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Dynamic properties of the multicomponent Bose fluid

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The hydrodynamic properties of an m-component Bose fluid are explored, with attention to the consequences of the invariance of the Hamiltonian under the group U(m) of unitary transformations in the component space. It is pointed out that the hydrodynamics of the multicomponent systems ($m \geq 2$) differs in significant respects from that of the ordinary superfluid ($m = 1$). For example, in the superfluid phase, the multicomponent system has propagating modes with $\omega \propto k^2$ at long wavelengths, similar to the spin wave in a Heisenberg ferromagnet. In the normal phase, the multicomponent system has $m^2 - 1$ new diffusive modes, corresponding to the conserved generators of the group SU(m), whose fluctuations diverge in the critical region. On the basis of dynamic scaling and the mode-mode coupling approach applied to these new modes, we predict a dynamic critical exponent $z = \varphi/\nu$, where φ is the cross-over exponent for a symmetry-breaking perturbation of the axial type.

I. INTRODUCTION

The static and dynamic properites of the m-component Bose fluid have been studied recently by a number of authors,[1-13] with emphasis on the critical behavior near the superfluid transition. One of the reasons for this interest is that many properties of the system simplify in the limit $m \to \infty$, and techniques have been developed for calculating various properties (e.g., critical exponents) as a series expansion in $(1/m)$.[14] Thus, the large-m system may be a useful tool for testing and expanding various general ideas about critical phenomena, dynamics, etc. Furthermore, one may hope that various specific large-m results may be extrapolated in some manner to $m = 1$, so that one may thereby study the properties of an ordinary Bose liquid, such as liquid He^4.

The purpose of the present article is to point out that the hydrodynamics of the multicomponent system differs from that of helium[15] in an important respect, and that the critical dynamics should differ accordingly. In particular, the m-component system has a number of conserved densities not present in helium—namely the generators of the group SU(m),[16] under which the Hamiltonian is invariant. (These new conserved variables include, among others, fluctuations in the relative densities of the various components.) Because the susceptibilities for these quantities are found to diverge fairly strongly at T_c, the associated diffusion rates in the normal phase tend to be slower near T_c than thermal diffusion, which is the important diffusive mode in the one-component system. In the superfluid phase, the conservation laws for the generators of SU(m) lead to the appearance of $m-1$ new propagating modes with an energy spectrum *quadratic* in the wave vector k, similar to the spin waves in a Heisenberg ferromagnet.[17] These modes again have a "characteristic frequency" small compared to that of ordinary second sound, in the vicinity of T_c.

The dynamic critical exponent of the order parameter at T_c may be guessed by applying the dynamic scaling hypothesis[18,19] to the new "spin-wave" modes found below T_c, in the same way that the exponent for helium was obtained from the temperature dependence of velocity of second sound near T_c. Alternatively, one may use scaling hypotheses and the mode-mode coupling theory of Kawasaki and others,[20,21] to deduce the scaling exponent from the nonlinear interaction of the order parameter fluctuations and the new SU(m) diffusive modes *above* T_c. In either case one predicts that

$$z = \varphi/\nu , \tag{1.1}$$

where ν is the exponent of the temperature dependence of the correlation length, φ is the "cross-over exponent" for an axially anisotropic perturbation studied by Riedel and Wegner,[22] Wilson,[23] Fisher and Pfeuty,[24] Hikami and Abe,[3] and others,[25] and z, the dynamic scaling exponent, is defined by the wave-vector dependence of the characteristic frequency of the order parameter at T_c[18]:

$$\omega_k \propto k^z . \tag{1.2}$$

Equation (1.1) should be contrasted with the results for the case $m = 1$[18,19]:

$$z = \tfrac{1}{2}(d + \alpha/\nu) \quad \text{for } \alpha > 0 , \tag{1.3a}$$

$$z = \tfrac{1}{2} d \quad \text{for } \alpha < 0 , \tag{1.3b}$$

where d is the spatial dimensionality ($2 < d < 4$) and α is the specific-heat exponent. Equation (1.3) is obtained, for example, by application of the dynamic scaling hypothesis to the frequency spectrum of second sound in the superfluid. A second-sound mode at long wavelengths is also found in the superfluid phase of the *multicomponent* system, with a frequency that scales with the exponent (1.3b). Since a finite fraction of the order-parameter fluctuations are found in the second-sound mode as well as in the spin-wave modes, it appears that there is at least a partial breakdown of the "restricted dynamic scaling hypothesis"[18] at long wavelengths below T_c.

Equation (1.1) agrees precisely with the result reported by Abe and Hikami[9] from a direct calculation of z to first order in $1/m$, for $2<d<4$. (The agreement between z and φ/ν, to first order in $1/m$ was in fact noted in Ref. 9.) On the other hand, Kondor and Szépfalusy,[7,8] and Suzuki and Tanaka[10] have found results which agree with those of Abe and Hikami, and of the present paper for $2<d<3$, but *disagree* with them for $3<d<4$. These authors assert[8,10] that Abe and Hikami's result is incorrect for $3<d<4$ due to an incorrect evaluation of a set of Feynman diagrams. Calculations of the dynamic critical exponent for $\epsilon \to 0$ at fixed m, by Suzuki and Igarashi and by Kondor and Szépfalusy, also disagree with both (1.1) and (1.3).[26]

Since our result (1.1) for the multicomponent system, like the result (1.3) for the ordinary Bose fluid, depends on a dynamic scaling hypothesis or on some other assumption that has not been rigorously established, the possibility certainly exists that our results are incorrect in part or all of the range of d and m. The present author believes, however, that there may be more reason to question the general validity of the approach used by the previous authors, as will be discussed in Sec. V.

Outline of the paper

In Sec. II, we discuss the symmetry and conservation properties of the multicomponent Bose fluid, and we note the forms of various static susceptibilities. In Sec. III we derive the dynamic modes of the normal and superfluid phases from a hydrodynamic point of view,[15,17,27–29] applicable in the limit of long wavelengths at arbitrary temperatures other than the transition temperature or absolute zero. Dynamics in the critical region are considered in Sec. IV, while discussion of the ground state and elementary excitation spectrum at $T=0$, is deferred to Appendix A. Comments on some previous studies of dynamics of the multicomponent Bose fluid are given in Sec. V. Expansions of the critical exponents entering (1.1) and (1.3), in powers of $1/m$ or of $\epsilon=4-d$, are listed in Appendix B.

II. SYMMETRIES AND STATIC PROPERTIES

A. Definition of the system

The Hamiltonian of the system under consideration has the form

$$H=\int d^dx\left(\frac{\hbar^2}{2m_b}\sum_{\alpha=1}^{m}[\vec\nabla\psi_\alpha^\dagger(\vec x)]\cdot[\vec\nabla\psi_\alpha(\vec x)]+V_0\rho(\vec x)\right)$$

$$+\frac{1}{2}\int d^dx\,d^dx'[u(\vec x-\vec x'):\rho(\vec x')\rho(\vec x):], \quad (2.1)$$

where $\psi_\alpha(\vec x)$ is the annihilation operator for a boson at point $\vec x$ and component α, and $\rho(\vec x)$, the total density of bosons at point $\vec x$, is the sum of the individual component densities $\rho_\alpha(\vec x)$:

$$\rho(\vec x)\equiv\sum_{\alpha=1}^{m}\rho_\alpha(\vec x), \quad (2.2)$$

$$\rho_\alpha(\vec x)\equiv\psi_\alpha^\dagger(\vec x)\psi_\alpha(\vec x). \quad (2.3)$$

The colons in (2.1) indicate normal ordering of the creation and annihilation operators, and the two-body interaction u is assumed to be short ranged. Note that the boson mass m_b and the interaction u are independent of the component indices. The one-body potential V_0 is an arbitrary constant introduced here for later convenience.

