Phonon viscosity and wide-angle phonon scattering in superfluid helium*

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(Received 26 August 1974)

Maris's numerical results for the viscosity of phonons with anomalous dispersion in superfluid He^4 are obtained by a simplified method which involves variational calculation of the eigenvalues of the completely continuous part of the collision integral operator in the phonon Boltzmann equation. We also obtain an analytic expression relating a sequence of wide-angle relaxation times to the phonon frequency anomaly.

I. INTRODUCTION

In superfluid helium 4 at saturated vapor pressure, there is increasing evidence that the dispersion relation for the phonon excitation spectrum is anomalous at long wavelengths; the phonon frequency increases slightly faster than linearly in the phonon wave vector.[1] Such an anomalous dispersion influences the interactions between phonons. In particular, the dominant phonon scattering process at low temperatures is the small-angle three-phonon collision, and wide-angle scattering is a slow process resulting from many back-to-back small-angle collisions. A central problem in discussing the hydrodynamics of the phonon system is to correctly identify and treat the long relaxation times characterizing wide-angle scattering.

In a recent series of papers,[2] Maris has calculated a number of transport properties of the phonons in superfluid helium, under the assumption of anomalous phonon dispersion. His results for the viscosity and for the first-sound velocity and attenuation are in striking agreement with experiment. Fundamental to Maris's method is a purely numerical solution of the phonon Boltzmann equation. In this note we shall show that Maris's result for the phonon viscosity may be obtained to within 10% by a very much simpler calculation, whose central result is a formula explicitly relating a sequence of wide-angle relaxation times to the phonon frequency anomaly.

We begin by consolidating in Sec. II some known features of the structure of the collision integral operator in the phonon Boltzmann equation, to obtain a compact formal representation of the collision integral and the phonon viscosity. In Sec. III a variational estimate is made of the low-lying eigenvalues of the completely continuous part of the collision integral. These eigenvalues form a sequence of wide-angle relaxation rates, whose temperature dependence we explicitly relate to the wave-vector dependence of the phonon-frequency anomaly. The wide-angle relaxation rates are measurable through several different experimental quantities, including the phonon viscosity (Sec. IV) and the velocity and attenuation of heat pulses.[3]

II. FORMAL EXPRESSION FOR THE COLLISION INTEGRAL

In this section we shall discuss the structure of the linearized three-phonon collision integral in the phonon Boltzmann equation. We rely heavily on the work of Buot,[4] Cercignani,[5] and Maris,[2] whose notation we follow. Our goal is to establish a basis for a variational treatment of the eigenvalue spectrum of the collision integral, and certain formal complications arise because this spectrum is continuous.

In the absence of superfluid motion, the Boltzmann equation describing the relaxation of a nonequilibrium distribution of phonons is

$$\left(\frac{\partial}{\partial t}+\vec{v}_{\vec{q}}\cdot\vec{\nabla}\right)N(\vec{q}\,;\vec{r}\,t)=C(N)\ . \tag{1}$$

$N(\vec{q},\vec{r}t)$ is the phonon distribution function, which we write in terms of the equilibrium Bose distribution $N^0(\vec{q})$ as

$$\begin{aligned}N(\vec{q},\vec{r}t)&=N^0(\vec{q})+[N^0(N^0+1)]^{1/2}\psi(\vec{q},\vec{r}t)\\&=N^0(\vec{q})+\frac{\psi(\vec{q},\vec{r}t)}{2\sinh(\frac{1}{2}\beta\hbar\omega_q)}\ ,\end{aligned}$$

where $\beta=(k_BT)^{-1}$, ω_q is the phonon frequency, and $\vec{v}_{\vec{q}}=\vec{\nabla}_{\vec{q}}\omega_q$. $C(N)$ is the collision integral operator.[2]

