

## Critical properties of many-component systems\*

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Critical properties are discussed for systems with order parameters given by  $n$  vectors  $\vec{S}_\alpha$ , each with  $m$  components. The Hamiltonian has an arbitrary symmetry for each vector separately, but there is a particular kind of coupling between them. It is shown that there is an integral representation for the partition function which reduces  $n$  to an explicit parameter in an averaged partition function for the  $m$ -component model. This leads to a simple discussion of properties of the system as a function of  $n$ . In particular, it is possible to give a coherent derivation of several known and new results without the aid of perturbation theory or the renormalization-group method. It is shown that, in certain special cases, the exponents are Gaussian when  $n$  is a negative even integer and that  $n = 0$  corresponds to the excluded-volume problem. The general case is shown to reduce to an arbitrary  $m$ -component model which is random when  $n = 0$  and constrained when  $n \rightarrow \infty$ . A direct derivation of the large- $n$  limit is given and leads to a variety of exactly solvable models. Expressions for the order  $n^{-1}$  correction are obtained in terms of correlation functions. This expansion is valid at all temperatures and for any order of transition, so that it is particularly suitable for considering tricritical phenomena.

### I. INTRODUCTION

In the recent development<sup>1</sup> of the theory of critical phenomena, exponents and related properties have been calculated as functions of the number of space dimensions ( $d$ ) and the number of degrees of freedom ( $n$ ) of the order parameter. It has proved possible<sup>1-4</sup> to evaluate the first few terms of expansions in powers of  $(4-d)$  or  $n^{-1}$  and to make exact statements about certain special cases where  $n$  is continued to negative even integers<sup>5,6</sup> or to zero.<sup>7,8</sup> These results have been derived for various continuous spin models, usually with fourth- or sixth-order Hamiltonians,<sup>4</sup> by using the renormalization-group method<sup>1,3,4</sup> or by studying every order in perturbation theory.<sup>2,5-8</sup>

The purpose of this paper is to describe a representation of the partition function which leads to an explicit and coherent derivation of the exact results<sup>5-8</sup> and the  $n^{-1}$  expansion, and allows us to generalize them. With this approach it is possible to avoid the complications of perturbation theory and the fixed-point assumption of the renormalization group, and, in addition, the  $n^{-1}$  expansion is valid for all temperatures and for any kind of transition. The method is applicable to quite general Hamiltonians and leads to an arbitrary  $m$ -component model which is random when  $n=0$  and constrained when  $n \rightarrow \infty$ . As a special case, we shall obtain the formal results of Abe's extension<sup>9</sup> of Stanley's discussion<sup>10</sup> of the classical Heisenberg model, which also has been solved explicitly for large  $n$  but involves a rather special constraint which does not have the flexibility necessary for a discussion of tricritical phenomena, metamagnets, coupled-order-parameter models, and other situa-

tions of interest.

The general system to be considered is a lattice model such that, at each site  $i$ , there are  $n$  vector variables  $\vec{S}_{i\alpha}$  each with  $m$  components. It will be shown that, for a wide variety of Hamiltonians, there is an integral representation of the statistical operator which allows the partition function to be cast into a form in which  $n$  appears solely as a parameter and not as a number of variables. Once this has been done it is relatively easy to discuss particular values of  $n$  and to obtain the expansion of the free energy in powers of  $n^{-1}$ .

In Sec. II, the method will be illustrated first for  $m=1$  and a particularly simple Hamiltonian

$$\mathcal{H} = \sum_{\alpha}^n \sum_{i,j}^N J_{ij} S_{i\alpha} S_{j\alpha} + \sum_i^N \left[ r \sum_{\alpha}^n S_{i\alpha}^2 + \frac{u}{4} \left( \sum_{\alpha}^n S_{i\alpha}^2 \right)^2 \right], \quad (1.1)$$

which has been studied extensively by the other methods.<sup>1-3,5</sup> Here  $J_{ij}$  is the exchange integral and it is assumed that the temperature  $T$  is absorbed into the parameters of the model. The partition function is

$$Z_n = \int_{-\infty}^{\infty} \prod_i^N \prod_{\alpha}^n dS_{i\alpha} e^{-\mathcal{H}}. \quad (1.2)$$

A discrete lattice has been assumed but it is easy to pass to the continuum limit. For this simple Hamiltonian, it is a Gaussian integral representation of the  $(\sum_{\alpha}^n S_{i\alpha}^2)^2$  term in  $\mathcal{H}$  which enables us to display the dependence of  $Z_n$  upon  $n$ , and, as an example of the method, it is shown, without using perturbation theory, that for  $n=-2$ ,  $Z_n$  reduces to

the Gaussian model<sup>5</sup> and  $n=0$  is equivalent to the excluded volume problem.<sup>7</sup>

In Sec. III,  $\mathcal{H}$  is generalized in a number of ways. The quadratic terms (including the exchange integral) of Eq. (1.1) are replaced by an arbitrary Hamiltonian  $\hbar[\tilde{S}_{i\alpha}]$  in which the variables are now  $m$ -component vectors at site  $i$ . The fourth-order terms of Eq. (1.1) are replaced by an arbitrary function  $f[\sum_{\alpha} g_p(\tilde{S}_{i\alpha})]$  of several variables ( $p=1, 2, \dots$ ). It is not assumed that  $\hbar$  or  $g_p$  are polynomials or that they are rotationally invariant in the space of the  $\tilde{S}_{i\alpha}$ . The introduction of a more general integral representation of the  $f$  term in the density matrix again explicitly exposes the dependence of  $Z_n$  upon  $n$ . As a particular illustration, it is shown that there are special cases in which the free energy and correlation functions are Gaussian for  $n$  a negative even integer.<sup>8</sup>

The general application of the representation is made in Sec. IV where it is shown that  $n=0$  corresponds to random  $m$ -component models and gives a relationship between averaging the partition function or the free energy for such systems. It is also shown that the  $n \rightarrow \infty$  limit may be obtained by a steepest descents calculation which gives a group of exactly solvable models, of particular interest for tricritical phenomena. For large but finite  $n$ , the order  $n^{-1}$  corrections are obtained in terms of correlation functions of the  $n \rightarrow \infty$  limit.