It may be readily verified that the following operators (in addition to the total momentum $\vec P$ and H itself) commute with the Hamiltonian and are therefore constants of the motion—the total number N_α of bosons of any given component α, the total number of bosons N, and the operators $N_{\alpha\beta}$ and $M_{\alpha\beta}\,(\alpha\neq\beta)$ defined by

$$N_{\alpha\beta}\equiv\int d^dx\,\rho_{\alpha\beta}(\vec x), \quad (2.4a)$$

$$M_{\alpha\beta}\equiv\int d^dx\,\mu_{\alpha\beta}(\vec x), \quad (2.4b)$$

$$\rho_{\alpha\beta}\equiv\tfrac{1}{2}(\psi_\alpha^\dagger\psi_\beta+\psi_\beta^\dagger\psi_\alpha)=\rho_{\beta\alpha}, \quad (2.4c)$$

$$\mu_{\alpha\beta}\equiv(1/2i)(\psi_\alpha^\dagger\psi_\beta-\psi_\beta^\dagger\psi_\alpha)=-\mu_{\beta\alpha}. \quad (2.4d)$$

The m^2 independent operators N_α, $N_{\alpha\beta}$, and $M_{\alpha\beta}\,(\alpha<\beta)$ are closed under commutation, forming a Lie algebra.[16] These operators do not commute with the fields ψ_α. In fact, the operators of the Lie algebra are the infinitesimal generators of the group U(m) of unitary transformations on the m-dimensional complex vector space formed by the fields ψ_α. Of the various operators in the Lie algebra, one linear combination, the total particle number

$$N=\sum_{\alpha=1}^{m}N_\alpha, \quad (2.5)$$

commutes with all the rest. The remaining m^2-1

orthogonal operators form the generators of the group SU(m), and are new symmetries of the multicomponent Bose system, not present for $m=1$.

Equation (2.1) may be interpreted as the nonrelativistic Hamiltonian for a set of identical spinor bosons, each having spin quantum number $s=(m-1)/2$, with a spin-independent two-particle interaction. The ordinary three-dimensional rotation group O(3) then has a representation in the spin space, and the Hamiltonian is invariant under the action of O(3). For $m>2$, the operators of O(3) are only a subgroup of the full symmetry group SU(m). For $m=2$, however, the group O(3) is equivalent to the full symmetry group SU(2), and the generators of SU(2) may be identified with the ordinary spin-angular-momentum operators. Thus we have, with the usual convention for a spin-$\frac{1}{2}$ field,

$$\rho_{12}(\vec{x})=\sigma_x(\vec{x}), \tag{2.6a}$$

$$\mu_{12}(\vec{x})=\sigma_y(\vec{x}), \tag{2.6b}$$

$$\tfrac{1}{2}[\rho_1(\vec{x})-\rho_2(\vec{x})]=\sigma_z(\vec{x}), \tag{2.6c}$$

where σ_x, σ_y, and σ_z are the three components of the spin density $\vec{\sigma}$.

B. Symmetry of the normal and superfluid states

At sufficiently high temperatures, for any fixed pressure or fixed density, it is clear that the thermal equilibrium state of the system defined by (2.1) is a "normal" fluid (liquid or gas), whose density matrix exhibits the full symmetry of the Hamiltonian. It follows that

$$\langle\psi_\alpha(\vec{x})\rangle=0, \tag{2.7a}$$

$$\langle\rho_{\alpha\beta}(\vec{x})\rangle=\langle\mu_{\alpha\beta}(\vec{x})\rangle=0, \tag{2.7b}$$

$$\langle\rho_\alpha(\vec{x})\rangle=(1/m)\langle\rho(\vec{x})\rangle\equiv(1/m)\rho. \tag{2.7c}$$

At sufficiently low temperatures, we do not expect the state of the system to exhibit the full symmetry of the Hamiltonian. In particular, for suitable choice of the interaction u and of the pressure, density, or chemical potential, we expect that the thermal equilibrium state of the system is a "superfluid state" in which the U(m) symmetry is broken by a Bose condensation in one of the zero-momentum states of ψ_α. Thus we have

$$\langle\psi_\alpha\rangle=\psi_0 u_\alpha, \tag{2.8}$$

where u_α are the components of an arbitrary complex unit vector $\hat{u}$ in an m-dimensional space, and ψ_0 is a positive quantity, whose value depends on the temperature and pressure of the system.

As a result of the broken symmetry, the density matrix of the superfluid state is not invariant under all operations of the group U(m). It is assumed that the system remains invariant, however, under the subgroup of U(m) consisting of all those transformations which leave the unit vector $\hat{u}$ unchanged. For example, when we consider the orientation $u_\alpha=\delta_{\alpha 1}$, a choice which we shall take as standard below, the density matrix is invariant under the subgroup generated by the operators N_α, $N_{\alpha\beta}$, and $M_{\alpha\beta}$ with $\alpha,\beta>1$. We shall exploit this symmetry property frequently below.

The generators of SU(m) need not have zero expectation value in the superfluid state. By symmetry, one expects

$$\langle\rho_{\alpha\beta}\rangle=2\sigma_0\,\mathrm{Re}\,u_\alpha^* u_\beta, \tag{2.9a}$$

$$\langle\mu_{\alpha\beta}\rangle=2\sigma_0\,\mathrm{Im}\,u_\alpha^* u_\beta, \tag{2.9b}$$

$$\langle\rho_\alpha-(1/m)\rho\rangle=2\sigma_0(u_\alpha^* u_\alpha-1/m), \tag{2.9c}$$

where σ_0 is a real number whose value depends on temperature and pressure. For the case $m=2$, the quantity σ_0 is the magnitude of the spin density in the ordered phase.

For a noninteracting boson system, or for a weakly interacting system not too close to the critical temperature, one has

$$\sigma_0\approx\tfrac{1}{2}\psi_0^2. \tag{2.10}$$

When the temperature approaches the critical temperature from below,[30] we expect that the "symmetry-breaking" constants ψ_0 and σ_0 will tend toward zero according to power laws, which we write as

$$\psi_0\sim(T_c-T)^\beta, \tag{2.11a}$$

$$\sigma_0\sim(T_c-T)^{\beta_x}. \tag{2.11b}$$

The exponents in (2.11) will depend on the space dimensionality d and the component number m. We remark that, in general, β_x is not precisely equal to 2β, as might have been suggested by the relation (2.10) for the noninteracting gas. (However, it will turn out that β_x does become equal to 2β in the limit $m\to\infty$.)

The overall phase of $\langle\psi_\alpha\rangle$ will not, in general, be independent of time. Just as in the ordinary superfluid, one has in a thermal equilibrium state

$$\langle\psi_\alpha\rangle(t)=e^{-i\mu t/\hbar}\langle\psi_\alpha\rangle(0), \tag{2.12}$$

where μ is the chemical potential.[17,28,31] In order to eliminate the resultant complications from our subsequent discussions, we shall assume that $\mu=0$ for the given thermal equilibrium state. This condition can be fulfilled, for any given set of physical parameters such as temperature and density, by a correct choice of the arbitrary constant potential V_0 in the Hamiltonian (2.1). Note, also, that the expectation values in (2.9) are independent of the overall phase of $\hat{u}$. The SU(m) densities are independent of time, in any homogeneous state, regardless of the choice of V_0.