For spatially uniform disturbances of the form $\psi(\vec{q},t)=\psi(\vec{q})e^{-t/\tau}$, the Boltzmann equation (1) reduces to the eigenvalue problem

$$(1/\tau)\psi(\vec{q})=-\tilde{C}(\psi(\vec{q}))\ , \tag{2}$$

where $\tilde{C}$ is the symmetrized collision-integral operator defined by[2]

$$\tilde{C}(\psi(\vec{q}))=2\sinh(\tfrac{1}{2}\beta\hbar\omega_q)C(N)\ .$$

The eigenvalue spectrum of $-\tilde{C}$ thus gives the spectrum of relaxation rates $1/\tau$ of different disturbances from equilibrium. $\tilde{C}$ is rotationally invariant, and the solutions of (2) are of the form[2]

$$\psi_{\nu l m}(\vec{q})=\psi_{\nu l}(q)Y_l^m(\theta,\phi)\ ,$$

where Y_l^m is a spherical harmonic. Energy and momentum conservation in phonon collisions imply a fourfold degenerate discrete eigenvalue $1/\tau=0$, with eigenfunctions

$$\psi_{000} = [\omega_q / \sinh(\tfrac{1}{2}\beta\hbar\omega_q)] Y_0^0(\theta, \phi)$$

and

$$\psi_{01m} = [q / \sinh(\tfrac{1}{2}\beta\hbar\omega_q)] Y_1^m(\theta, \phi) .$$

Otherwise the spectrum of $-\tilde{C}$ is known[2,4,6] to be purely continuous, extending down to $1/\tau = 0$.

Maris numerically solved the eigenvalue equation (2) by replacing $\tilde{C}$ by a matrix operator on a finite mesh of points in $\vec{q}$ space.[2] The matrix has of course a purely point spectrum, but the *integral* operator has a continuous spectrum, and to preserve this feature we shall discuss a somewhat different eigenvalue problem than (2). $\tilde{C}$ has the structure[2,4]:

$$\tilde{C}(\psi(\vec{q})) = -\Gamma(q)\psi(\vec{q}) + K(\psi(\vec{q})) . \tag{3}$$

$\Gamma(q)$ is the single-phonon relaxation rate due to small-angle phonon scattering, and K is a symmetric integral operator. Buot[4] has shown that K is relatively compact with respect to Γ, i.e., K is completely continuous in a Hilbert space where the inner product contains a weight factor $\Gamma(q)$. Thus the eigenvalue equation

$$K(\varphi_{nlm}(\vec{q})) = (1 - \lambda_{nl})\Gamma(q)\varphi_{nlm}(\vec{q}) \tag{4}$$

possesses a purely point spectrum of eigenvalues (which we write as $1 - \lambda_{nl}$), and orthonormal eigenfunctions:

$$(\Gamma\varphi_{nlm}, \varphi_{n'l'm'}) \equiv \int d\vec{q}\, \Gamma(q)\varphi^*_{nlm}(\vec{q})\varphi_{n'l'm'}(\vec{q}) = \delta_{nn'}\delta_{ll'}\delta_{mm'} .$$

K then has the spectral representation

$$K(\psi(\vec{q})) = \Gamma(q) \sum_{nlm} (1 - \lambda_{nl})\varphi_{nlm}(\vec{q})(\Gamma\varphi_{nlm}, \psi) ,$$

and thus from Eq. (3)

$$\tilde{C}(\psi(\vec{q})) = -\Gamma(q)\left[\psi - \sum_{nlm} (1 - \lambda_{nl})\varphi_{nlm}(\Gamma\varphi_{nlm}, \psi)\right] . \tag{5}$$

In particular,

$$\tilde{C}(\varphi_{nlm}) = -\lambda_{nl}\Gamma(q)\varphi_{nlm}(\vec{q}) . \tag{6}$$

Let the index n denote the number of radial nodes of φ_{nlm}. Two nodeless eigenfunctions are apparent from energy and momentum conservation:

$$\varphi_{000}(\vec{q}) = [\omega_q / \sinh(\tfrac{1}{2}\beta\hbar\omega_q)] Y_0^0(\theta\phi) , \quad \lambda_{00} = 0 ,$$

$$\varphi_{01m}(\vec{q}) = [q / \sinh(\tfrac{1}{2}\beta\hbar\omega_q)] Y_1^m(\theta\phi) , \quad \lambda_{01} = 0 .$$

