Statistical mechanics of Ginzburg-Landau fields for weakly coupled chains*

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The free energy of a Ginzburg-Landau field describing a system of weakly coupled chains in a plane is identified with the ground-state energy of a linear array of quantum-mechanical anharmonic oscillators. The equivalent Hamiltonian is simplified for both real and complex fields using a truncated basis of states of the uncoupled oscillators. For the real field, the reduced Hamiltonian is solved, and the system is shown to have a logarithmic divergence in the specific heat similar to the anisotropic, two-dimensional Ising model.

I. INTRODUCTION

In this paper we consider the statistical mechanics of a system of interacting chains laid out in a plane. Our purpose is to apply what is known about one-dimensional systems to the study of phase transitions in two dimensions. Each of the chains is described by a continuous order parameter $\psi_i(x)$ which will be taken as real or complex. The energy of a given configuration for each chain is assumed to be given by a Ginzburg-Landau ψ^4 functional[1] and the interchain-coupling energy density is proportional to $|\psi_{i+1} - \psi_i|^2$. The canonical partition function is then a functional integral over all $\psi_i(x)$ fields.

In one dimension a transfer-matrix technique allows the functional integral to be reduced to the problem of finding the lowest eigenvalue of an equivalent Hamiltonian. This problem has been solved numerically for the cases of real and complex ψ fields.[2]

In one dimension, the equivalent Hamiltonian describes the quantum mechanical problem of an anharmonic oscillator. For the coupled chains, it will turn out that the equivalent eigenvalue problem corresponds to an array of linearly coupled anharmonic oscillators. The ground-state energy of the array is proportional to the free energy of the two-dimensional statistical-mechanical problem.

We simplify this problem for weakly coupled chains by expressing the Hamiltonian in a truncated basis of states made up of only the low-lying states of the individual oscillators. This is reasonable as long as the coupling energy is small enough to mix higher states only weakly. For the real field, only two states are important, and a fermion representation[3] can be used to solve the problem. The result is very similar to the anisotropic two-dimensional Ising[4] model, as one would expect. For the complex field, the important states can be expressed as eigenfunctions of a plane rotor, or with further truncation, as states of a spin-one system, but we have not been able to solve the resulting problem. Nevertheless, we believe that this type

of formulation gives insight into the problem of a complex order parameter in two dimensions. It also suggests alternative approximate solutions such as variational calculations for the ground state which we are presently investigating.

II. FORMULATION OF THE PROBLEM

We consider a system of M chains of length L. The value of the order parameter at point x on the jth chain is represented by $\psi_j(x)$. Then the energy functional is given by

$$E[\psi] = \sum_{j=1}^{M} \int_0^L dx \left(a|\psi_j|^2 + b|\psi_j|^4 \right.$$
$$\left. + c_x \left|\frac{d\psi_j}{dx}\right|^2 + c_y|\psi_{j+1} - \psi_j|^2 \right), \quad (2.1)$$

where $\psi_{M+1} \equiv \psi_1$. This functional is like a free energy in that it contains the entropy contributions of all the degrees of freedom of the system except those associated with the ψ field. That is, we imagine that we have already integrated all the other degrees of freedom out of the canonical partition function. As a result, the coefficients of Eq. (2.1) may depend on the temperature; in particular, the important temperature dependence in the problem is in the parameter a, which vanishes linearly at the mean-field transition temperature T_{c0}.

$$a = a'(T - T_{c0})/T_{c0} . \quad (2.2)$$

The canonical partition function for the system is then given by

$$Z = \int \delta\psi \, e^{-\beta E[\psi]} , \quad (2.3)$$

where the symbol $\int \delta\psi$ means "sum over all possible functions ψ," and $\beta = 1/kT$. The method by which the functional integral is given precise mathematical meaning is discussed by various authors.[5] Here we will evaluate the functional integral by solving an equivalent eigenvalue problem. The procedure for treating functional integrals, used by Feynman to generate Schrödinger's equation from an "integral over paths," is a continuum general-

ization of the well-known transfer-matrix technique. It was used by Scalapino, Sears, and Ferrell[2] to generate the equivalent eigenvalue problem for a single chain; the extension to many chains is straightforward, and we will simply state the result.