The discussion of the general model is not much more difficult than that of the simpler case given in Eq. (1.1). In contrast, the renormalization-group method is much more complicated both formally, in deriving equations to be solved, and practically in specific problems where the stability of large numbers of fixed points must be considered.

Most of the discussion in this paper is concerned with formal considerations. Applications to tricritical phenomena, coupled-order-parameter models and time-dependent problems will be made in separate publications.<sup>11</sup>

## II. FOURTH-ORDER SPHERICALLY SYMMETRIC MODEL

In this section the method will be illustrated for the Hamiltonian given in Eq. (1.1) and discussion of the more general case will be deferred to Secs. III and IV. As an application, it will be shown how two familiar results—the equivalence of  $n=-2$  to the Gaussian model<sup>5,6</sup> and  $n=0$  to the excluded-volume problem<sup>7</sup>—may be derived in a coherent fashion without the use of perturbation theory.

The object is to rewrite the partition function  $Z_n$  of Eq. (1.2) in a form in which  $n$  appears explicitly as a parameter. This is achieved by introducing

an integral representation of the fourth order terms in the exponent of  $e^{-\mathcal{H}}$ :

$$\exp\left[-\frac{u}{4}\left(\sum_{\alpha} S_{i\alpha}^2\right)^2\right] = (\pi u)^{-1/2} \int_{-\infty}^{\infty} d\psi_i \times \exp\left(-\frac{\psi_i^2}{u} + i\psi_i \sum_{\alpha} S_{i\alpha}^2\right). \quad (2.1)$$

Here, we have assumed  $u > 0$ , which is necessary for convergence.

This result may be proved in the usual way by changing the variable of integration to  $(\psi_i - i(\frac{1}{2}u) \times \sum_{\alpha} S_{i\alpha}^2)$ . Substituting Eqs. (1.1) and (2.1) into Eq. (1.2) and changing the order of integration over  $\psi_i$  and  $S_{i\alpha}$  we obtain

$$Z_n = (\pi u)^{-N/2} \int_{-\infty}^{\infty} \prod_i d\psi_i \times \exp\left(-\frac{1}{u} \sum_i \psi_i^2 - n\Phi_G[\psi_i]\right), \quad (2.2)$$

where

$$e^{-n\Phi_G[\psi_i]} = \int_{-\infty}^{\infty} \prod_i^N \prod_{\alpha}^n dS_{i\alpha} \times \exp\left[\sum_{\alpha} \left(-\sum_{i,j} J_{ij} S_{i\alpha} S_{j\alpha} + \sum_i^N (r + i\psi_i) S_{i\alpha}^2\right)\right]. \quad (2.3)$$

The point now is that since the exponent on the right-hand side of Eq. (2.3) is a sum over  $\alpha$ , the integral factors into a product of  $n$   $N$ -dimensional integrals, and since  $\alpha$  is merely a suffix on the integration variable, all  $n$  integrals are identical. That is,

$$e^{-\Phi_G[\psi_i]} = \int_{-\infty}^{\infty} \prod_i^N dS_i \times \exp\left(-\sum_{i,j} J_{ij} S_i S_j + \sum_i^N (r + i\psi_i) S_i^2\right) \quad (2.4)$$

and  $\Phi_G[\psi_i]$  is independent of  $n$ . Thus the  $n$  dependence of  $Z_n$  is completely exposed in Eq. (2.2), where  $n$  is merely a parameter in the exponent. This makes it relatively simple to study particular values and limits of  $n$ . Since the exponent in Eq. (2.4) is quadratic in the  $S_i$ , the integral can be evaluated to give

$$\Phi_G[\psi_i] = \frac{1}{2} \ln \det(\hat{J} + r + i\hat{\psi}) - (\frac{1}{2}N) \ln \pi = \frac{1}{2} \text{Tr} \ln(\hat{J} + r + i\hat{\psi}) - (\frac{1}{2}N) \ln \pi. \quad (2.5)$$

Here  $\hat{J}$  and  $\hat{\psi}$  are matrices with elements  $J_{ij}$  and  $\psi_i \delta_{ij}$ , respectively, and the suffix  $G$  has been put on  $\Phi_G$  to indicate that it is such a Gaussian free energy. Equation (2.5) may be proved by changing to a representation in which  $(\hat{J} + r + i\hat{\psi})$  is diagonal and evaluating the integral in Eq. (2.4), and then the determinant and the trace in terms of the eigenvalues. The cases  $n=0$  and  $n=-2$  will now be discussed separately.

#### A. Gaussian model and $n=-2$

Balian and Toulouse<sup>5</sup> have shown that, order by order in an expansion in powers of  $u$ , the partition function is the same as that of the Gaussian model when  $n=-2$ . An alternative derivation was given by Fisher.<sup>6</sup> It will now be shown that this result may be obtained from Eqs. (2.2) and (2.5) directly, without use of expansions.