All other λ_{nl}'s are positive, as is apparent from (6) and the fact that $\tilde{C}$ is negative semidefinite. Equation (6) also shows that if $\lambda_{nl} \ll 1$ the *time scale* of relaxation of φ_{nlm} is slow. Maris's numerical work shows the presence of such slowly relaxing disturbances, which he interpreted roughly as follows: Consider a disturbance whose (nodeless) radial distribution of surplus phonons is nearly conserved in collisions, $\varphi_{0l}(q)\sinh(\tfrac{1}{2}\beta\hbar\omega_q) \approx \omega_q$ or q. The relaxation is then primarily an angular relaxation, involving the scattering of surplus phonons from directions where $Y_l^m(\theta\phi)$ is positive to directions where it is negative. If l is small enough so that Y_l^m varies slowly over the small angles between three interacting phonons, the angular relaxation must couple phonons over wide angles via many successive small-angle scatterings. Angular relaxation is then a slow diffusionlike process. Thus for each small enough l, there is one nodeless radial eigenfunction $\varphi_{0l}(q)$ with a small λ_{0l}.

III. APPROXIMATE WIDE-ANGLE RELAXATION RATES

Since the eigenvalue problem for K, Eq. (4), has a purely point spectrum, the simple variational principle

$$1 - \lambda_{0l} \geq (\phi_{0lm}, K(\phi_{0lm})) / (\Gamma\phi_{0lm}, \phi_{0lm})$$

holds for the smallest λ; $\phi_{0lm}(\vec{q})$ is a radially nodeless trial function approximating the eigenfunction φ_{0lm}. Dropping the subscript 0 and rearranging, using (3), we obtain

$$\lambda_l \leq (\phi_{lm}, -\tilde{C}(\phi_{lm})) / (\Gamma\phi_{lm}, \phi_{lm}) . \tag{7}$$

We seek to minimize the right-hand side. Using Maris's expression[2] for the collision integral and with

$$\phi_{lm}(\vec{q}) \equiv [\Phi_l(q) / \sinh(\tfrac{1}{2}\beta\hbar\omega_q)] Y_l^m(\theta\phi) , \tag{8}$$

the numerator of (7) is found to be

$$(\phi_{lm}, -\tilde{C}(\phi_{lm})) = \frac{\hbar c_0(u_0+1)^2}{16\pi\rho} \iint_0^\infty dq\, dq'\, q^2 q'^2 \int_{-1}^{1} d(\cos\theta)\, \delta(\omega_q - \omega_{q'} - \omega_{q''}) \frac{qq'q''}{\sinh(\tfrac{1}{2}\beta\hbar\omega_q)\sinh(\tfrac{1}{2}\beta\hbar\omega_{q'})\sinh(\tfrac{1}{2}\beta\hbar\omega_{q''})}$$
$$\times \{\Phi_l^2(q) + \Phi_l^2(q') + \Phi_l^2(q'') - 2\Phi_l(q)\Phi_l(q')P_l(\cos\theta) - 2\Phi_l(q)\Phi_l(q'')P_l(\cos\theta') + 2\Phi_l(q')\Phi_l(q'')P_l(\cos\theta'')\} . \tag{9}$$

Here c_0, u_0, and ρ are the zero-temperature sound velocity, Grüneisen constant, and density of liquid helium; $q'' = |\vec{q} - \vec{q}'|$; θ, θ', and θ'' are, respectively, the angles between $\vec{q}$ and $\vec{q}'$, $\vec{q}$ and $\vec{q} - \vec{q}'$, and $\vec{q}'$ and $\vec{q} - \vec{q}'$; the P_l's are Legendre polynomials; we have assumed that the radial function $\Phi_l(q)$ is real. We simplify (9) as follows:

(a) Since energy conservation restricts θ, θ', and θ'' to small angles, we expand the three Legendre polynomials in powers of 1 – cos to second order:

$$P_l(\cos\theta) \cong 1 - L^2(1-\cos\theta) + \tfrac{1}{4}L^2(L^2-1)(1-\cos\theta)^2 ,$$

$$L^2 \equiv \tfrac{1}{2}l(l+1) ,$$

and similarly for the other two.