It should be emphasized that because there are many constants of the motion other than the energy and the density, the thermal equilibrium states discussed above are *not* the only possible equilibrium states of the model. For example, there must exist an equilibrium state (i.e., a state of infinite lifetime) corresponding to any specified values of the quantities N_α, $N_{\alpha\beta}$, and $M_{\alpha\beta}$, even at temperatures well above T_c, where the *thermal equilibrium state* (i.e., the state of minimum free energy) exhibits the full symmetry of the Hamiltonian.[32] The situation is similar to a Heisenberg magnetic system, for example, where there are equilibrium states corresponding to arbitrary values of the magnetization and temperature, although the true "thermal equilibrium" state (in zero magnetic field) has a magnetization determined by the temperature.[17]

The various infinite-lived equilibrium states may be understood as states which minimize the free energy, at a given temperature (or maximize the entropy, at a given energy) subject to the *constraint* that various of the concerned quantities have specified values. (Equivalently, these are states which would minimize the unconstrained free energy if appropriate "applied fields," coupling to the conserved quantities, were added to the Hamiltonian.) In addition, there will be a wide variety of states having finite, but slow, time-dependent state, where the densities of the conserved quantities vary from point to point on a long wavelength scale, and where nonconserved quantities have generally relaxed to the values required by a condition of local equilibrium.

The time dependence of slowly varying states is the subject matter of hydrodynamics and will be discussed in Sec. III. First, however, we must consider the static response of the thermal equilibrium system, when various inhomogeneous applied fields are added to the Hamiltonian.

C. Susceptibilities and static critical behavior

Let us define $\chi(A, B; \vec{k})$ as the linear response of the variable $A(\vec{x})$, to a static perturbation in the Hamiltonian of the form

$$\delta H = \int d^d x' [h e^{i\vec{k}\cdot\vec{x}'} B(\vec{x}') + \text{H.c.}], \tag{2.13}$$

where B is an operator density and h is an infinitesimal constant. We shall abbreviate $\chi(A, A; \vec{k})$ by $\chi(A; \vec{k})$ and we shall drop the argument $\vec{k}$ where convenient.

For the order parameter ψ, at and above T_c, we have, by symmetry,

$$\chi(\text{Re}\psi_\alpha, \text{Re}\psi_\beta) = \chi(\text{Im}\psi_\alpha, \text{Im}\psi_\beta) \equiv \chi_\psi(k)\delta_{\alpha\beta},$$
$$\chi(\text{Re}\psi_\alpha, \text{Im}\psi_\beta) = 0. \tag{2.14}$$

At T_c, $\chi_\psi(k)$ diverges in the limit $k \to 0$ as

$$\chi_\psi(k) \sim 1/k^{2-\eta}, \tag{2.15}$$

which may be taken as the definition of the exponent η. At any fixed temperature above T_c, the susceptibility χ_ψ is finite at $k=0$. According to the static scaling theory,[33] we may write

$$\chi_\psi(0) \sim 1/\kappa^{2-\eta}, \tag{2.16a}$$

where κ is the reciprocal of the correlation length. In turn, κ varies near T_c as

$$\kappa \propto (T - T_c)^\nu, \tag{2.16b}$$

which may be taken as the definition of the exponent ν. [Note: In (2.16b) and elsewhere in this paper, $T - T_c$ is understood to be measured along a path of *constant pressure*.]

Next consider the generators of the group U(m). At or above T_c, we have by symmetry,[34]

$$\chi(\rho_\alpha, \rho_\beta) = 2\chi_1(\vec{k})(\delta_{\alpha\beta} - 1/m) + (\rho^2/m^2)\chi_2(\vec{k}), \tag{2.17a}$$

$$\chi(\rho_{\alpha\beta}, \rho_{\alpha'\beta'}) = \chi_1(k)(\delta_{\alpha\alpha'}\delta_{\beta\beta'} + \delta_{\alpha\beta'}\delta_{\beta\alpha'}), \tag{2.17b}$$

$$\chi(\mu_{\alpha\beta}, \mu_{\alpha'\beta'}) = \chi_1(k)(\delta_{\alpha\alpha'}\delta_{\beta\beta'} - \delta_{\alpha\beta'}\delta_{\beta\alpha'}). \tag{2.17c}$$

Cross susceptibilities such as $\chi(\rho_\alpha, \rho_{\beta\gamma})$ or $\chi(\rho_{\alpha\beta}, \mu_{\alpha'\beta'})$ vanish. Note that

$$\chi(\rho; \vec{k}) = \rho^2 \chi_2(\vec{k}), \tag{2.18}$$

so that χ_2 is the isothermal compressibility of the system. According to the scaling theory we have

$$\chi_2(0) \sim \text{const} + \text{const} \times \kappa^{-\alpha/\nu}, \quad T > T_c, \tag{2.19}$$

$$\chi_2(k) \sim \text{const} + \text{const} \times k^{-\alpha/\nu}, \quad T = T_c, \tag{2.20}$$

where α is the exponent of the singular temperature dependence of the specific heat C_p.

The function χ_1 is expected to diverge more strongly at T_c than χ_2. According to the static scaling theory, one may write

$$\chi_1(k) \sim k^{-x}, \quad T = T_c, \tag{2.21a}$$

$$\chi_1(0) \sim \kappa^{-x}, \quad T > T_c, \tag{2.21b}$$

$$x = (2\varphi/\nu) - d, \tag{2.22}$$

where φ is the "cross-over exponent" for a perturbation proportional to $[\rho_\alpha - (1/m)\rho]$.[33] According to the current ideas of universality,[14] φ is the same as the cross-over exponent for a perturbation $[\psi_\alpha^2 - (1/2m)|\psi^2|]$ in a classical model with $2m$ real components to the order parameter, which has been studied by many authors.[3,22-25,33]

The exponents η, ν, and φ are related to the exponents β and β_x of Eq. (2.22) by the scaling laws[33]

$$2\beta = (d - 2 + \eta)\nu , \qquad (2.23)$$

$$\beta_x = d\nu - \varphi . \qquad (2.24)$$

In order to understand the linear-response functions below T_c, it is necessary to consider the cost in free energy of a long-wavelength variation in the direction of orientation of the order parameter. We shall assume that

$$\langle \psi_\alpha(\vec{x})\rangle/\psi_0 = u_\alpha(\vec{x}) = u_\alpha^0 + \delta u_\alpha(\vec{x}) , \qquad (2.25)$$

where $\hat{u}(\vec{x})$ and $\hat{u}^0$ are unit vectors, and δu_α is small. For simplicity, we shall choose

$$u_\alpha^0 = \delta_{1\alpha} . \qquad (2.26)$$

By symmetry, the free energy is unchanged if δu_α is any constant vector, independent of $\vec{x}$. [Of course, δu_1 must be purely imaginary, since $\hat{u}(\vec{x})$ is defined to have unit magnitude.] Thus, to lowest order, we expect that the excess free energy δF will be proportional to $|\nabla \hat{u}|^2$. More precisely, we may write

$$\delta F = \frac{\hbar^2}{m_b} \int d^d x \left(\tfrac{1}{2}\rho_s |\vec{\nabla} u_1|^2 + \tfrac{1}{2}\rho_s' \sum_{\alpha>1} |\vec{\nabla} u_\alpha|^2 \right), \qquad (2.27)$$

where ρ_s and ρ_s' are constants. In the case $m = 1$, the quantity ρ_s is the usual superfluid density, and one may make a similar identification for the case $m > 1$.

Symmetry of the Hamiltonian under the group $U(m)$ does *not* require ρ_s to be equal to ρ_s'. In the vicinity of the critical point, however, we expect that all singular static and thermodynamic properties are invariant under the larger group $O(2m)$, so that $\rho_s/\rho_s' \to 1$, as $T \to T_c$.[14] In the limit of a weakly interacting system, or for the spherical-model limit ($m \to \infty$), one finds $\rho_s = \rho_s' = \hbar^2\psi_0^2/2m_b$, for arbitrary temperature below T_c.