Substitution of the first equality of Eq. (2.5) into Eq. (2.2) gives, for  $n=-2$ ,

$$Z_{-2} = (\pi^3 u)^{-N/2} \int_{-\infty}^{\infty} \prod_i^N d\psi_i \times \left( \exp -\frac{1}{u} \sum_i^N \psi_i^2 \right) \det[\hat{J} + r + i\hat{\psi}]. \quad (2.6)$$

Now  $\hat{\psi}$  is a diagonal matrix and it is clear that, regardless of the form of  $\hat{J}$ , each  $\psi_i$  occurs at most linearly in every term of the expansion of the determinant. But, since the exponent does not couple the  $\psi_i$ 's, then, whenever a  $\psi_i$  appears, the integral over that  $\psi_i$  is an integral of an odd function and hence it vanishes. Thus  $\hat{\psi}$  may be deleted from the determinant in Eq. (2.6) without changing  $Z_{-2}$ . But then the  $\psi_i$  integrals may be completed trivially since they do not concern the determinant. Then again writing  $\det(\hat{J} + r)$  as  $\exp[\text{Tr} \ln(\hat{J} + r)]$ , the free energy per degree of freedom is

$$F_{-2} = -\ln Z_{-2} / -2 = \frac{1}{2} \text{Tr} \ln(\hat{J} + r) - (\frac{1}{2}N) \ln \pi, \quad (2.7)$$

which is the free energy of the Gaussian model. The same result follows for the  $S_i S_j$  correlation functions by differentiating  $F_{-2}$  with respect to  $J_{ij}$ . In Sec. III, we shall discuss the generalization of this result to  $n=-4, -6, \dots$  etc., for higher-order weight functions and derive the results of Fisher.<sup>6</sup>

At this point, it is plausible that there might be a useful expansion of critical exponents in powers of  $(n+2)$  and, indeed, from Eqs. (2.2) and (2.5), it is straightforward to write down the expansion of  $Z_n$  formally. However, it is not easy to evaluate the successive terms. The series is in powers of  $\text{Tr} \ln(\hat{J} + r + i\hat{\psi})$  and there is a weight factor  $\det(\hat{J} + r + i\hat{\psi})$  in addition to  $\exp[-(1/u) \sum_i^N \psi_i^2]$ . We shall now see that even the first-order term is

more complicated than the excluded-volume problem, which has not been solved explicitly.

#### B. Excluded-volume problem and $n=0$

The excluded volume problem for polymer chains concerns the calculation of the root mean square distance between the ends of a polymer chain with  $M$  links. In the usual model, the links of the chain are freely pivoted and the excluded volume is the volume of a hard-sphere interaction between the monomers situated at the pivots. The particular representation of this problem which we shall obtain was derived by Edwards<sup>12</sup> in the limit that the length of the links tends to zero.

The relationship between the excluded volume problem and the partition function of Eqs. (1.1) and (1.2) for  $n=0$  was first pointed out by de Gennes<sup>7</sup> who gave a proof order by order in perturbation theory. Equations (2.2) and (2.5) enable us to show this directly without use of expansions.

From Eq. (2.2), the free energy,  $F_n$ , per degree of freedom becomes (as  $n \rightarrow 0$ )

$$F_0 = -\lim_{n \rightarrow 0} \frac{1}{n} \ln Z_n = (\pi u)^{-N/2} \int_{-\infty}^{\infty} \prod_i^N d\psi_i \Phi_G(\psi_i) \times \exp\left(-\frac{1}{u} \sum_i^N \psi_i^2\right). \quad (2.8)$$

Then the correlation function

$$G_{ij}^n = \frac{1}{n} \sum_{\alpha}^n \langle S_{i\alpha} S_{j\alpha} \rangle = \frac{1}{n} \frac{\partial F_n}{\partial J_{ij}} \quad (2.9)$$

may be obtained from Eqs. (2.5) and (2.8), for  $n=0$ ;

$$G_{ij}^0 = (\pi u)^{-N/2} \int_{-\infty}^{\infty} \prod_k^N d\psi_k \times \exp\left(-\frac{1}{u} \sum_k^N \psi_k^2\right) [(\hat{J} + r + i\hat{\psi})^{-1}]_{ji}. \quad (2.10)$$

It is this quantity which is related to the probability of finding one end of the polymer at point  $j$  given that the other end is at point  $i$ . To see this, it is simplest to go to the continuum limit, which may be obtained by diagonalizing  $J_{ij}$  by Fourier transformation. For a simple cubic lattice and near-neighbor coupling of strength  $J$ , the Fourier transform is  $g_q = J(3 - q^2 a^2 / 2)$  for small  $(qa)$ . For  $J < 0$  (ferromagnetic coupling) that is also the Fourier transform of  $3J - [\frac{1}{2}(|J|a^2)] \nabla^2$  and hence  $[(J + r + i\hat{\psi})^{-1}]_{ji}$  becomes a propagator from point  $\vec{r}_i$  to  $\vec{r}_j$  (the locations of lattice points  $i$  and  $j$ ).

This is essentially the form obtained by Edwards.<sup>12</sup> His result was actually expressed as a Feynman path integral<sup>13</sup> for the Laplace transform with respect to  $(r+3J)$  of the propagator:

$$\begin{aligned} \langle \vec{r}_j | \exp \left[ -t \left( -\frac{|J|a^2}{2} \nabla^2 + i\psi \right) \right] | \vec{r}_i \rangle \\ = \mathfrak{N} \int_{\vec{R}(0)=\vec{r}_i}^{\vec{R}(t)=\vec{r}_j} \delta \vec{R} \exp \left\{ \int_0^t d\tau \right. \\ \left. \times \left[ -\frac{Ja^2}{2} \left( \frac{d\vec{R}}{d\tau} \right)^2 + i\psi(\vec{R}(\tau)) \right] \right\}. \end{aligned} \quad (2.11)$$

Here  $\mathfrak{N}$  is a normalization constant and the integral is over all space-time paths connecting the end points.