(b) We write the phonon frequency as $\omega_q = c_0[q + \Delta(q)]$. $\Delta(q)$ is the anomalous part, and is small and positive for small q. An explicit form for $\Delta(q)$ need not yet be assumed. The energy conservation condition $\omega(q) = \omega(q') + \omega(|\vec{q} - \vec{q}'|)$ then gives, to lowest order,

$$1 - \cos\theta = [(q-q')/qq'][\Delta(q) - \Delta(q') - \Delta(q-q')] + O(\Delta^2) .$$

With the geometrical approximations

$$1 - \cos\theta' \cong [q'^2/(q-q')^2](1-\cos\theta) ,$$

$$1 - \cos\theta'' \cong [q^2/(q-q')^2](1-\cos\theta) ,$$

the curly bracket in (9) can then be expressed entirely in terms of Φ_l's and Δ's.

(c) We write the radial function $\Phi_l(q)$ as a series in powers of $\Delta(q)$ and try to minimize the curly bracket in (9). It may be verified that the choice

$$\Phi_l(q) = q - (L^2-1)\Delta(q) + O(L^2(L^2-1)\Delta^2) \qquad (10)$$

makes the curly bracket zero for $l = 0$, 1 and of order Δ^2 for $l \geq 2$, while the denominator in (7) is of zeroth order in Δ. To present results we rewrite (7) as

$$\lambda_l \leq (1/\tau_l)/(1/\tau_\parallel) ,$$
$$1/\tau_l \equiv (\phi_{lm}, -\tilde{C}(\phi_{lm}))/(\phi_{lm}, \phi_{lm}) , \qquad (11)$$
$$1/\tau_\parallel \equiv (\Gamma\phi_{lm}, \phi_{lm})/(\phi_{lm}, \phi_{lm}) .$$

Equations (8)–(10) then give (to order Δ^2):

$$\frac{1}{\tau_l} = \frac{15\hbar(u_0+1)^2}{8\pi^5\rho}\left(\frac{2k_B T}{\hbar c_0}\right)^{-5} L^2(L^2-1)$$
$$\times \iint dq\, dq' \frac{q^2 q'^2 (q-q')^2}{\sinh(\frac{1}{2}\beta\hbar c_0 q)\sinh(\frac{1}{2}\beta\hbar c_0 q')\sinh(\frac{1}{2}\beta\hbar c_0(q-q'))} \tfrac{1}{2}[\Delta(q) - \Delta(q') - \Delta(q-q')]^2 . \qquad (12)$$

The double integral is over the domain

$$q \geq q' \geq 0, \quad \Delta(q) - \Delta(q') - \Delta(q-q') \geq 0 ,$$

which follows from the energy – conservation δ function in (9).

The inner product $(\Gamma\phi_{lm}, \phi_{lm})$ is given by the right-hand side of (9) but without the last three terms involving Legendre polynomials in the curly bracket. With (8) and (10), we then obtain an expression for $1/\tau_\parallel$ which (to zeroth order in Δ) is identical to the right-hand side of (12) except for the replacement

$$L^2(L^2-1)\tfrac{1}{2}[\Delta(q) - \Delta(q') - \Delta(q-q')]^2 \rightarrow q^2 + q'^2 + (q-q')^2 .$$

The domain of integration is the same as for (12).