We remark that (2.27) applies to the state of lowest possible free energy, consistent with the given variation of $\hat{u}(\vec{x})$. That state will clearly be a state of local equilibrium, in which for example, the local values of $\rho_\alpha(\vec{x})$, $\rho_{\alpha\beta}(\vec{x})$, and $\mu_{\alpha\beta}(\vec{x})$ are all related to $\hat{u}(\vec{x})$ by Eqs. (2.9). Under these conditions, the second term in (2.27) may be written in the alternate form

$$\frac{\hbar^2\rho_s'}{2m_b} \sum_{\alpha>1} |\vec{\nabla} u_\alpha|^2 = \frac{\hbar^2\rho_s'}{8\sigma_0^2 m_b} \sum_{\alpha>1} [(\vec{\nabla}\rho_{1\alpha})^2 + (\vec{\nabla}\mu_{1\alpha})^2] . \qquad (2.28)$$

Let us now consider a situation in which a weak field is applied at wave vector k coupling to the variable $\mathrm{Im}\psi_1$.[28] If the field is sufficiently weak, and the wavelength sufficiently long so that the system remains in local equilibrium, then the energy of interaction with the field will be given by

$$- \int (h e^{i\vec{k}\cdot\vec{x}} + \mathrm{c.c.})\, \mathrm{Im}\langle \psi_1(\vec{x})\rangle d^d x = - \int (h e^{i\vec{k}\cdot\vec{x}} + \mathrm{c.c.})\, \psi_0\, \mathrm{Im} u_1(\vec{x}) d^d x . \qquad (2.29)$$

Then, minimization of the sum of (2.29) and (2.27) with respect to $u_1(\vec{x})$ leads to the result[28]

$$\chi(\mathrm{Im}\psi_1; k) = \frac{m_b}{\hbar^2} \frac{\psi_0^2}{\rho_s k^2} . \qquad (2.30)$$

Similarly, we see that for $\alpha > 1$

$$\chi(\mathrm{Re}\psi_\alpha; k) = \chi(\mathrm{Im}\psi_\alpha; k) = \frac{m_b}{\hbar^2} \frac{\psi_0^2}{\rho_s' k^2} , \qquad (2.31a)$$

$$\chi(\rho_{1\alpha}; k) = \chi(\mu_{1\alpha}; k) = \frac{m_b}{\hbar^2} \frac{4\sigma_0^2}{\rho_s' k^2} . \qquad (2.31b)$$

The behavior of ρ_s and ρ_s', in the vicinity of T_c is determined by the static scaling laws to be[35]

$$\rho_s \approx \rho_s' \propto \kappa^{d-2} \propto (T_c - T)^{(d-2)\nu} . \qquad (2.32)$$

Note that with this temperature dependence, the susceptibilities (2.30) and (2.31), when evaluated for wave vector k equal to κ, have the same temperature dependence as the corresponding susceptibilities above T_c.

The form of the longitudinal order-parameter susceptibility $\chi(\mathrm{Re}\psi_1; k)$ is somewhat uncertain. It is *believed,* however, that there will be a divergence in the longitudinal susceptibility at long wavelengths, for $d \leq 4$, due to the occurrence of fluctuations in the value of $\mathrm{Re}\psi_1$ proportional to the *square* of the amplitudes of the long-wavelength fluctuations in ψ_α, for $\alpha > 1$, and in $\mathrm{Im}\psi_1$. On this basis one predicts that for $k \to 0$, below T_c, and $2 < d < 4$,[36,5]

$$\chi(\mathrm{Re}\psi_1; k) \sim \frac{\mathrm{const}}{k^{4-d}}\, T \left[2(m-1)\left(\frac{\psi_0}{\rho_s'}\right)^2 + \left(\frac{\psi_0}{\rho_s}\right)^2 \right] . \qquad (2.33)$$

[Note that this divergence is weaker than that in the transverse susceptibilities (2.30) and (2.31a).] Similarly, one expects a divergence in the "longitudinal susceptibility" for generators of $SU(m)$, of the form

$$\chi(\rho_\alpha, \rho_\beta; k) \sim \frac{\mathrm{const}}{k^{4-d}}\, T \left(\frac{\sigma_0}{\rho_s'}\right)^2 \times [m\,\delta_{\alpha 1}\delta_{\beta 1} - \delta_{\alpha 1} - \delta_{\beta 1} + \delta_{\alpha\beta}] + \chi'(\rho_\alpha, \rho_\beta) , \qquad (2.34)$$

where χ' is finite as $k \to 0$.

III. HYDRODYNAMIC PROPERTIES

In this section we shall discuss the dynamic properties of the normal and superfluid phases, from a hydrodynamic point of view,[15,17,27-29] valid in the limit of long wavelengths and low frequencies. In particular, hydrodynamics requires that the wavelength of the excitation under consideration be long compared to all important characteristic lengths of the system, including the correlation length κ^{-1}, which becomes large near T_c, and the mean free path of thermal excitations, which becomes large near $T=0$. Some results of hydrodynamics, however, such as the spin-wave and phonon spectra, are expected to remain valid even at $T=0$.

The principal ingredients of the hydrodynamic theory are the symmetry and conservation laws of the system, and assumptions about the form of various static susceptibilities. Furthermore, one assumes that for sufficiently long wavelength variations in the parameters of the system one can write the equations of motion as a gradient expansion in a properly chosen set of variables. To lowest order in the wave vector, the state of the system may be understood in terms of local equilibrium.

A. Hydrodynamics in the normal phase

A state of local equilibrium in the normal phase of the Bose fluid is described by the densities of the conserved quantities; namely $\rho_\alpha(\vec{x})$, $\rho_{\alpha\beta}(\vec{x})$, and $\mu_{\alpha\beta}(\vec{x})$, together with the energy density $\epsilon(\vec{x})$ and the d components of the momentum density $\vec{g}(\vec{x})$, all of which are assumed to be slowly varying functions of position. (Note that there are m^2+d+1 independent variables in all.) Since the values of all other quantities are supposed to relax to their local equilibrium values in a "microscopic" time, the values of all quantities of interest are determined by the conserved fields. In particular, the time derivatives of the conserved quantities must be expressible as functionals of the densities themselves.[17]

We shall be concerned here only with the equations of motion linear in the deviations from equilibrium, valid for small fluctuations. By symmetry, it is clear that in the normal state there can be no coupling in the linear equations of motion between the different generators of SU(m), nor between any of these variables and the variables ρ, ϵ, or $\vec{g}$. Thus long-wavelength variations in the SU(m) densities obey equations of motion of the form,

$$\frac{\partial}{\partial t}\left(\rho_\alpha-\frac{1}{m}\rho\right)=D\nabla^2\left(\rho_\alpha-\frac{1}{m}\rho\right), \tag{3.1a}$$

$$\frac{\partial}{\partial t}\rho_{\alpha\beta}=D\nabla^2\rho_{\alpha\beta}, \tag{3.1b}$$

$$\frac{\partial}{\partial t}\mu_{\alpha\beta}=D\nabla^2\mu_{\alpha\beta}, \tag{3.1c}$$

where D is a diffusion constant. Note that terms linear in $\vec{\nabla}$ must be absent from the right-hand sides of (3.1), by virtue of spatial isotropy, and terms of zeroth power in $\vec{\nabla}$ are absent because of the conservation of the SU(m) densities. These equations lead to the existence of m^2-1 independent diffusive modes, each with a relaxation rate

$$\gamma_k=Dk^2. \tag{3.2}$$

The diffusion constant D may be written in the form

$$D=\lambda/\chi_1, \tag{3.3}$$

where λ is the "transport coefficient" for the SU(m) densities.