The partition function is given by

$$Z = e^{-\beta F} = \sum_n e^{-E_n L} , \qquad (2.4)$$

where F is the free energy, and the E_n's are the eigenvalues of the equivalent Hamiltonian

$$H = \sum_{j=1}^{M} \left(-\frac{1}{4\beta c_x} \frac{\partial^2}{\partial \psi_j^2} + \beta a |\psi_j|^2 \right.$$
$$\left. + \beta b |\psi_j|^4 + \beta c_y |\psi_{j+1} - \psi_j|^2 \right) . \qquad (2.5)$$

Here the ψ_j's are independent variables and not functions. Again, $\psi_{M+1} \equiv \psi_1$. If ψ_j is complex, then $\partial^2/\partial\psi_j^2$ means $\partial^2/\partial(\text{Re}\psi_j)^2 + \partial^2/\partial(\text{Im}\psi_j)^2$.

We see from Eq. (2.4), that in the thermodynamic limit ($L \to \infty$), the free energy is dominated by the lowest eigenvalue of the equivalent Hamiltonian. This is analogous to the treatment of a discrete lattice in which the free energy is dominated by the largest eigenvalue of a transfer matrix.

III. FERMION REPRESENTATION FOR THE REAL FIELD

When ψ is real, the Hamiltonian in Eq. (2.5) is equivalent to the quantum-mechanical problem of M linear anharmonic oscillators coupled by springs. We therefore discuss the problem in the language of quantum mechanics. The problem of a single such oscillator has been studied in detail by Scalapino, Sears, and Ferrell.[2] Above the critical region ($a > 0$), the low-lying "energy" levels are dominated by the harmonic term. As the parameter a approaches zero, the anharmonic term becomes more and more important. When a becomes negative, the potential has two minima, and tunneling between the two wells splits the "energies" of the two lowest eigenstates slightly. As a decreases further, the two wells become deeper; tunneling between them and, therefore, the splitting of the two lowest "energy" levels decrease exponentially to zero. At the same time, all of the higher excited states are being driven to higher and higher "energies."

For weakly coupled chains the two-dimensional phase transition will occur in the critical region below T_{c0}. (By weakly coupled, we mean that at the two-dimensional transition, $c_y \ll c_x/\xi^2$, where ξ is the coherence length along one chain.) In this region we know that the two lowest eigenvalues of the uncoupled oscillators are close together, and all of the other eigenvalues are much larger. It seems reasonable, therefore, to diagonalize the full equivalent Hamiltonian in the truncated basis

of states in which each oscillator is restricted to its lowest two states. This should certainly be a good approximation as long as the coupling is weak enough so that the coupling "energy" is small compared to the splitting between the ground state of the uncoupled oscillator and the first state neglected. We shall show that these conditions are consistent with the results.

Since only two states are to be considered for each oscillator, they can be represented by the presence or absence of a fermion. We define fermion operators $c_j^\dagger$ and c_j in the usual way and construct an effective Hamiltonian which will have all the same matrix elements as the original Hamiltonian in the truncated basis of states. In that truncated basis, the only nonvanishing matrix elements of ψ_n are those which couple opposite states of the nth oscillator and the same state for each other oscillator.

The effective Hamiltonian can be written

$$H_{\text{eff}} = \sum_{j=1}^{M} \{ \epsilon c_j^\dagger c_j + \Delta [1 - (c_j^\dagger - c_j)$$
$$\times (c_{j+1}^\dagger + c_{j+1})] \} , \qquad (3.1)$$

where ϵ is the splitting between the two levels of a single oscillator and Δ is given by

$$\Delta = 2\beta c_y |\langle 1 | \psi | 0 \rangle|^2 = 2\beta c_y \langle 0 | \psi^2 | 0 \rangle , \qquad (3.2)$$

with the matrix element evaluated between states of a single oscillator. The Hamiltonian of Eq. (3.1) has a temperature-dependent zero point which can be removed by adding ME_0 to it. As it stands, it is very similar, but not identical, to the Hamiltonian treated by Schultz, Mattis, and Lieb[3] in their solution of the two-dimensional Ising model.