The central result of this work is Eq. (12), relating the phonon-frequency anomaly to a sequence of inverse lifetimes $1/\tau_l$. In Sec. IV we shall show that τ_2 is the viscosity lifetime. The lifetimes τ_l characterize the relaxation, by wide-angle scattering of phonons, of disturbances whose radial distributions of surplus phonons are nearly conserved in collisions and whose angular distributions vary slowly. Except for $l = 0$, 1, $1/\tau_l$ is not a discrete eigenvalue of the collision integral $-\tilde{C}$. Its interpretation is suggested by Eq. (6), along with (11): the continuous spectrum of $-\tilde{C}$ has a high density of eigenvalues not only near $1/\tau_\parallel$, the inverse lifetime of a thermal phonon due to small-angle scattering, but also near the smaller inverse lifetimes $1/\tau_l$. This is at least true for small enough l so that the approximations following Eq. (9) are valid.

This interpretation is borne out by comparison with Maris's numerical solution[2] of the eigenvalue equation for $\tilde{C}$. We have numerically evaluated the double integral in (12), and the similar integral in the expression for $1/\tau_\parallel$, at 0.25 °K for the dispersion relation

$$\omega_q = c_0\left[q + \gamma q^3 \frac{1-(q/q_A)^2}{1+(q/q_B)^2}\right] , \qquad (13)$$

with $\gamma = 1.11$ Å^2, $q_A = 0.5418$ Å^{-1}, $q_B = 0.3322$ Å^{-1} (this is Maris's dispersion relation D). Other parameters were chosen in agreement with Maris. Results for $1/\tau_l$ are shown in Fig. 1, along with the smallest eigenvalue for each l which Maris found for a finite matrix approximation to $\tilde{C}$. $1/\tau_2$ from (12) is 15% larger than Maris's lowest $l = 2$ eigenvalue. For larger l, the sequence (12) increasingly diverges from Maris's eigenvalues, due to the gradual failure of the approximations following

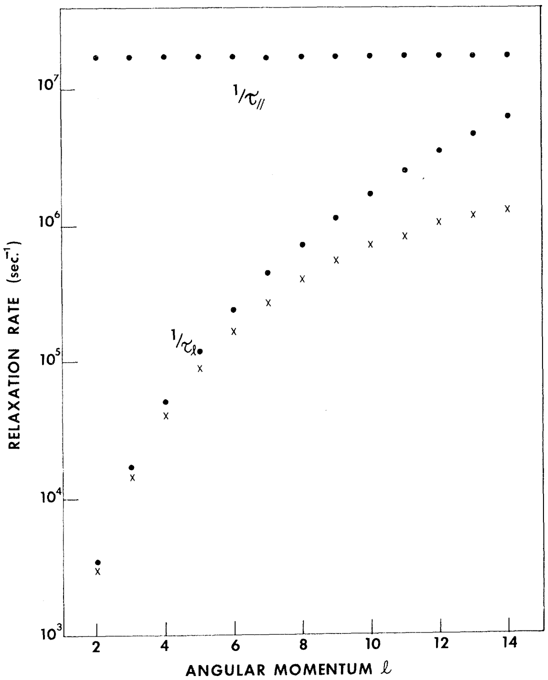

FIG. 1. Relaxation rates vs angular momentum. ● denote wide- and small-angle relaxation rates $1/\tau_l$ and $1/\tau_{\parallel}$ as discussed in Sec. III; $1/\tau_{\parallel}$ is independent of l in the approximations of Sec. III. × denotes the lowest eigenvalues for each l found by Maris (Ref. 2). All calculations are for Maris's dispersion relation D and 0.25 °K.

Eq. (9). The 15% difference between $1/\tau_2$'s at 0.25 °K seems to be about the *worst* possible case; one can show that (12) is asymptotically exact as $T\to 0$ °K, and in Sec. IV we shall find that the agreement between our $1/\tau_2$ and Maris's is also better at temperatures *above* 0.25 °K. This latter circumstance is probably due to the fact that $1/\tau_2$ is less strongly temperature dependent at higher temperatures, so that slight differences in temperature dependence between $1/\tau_2$'s affect their magnitudes less. However, our $1/\tau_2$ and Maris's lowest $l=2$ eigenvalue are not defined by the same eigenvalue problem, and it is an open question whether they should *in principle* be equal, even if our variational calculation could be done exactly.

