int\frac{d\Omega\,d\Omega'}{(4\pi)^2}f_i^*(\Omega')$$
$$\times\Lambda_{ij}(\Omega',\Omega)f_j(\Omega),$$
$$\Delta^2(\Omega)=|\Delta_\uparrow(\Omega)|^2=|\Delta_\downarrow(\Omega)|^2=\Delta^2|f|^2, \tag{A6}$$
$$\Delta_1=(\Delta_\uparrow-\Delta_\downarrow)/\sqrt{2}i,\quad\Delta_2=(\Delta_\downarrow+\Delta_\uparrow)/\sqrt{2}\ ,$$

and

$$\Delta_3=0.$$

In the above derivation we made use of the gap equation:

$$1=\pi TgN(0)\sum_n\int\frac{d\Omega}{4\pi}\frac{|f|^2}{[\omega_n^2+\Delta^2(\Omega)]^{1/2}}\,. \tag{A7}$$

Finally, assuming that

$$\Delta_\uparrow=\Delta e^{i\phi_\uparrow}f$$

and

$$\Delta_\downarrow=\Delta e^{i\phi_\downarrow}f\ ,$$
$$f=(\tfrac{3}{2})^{1/2}(x+iz) \tag{A8}$$

for the axial solution, we obtain Eq. (3).

APPENDIX B: CALCULATION OF THE SUPERFLUID DENSITIES ASSOCIATED WITH THE UP-SPIN CURRENT AND THE DOWN-SPIN CURRENT

The up-spin current and the down-spin current are expressed in terms of the electron field operators

$$\vec{j}_\uparrow(\vec{r})=(1/2i)[\psi_\uparrow^+(\vec{r})\vec{\nabla}\psi_\uparrow(\vec{r})-(\vec{\nabla}\psi_\uparrow^+(\vec{r}))\psi_\uparrow(\vec{r})]$$

and (B1)

$$\vec{j}_\downarrow(\vec{r})=(1/2i)[\psi_\downarrow^+(\vec{r})\vec{\nabla}\psi_\downarrow(\vec{r})-(\vec{\nabla}\psi_\downarrow^+(\vec{r}))\psi_\downarrow(\vec{r})]\ ,$$

respectively. The static part of the retarted product of the current correlation function is then given as

$$\left\langle[j_\uparrow^i,j_\uparrow^j]\right\rangle(0,0)=T\sum_n\int\frac{d^3p}{(2\pi)^3}\mathrm{Tr}\left\{\frac{1+\sigma_3}{2}p_iG(\vec{p},\omega_n)\left(\frac{1+\sigma_3}{2}\right)p_jG(\vec{p},\omega_n)\right\}, \tag{B2}$$

where G is the Green's function already defined in (A3). Substituting the Green's function defined in (A3), it is easy to carry out the integral over d^3p and we find

$$\langle[j_\uparrow^i,j_\uparrow^j]\rangle(0,0)=\pi TN(0)\int\frac{d\Omega}{4\pi}\sum_n\frac{p_ip_j|\Delta_\uparrow(\Omega)|^2}{(\omega_n^2+\Delta^2(\Omega))^{3/2}}$$
$$=\tfrac{3}{2}mNT\sum_n\int\frac{d\Omega}{4\pi}\left(k_ik_j\frac{\Delta^2(\Omega)}{(\omega_n^2+\Delta^2(\Omega))^{3/2}}\right)$$

and

$$\vec{k}=\vec{p}/|\vec{p}|\ . \tag{B3}$$

Equation (B3) is identical to Eq. (10) in the text.

We have an identical expression for $\langle[j_\downarrow^t j_\downarrow^t]\rangle(0,0)$, since

$$|\Delta_\uparrow(\Omega)|^2 = |\Delta_\downarrow(\Omega)|^2.$$

APPENDIX C: EXTENSION TO THE TRANSVERSE RESONANCE

Generally speaking, the transverse resonance involves an additional order parameter Δ_0, which vanishes identically in the A phase in the presence of a magnetic field along the z axis. However, we can still exploit a close analogy between the longitudinal resonance in the present consideration. We have treated $\frac{1}{2}\phi = \theta_1$ as the conjugate variable to S_z. Although ϕ is interpreted as the difference to two phases $\phi_\uparrow$ and $\phi_\downarrow$, we can view θ_1 as an angle between $\vec{d}$ (i.e., the spin vector of the condensed pair) and $\vec{L}$ the symmetry axis of the orbital wave function f defined in (A8). In the equilibrium configuration (including the dipolar energy), we assumed both $\vec{L}$ and $\vec{d}$ are along the y axis. For the longitudinal resonance the $\vec{d}$ vector rotates in the x-y plane with angle θ_1 from the y axis. Let us consider the case when an external rf field is applied along the x axis. This exerts a torque on the $\vec{d}$ vector, and in the present case the $\vec{d}$ vector begins to rotate around the x axis. Then S_x (the x component of the total spin) and θ_2 (the angle between the $\vec{d}$ and $\vec{L}$ vectors in the y-z plane) are the conjugate variables, as are S_z and θ_1. However, since S_x couples with S_y through the Larmor term we have to take into account the motion of S_y as well. With these preliminaries we can write down the following set of equations for this particular case:

$$\dot{S}_x = \omega_L S_y + \frac{\partial H}{\partial \theta_2} = \omega_L S_y + \frac{\partial E_d(\theta_2)}{\partial \theta_2}\ ,$$

$$\dot{S}_y = -\omega_L S_x + \frac{\partial H}{\partial \theta_3} = -\omega_L S_x\ , \tag{C1}$$

and

$$\dot{\theta}_2 = -\frac{\partial H}{\partial S_x} = \Delta\omega_L^x - X_0^{-1} S_x\ ,$$

where

$$E_d(\theta_2) = -(\pi\gamma^2/20g^2)\Delta^2(T)[1+3\cos(2\theta_2)] \tag{C2}$$

where $\Delta\omega_L^x = |\gamma|(\Delta H_x)$, and ΔH_x is the change in the transverse component of the field. Eliminating S_x and S_y from the above equation, we have for $\psi = 2\theta_2$:

$$\frac{\partial}{\partial t}(\ddot{\psi} + \omega_L^2\psi + \Omega_l^2 \sin\psi) = 2\omega_l^2(\Delta\omega_L^x). \tag{C3}$$

In the absence of the transverse rf field, ψ obeys a simple differential equation

$$\ddot{\psi} + \omega_L^2\psi + \Omega_l^2 \sin\psi = 0. \tag{C4}$$

Especially when $|\psi| \ll 1$, (C4) reduces to the equation previously obtained by Leggett.[4] In particular, Eq. (C4) has an oscillatory solution with the transverse resonance frequency ω_t when $|\psi| \ll 1$,

$$\omega_t^2 = \omega_L^2 + \Omega_l^2. \tag{C5}$$

In general, Eq. (C3) can be used to describe nonlinear transverse response of the magnetization.

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