Now substitute Eq. (2.11) into the Laplace transform of Eq. (2.10) and the integral over the  $\psi$ 's may be carried out explicitly [going from right to left in Eq. (2.1)] to obtain the Laplace transform  $\mathfrak{g}_{ji}^0$  of  $G_{ji}^0$ :

$$\begin{aligned} \mathfrak{g}_{ji}^0 = \mathfrak{N} \int_{\vec{R}(0)=\vec{r}_i}^{\vec{R}(t)=\vec{r}_j} \delta \vec{R} \\ \times \exp \left[ - \int_0^t d\tau \frac{|J|a^2}{2} \left( \frac{d\vec{R}(\tau)}{d\tau} \right)^2 \right. \\ \left. - \frac{u}{4} \int_0^t d\tau_1 d\tau_2 \delta(\vec{R}(\tau_1) - \vec{R}(\tau_2)) \right]. \end{aligned} \quad (2.12)$$

But this is precisely the probability of finding one end of a polymer of length  $t$  at point  $\vec{r}_j$  given that the other end is at  $\vec{r}_i$ , in the continuum limit.<sup>12</sup> We have a  $\delta$ -function potential between monomers but this is supposed to be inessential in the excluded volume problem and it is in any case simple to extend the derivation to an arbitrary interaction.

As pointed out by de Gennes,<sup>7</sup> this identity enables us to use the evaluation of critical exponents<sup>1</sup> in powers of  $(4-d)$  for arbitrary  $n$ , to find how the root mean square length of a chain grows with the number of links.

### III. MORE-GENERAL HAMILTONIANS

In this section, it will be shown how the method may be extended to include a rather wide class of Hamiltonians. As one illustration we shall show that, in special cases, there are Gaussian exponents and correlation functions for  $n$  a negative even integer, as pointed out by Fisher.<sup>6</sup> The main application will be to derive a general relationship between  $n=0$  and random models and  $n=\infty$  and constrained models. This will be described in Sec. IV.

The general form of Hamiltonian which will be considered is

$$\begin{aligned} \mathcal{H} = \sum_{\alpha}^n \left\{ h_m[\vec{S}_{i\alpha}] - \sum_i^N \vec{H}_i \cdot \vec{S}_{i\alpha} \right\} \\ + \sum_i^N f \left( \sum_{\alpha}^n g_p(\vec{S}_{i\alpha}) \right). \end{aligned} \quad (3.1)$$

Here, the variables at each site are the  $n$  vectors  $\vec{S}_{i\alpha}$ ,  $\alpha=1, 2, \dots, n$ , each with  $m$  components. The Hamiltonian  $h$  replaces the  $rS_{i\alpha}^2$  term in Eq. (1.1) and is quite arbitrary. It contains the sum over sites  $i$  but *not* the sum over  $\alpha$ . A particular example to keep in mind would be

$$h_m(\vec{S}_{i\alpha}) = \sum_{i,j}^N J_{ij} \vec{S}_{i\alpha} \cdot \vec{S}_{j\alpha} + \sum_i^N g_0(\vec{S}_{i\alpha}^2), \quad (3.2)$$

but this is not essential.

It is not assumed that  $h_m(\vec{S}_{i\alpha})$  is rotationally invariant in the subspace of the  $\vec{S}_{i\alpha}$  vectors. The Gaussian free energy  $\Phi_G$  of Eqs. (2.2) and (2.4) will be replaced by the free energy of a general  $m$ -component system related to  $h_m(\vec{S}_{i\alpha})$  and it could be an Ising,  $X$ - $Y$ , or Heisenberg model, etc. The function  $f[\sum_{\alpha}^n g_p(\vec{S}_{i\alpha})]$  in Eq. (3.1) replaces  $(\frac{1}{4}u) \times (\sum_{\alpha}^n \vec{S}_{i\alpha}^2)^2$  in Eq. (1.1) and it is this generalization which requires a change in derivation because the integral representation (2.1) is no longer appropriate. It is possible to have an arbitrary number of variables  $\sum_{\alpha}^n g_p(\vec{S}_{i\alpha})$  for  $f$ , and this is necessary for considering  $n \rightarrow \infty$  limits for certain kinds of coupled order parameter models. We shall describe the method for one variable ( $b=1$ ) for which the equations are less cumbersome. The extension will be obvious. Without loss of generality, it will be assumed that  $f(0)=0$ .

A magnetic field also has been included in Eq. (3.1). The vector  $\vec{H}_i$  is assumed to have all components equal to  $H_i$ , which would be a constant (independent of  $i$ ) for a uniform field but site dependent for a staggered field. Notice that the field is applied in the  $(1, 1, 1, \dots, 1)$  direction. This is different from the usual convention of applying  $n^{1/2}H$  to one of the order parameters. The two are equivalent if  $h_p(\vec{S}_{i\alpha})$  and  $g_p(\vec{S}_{i\alpha})$  are quadratic in  $\vec{S}_{i\alpha}$  (i.e.,  $\Phi$  is a Gaussian) but the form used in Eq. (3.1) is essential in the general case for large  $n$ , as will be seen in Sec. IV.

As examples of Eq. (3.1) which may arise in practice, the case  $m=1$ ,  $g_0(S_{i\alpha}) = rS_{i\alpha}^2 + vS_{i\alpha}^4$ ,  $f$  and  $g_1$  quadratic leads to the fourth-order cubic symmetry Hamiltonian discussed by Wallace,<sup>14</sup> Aharony,<sup>15</sup> Rudnick<sup>16</sup> and by Grinstein and Luther<sup>8</sup> using the renormalization-group method. Such a form is relevant for displacive phase transitions. Various functions  $f[\sum_{\alpha}^n g(\vec{S}_{i\alpha})]$  may be obtained from discrete spin Ising models by making Gaussian integral representations of the exchange term.<sup>17</sup>

When  $f$ ,  $g_1$ , or  $h$  depend upon a parameter  $\Delta$  it may be possible to move from second- to first-order phase transitions, by varying  $\Delta$ , thus producing tricritical points. Many kinds of coupled order parameter models may be described by Eq. (3.1).