IV. EVALUATION OF THE VISCOSITY LIFETIME

According to Maris,[2] the normal-fluid viscosity η of liquid helium in the temperature range below about 0.6 °K, where the phonon contribution dominates, is given approximately by

$$\eta=\tfrac{1}{15}c_v T\tau_2 \;, \tag{14}$$

with c_v the specific heat per unit volume. As discussed above, Maris's lowest eigenvalue and our $1/\tau_2$ given by (12) are not in principle the same quantity, but we show in the Appendix that Eq. (14) is also a valid approximation for the viscosity with our τ_2. Consequently, we have evaluated τ_2 from (12) as a function of temperature, and shall compare with Maris's calculation and with Whitworth's measurements[7] of the viscosity lifetime $15\eta/c_v T$. Two different models for the anomalous dispersion were investigated, Maris's dispersion relation D, Eq. (13), and the polynomial dispersion relation

$$\omega_q=c_0(q+\gamma q^3-\delta q^5)\;.$$

Figure 2 shows the results. Maris's dispersion relation D fits Whitworth's data well in his calculation, and it continues to give a good fit in our evaluation of τ_2, which differs from Maris's value by less than 10% over the entire temperature range shown. The polynomial dispersion relation fits with $\gamma=0.72$ Å^2, $\delta=3.7$ Å^4, as shown in Fig. 2, but unique values for γ and δ are not very reliably determined by a fit to Whitworth's data. In particular, δ enters the fit only in the combination $(\gamma/\delta)^{1/2}$, which acts like a cutoff wave vector controlling the temperature dependence of τ_2, and Whitworth's data cover too limited a temperature range for an unambiguous determination. Our parameters $\gamma=0.72$ Å^2, $(\gamma/\delta)^{1/2}=0.44$ Å^{-1} may be compared to $\gamma=0.62$ Å^2 determined from an analysis of specific heat data by Zasada and Pathria[8] (who set $\delta\equiv 0$ in the wave vector range of interest here), and the value $(\gamma/\delta)^{1/2}=0.56$ Å^{-1} estimated by Jäckle and Kehr.[9]

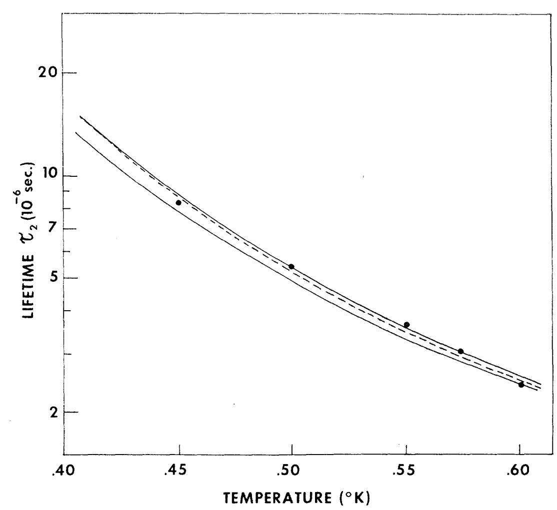

FIG. 2. Viscosity lifetime vs temperature. Points are the measurements of Whitworth (Ref. 7). Upper and lower solid curves are, respectively, Maris's calculation (Ref. 2) and the result of Eq. (12), both for Maris's dispersion relation D. Dashed curve is the result of Eq. (12) for the polynomial dispersion relation with parameters as in Sec. IV.

Finally, we wish to emphasize Maris's observation that the *temperature* dependence of the viscosity lifetime τ_2 probes the *wave vector* dependence of the frequency anomaly $\Delta(q)$. Equation (12) provides an explicit connection between the two, and is simple to evaluate numerically. Thus new measurements of the viscosity over a wider temperature range could readily be used to determine the form of the anomaly more precisely.