The remaining variables $\vec{g}$, ρ, and ϵ obey the usual hydrodynamic equations for a normal fluid. The transverse components of $\vec{g}$ relax at a rate $D_g k^2$ while ϵ, ρ, and the longitudinal part of g, couple to a sound wave of frequency ck and a thermal diffusion mode with relaxation rate $D_T k^2$. The diffusion coefficients and the sound velocity are given by the usual formulas[27,29]

$$D_g=\overline{\eta}/\rho m_b, \tag{3.4a}$$

$$D_T=\lambda_T/\rho C_p, \tag{3.4b}$$

$$c=(m_b\rho\chi_s)^{-1/2}, \tag{3.4c}$$

where $\overline{\eta}$ is the viscosity, λ_T the thermal conductivity, C_p the specific heat at constant pressure, and χ_s the isentropic compressibility.

B. Hydrodynamics in the superfluid phase

Below T_c, the equilibrium state of the system requires specification of the orientation of the unit vector $\hat{u}$, as well as the values of the conserved quantites. Note, however, that the orientation of $\hat{u}$ is not independent of the SU(m) densities. From (2.9) it is clear that the values of ρ_α, $\rho_{\alpha\beta}$, and $\mu_{\alpha\beta}$ determine the equilibrium orientation of $\hat{u}$ except for an overall phase factor.[37] Since the order parameter ψ is not a conserved quantity, we expect that the orientation of $\hat{u}(\vec{x})$ should relax in a microscopic time to its equilibrium orientation. Thus, in the case of small long-wavelength deviations from the thermal equilibrium state, we may uniquely describe the system by specifying the following variables as functions of space: ϵ, $\vec{g}$, ρ_α, $\rho_{\alpha\beta}$, $\mu_{\alpha\beta}$, and the phase $\varphi(\vec{x})\approx \mathrm{Im}\,u_1(\vec{x})$. (We have now chosen, as usual, $u^0_\alpha=\delta_{1\alpha}$.) We shall write the time derivatives of these quantities as linear functionals of the variables themselves, and once again we shall make a gradient expansion for long-wavelength variations.

1. Spin-wave modes

It is clear from the symmetry of the system that the linearized equations of motion for $\rho_{1\alpha}$ and

$\mu_{1\beta}$ can be coupled together only if $\alpha = \beta$, and are uncoupled from all other variables. As a result, we may write

$$\frac{\partial \rho_{1\alpha}}{\partial t} = A\nabla^2\rho_{1\alpha} + B\nabla^2\mu_{1\alpha} + \mathcal{O}(\nabla^4\rho_{1\alpha};\nabla^4\mu_{1\alpha}), \quad (3.5a)$$

$$\frac{\partial \mu_{1\alpha}}{\partial t} = A\nabla^2\mu_{1\alpha} - B\nabla^2\rho_{1\alpha} + \mathcal{O}(\nabla^4\rho_{1\alpha}, \nabla^4\mu_{1\alpha}). \quad (3.5b)$$

The operators $\rho_{1\alpha}$ and $\mu_{1\alpha}$ obey the commutation relations

$$[\rho_{1\alpha}(\vec{x}), \mu_{1\alpha}(\vec{x}')] = \tfrac{1}{2} i [\rho_1(\vec{x}) - \rho_\alpha(\vec{x})]\delta(\vec{x} - \vec{x}'). \quad (3.6)$$

If one neglects dissipative processes, then one expects, formally,[31]

$$\frac{\partial \rho_{1\alpha}(\vec{x})}{\partial t} = \frac{\sigma_0}{\hbar}\frac{\delta E}{\delta \mu_{1\alpha}(\vec{x})}, \quad (3.7a)$$

$$\frac{\partial \mu_{1\alpha}(\vec{x})}{\partial t} = -\frac{\sigma_0}{\hbar}\frac{\delta E}{\delta \rho_{1\alpha}(\vec{x})}, \quad (3.7b)$$

where E is the energy of the system, and the derivative is taken at constant entropy. Comparing (3.7) and (2.28) with (3.5), one is led to the result

$$A = 0, \quad (3.8a)$$

$$B = -\hbar\rho_s'/4m_b\sigma_0. \quad (3.8b)$$

Equation (3.5) then has as its solution, in the limit of long wavelengths, a propagating mode with real frequency

$$\omega_k = (\hbar/2m^*)k^2, \quad (3.9a)$$

where

$$m^* = 2m_b\sigma_0/\rho_s' \quad (3.9b)$$

is the "effective mass" of the frequency spectrum.

These modes are closely analogous to the spin-wave mode of a Heisenberg ferromagnet. In fact, for the case $m = 2$, the variables ρ_{12} and μ_{12} are simply the x and y components of the spin density $\vec{\sigma}(x)$, and (3.9) is identical to the standard Landau-Lifshitz formula for the frequency spectrum of a ferromagnet. [See, for example, Eq. (7.9) of Ref. 17.[38]]

Our heuristic derivation of (3.8) can be justified by an argument exactly analogous to the one used in Ref. 17 for the Heisenberg ferromagnet, in which the absence of dissipation is not assumed *a priori*. If one carries out the indicated expansion in (3.5), then the terms of order ∇^4 will presumably introduce a spin-wave damping of order k^4. On the other hand, microscopic spin-wave calculations for the Heisenberg ferromagnet (in three dimensions) suggest that there is a singular contribution to the spin-wave damping which leads to a decay rate proportional to $k^4 \ln^2 k$.[39] It seems possible that a similar term may be present in the present case of the multicomponent Bose gas. In either case, however, the damping rate of the "spin-wave mode" will be infinitesimal compared to the real part of the frequency, in the limit $k \to 0$.

In the vicinity of T_c, the effective mass m^* given by (3.9b) approaches zero according to

$$m^* \propto (T_c - T)^{-\varphi + 2\nu} \propto \kappa^{2-\varphi/\nu}. \quad (3.10)$$

In the limit of a weakly interacting Bose system for any fixed value of $T < T_c$, or in the spherical-model limit ($m \to \infty$), the effective mass m^* becomes independent of temperature, and is just equal to m_b, the Bose mass of the bosons. Note also that the exponent $2 - \varphi/\nu$ in (3.10) approaches zero in the limit where the component number m becomes large [see Eq. (B2)].

2. *Diffusive modes*

Next consider the variable $\rho_{\alpha\beta}$, with $\beta > \alpha > 1$. It follows from the symmetry of the system that the linear equation of motion for $\rho_{\alpha\beta}$ is uncoupled from all other variables, and therefore the lowest term in a gradient expansion must have the form

$$\partial\rho_{\alpha\beta}/\partial t = D\nabla^2\rho_{\alpha\beta}, \quad (3.11a)$$

$$D \equiv \lambda/\chi(\rho_{\alpha\beta}). \quad (3.11b)$$

We have already remarked that we expect $\chi(\rho_{\alpha\beta})$ to be finite in the limit $k \to 0$, at any fixed temperature below T_c. Thus, if the transport coefficient λ is also finite, as seems most likely, the diffusion constant D is itself finite, at fixed temperature, and the variable $\rho_{\alpha\beta}$ will relax at a rate proportional to k^2, for $k \to 0$. Using the symmetry of the system we also find, for $\beta > \alpha > 1$

$$\partial\mu_{\alpha\beta}/\partial t = D\nabla^2\mu_{\alpha\beta}, \quad (3.12a)$$

$$\partial(\rho_\alpha - \rho_\beta)/\partial t = D\nabla^2(\rho_\alpha - \rho_\beta). \quad (3.12b)$$

The diffusion constant here is the same as in (3.11).