Apart from the last term, the Hamiltonian in Eq. (3.1) is a sum over  $\alpha$ , and in order to achieve the extension of the results of Sec. II, it is necessary to make an integral representation of  $f[\sum_{\alpha} g_1(\vec{S}_{i\alpha})]$ . This is achieved by writing

$$\begin{aligned} & \exp\left[-f\left(\sum_{\alpha} g_1(\vec{S}_{i\alpha})\right)\right] \\ &= \int_{-\infty}^{\infty} dt_i e^{-f(t_i)} \delta\left(t_i - \sum_{\alpha} g_1(\vec{S}_{i\alpha})\right) \\ &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt_i \int_{-i\infty}^{i\infty} d\psi_i \\ & \times \exp\left[-f(t_i) + \psi_i\left(t_i - \sum_{\alpha} g_1(\vec{S}_{i\alpha})\right)\right]. \end{aligned} \quad (3.3)$$

Now if Eq. (3.1) is substituted into Eq. (1.2) and the representation (3.3) is used, the resulting integral again factors into a product over  $\alpha$ , giving  $n$  identical integrals just as in Eq. (2.3) and the  $n$ th power of a partition function  $e^{-\Phi_m[\psi_i]}$  analogous to Eq. (2.4):

$$\begin{aligned} Z_n &= (2\pi i)^{-N} \int_{-\infty}^{\infty} \prod_i dt_i \int_{-i\infty}^{i\infty} \prod_i d\psi_i \\ & \times \exp\left(-\sum_{i=1}^N [f(t_i) - t_i \psi_i] - n\Phi_m[\psi_i]\right), \end{aligned} \quad (3.4)$$

where

$$\begin{aligned} e^{-\Phi_m[\psi_i]} &= \int_{-\infty}^{\infty} \prod_i d\vec{S}_i \\ & \times \exp\left(-h_m(\vec{S}_i) + \sum_i [\vec{H}_i \cdot \vec{S}_i - \psi_i g_1(\vec{S}_i)]\right). \end{aligned} \quad (3.5)$$

Once again  $\Phi_m[\psi_i]$  is independent of  $n$ , which is now just a parameter in Eq. (3.4).

The form of Sec. II may be obtained by taking  $f(t_i) = (\frac{1}{4}u)t_i^2$ . Then the  $t_i$  integrals in Eq. (3.4) may be carried out<sup>18</sup> to give  $\exp[-(1/u)\sum_i \psi_i^2]$  as a weight for the  $\psi_i$  integrals as in Eq. (2.2). [Note that  $\psi_i$  is integrated along the imaginary axis in Eq. (3.4) and along the real axis in Eq. (2.2)].

Equations (3.4) and (3.5) are the principal results of this section, and they will be used in Sec. IV to discuss the  $n \rightarrow 0$  and  $n \rightarrow \infty$  limits. Before doing this, we give an example of a more special application of these formulas. Suppose  $h_m$  is given by Eq. (3.2) with  $g_0 = \nu \vec{S}_{i\alpha}^2$ , and  $f$  is an arbitrary

function but  $g_1 = \vec{S}_{i\alpha}^2$ . Then, from Eqs. (3.2) and (3.5),  $\Phi_m[\psi_i]$  is a Gaussian free energy given by Eq. (2.5) for  $\vec{H}_i = 0$  and  $m = 1$ . In this case, it is possible to show that, for some  $f(t_i)$ , the free energy and correlation functions are Gaussian at  $n = -2, -4, -6, \dots$ . The proof is similar to that of Sec. II A. As in Eq. (2.6), when  $n = 2p$ ,  $\exp(2p\Phi_G)$  is the  $p$ th power of a determinant and is a polynomial of order  $p$  in each  $\psi_i$ . For a factor  $\psi_i^q$ , Eq. (3.4) contains

$$\begin{aligned} & \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt_i \int_{-i\infty}^{i\infty} d\psi_i \exp[-f(t_i) + t_i \psi_i] \psi_i^q \\ &= \int_{-\infty}^{\infty} dt_i e^{-f(t_i)} \frac{d^{(q)}}{dt_i^q} \delta(t_i) \\ &= (-1)^q \left[ \frac{d^{(q)}}{dt_i^q} e^{-f(t_i)} \right]_{t_i=0}. \end{aligned} \quad (3.6)$$

Thus, if the  $p$ th derivative of  $\exp[-f(t_i)]$  vanishes, only the zeroth-order term in the expansion of  $\exp[-n\Phi_G(\psi_i)]$  is nonvanishing and  $\psi_i$  may be deleted from  $\Phi_G(\psi_i)$  and

$$\begin{aligned} F_{-2p} &= \frac{1}{2p} \ln Z_{-2p} \\ &= \Phi_G(0). \end{aligned} \quad (3.7)$$

That is, the free energy is Gaussian. As in Sec. II, differentiation with respect to  $J_{ij}$  shows that the correlation functions also are Gaussian. An example of an  $f(t_i)$  satisfying this requirement is a power series whose lowest power is  $t_i^{p+1}$ . This result was first obtained by Fisher.<sup>6</sup>

One other important special case is  $\exp[-f(t)] = \delta(t-n)$ ,  $g_1(\vec{S}_i) = \vec{S}_i^2$ , and  $h_m$  given by Eq. (3.2) with  $g_0 = 0$  and  $m = 1$ , which gives Stanley's  $n$ -vector model.<sup>10</sup> In this instance, the  $t_i$  integral in Eq. (3.4) may be carried out and the weight function for the  $\psi_i$  integrals is  $\exp(\sum_i n i \psi_i)$ . This model cannot be continued to  $n \leq 0$  since the  $f(t)$  factor no longer ensures convergence. It is also less flexible than the general form. For large  $n$ , however, it does give the same expansion of exponents in powers of  $n^{-1}$  as Eq. (1.1) and it is identical to the spin- $\frac{1}{2}$  Ising model for  $n = 1$ . The large  $n$  expansion to be derived in Sec. IV may be adapted to give Abe's formal result,<sup>9</sup> as a special case.