Periodic boundary conditions require different definitions of c_{M+1}, $c_{M+1}^\dagger$ depending on whether H_{eff} acts on a state containing an even or an odd number of particles. We define

$$c_{M+1} \equiv (-1)^{1 + \sum_{i=1}^{M} n_i} c_1$$

and

$$c_{M+1}^\dagger \equiv (-1)^{1 + \sum_{i=1}^{M} n_i} c_1^\dagger . \qquad (3.3)$$

Thus we have two distinct Hamiltonian operators; one, H^{even}, to act on states with an even number of particles; the other, H^{odd}, to act on states of an odd number of particles. It should be noted that the Hamiltonian in Eq. (3.1) does not conserve particle number, but it does preserve the evenness or oddness of the particle number. We must discard eigenstates of H^{even} having an odd particle number and eigenstates of H^{odd} having an even particle number.

This peculiarity (which also occurred in the work of Schultz, Mattis, and Lieb) and the relative minus sign between $c_j^\dagger$ and c_j occur because we are mimicking the matrix elements of commuting quantities (ψ_i, ψ_j) with anticommuting operators $(c_i, c_j^\dagger)$, and we must keep track of the minus signs by hand. The duplicity in the definition of c_{M+1}, $c_{M+1}^\dagger$, and therefore H_{eff}, although it does come from the boundary conditions, will be crucial to what follows.

IV. SOLUTION OF THE FERMION HAMILTONIAN

The method of solution of the Hamiltonian of Eq. (3.1) follows closely Schultz, Mattis, and Lieb,[3] except that they work with a transfer matrix rather than a Hamiltonian.

The first step is to Fourier transform the operators,

$$b_k = \frac{e^{i\pi/4}}{\sqrt{M}} \sum_{j=1}^{M} e^{-ikj} c_j ,$$

$$b_k^\dagger = \frac{e^{-i\pi/4}}{\sqrt{M}} \sum_{j=1}^{M} e^{ikj} c_j^\dagger . \qquad (4.1)$$

The factors $e^{\pm i\pi/4}$ are just a convenience; they make the coefficients in the Hamiltonian real.

For states with even numbers of particles, the allowed values of k are

$$k = \pm \frac{\pi}{M}, \pm \frac{3\pi}{M}, \ldots \pm \frac{(M-1)\pi}{M} , \qquad (4.2a)$$

while for an odd number of particles, the allowed k's are

$$k = 0, \pm \frac{2\pi}{M}, \pm \frac{4\pi}{M}, \ldots \pm \frac{(M-2)\pi}{M}, \pi . \qquad (4.2b)$$

In this representation, the two Hamiltonians are given by

$$H^{\text{even}} = \sum_{k>0} [(\epsilon - 2\Delta \cos k)(b_k^\dagger b_k + b_{-k}^\dagger b_{-k})$$

$$+ 2\Delta + 2\Delta \cos k + 2\Delta \sin k\, b_k^\dagger b_{-k}^\dagger$$

$$- 2\Delta \sin k\, b_k b_{-k}] \qquad (4.3a)$$

and

$$H^{\text{odd}} = (\epsilon - 2\Delta)b_0^\dagger b_0 + 2\Delta + (\epsilon + 2\Delta)b_\pi^\dagger b_\pi$$

$$+ \sum_{0<k<\pi} [(\epsilon - 2\Delta \cos k)(b_k^\dagger b_k + b_{-k}^\dagger b_{-k})$$

$$+ 2\Delta + 2\Delta \cos k + 2\Delta \sin k\, b_k^\dagger b_{-k}^\dagger$$

$$- 2\Delta \sin k\, b_k b_{-k}] . \qquad (4.3b)$$

Each Hamiltonian is now block diagonalized; that is, operators of a given k are coupled only to those of $-k$. We can write

$$H = \sum_k H_k$$

and diagonalize each H_k. It turns out that the ground state of each H_k corresponds to a mixture of the zero particle and two particle states. Thus, the ground state of H^{even} can be constructed by putting each subspace in its ground state and adding up the "energies," i.e., $E^{\text{even}} = \sum_k \epsilon_0(k)$.