Altogether, these diffusive modes represent $m^2 - 2m$ independent degrees of freedom. These modes are absent in the case $m = 2$.

3. *Remaining variables*

The remaining variables in the superfluid state are φ, $\vec{g}$, ϵ, ρ, and ρ_1. These variables are analogous to those which describe the state of a superfluid mixture of He^3 and He^4 (Ref. 15): the variable ρ_1 corresponds to the density of He^4 atoms while the remaining variables are just those which characterize an ordinary one-component superfluid.

We may define a superfluid velocity $\vec{v}_s$ and supercurrent density $\vec{j}_s$ by

$$\vec{v}_s = (\hbar/m_b)\nabla\varphi\,, \tag{3.13}$$

$$\vec{j}_s = \rho_s \vec{v}_s\,. \tag{3.14}$$

If one considers a state in which the normal-fluid velocity is zero, which by definition is the state of minimum free energy for a given, constant $\vec{v}_s$, then one finds a momentum density

$$\vec{g} = m_b \vec{j}_s\,. \tag{3.15}$$

Furthermore, one finds that the supercurrent entirely comes from a flow of particles in the component state $\alpha = 1$, so that the supercurrent contribution to $d\rho_1/dt$ is simply $-\vec{\nabla}\cdot\vec{j}_s$.

If one takes a naive viewpoint, and ignores the possibility of divergences in longitudinal susceptibilities such as (2.34) or in the associated transport coefficients, then the normal modes of the variables under consideration will indeed be the same as in superfluid He^3-He^4 mixtures. (The naive viewpoint is probably correct for $d > 4$.) The variables ρ, ρ_1, ϵ, φ, and the longitudinal part of $\vec{g}$ will appear in two propagating modes, first and second sound, with real frequencies $c_1 k$ and $c_2 k$, as well as in one diffusive mode, with a relaxation rate $D'k^2$. [Formulas for c_1 and c_2, in terms of various thermodynamic parameters, may be found in Sec. 24 of Ref. 15.] The transverse components of $\vec{g}$ relax at a rate $\overline{\eta}k^2/m_b(\rho - \rho_s)$, where $\overline{\eta}$ is the "normal-fluid viscosity."

If a divergent susceptibility of the form (2.34) is inserted in the formulas for He^3-He^4 mixtures, the first- and second-sound velocities remain finite.[40] The diffusion constant D', however, may be expressed as the ratio of a transport coefficient to a susceptibility that *diverges* at $k = 0$.[40] If one assumes finite transport coefficients, D' is found to vanish as k^{4-d}, so that the corresponding relaxation rate of a fluctuation in ρ_1, would be predicted to go as k^{6-d}. This result seems extremely unlikely however, since the divergence in (2.34) comes essentially from the contributions of pairs of long-wavelength "spin waves" with total wave vector $\vec{k}$ for the pair, and the interaction between the spin waves is presumed to be small at long wavelengths. It seems most likely therefore, that the divergent portion of the correlation function $\langle \rho_1(k, t)\rho_1(-k, 0)\rangle$ will decay in a nonexponential fashion with a rate characteristic of the contributing spin-wave frequencies, i.e., a rate of order $(\hbar/m^*)k^2$.[41] It follows that the transport coefficient must vanish as $k \to 0$, or more precisely, that the local equilibrium expansion is inapplicable in this situation.

The second-sound velocity c_2, in the vicinity of T_c, may be written in the form

$$c_2 \approx (\rho_s K)^{1/2}\,, \tag{3.16}$$

where K, a complicated combination of thermodynamic functions, is expected to remain finite at T_c.[40] Thus we have

$$c_2 \propto \kappa^{(d-2)/2}\,. \tag{3.17}$$

IV. CRITICAL DYNAMICS

Consider the behavior of the order-parameter correlation function, $\sum_\alpha \langle \psi_\alpha^\dagger(\vec{k}, t)\psi_\alpha(\vec{k}, 0)\rangle$, in the limit $k \to 0$, for fixed temperature slightly below T_c. As a consequence of (2.30) and (2.31a) and of the results of Sec. III, the correlation function will have two predominant parts—one, with weight $\propto 2(m-1)/\rho_s' k^2$, will occur at the spin-wave frequency $\hbar k^2/2m^*$, while the second, with weight $\propto 1/\rho_s k^2$, will occur at the second-sound frequency $c_2 k$.[17,28] If we apply the dynamic scaling hypothesis[18,19] to the spin-wave frequency (3.9) and (3.10), we are led to predict a dynamic scaling exponent

$$z = \varphi/\nu\,. \tag{4.1}$$

Note that the second-sound frequency, scales with a characteristic exponent $z_2 = \frac{1}{2}d$, which *differs* from (4.1). Since the order-parameter correlation function below T_c has a finite fraction of its weight in both the second-sound and spin-wave modes, this implies a violation of "restricted dynamic scaling",[18] in the limit of long wavelengths below T_c. At least in the case of large m, however, there is more weight in the spin-wave mode than in the second-sound mode, so that it seems likely that (4.1) should be the correct scaling exponent for fluctuations in the order parameter at and above T_c—*assuming* that the correlation functions at and above T_c can indeed be characterized by a single dynamic scaling exponent.

The dynamic scaling exponent may also be predicted directly above T_c, by means of the mode-mode coupling theories, using a different kind of dynamic scaling assumption. According to a mode-mode coupling analysis similar to the discussion of helium in Ref. 20, there will be contribution $\delta\lambda$ to the transport coefficient λ for the SU(m) densities, defined in Eqs. (3.1) and (3.3), of the form

$$\delta\lambda = \text{const}\times\kappa^{d-2}/\Gamma\,, \tag{4.2}$$

where Γ is the characteristic relaxation rate of the order parameter,

$$\Gamma \propto \kappa^z\,. \tag{4.3}$$

This contribution arises from the process in which the SU(m) fluctuation at $k \approx 0$, interacts with a pair of order-parameter fluctuations having wave vectors near κ. Similarly, there will be a contribution to Γ of the form

$$\delta\Gamma = \text{const}\times\kappa^d/(\Gamma + D\kappa^2)\chi_1\,, \tag{4.4}$$

where $D = \lambda/\chi_1$ is the diffusion constant for the SU(m) densities. The contribution (4.4) comes from a process in which the order-parameter fluctuation at $k = 0$ couples to the product of an SU(m) fluctuation and an order-parameter fluctuation at wave vector κ.

If one makes the scaling assumption that Γ and $D\kappa^2$ have the same order of magnitude, and that $\Gamma \approx \delta\Gamma$ and $\lambda \approx \delta\lambda$, then one finds from either (4.3) or (4.4), that $z = \varphi/\nu$, in agreement with (4.1), and that

$$\lambda \propto \kappa^{-y}, \tag{4.5a}$$

$$y = 2 - d + z \,. \tag{4.5b}$$

In a recent paper,[42] Hohenberg, Siggia, and the present author employed a recursion relation approach to calculate the critical exponents for divergent transport coefficients in a number of simple models, correct to first order in $\epsilon = 4 - d$. These models included a simplified model of the superfluid transition of a one-component Bose fluid, where the scaling results were confirmed. One can generalize that model to a multicomponent system in which the order-parameter fluctuations are coupled to diffusive modes for both the SU(m) densities *and* the overall density ρ. (The density diffusion represents the contribution of the thermal diffusive mode in the present Bose system.) A recursion-relation study of this model indicates that *in the limit* $d \to 4$ the dynamic scaling exponent (at and above T_c) is determined by coupling to the SU(m) diffusion modes, and is indeed given by (4.1), *provided that m is greater than* 2.213. A thorough renormalization group analysis of the model has not been completed, however.