The extension of Eqs. (3.3), (3.4), and (3.5) to the situation where  $f[\sum_{\alpha} g_p(\vec{S}_{i\alpha})]$  is a function of several variables (i.e.,  $p \neq 1$ ) is straightforward—it is merely necessary to introduce a  $\delta$ -function factor as in Eq. (3.3), for each of the variables.

#### IV. RANDOM MODELS ( $n=0$ ) AND LARGE- $n$ LIMIT

In this section we shall consider two limits,  $n \rightarrow 0$  which will lead to various random models<sup>8</sup> and  $n \rightarrow \infty$  which gives the corresponding constrained models. For large but finite  $n$ , it will be shown that critical properties and expansions in  $n^{-1}$  may be obtained directly without use of the renormalization-group method, or perturbation theory.

##### A. Random models

From Eq. (3.4), the free energy per particle becomes, in the limit  $n \rightarrow 0$ ,

$$\begin{aligned} F_0 &= -\lim_{n \rightarrow 0} \frac{1}{n} Z_n \\ &= (2\pi i)^{-N} \int_{-\infty}^{\infty} \prod_i^N dt_i \int_{-\infty}^{\infty} \prod_i^N d\psi_i \\ &\quad \times \exp \left( - \sum_i^N [f(t_i) - t_i \psi_i] \right) \Phi_m(\psi_i). \end{aligned} \quad (4.1)$$

From Eqs. (3.5) and (4.1), it can be seen that  $F_0$  is the free energy of a random  $m$ -component model with a contribution  $[-\psi_i g_1(\tilde{S}_i)]$  at each site  $i$  whose distribution is defined by  $f(t_i)$ . This is appropriate for a system with random impurities and the critical exponents may be obtained from those of the corresponding  $mn$ -component model by setting  $n=0$ . The connection between  $n=0$  and random models was derived by Grinstein and Luther<sup>8</sup> order by order in perturbation theory, and they have used this to obtain critical exponents for a number of random models by using an expansion in powers of  $(4-d)$ .

Evidently this derivation of Eq. (4.1) is much simpler than perturbation theory. Moreover, the result clearly can be generalized. For disordered systems, it is necessary to average the free energy, although averaging the partition function is often simpler. It is clear that by reversing the derivation of Eq. (4.1) one can always average the partition function of an  $n$ -component model, discuss the properties for arbitrary  $n$  and finally let  $n \rightarrow 0$ . This is true even when the averaging is not of the form given in Eq. (4.1), for example, if parameters in the Hamiltonian are random.

The excluded-volume problem is not random in the same sense as the  $m$ -component models of Eq. (4.1) and its relationship to  $n=0$  systems is more special. In the excluded-volume problem, the  $\delta$ -function interaction in Eq. (2.12) has been replaced by a random-single-particle potential as in Eqs. (2.10) and (2.11). This can be done for any system with two-body forces.<sup>19</sup> Thus, if  $G_{AB}$  is the correlation function for two quantities  $A$  and  $B$ , it

is always possible to write  $\text{Tr } e^{-\beta n G_{AB}} = Z G_{AB}$  as the average of  $Z_M G_{AB}^M$  for some random model. This is not the average of the correlation function  $G_{AB}^M$  because of the factor  $Z_M$ , but in special cases  $Z_M G_{AB}^M$  may be the correlation function for a different model. This is what happens in the excluded-volume problem.

##### B. Large- $n$ limit

The representation (3.4) makes it quite straightforward to determine  $Z_n$  for large  $n$ , since the integrals over  $t_i$  and  $\psi_i$  may be evaluated by the method of steepest descents. We shall find the first two terms in the expansion in powers of  $n^{-1}$ . In leading order, the free energy is that of a general constrained  $m$ -component system. When the unconstrained model is solvable, the freedom to choose  $f(t)$  leads to a wide variety of solvable models. These are of particular interest for tricritical phenomena. Special cases of the results which we shall obtain have been derived by the renormalization-group method or by studying every order of perturbation theory,<sup>1,2,4</sup> but the integral representation method is much simpler to use in general.

In order for the large  $n$  limit to be well defined, it is necessary that

$$f(n\tau) = n\varphi(\tau), \quad (4.2)$$

with  $\varphi(\tau)$  independent of  $n$  for large  $n$ . For example, for the fourth-order Hamiltonian (1.1),  $f(t) = (\frac{1}{4}u)t^2$  and it is necessary that  $nu$  is finite as  $n \rightarrow \infty$ , as is usually assumed. Equation (4.2) is the generalization of this condition. To exploit this equation, write  $t_i = n\tau_i$  in Eq. (3.4) and obtain

$$\begin{aligned} Z_n &= \left( \frac{n}{2\pi i} \right)^N \int_{-\infty}^{\infty} \prod_i^N d\tau_i \int_{-\infty}^{\infty} \prod_i^N d\psi_i \\ &\quad \times \exp \left( -n \sum_i^N [\varphi(\tau_i) - \tau_i \psi_i] - n\Phi_m(\psi_i) \right). \end{aligned} \quad (4.3)$$

Now the saddle-point equations are obtained by maximizing the exponent:

$$\psi_i = \varphi'(\tau_i), \quad (4.4)$$

$$\tau_i = \frac{\partial \Phi_m}{\partial \psi_i}, \quad (4.5)$$

and  $\tau_i$  can be eliminated at once by substituting Eqs. (4.5) into Eqs. (4.4) to obtain

$$\psi_i = \varphi' \left( \frac{\partial \Phi_m}{\partial \psi_i} \right). \quad (4.6)$$