The ground state of H^{odd} must contain an odd number of particles. Thus, at least one of the subspaces must contain an odd number of particles and the energetically favored choice is $k = 0$. The ground-state energy of H^{odd} differs from the ground-state energy of H^{even} for two reasons: first, the creation energy of the odd particle; second, the shift in energy of each $\epsilon_0(k)$ due to the difference in the allowed k's. These shifts are each of order $1/M$, but there are M of them, so they contribute in the same order $[0(1)]$ as the particle creation energy.

When all the contributions are worked out the results are

$$E^{\text{even}} = ME_0 + \frac{M}{2}\epsilon + M\Delta - \frac{M}{\pi}(\epsilon + 2\Delta)$$

$$\times \int_0^{\pi/2} dy \left(1 - \frac{8\epsilon\Delta}{(\epsilon + 2\Delta)^2} \sin^2 y\right)^{1/2} , \qquad (4.4a)$$

$$E^{\text{odd}} = E^{\text{even}} + \epsilon - 2\Delta . \qquad (4.4b)$$

E_0 is the ground-state energy of the single oscillator with respect to the zero of Eq. (2.1); it should be included in the result since it is temperature dependent.

We know from the work of Scalapino, Sears, and Ferrell[2] that ϵ is a monotonic increasing function of temperature, and Δ is a monotonic decreasing function of temperature. Thus at high temperatures, E^{even} is the lowest eigenvalue, and at low temperatures E^{odd} is the lowest eigenvalue. Their crossing, $\epsilon = 2\Delta$, defines a transition temperature, T_c. At T_c, the argument of the elliptic integral is unity, and it is a relative maximum in both ϵ and Δ. Figure 1 shows the behavior of ϵ and Δ as functions of temperature and also shows Δ for various values of c_y.

Also, since Δ is the relevant measure of coupling strength, and since at $T_c < T_{c0}$, ϵ is small compared to the spacing between the ground state and first neglected state (second excited level) of the uncoupled oscillators, the coupling near T_c only weakly mixes in higher states, and our truncation of the basis of states is consistent.

The free energy (also E^{even} and E^{odd}) does not have a Taylor series about T_c, but ϵ and Δ do. If we expand them in $t = (T - T_c)/T_c$:

$$\epsilon(t) = \epsilon_0 + \epsilon_1 t + \tfrac{1}{2}\epsilon_2 t^2 + \cdots , \qquad (4.5a)$$

$$\Delta(t) = \Delta_0 + \Delta_1 t + \tfrac{1}{2}\Delta_2 t^2 + \cdots , \qquad (4.5b)$$

then $\epsilon_0 = 2\Delta_0$ and we can substitute the series into

Eq. (4.4a) or (4.4b). Recalling that $\beta F = E^{\text{even}} L$ (above T_c) or $\beta F = E^{\text{odd}} L$ (below T_c), we can differentiate twice with respect to t and examine the result asymptotically as $t \to 0$. Using the asymptotic behavior of elliptic integrals of the second kind,[6] we find that the specific heat is given by

$$C \sim MLk \frac{(\epsilon_1 - 2\Delta_1)^2}{4\pi\Delta_0}(-\ln|t|) + O(1) . \qquad (4.6)$$

V. PLANE ROTOR REPRESENTATION FOR THE COMPLEX FIELD

When ψ is complex, the Hamiltonian in Eq. (2.5) is equivalent to the quantum-mechanical problem of a system of coupled two-dimensional anharmonic oscillators. If we write $\psi = re^{i\varphi}$, then the equivalent eigenvalue problem for a single, uncoupled oscillator is

$$\frac{-1}{4\beta c_x}\left(\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r}\frac{\partial\Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\varphi^2}\right)$$
$$+ \beta ar^2\Psi + \beta br^4\Psi = E\Psi . \qquad (5.1)$$