V. COMPARISON WITH MICROSCOPIC CALCULATIONS

Szépfalusy and Kondor[6] have examined the mode structure below T_c of a "dynamic spherical model," which describes the Bose system in the limit $m \to \infty$. Their analysis only considers the modes that appear in the correlation function for the total density and for the fluctuations of the magnitude and overall phase of the order parameter. At $T = 0$, they find a phonon spectrum at long wavelengths, in agreement with the results of Appendix A of the present paper. At finite temperatures, however, the damping of their sound waves is not negligible in the long-wavelength limit. Neither do they find the second-sound modes found in Sec. III of the present paper. In the limit $m \to \infty$, however, the scattering between the bosons vanishes, and mean free paths become infinite. Thus it should not be surprising if hydrodynamics does not apply, and in particular, if second sound does not exist in that case.

Szépfalusy and Kondor did not find spin-wave-like modes in their analysis, because they did not investigate excitations of the required symmetry. It is easy to establish that such modes exist, however, and that they behave like noninteracting bosons at all temperatures below T_c, in the spherical-model limit.

Ma and Senebetu[12] have also studied the spectrum of total-density fluctuations in the limit of large m, from a microscopic point of view. They have carried out their calculations to the leading order in $1/m$ for which collisions occur, summing all diagrams necessary to obtain the correct hydrodynamic form, in the limit of long wavelengths. In contrast to Ref. 6, Ma and Senebetu find well-defined first- and second-sound modes, with damping proportional to k^2, in the limit of long wavelengths, at finite temperatures below T_c. In the normal phase, above T_c, they obtain the usual sound wave and thermal diffusion modes.

The value of the thermal conductivity found by Ma and Senebetu remains finite as $T \to T_c^+$, in contrast to the results of a mode-mode coupling analysis of the m-component system, which suggest that the thermal conductivity λ_T should diverge at T_c in the same manner as the SU(m) transport coefficients λ, described in Eq. (4.5). As Ma and Senebetu point out, however, the divergent portion of λ_T might be of higher order in $1/m$ than the nondivergent portion, in which case it would not be included in their calculations.

Suzuki[11] has also studied the thermal conductivity in the normal phase, using a microscopic approach designed to be correct in the limit of large m. He concludes that there is a divergence in λ_T, which may be written in the form

$$\lambda_T \propto \kappa^{-y}, \tag{5.1a}$$

$$y = 2 - d + z \,, \tag{5.1b}$$

with possible corrections of order $(1/m)^2$. Although Suzuki's evaluation of z disagrees with ours for $3 < d < 4$, relation (5.1) coincides with the form of Eq. (4.5), as predicted by mode-mode coupling.

Equation (5.1) is applicable to helium as well as the multicomponent system. A relation equivalent to (5.1b) has also been obtained by Yamashita and Tsuneto[43] from a microscopic analysis of the one-component system, up to second order in $\epsilon = 4 - d$.

As was mentioned in the Introduction, a number of authors have calculated dynamic critical exponents for the quantum-mechanical m-component fluid, from an analysis of a Feynman graph expansion designed to be valid in the limit $m \to \infty$[7-10] or the limit $d \to 4$.[26] Although these calculations agree with our result (1.1), at least to first order in $1/m$, for $2<d<3$, they apparently disagree with (1.1) and with (1.3) to first order in $1/m$ for $3<d<4$, and to first order in $4-d$ for arbitrary m.[44]

In Ref. 36, where (1.3) was obtained from a renormalization group analysis of a simplified model of the one-component Bose system, some reasons were already given why a dynamic critical exponent calculated from a Feynman graph analysis in the limit $\epsilon\to 0$ might have been incorrect. Similar reasons can be given for questioning the calculations that have been carried out in the limit $m\to\infty$. In all of those calculations, the authors have extracted their exponents by a matching condition from a term proportional to $(1/m)k^2\ln k$, in the $1/m$ expansion of some quantity, such as the characteristic frequency ω_k for fluctuations in the order parameter T_c. The $1/m$ expansion in turn involves a systematic resummation of the Feynman diagrams for the interacting Bose system and is valid in the limit $1/m\to 0$, for fixed, nonzero k. For critical phenomena, however, we are interested in the behavior of ω_k as $k\to 0$, for fixed value of m. The matching condition can be used to extract the true critical behavior only if there are no "slow transients" in the renormalization group for the dynamics at large m. The absence of slow transients in the dynamical renormalization group for the Bose system, in the limit $m\to\infty$, has not been demonstrated as far as the present author is aware.

In contrast with this, the absence of slow transients in the *static* renormalization group, for large m, has been established.[1] As a consequence, one can justify the calculation of static critical exponents such as η and ν, to arbitrary order in $1/m$ by a straightforward matching condition. A more complicated procedure is necessary to calculate static critical exponents in the limit $\epsilon\to 0$, at fixed m.[23] In the latter case, the presence of a slow transient prevents the matching condition from being applicable, except when the interaction strength u_0 is chosen in a special way to eliminate the slow transient.

The possibility of additional slow transients in the dynamical renormalization group, not present in the static case, is illustrated by several examples[42,45] among the models that have been studied in the limit $\epsilon\to 0$. In particular, the recursion relations for the simplified model of helium in Ref. 42 were found to have *five* slow transients for $d\to 4$, and these transients cannot all be eliminated by any choice of the single parameter u_0.

The present situation may also be contrasted with the case of the simple stochastic time-dependent Ginzburg-Landau model (without energy conservation) discussed in a number of recent papers.[45–47] As was mentioned in Ref. 46, a renormalization group analysis in the limit of $d\to 4$ shows that there are *no new slow transients* present in the dynamic renormalization group for *that model*, so that the correct dynamic exponent could be obtained via a matching condition on the ϵ expansion after the correct choice of the single parameter u_0.

Note added. The limit $m\to\infty$ is often referred to as the "spherical-model limit,"[6,11,12] because the static correlation functions for the order parameter of the $m=\infty$ Bose system, like those of an $n=\infty$ spin system, are essentially the same as for the Berlin-Kac spherical model.[48,49] On the other hand, certain other properties of the multicomponent spin or boson systems, such as the correlation functions

$$\sum_{\alpha,\alpha'}\langle|\psi_\alpha(\vec{x})|^2|\psi_{\alpha'}(\vec{x}')|^2\rangle,$$

are different from those in the spherical model. A more precise analog of the spherical model is the ordinary ($m=1$) *noninteracting* Bose gas, with a fixed number of particles per unit volume.[50]

A second point to bear in mind, is that there are a number of ways to introduce dynamics into the spherical model, which lead to quite different dynamic behaviors. In addition to the trivial dynamics of the noninteracting Bose gas, the boson dynamics of Refs. 6, 11, and 12, and the stochastic dynamics of Refs. 45–47, the literature includes models with a phonon dynamics, appropriate to a displacive transition,[51,52] and a spin dynamics, based on the commutation relations of the Heisenberg ferromagnet.[53]

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APPENDIX A: GROUND-STATE AND ELEMENTARY EXCITATIONS

As is well known, the lowest eigenstate of the Schrödinger equation, for a system of identical particles, has a totally symmetric spatial wave function.[54] This can be achieved for our boson system by requiring the state to be totally symmetric in the spin variables, i.e., spins fully aligned. We may choose this alignment so that all the bosons are in the $\alpha=1$ subspace—i.e., $\rho_1=\rho$, and $\rho_\alpha=0$ for $\alpha>1$. For the case $m=2$, this implies $\langle\sigma_z\rangle=\frac{1}{2}\rho$, $\langle\sigma_x\rangle=\langle\sigma_y\rangle=0$. More generally, we have

$$\sigma_0=\tfrac{1}{2}\rho \tag{A1}$$

at $T=0$, where σ_0 is defined by (2.9). Note that the spatial wave function and energy of the ground state are precisely the same as for a one-component boson system with the given interaction and mass.