The procedure now is to solve Eq. (4.6) for the  $\psi_i$  and then to find the  $\tau_i$  from Eq. (4.5). These equations have a solution with  $\psi_i = \psi$  and  $\tau_i = \tau$  for all  $i$  and it will be assumed that this gives the maxi-

mum.<sup>20</sup> If there are two such solutions with equal values of the exponent, there is a first-order phase transition and there may be a two-phase region with  $(\psi, \tau)$  taking different values in different regions of space. When  $\psi$  and  $\tau$  are known, the free energy per degree of freedom  $F_n$ , is given by the exponent in (4.3) when  $n \rightarrow \infty$ ;

$$F_\infty = N[\varphi(\tau) - \tau\psi] + \Phi_m(\psi) \quad (4.7)$$

and this is exact. Since the right-hand side is stationary with respect to variations of  $\tau$  and  $\psi$ , the magnetization per degree of freedom is

$$M = - \left( \frac{\partial \Phi_m}{\partial H} \right)_\psi. \quad (4.8)$$

Here,  $H_i = H = \text{constant}$  has been used in Eq. (3.5). Thus  $M$  is the magnetization for the background model, which may be Ising for  $m=1$ , or  $X$ - $Y$  for  $m=2$ , etc. It is now clear why it was necessary to take  $H$  in the  $(1, 1, 1, \dots, 1)$  direction in Eq. (3.1). With the more usual convention of taking a field  $n^{1/2}H$  coupled to only one of the  $\tilde{S}_i^\alpha$  at each site, the magnetization would have been  $n^{-1}$  times the magnetization of  $\Phi_m$  in a field  $n^{1/2}H$ . For  $n \rightarrow \infty$ , this would give the large  $H$  limit of the magnetization, which is not proportional to  $n$  in general. The Gaussian model is an exception to this since  $\Phi_G$  is exactly quadratic in  $H$ , and either assumption for the field gives the same result.

Equations (4.4) and (4.5) are equations of constraint on  $\Phi_m$  and, as shown by Fisher,<sup>21</sup> they lead to renormalized critical exponents. If  $\alpha_m, \gamma_m, \nu_m$  are the exponents of  $\Phi_m$  for the specific heat, susceptibility and coherence length, respectively, then, if  $H=0$  and the constraint does not induce singularities, the corresponding exponents of  $F_\infty$  are

$$\begin{aligned} \alpha &= \frac{-\alpha_m}{1 - \alpha_m}, \\ \gamma &= \frac{\gamma_m}{1 - \alpha_m}, \\ \nu &= \frac{\nu_m}{1 - \alpha_m}. \end{aligned} \quad (4.9)$$

This is true<sup>22</sup> for any  $m$ -component system with  $\alpha_m > 0$ . If  $\Phi_m$  were Gaussian, the renormalized exponents would be those of the spherical model, as obtained by Stanley,<sup>10</sup> Ferrel and Scalapino,<sup>2</sup> Ma,<sup>3</sup> and by Abe.<sup>9</sup> When  $\Phi_m$  is an Ising free energy, Eq. (4.9) gives the results obtained by Aharony<sup>15</sup> and by Rudnick.<sup>16</sup>

It should be emphasized that the derivation of Eq. (4.7) for  $F_\infty$  is not restricted to  $T > T_c$  and that it also works for first-order phase transitions. Thus, if either  $h(\tilde{S}_{i\alpha})$  or  $f(t)$  depends upon some

parameter  $\Delta$ , such as the crystal-field splitting in the spin-1 model of Blume, Emery, and Griffiths,<sup>23</sup> the system may have a tricritical point and hence a first-order phase transition region. The value of Eqs. (4.4), (4.5), and (4.7) is that they are valid for the entire phase diagram and may be used to study the approach to the tricritical point from all sides. Applications to the BEG model<sup>23</sup> and metamagnets will be described in another publication.<sup>11</sup>

### C. Order $n^{-1}$ corrections

We now consider corrections to  $F_\infty$  when  $n$  is finite. To obtain order  $n^{-1}$ , it is necessary to expand the exponent in Eq. (4.3) about the saddle point. Conditions (4.4) and (4.5) ensure that first-order terms vanish and, to second order, the exponent is

$$\begin{aligned} E &= -nF_\infty - n \sum_i \left( \frac{1}{2} \varphi''(\tau) (\tau_i - \tau)^2 \right. \\ &\quad \left. - (\tau_i - \tau)(\psi_i - \psi) \right) + \frac{n}{2} \sum_{i,j} C_{ij} (\psi_i - \psi)(\psi_j - \psi), \end{aligned} \quad (4.10)$$

where

$$\begin{aligned} C_{ij} &= - \left( \frac{\partial^2 \Phi_m}{\partial \psi_i \partial \psi_j} \right)_0 \\ &= \langle g_1(\tilde{S}_i) g_1(\tilde{S}_j) \rangle_0 - \langle g_1(\tilde{S}_i) \rangle_0 \langle g_1(\tilde{S}_j) \rangle_0. \end{aligned} \quad (4.11)$$

Here the subscripts zero indicate thermal averages taken in the  $m$ -component distribution of Eq. (3.5) with all  $\psi_i$  set equal to  $\psi$ .