A trial solution of the form

$$\Psi = (f(r)/\sqrt{r})e^{\pm im\varphi} \qquad (5.2)$$

yields the radial equation,

$$\frac{-1}{4\beta c_x}f''(r) + \left(\beta ar^2 + \beta br^4 + \frac{m^2 - \frac{1}{4}}{4\beta c_x r^2}\right)f(r) = Ef(r) . \qquad (5.3)$$
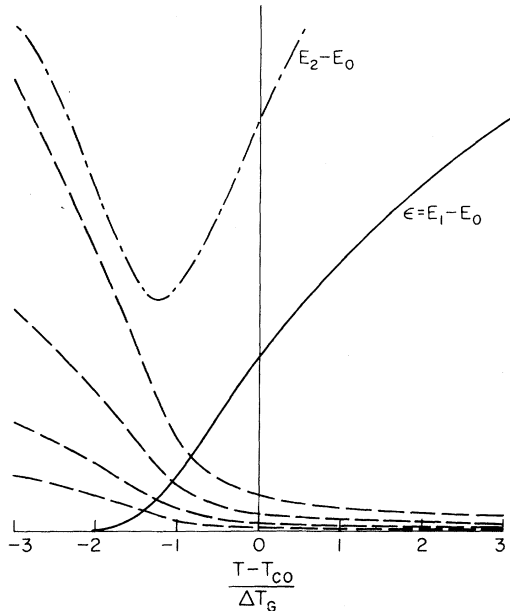

FIG. 1. Solid line shows the temperature dependence of ϵ. Dashed lines show the behavior of 2Δ for various values of c_y. Each curve corresponds to a value of c_y half that of the curve above it. Dashed and dotted curve shows the splitting between the lowest eigenvalue and the second excited state (first neglected state) as a function of temperature.

We need to know which eigenstates of Eq. (5.1) will contribute significantly to the ground state of the full system. We therefore examine the eigenvalue spectrum of Eq. (5.3). For a given value of m, this equation is equivalent to a single quantum-mechanical particle in the effective potential

$$V = \beta ar^2 + \beta br^4 + \frac{m^2 - \frac{1}{4}}{4\beta c_x r^2} . \qquad (5.4)$$

As in the real-field case, the two dimensional ordering of weakly coupled chains should occur at a temperature below T_{c0}. The parameter a will be negative in the region of interest, and the first two terms in the potential of Eq. (5.4) will dominate, at least for reasonably small values of m. We then find the minimum of the potential (to first order in the $m^2 - \frac{1}{4}$ term) by first finding the minimum of $\beta ar^2 + \beta br^4$ and then treating the $m^2 - \frac{1}{4}$ term as a perturbation. This approximation will be consistent as long as

$$\frac{\beta^2|a|^3c_x}{2b^2} \gg m^2 . \qquad (5.5)$$

In order to understand this criterion physically, we introduce the Ginzburg critical region ΔT_G. ΔT_G is the temperature interval around T_{c0} in which the thermal energy kT_{c0} is sufficient to switch a section of the chain, one mean-field coherence length long, from the ordered to the disordered state or vice versa. ΔT_G can be estimated by setting kT_{c0} equal to the mean-field free energy per unit length times the mean-field coherence length, $(c/|a|)^{1/2}$ at $T - T_{c0} = \Delta T_G$. This yields the relation

$$|a|^3\beta_c^2 c_x/b^2 = 16(|T - T_{c0}|/\Delta T_G)^3 . \qquad (5.6)$$

Combining Eqs. (5.6) and (5.5) gives the condition

$$8|(T - T_{c0}/\Delta T_G|^3 \gg m^2 . \qquad (5.7)$$

Thus, as long as the two-dimensional transition temperature is several times ΔT_G below T_{c0}, a reasonable number of the lowest m values will be treated accurately in this approximation.