Note added in proof. A variational calculation of the phonon viscosity has been reported by Jäckle and Kehr (see footnote 10 of Ref. 9), who obtained parameters in close agreement with ours for the polynomial-dispersion-relation case. I apologize for having overlooked the previous work of Jäckle and Kehr.

APPENDIX

The phonon contribution to the normal-fluid first viscosity tensor may be found by solving the inhomogeneous equation

$$\overleftrightarrow{\chi}(\vec{q}) = -\tilde{C}(\overleftrightarrow{\psi}(\vec{q})) . \qquad \text{(A1)}$$

Here $\overleftrightarrow{\chi}$ is a traceless symmetric second-rank tensor with Cartesian components

$$\chi_{ij}(\vec{q}) = \frac{\hbar\vec{q}\cdot\vec{v}_{\vec{q}}}{4k_B T\sinh(\frac{1}{2}\beta\hbar\omega_q)}\left(\frac{q_i q_j - \frac{1}{3}\delta_{ij}q^2}{q^2}\right).$$

The viscosity tensor η_{ijkl} and the bulk viscosity η are then given by

$$\eta_{ijkl} = (2\pi)^{-3}4k_B T(\chi_{ij}, \psi_{kl}) ,$$
$$\eta = \eta_{xzxz} = \eta_{yzyz} . \qquad \text{(A2)}$$

We define $l=2$ spherical tensors whose components χ_m and ψ_m are linear combinations of the five independent components of $\overleftrightarrow{\chi}$ and $\overleftrightarrow{\psi}$; in particular for $m=1$,

$$\chi_1 = -\frac{1}{\sqrt{2}}(\chi_{xz} + i\chi_{yz}) = \sqrt{\frac{4\pi}{15}}\,\frac{\hbar\vec{q}\cdot\vec{v}_{\vec{q}}}{4k_B T\sinh(\frac{1}{2}\beta\hbar\omega_q)}Y_2^1(\theta\phi) . \qquad \text{(A3)}$$

Equation (A1) then becomes

$$\chi_m = -\tilde{C}(\psi_m) ;$$

with (5) above for $\tilde{C}$ the solution is

$$\psi_m = \frac{1}{\Gamma}\chi_m - \sum_n\left(1 - \frac{1}{\lambda_{n2}}\right)\varphi_{n2m}(\varphi_{n2m}, \chi_m) . \qquad \text{(A4)}$$

A short calculation shows that (A2) is equivalent to

$$\eta = (2\pi)^{-3}4k_B T(\chi_1, \psi_1)$$
$$= (2\pi)^{-3}4k_B T\left[\left(\frac{1}{\Gamma}\chi_1, \chi_1\right) - \sum_n\left(1 - \frac{1}{\lambda_{n2}}\right)|(\chi_1, \varphi_{n21})|^2\right] , \qquad \text{(A5)}$$

using (A4) for ψ_1. Equation (A5) is an exact formal expression for the phonon viscosity. If $\lambda_{02} \ll 1$, as discussed in Sec. II above, one term in the sum dominates the expression, and

$$\eta \cong (2\pi)^{-3}4k_B T|(\chi_1, \varphi_{021})|^2/\lambda_{02} . \qquad \text{(A6)}$$

We evaluate this expression using (A3) for χ_1 and the variational results (11) for λ_{02} and

$$\varphi_{021} \cong \phi_{21}/(\Gamma\phi_{21}, \phi_{21})^{1/2} ,$$

with ϕ_{21} given by (8) and (10). If we take $\omega_q \cong c_0 q$ and keep only terms of lowest (zeroth) order in the frequency anomaly in the inner products, (A6) gives

$$\eta \cong \tfrac{1}{15}c_v T\tau_2 ,$$

where c_v is the specific heat per unit volume,

$$c_v = (2\pi)^{-3}\int d\vec{q}\,(\hbar\omega_q)^2/4k_B T^2\sinh^2(\tfrac{1}{2}\beta\hbar\omega_q) ,$$

and τ_2 is given by Eq. (12) above.

*Supported by an Arizona State University Faculty Grant in Aid.

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