If Eq. (4.10) is substituted into Eq. (4.3) the resulting Gaussian integral may be evaluated to give

$$\begin{aligned} F_n &= -\frac{1}{n} \ln Z_n \\ &= F_\infty + \frac{1}{2n} \text{Tr} \ln [1 + \varphi''(\tau) \hat{C}], \end{aligned} \quad (4.12)$$

where  $\hat{C}$  is the matrix with elements  $C_{ij}$  given in Eq. (4.11). Equation (4.12) is correct to order  $n^{-1}$ . Higher orders in  $n^{-1}$  may be obtained systematically by including higher powers of  $(\psi_i - \psi)$  and  $(\tau_i - \tau)$  in Eq. (4.10) and expanding the exponential in Eq. (4.3). In the special case that  $\Phi_m$  is Gaussian and  $g_1(\tilde{S}_i)$  is quadratic in the  $\tilde{S}_i$ , it can be seen that  $C_{ij}$  is the  $\tilde{S}_i^2 \tilde{S}_j^2$  correlation function, and Eq. (4.12) gives the result of Ferrel and Scalapino,<sup>2</sup> who showed that, in that case,  $F_\infty$  is given by Hartree theory and the order  $n^{-1}$  correction is the screening approximation. It is not surprising that these two approximations are obtained by introducing the integral representation (2.1) and making the approximation (4.12) since it was shown by Hubbard<sup>19</sup>

that this is true for a fourth-order quantum mechanical Hamiltonian. In that case, the field did not have many components and the approximation required a different justification.

The extension of all of these results to a situation where  $f$  is a function of several variables is straightforward. It merely involves the introduction of an equivalent number of sets of variables  $t_i$ .

Equation (4.12) may be used to obtain critical exponents to order  $n^{-1}$ . A detailed evaluation for the case that  $\Phi_m$  is Gaussian and  $g_1(\vec{S}_i)$  is quadratic has been given by Ferrel and Scalapino<sup>2</sup> and by Abe.<sup>9</sup> We conclude by indicating how the derivation of the specific-heat exponent  $\alpha$  would go in general and showing what has to be known about  $\Phi_m$ . For a translationally invariant system,  $C_{ij}$  may be diagonalized to  $C(\vec{q})$  by Fourier transformation and, using this in Eq. (4.12) after differentiation with respect to  $T$  to get the entropy  $S$ , we obtain

$$S_n = S_\infty - \frac{1}{2n} \sum_{\vec{q}} [1 + \varphi''(\tau)C(\vec{q})]^{-1} \times \frac{\partial}{\partial T} [\varphi''(\tau)C(\vec{q})]. \quad (4.13)$$

From Eqs. (4.9), the nonanalytic part of  $S_\infty$  is of the form  $A(T - T_{c\infty})^{1/(1-\alpha_m)}$  when  $T$  is close to the transition temperature  $T_{c\infty}$  for  $n \rightarrow \infty$ .

We now assume a scaling form  $C(\vec{q}) = q^\lambda \gamma(q/\kappa)$  for  $q$  small and  $T \approx T_{c\infty}$ , where  $\kappa$  is the inverse coherence length,  $\gamma$  is finite at  $q=0$ , and the most important temperature dependence resides in  $\kappa$ . Then, for  $\lambda < 0$ , and small  $q$ ,  $\varphi''(\tau)C(q) \gg 1$ , and since  $\varphi''(\tau)$  is assumed not to have a strong temperature dependence,

$$S_n = A(T - T_{c\infty})^{1/(1-\alpha_m)} + \frac{1}{2n} \sum_{\vec{q}} \frac{\gamma'(q/\kappa)}{\gamma(q/\kappa)} \frac{q}{\kappa^2} \frac{d\kappa}{dT}. \quad (4.14)$$

If this sum is turned into an integral and  $q$  is set equal to  $Q\kappa$ , the last term in Eq. (4.14) becomes  $\kappa^{d-1} d\kappa/dT$  times an integral which usually has a  $\log \kappa$  divergence. Then, using the scaling law  $2 - \alpha_m = d\nu_m$  together with Eqs. (4.9), it follows that  $\kappa^{d-1}(d\kappa/dT)\log \kappa$  is proportional to

$$|T - T_{c\infty}|^{1/(1-\alpha_m)} \log |T - T_{c\infty}|$$

and Eq. (4.14) gives the first term in the expansion of the exponent  $\alpha$  in powers of  $n^{-1}$ :

$$1 - \alpha = 1/(1 - \alpha_m) + B/n. \quad (4.15)$$

In general, it is necessary to know both the constant  $A$  and some details of  $\gamma(q/\kappa)$  in order to evaluate  $B$ . These are known when  $\Phi_m$  is Gaussian and were used by Ferrel and Scalapino<sup>2</sup> and by Abe.<sup>9</sup>

The coefficient  $B$  might also be evaluated when  $\Phi_m$  is the two-dimensional Ising model. By making a Gaussian integral representation<sup>17</sup> of the exchange term, it can be rewritten as a continuous-spin Ising model of the form (3.5) with  $m=1$  and  $h_m$  given by Eq. (3.2) with  $g_0 = \ln 2 \cosh 2S_i$ . If  $g_1(\vec{S}_i) = \vec{S}_i^2$ , this term in Eq. (3.5) can formally be absorbed into the exchange integral, and the free energy found by using Onsager's solution.<sup>24</sup>

## V. CONCLUSION

We have shown that, for a very wide class of models, the use of integral representations considerably simplifies the discussion of the properties of the system as a function of  $n$ . It enables us to give a simple and coherent derivation of a number of exact results without use of perturbation theory or the renormalization-group method and to obtain a direct derivation of the free energy as a series in  $n^{-1}$  which is valid at all temperatures and for first- and second-order transitions. It is clear that this method has a number of potential applications and, in particular, tricritical phenomena, coupled-order-parameter models, and certain dynamical properties will be considered in forthcoming publications.<sup>11</sup>

*Note added in proof.* After this paper had been accepted for publication, Dr. M. E. Fisher informed me that he had derived independently the large  $n$  results of Sec. IV, using central limit theorems.

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