Expanding the potential about the minimum and using the harmonic approximation, we find the lowest eigenvalues are given by

$$E_n(m) \approx -\frac{\beta a^2}{4b}\left[1 - \frac{m^2 - \frac{1}{4}}{8}\left(\frac{\Delta T_G}{|T - T_{c0}|}\right)^3\right]$$
$$+ (n + \tfrac{1}{2})2\left(\frac{-a}{2c}\right)^{1/2}\left[1 + \frac{3(m^2 - \frac{1}{4})}{32}\left(\frac{\Delta T_G}{|T - T_{c0}|}\right)^3\right] . \qquad (5.8)$$

Here the first term is the value of the potential at the minimum, and the second is the "vibrational energy" reflecting the localization of the wave function about the minimum. We think of n as a radial or vibrational quantum number and m as a rotational quantum number.

Recalling that $a = a'(T - T_{c0})/T_{c0}$, we can make

the temperature dependence of the terms in Eq. (5.8) explicit;

$$E_n(m) \approx -\frac{\beta a'^2}{4b}\left(\frac{\Delta T_G}{T_{c0}}\right)^2\left(\frac{T-T_{c0}}{\Delta T_G}\right)^2$$
$$+\,(n+\tfrac{1}{2})\,2\left[\left(\frac{a'}{2c}\right)^{1/2}\right]\left(\frac{\Delta T_G}{T_{c0}}\right)^{1/2}\left(\frac{|T-T_{c0}|}{\Delta T_G}\right)^{1/2}$$
$$+\,\frac{\beta a'^2}{4b}\,\frac{m^2-\tfrac{1}{4}}{8}\left(\frac{\Delta T_G}{T_{c0}}\right)^2\frac{\Delta T_G}{|T-T_{c0}|}$$
$$+\,(n+\tfrac{1}{2})\,2\left[\left(\frac{a'}{2c}\right)^{1/2}\right]\frac{3(m^2-\tfrac{1}{4})}{32}\left(\frac{\Delta T_G}{T_{c0}}\right)^{1/2}\left(\frac{\Delta T_G}{|T-T_{c0}|}\right)^{5/2}.$$

$$(5.9)$$

The first term represents the depth of the well and it increases as $(T-T_{c0})^2/\Delta T_G^2$ as T decreases. The second term represents the principal vibrational contribution to the eigenvalue, and it increases as $(|T-T_{c0}|/\Delta T_G)^{1/2}$; that is, as T decreases below T_{c0} the "vibrational energy levels" split farther apart. The third term represents the principal rotational contribution. While the vibrational levels are splitting farther apart, the rotational levels are coming together with splittings decreasing as $(|T-T_{c0}|/\Delta T_G)^{-1}$. For $m>0$, the rotational levels are doubly degenerate. The last term falls off rapidly compared to other rotational and vibrational terms and therefore is not important.

In the region which we expect to show a two-dimensional phase transition for weakly coupled chains, the eigenvalue spectrum of Eq. (5.1) should look schematically as shown in Fig. 2. The situation is very similar to the energy-level structure of a diatomic molecule, in which the electronic structure produces an effective potential wherein
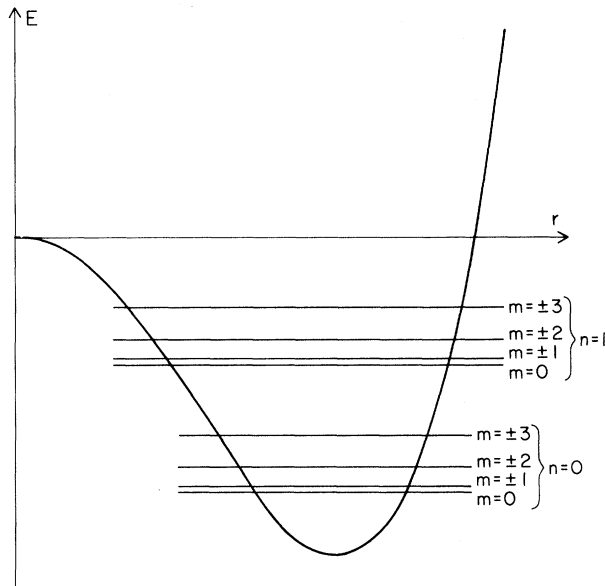

FIG. 2. Schematic diagram of the eigenvalue spectrum of Eq. (5.1).