Elementary excitations from the ground state are of two kinds. If all particles remain in the $\alpha=1$ space in the excited state, then the wave function and the energy spectrum are the same as for a one-component Bose system. In particular, the excitations on this branch are phonons at long wavelengths, with a sound velocity determined by the bulk modulus and the density of the liquid, as in Eq. (3.4c) above.

The remaining elementary excitations will have $N-1$ bosons with $\alpha=1$, and one boson with $\alpha\neq 1$. For small values of the wave vector k, these excitations may be described as a single "impurity" quasiparticle of momentum k and the given $\alpha\neq 1$, added to the ground state of $N-1$ bosons with $\alpha=1$. For small values of k such an excitation will have an infinite lifetime, and will have an energy quadratic in k[15]

$$\epsilon_k = \hbar^2 k^2/2m^* . \tag{A2}$$

In an m-component Bose system, there will clearly be $m-1$ degenerate excitation branches of this form. These free-particle-like branches may also be interpreted as ferromagnetic magnons in the space of SU(m).

In the limit of a weakly interacting system, the effective mass m^* will become the same as the Bose mass m_b. In a strongly interacting fluid, however, we expect $m^* > m_b$, as is found[15] for the effective mass of an He^3 impurity in liquid He^4.

As in the case of a Heisenberg ferromagnet or antiferromagnet, we expect that the "elementary excitation" spin waves obtained at $T=0$ for finite k, will go over smoothly to the hydrodynamic spin waves derived for $k\to 0$, at finite T. Then the result $m^*>m_b$, combined with (5.1) and (3.9b) indicates that $\rho_s<\rho$ at $T=0$ for the strongly interacting case. On the other hand, one can show that $\rho_s=\rho$ at $T=0$ in the multicomponent system as in ordinary superfluid helium, as a result of Gallilean invariance. It follows that $\rho_s' \neq \rho_s$ at low temperatures.

APPENDIX B: EXPANSIONS OF THE EXPONENTS IN THE LIMIT $4\to d$ OR $m\to\infty$

The ratio φ/ν, which appears in (1.1), has the known expansions[3,9,23,25]

$$\frac{\varphi}{\nu} = 2 - \frac{\epsilon}{m+4} + \frac{\epsilon^2(m^2-9m+14)}{4(m+4)^3} + O(\epsilon^3), \tag{B1}$$

$$\frac{\varphi}{\nu} = 2 - \frac{1}{m}\frac{2^{d-1}\Gamma(\frac{1}{2}(d-1))}{d\Gamma(\frac{1}{2}d)\Gamma(2-\frac{1}{2}d)\Gamma(\frac{3}{2})\Gamma(\frac{1}{2}d-1)} + O(1/m^2), \tag{B2}$$

where $\epsilon=4-d$.[14] Equation (B1) is valid for $\epsilon\to 0$ at fixed m, whereas (B2) applies in limit $m\to\infty$ for fixed d in the range $2<d<4$. The two formulas agree up to order ϵ^2/m in the simultaneous limit $\epsilon\to 0$ and $m\to\infty$. The coefficient of $1/m$ in (B2) is equal to $\frac{16}{3}\pi^{-2}$ at $d=3$. Equation (A2) may also be written[9]

$$\varphi/\nu = 2 - [4/(4-d)]\eta + O(1/m^2), \tag{B3}$$

where η is the correlation exponent, defined by (2.14).

The ratio α/ν, which enters (1.3a), has the expansion[23]

$$\frac{\alpha}{\nu} = \frac{1}{5}\epsilon - \frac{7}{25}\epsilon^2 + O(\epsilon^3), \quad \text{for } m=1 . \tag{B4}$$

Although α is positive for small ϵ, it is believed that α is slightly negative for $d=3$.[55] The exponent α is believed to be negative for $m\geq 2$, for all $2<d<4$.

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[14]The static critical properties of the m-component Bose system are believed to be the same as those of a classical system in which the order parameter has $2m$ real components. This has been confirmed by a number of calculations, including the direct investigation of the quantum-mechanical system carried out in Ref. 9, for large m. From the point of view of the Wilson theory of critical phenomena (cf. Refs. 23, 24, 33, and 45 below) the perturbation which reduces the symmetry group from $O(2m)$, for the classical system, to $U(m)$, for the quantum-mechanical system, is believed to be "irrelevant" near the critical point. The first term in the "free-energy functional" which distinguishes between the two cases is a term fourth order in the order parameter and second order in wave vector, which is irrelevant at least for d near four dimensions. The groups $O(2m)$ and $U(m)$ are identical for $m=1$.

[15]See for example, I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity* (Benjamin, New York, 1965).

[16]The group $SU(m)$ is the group of unitary, unimodular (i.e., unit determinant) transformations on an m-dimensional complex vector space. Other groups used in this paper are the group $U(m)$ of *all* unitary transformations in an m-dimensional space and the group $O(n)$ of rotations (including inversions) in a real, n-dimensional space. See R. Gilmore, *Lie Groups and Some of Their Applications* (Wiley, New York, 1974).

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[30]We assume in this paper, that the transition between the superfluid and normal states is second order. Of course, this need not always be so. For example, helium, at low pressures, has a *first-order* transition from the superfluid liquid directly to a normal vapor state.

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[32]At temperatures below T_c, the equilibrium states corresponding to some values of the parameters N_α, $N_{\alpha\beta}$, and $M_{\alpha\beta}$ will be inhomogeneous. For infinitesimal deviations from "thermal equilibrium," however, which will be the regime of principal interest to us, the spontaneous development of inhomogeneities should not occur in a finite time.

[33]See for example M. E. Fisher, in *Collective Properties of Physical Systems, Proceedings of the Twenty-Fourth Nobel Symposium*, edited by B. Lundqvist and S. Lundqvist (Academic, New York, 1973), p. 16.

[34]The occurrence of the same factor $\chi_1(k)$ in three places in (2.17) can be most easily understood for the case $m=2$, where the three occurrences simply reflect the equality of the susceptibilities for σ_x, σ_y, and σ_z.

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[37]More precisely, let us consider the matrix $S(\vec{x})$, whose matrix elements are given by

$$S_{\alpha\beta}=\rho_\alpha-(1/m)\rho, \quad \text{for } \alpha=\beta,$$

$$S_{\alpha\beta}=\rho_{\alpha\beta}+i\mu_{\alpha\beta}, \quad \text{for } \alpha\neq\beta.$$

For small deviations from local equilibrium, $S(\vec{x})$ will have one eigenvalue approximately equal to $2\sigma_0(m-1)/m$, and $m-1$ eigenvalues approximately equal to $-2\sigma_0/m$. *The unit vector* $\hat{u}(\vec{x})$ *will be an eigenvector of* $S(\vec{x})$, *belonging to the first eigenvalue.*

[38]The coefficient $\rho_s' m_b/\hbar^2$ in the present paper differs by a factor of 4 from the stiffness constant ρ_s for the Heisenberg ferromagnet, defined in Ref. 17. Note that the angle of rotation of $\vec{\sigma}$ from the z axis is *twice* the deviation of $\hat{u}(x)$ from $\hat{u}_0$.

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