the vibrational states are widely spaced and rotational states lie close together. As long as the coupling between the oscillators in Eq. (2.5) is weak, the ground state of the whole system will include primarily several rotational states of the lowest vibrational level of the uncoupled oscillator. Higher vibrational states will contribute only weakly, and we can truncate the basis of states to include only the rotational states of the lowest vibrational level ($n=0$). Since the separation of the various rotational states decreases as $|T-T_{c0}|^{-1}$, rather than exponentially, the separation remains finite in the region of interest, and we expect small values of m to be most important, so that the approximations made in arriving at Eq. (5.8) are consistent.

We can regroup the terms of Eq. (5.8) to put the energy in the form

$$E(m) = E_0 + (1/2I)m^2 . \qquad (5.10)$$

In this form, the spectrum is that of a plane rotor with the zero of "energy" set at E_0. Both E_0 and I are temperature dependent.

We approximate the full Hamiltonian of Eq. (2.5) by a system of coupled plane rotors. We write

$$\psi_j = r_0 e^{i\varphi_j} ,$$

with r_0 the minimum of the effective potential for $m=0$. Then the effective Hamiltonian takes the form

$$H_{eff} = ME_0 + \sum_{j=1}^{M}\left\{ -\frac{1}{2I}\frac{\partial^2}{\partial\varphi_j^2} \right.$$
$$\left. +\,2\beta c_y r_0^2[1-\cos(\varphi_{j+1}-\varphi_j)] \right\}. \quad (5.11)$$

Since the spacing between the ground state of the uncoupled oscillator and the lowest states with $m=\pm 2$ is four times the spacing between the ground state and the lowest states with $m=\pm 1$, we expect that all of the important information about the transition will be retained if we further truncate the basis of states to include only the three lowest states ($n=0$, $m=0,\pm 1$). Then the effective Hamiltonian can be written in terms of a system of coupled spins with $S=1$. The effective Hamiltonian becomes

$$H_{eff} = \sum_{j=1}^{M}\left\{ [\epsilon + 2\beta c_y(\rho_1-\rho_0)](s_j^z)^2 \right.$$
$$\left. +\,2\beta c_y\rho_0 - \beta c_y\mu^2(s_{j+1}^x s_j^x + s_{j+1}^y s_j^y) \right\}, \quad (5.12)$$

where ϵ is the splitting between the ground state and the first excited states of the uncoupled oscillators, and ρ_0, ρ_1, and μ are given by

$$\rho_1 = \langle 1|\psi^2|1\rangle , \qquad (5.13a)$$

$$\rho_0 = \langle 0|\psi^2|0\rangle , \qquad (5.13b)$$

$$\mu = \langle 1|\psi|0\rangle . \qquad (5.13c)$$

The numbers in the brackets are the m values of the states.

VI. SUMMARY AND COMPARISON TO THE ISING MODEL

We have treated the statistical mechanics of Ginzburg-Landau fields for a two-dimensional system made up of weakly coupled one-dimensional chains. For both real and complex order parameters, we reduced the problem of finding the free energy to that of solving for the quantum-mechanical ground state at an equivalent Hamiltonian of coupled anharmonic oscillations. Then we used the weakness of the coupling to simplify the problem further by considering only low-lying eigenstates of the uncoupled oscillators. For the complex order parameter, we exhibited the equivalent Hamiltonian in several truncated forms. For the real field, we were able to solve the two-state truncated Hamiltonian.

The free energy of the system is equal to the free energy of the uncoupled chains plus additional terms reflecting the coupling energy [see Eq. (4.4)]. Thus the specific heat shows the large bump which Scalapino, Sears, and Ferrell[2] found for the individual chains. In addition, at a somewhat lower temperature, the weakly coupled chains show a logarithmic singularity coming from the coupling-energy terms. This is very similar to the anisotropic, two-dimensional Ising model,[4] in which the specific heat shows a large gentle hump (like the one-dimensional Ising model) and, at a lower temperature corresponding to two-dimensional ordering, a logarithmic singularity.

ACKNOWLEDGMENTS

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